1) $y'' - y' - 6y = 6x^3 + 26\sin 2x$

Homo:

$r^2 - r - 6 = 0$

$(r-3)(r+2) = 0$

$r = 3, -2$

$y_H = C_1 e^{3x} + C_2 e^{-2x}$

Particular: Method of undet. coeff.

$y_{p1} = Ax^3 + Bx^2 + Cx^1 + D$

$y'_{p1} = 3Ax^2 + 2Bx + C$

$y''_{p1} = 6Ax + 2B$

$6Ax + 2B - 3Ax^2 - 2Bx - C - 6Ax^3 - 6Bx^2 - 6Cx - 6D$

$= 6x^3$

$(6A)x^3 + (-3A - 6B)x^2 + (-2B + 6A - 6C) + (-6D + 2B - C)$

$= 6x^3$

$A = -1$

$-3A - 6B = 0 \Rightarrow B = \frac{1}{2}$

$-2B + 6A - 6C = 0$

$-6 - 1 = 6C \quad C = -7/6$

$-6D + 2B - C = 0$

$6D = 1 + 7/6$

$6D = 13/6 \quad D = 13/36$

$y_{p1} = -x^3 + \frac{1}{2}x^2 - 7/6\,x + 13/36$

$$y_{p2} = A\sin(2x) + B\cos(2x)$$
$$y'_{p2} = 2A\cos(2x) - 2B\sin(2x)$$
$$y''_{p2} = -4A\sin(2x) - 4B\cos(2x)$$

$$-4A\sin(2x) - 4B\cos(2x) - 2A\cos(2x) + 2B\sin(2x)$$
$$-6A\sin(2x) - 6B\cos(2x) = 26\sin(2x)$$

$$(-4A + 2B - 6A) = 26$$
$$(-4B - 2A - 6B) = 0$$

let this be Y $\rightarrow$ $\begin{pmatrix} -10 & 2 \\ -2 & -10 \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 26 \\ 0 \end{pmatrix}$

$$\det(Y) = 100 + 4 = 104$$

$Y^{-1} = \frac{1}{104}\begin{pmatrix} -10 & -2 \\ 2 & -10 \end{pmatrix}$ by Cramer's rule

$$\begin{pmatrix} A \\ B \end{pmatrix} = Y^{-1}\begin{pmatrix} 26 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} -260/104 \\ 52/104 \end{pmatrix} = \begin{pmatrix} -5/2 \\ 1/2 \end{pmatrix}$$

So $y_{p2} = -\frac{5}{2}\sin(2x) + \frac{1}{2}\cos(2x)$

$$y(x) = C_1 e^{3x} + C_2 e^{-2x} + \left(-x^3 + \tfrac{1}{2}x^2 - 7/6\, x + 13/36\right) + \left(-\tfrac{5}{2}\sin(2x) + \tfrac{1}{2}\cos(2x)\right)$$

2) Green's function soln of

$y'' + y' - 2y = f(x)$ $\quad y(0)=0 \quad y'(1)=0$

$f(x) = 3\sin x$

looking for $G(x,z)$ s.t.

$$\frac{\partial^2 G}{\partial x^2} + \frac{\partial G}{\partial x} - 2G = \delta(x-z)$$

$G(0,z) = 0$

$\frac{\partial G}{\partial x}(1,z) = 0$

$\frac{\partial G_R}{\partial x} - \frac{\partial G_L}{\partial x} = \frac{1}{a_2(x)} = 1$

$G_L(x,z) = G_R(x,z)$

y_H:

$r^2 + r - 2 = 0$

$(r+2)(r-1) = 0$

$r = 1, -2$

$G(x,z) = Ae^x + Be^{-2x}$

Split at $x = z$:

$x < z$:

