Surface integral (Powers & Sen 2.29)

Compute

$$I \;=\; \iint_S \mathbf{F}\cdot\mathbf{n}\,dS, \qquad \mathbf{F} = (x^2, y^2, z^2),$$

over the surface $S:\; z = x - y$ with $(x,y) \in [0,1]\times[0,1]$.

Parametrization. Let $\mathbf{r}(x,y) = (x,\; y,\; x-y)$. Then

$$\mathbf{r}_x = (1,0,1), \qquad \mathbf{r}_y = (0,1,-1), \qquad \mathbf{r}_x \times \mathbf{r}_y = (-1,\,1,\,1).$$

Hence the vector area element is

$$d\mathbf{S} = (\mathbf{r}_x \times \mathbf{r}_y)\,dx\,dy = (-1,1,1)\,dx\,dy.$$

Field on the surface. On S, $z = x - y$, so

$$\mathbf{F}(\mathbf{r}(x,y)) = (x^2,\; y^2,\; (x-y)^2).$$

Integrand.

$$\mathbf{F}\cdot d\mathbf{S} = (x^2, y^2, (x-y)^2)\cdot(-1,1,1)\,dx\,dy = \big(-x^2 + y^2 + (x-y)^2\big)\,dx\,dy = \big(2y^2 - 2xy\big)\,dx\,dy.$$

Integral over the unit square.

$$\begin{aligned} I &= \int_0^1\!\int_0^1 \big(2y^2 - 2xy\big)\,dx\,dy = \int_0^1 \left[2y\left(y - \tfrac{1}{2}\right)\right]dy \\ &= \int_0^1 \big(2y^2 - y\big)\,dy = \left[\tfrac{2}{3}y^3 - \tfrac{1}{2}y^2\right]_0^1 = \tfrac{2}{3} - \tfrac{1}{2} = \boxed{\tfrac{1}{6}}. \end{aligned}$$