$G_L(x,z) = Ae^x + Be^{-2x}$ $\qquad$ $G_R(x,z) = Ce^x + De^{-2x}$

BC1: $G_L(0,z) = 0$

$0 = A + B \quad B = -A$

$G_L = A(e^x - e^{-2x})$

BC2: $\partial_x G(1,z) = 0$

$\partial_x G_R = Ce^x - 2De^{-2x}$

$0 = Ce - 2De^{-2}$

$2De^{-2} = Ce$

$D = \frac{Ce^3}{2}$ $\Rightarrow$ $G_R = Ce^x + \frac{Ce^{3-2x}}{2}$

Condition 1: at $x=z$, $G_L = G_R$

$$Ce^{z} + \frac{C}{2}e^{3-2z} = A(e^{z} - e^{-2z})$$

$$-A(e^{z} - e^{-2z}) + C(e^{z} + \tfrac{1}{2}e^{3-2z}) = 0$$

Condition 2: $\partial_x G_R - \partial_x G_L = 1$

$$\partial_x G_R = Ce^{z} + \frac{Ce^{3}}{2}(-2)e^{-2z}$$

$$\partial_x G_R = Ce^{z} - Ce^{3-2z}$$

$$\partial_x G_L = A(e^{z} + 2e^{-2z})$$

$$C(e^{z} - e^{3-2z}) - A(e^{z} + 2e^{-2z}) = 1$$

$$\begin{pmatrix} -e^{z} + e^{-2z} & e^{z} + \frac{1}{2}e^{3-2z} \\ -e^{z} - 2e^{-2z} & e^{z} - e^{3-2z} \end{pmatrix} \begin{pmatrix} A \\ C \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

matrix W

let

$$X = e^{z}, \quad Y = e^{-2z}, \quad V = e^{3}$$

$$W = \begin{pmatrix} -X + Y & X + \frac{1}{2}VY \\ -X - 2Y & X - VY \end{pmatrix}$$

take W^{-1} in MATLAB (code appended)

s.t. $\begin{pmatrix} A \\ C \end{pmatrix} = W^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$\Rightarrow A = \frac{-2e^{3z} + e^{3}}{3(2e^{z} + e^{3+z})}$$

$$C = \frac{-2e^{3z} - 1}{3(2e^{z} + e^{3+z})}$$

plugging back into MATLAB,

$$G = \frac{1}{3(2e^{z} + e^{3+z})} \begin{cases} (2e^{3z} + e^{3})(e^{-2x} - e^{x}) & x < z \\ -(2e^{x} + \frac{1}{2}e^{-2x+3})(e^{3z} - 1) & x > z \end{cases}$$

for $z \in (0, 1)$

$\Rightarrow y(x) =$

$$\frac{1}{3(2e^{z} + e^{3+z})} \int_0^x (2e^{3z} + e^{3})(e^{-2x} - e^{x}) f(z)\,dz$$

$$+ \frac{1}{3(2e^{z} + e^{3+z})} \int_x^1 (2e^{x} + \frac{1}{2}e^{-2x+3})(e^{3z} - 1) f(z)\,dz$$

If $f(x) = 3\sin x$,

$$y(x) = \frac{1}{2e^{z} + e^{3+z}} \left\{ \int_0^x (2e^{3z} + e^{3})(e^{-2x} - e^{x}) \sin(z)\,dz - \int_x^1 (2e^{x} + \frac{1}{2}e^{-2x+3})(e^{3z} - 1) \sin(z)\,dz \right\}$$

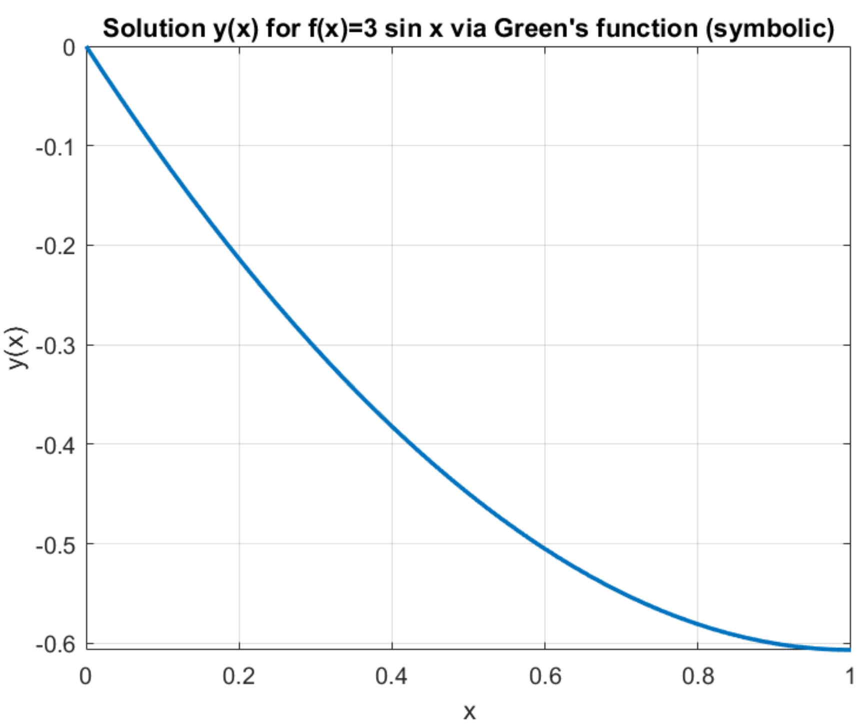
$\Rightarrow$ take x-terms out of integration bounds

$$y(x) = \frac{1}{2e^{2}+e^{3+2}} \left\{ \left(e^{-2x} - e^{x}\right) \int_x^1 \left(2e^{3z} + e^{3}\right) \sin z \, dz + \left(2e^{x} + \frac{1}{2} e^{-2x+3}\right) \int_0^x \left(e^{3z} - 1\right) \sin z \, dz \right\}$$

$\Rightarrow$ MATLAB to integrate

$\Rightarrow$

$$y(x) = \frac{\left[-9e^{2} - 3e^{9} - 6e^{3x} + 6e^{2x}\cos(x) + 18e^{2x}\sin x - 9e^{3x+2} + 3e^{2x+3}\cos x + 9e^{2x+3}\sin x \right]}{20e^{2x} + 10e^{2x+3}}$$



3) $f(x) = \dfrac{x^3/15 + x}{\frac{2x^2}{5} + 1}$ is padé for $\tanh(x)$ about $x=0$ with cubic num, quadratic den.

First take taylor series of $\tanh(x)|_{x_0}$

$y(x) = \tanh(x)$ to $M+N+1$

$\Rightarrow$ known result

$$T(x) = x - \frac{x^3}{3} + \frac{2x^5}{15} + O(x^7)$$

$$P_2^3 = \frac{a_0 + a_1 x + a_2 x^2 + a_3 x^3}{1 + b_1 x + b_2 x^2} = x - \frac{x^3}{3} + \frac{2x^5}{15}$$

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 = (1 + b_1 x + b_2 x^2)\tanh(x)$$

Expand RHS:

$$\tanh(x) = x - \frac{x^3}{3} + \frac{2x^5}{15}$$

$$b_1 x \tanh(x) = b_1\left(x^2 - \frac{x^4}{3} + \frac{2x^6}{15}\right)$$

$$b_2 x^2 \tanh(x) = b_2\left(x^3 - \frac{x^5}{3} + \frac{2x^7}{15}\right)$$

$\underline{x^n}$

$n=0$ $\quad a_0 = 0$

$n=1$ $\quad a_1 = 1$

$n=2$ $\quad a_2 - b_1 = 0$

$\quad a_2 = b_1$

$n=3$ $\quad a_3 + \frac{1}{3} - b_2 = 0$

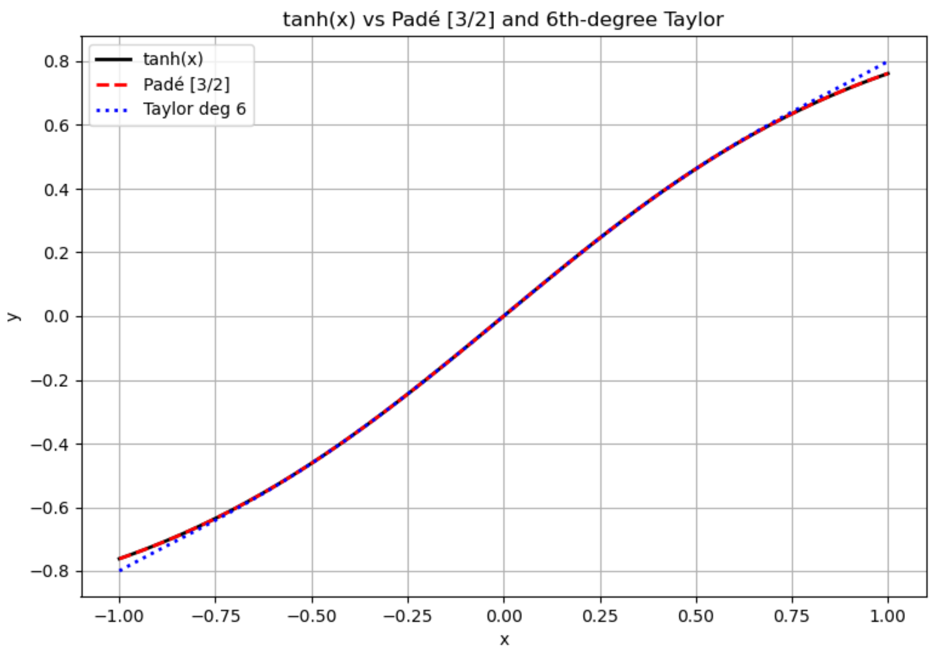
$\quad a_3 = b_2 - 1/3$

$n=4$ $\quad 0 = -\frac{1}{3} b \qquad b_1 = 0 \quad \Rightarrow a_2 = 0$

$n=5$ $\quad 0 = -\frac{1}{3} b_2 + \frac{2}{15}$ $\quad b_2 = 2/5$ $\quad a_3 = \frac{2}{5} - \frac{1}{3}$

$a_3 = 1/15$

$$\boxed{P_2^3 = \frac{x + 1/15\, x^3}{1 + 2/5\, x^2}}$$



4) Find first 4 nonzeros in power series soln around $x=1$
of $xy'' - y = 0 \quad y(1)=0 \quad y'(1)=1$

$$y(x) = \sum_{n=0}^{\infty} a_n t^n \quad y' = \sum_{n=1}^{\infty} n a_n t^{n-1} \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n t^{n-2}$$

Solving about $x=1 \Rightarrow x = 1+t$

shift $\Rightarrow$ $y = \sum_{n=0}^{\infty} a_n t^n \quad y' = \sum_{n=0}^{\infty} (n+1) a_{n+1} t^n$

$$y'' = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} t^n$$

$$(1+t) y'' - y = 0$$

$$\Rightarrow \sum ((1+t)(n+2)(n+1) a_{n+2} t^n - a_n t^n = 0$$

$$= \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} t^n + (n+2)(n+1) a_{n+2} t^{n+1} - a_n t^n = 0$$

shift this up 1 to match ks

$$= \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} t^n + \sum_{n=1}^{\infty} (n+1)(n) a_{n+1} t^n$$

$$- \sum_{n=0}^{\infty} a_n t^n = 0$$

$n=0$:

$$0 = 2a_2 + a_0$$

$$a_0 = 2a_2$$

$n \geq 1$: $\quad (n+2)(n+1) a_{n+2} + (n+1)(n) a_{n+1} - a_n = 0$

$$\Rightarrow \quad a_{n+2} = \frac{a_n - n(n+1) a_{n+1}}{(n+2)(n+1)}$$

we have $y(1) = 0 \quad y'(1) = 1$

and

$$y(x) = \sum_{n=0}^{\infty} a_n (x-1)^n = \sum_{n=0}^{\infty} a_n t^n$$

$$y(t) = a_0 + a_1 t + a_2 t^2 + \dots$$

$$y(1) = y((t=0)) = a_0 = 0 \quad \Rightarrow a_0 = 0$$

$$y'(t) = a_1 + 2a_2 t + \dots = 1 \quad \Rightarrow a_1 = 1$$

$$a_0 = 2a_2 \quad \Rightarrow a_2 = 0$$

$n=1$

$$a_3 = \frac{a_1 - \cancel{(2)\overset{0}{a_2}}}{6} = \frac{1}{6} = a_3$$

$n=2$

$$a_4 = \frac{a_2 - 6a_3}{12} = -\frac{1}{12} = a_4$$

$n=3$ $$a_5 = \frac{a_3 - 12a_4}{20} = \frac{7}{120}$$

$$y(x) = a_1 t + \underbrace{a_2 t^2}_{0} + a_3 t^3 + a_4 t^4 + a_5 t^5$$

$$= (x-1) + \frac{1}{6}(x-1)^3 - \frac{1}{12}(x-1)^4 + \frac{7}{120}(x-1)^5$$