1) $f_0 = e^{-t^2/2}$ $f_1 = te^{-t^2/2}$ $f_2 = t^2 e^{-t^2/2}$ in L_2; $\langle f, g\rangle = \int_{-\infty}^{\infty} f(t)g(t)dt$

$$\int_{-\infty}^{\infty} e^{-t^2/2} dt = \sqrt{2\pi}$$

1. $g_0 = f_0$

$$\|g_0\|^2 = \langle f_0, f_0\rangle = \int_{-\infty}^{\infty} e^{-t^2} dt = \sqrt{\pi}$$

2. $g_1 = f_1 - \dfrac{\langle f_1, g_0\rangle}{\langle g_0, g_0\rangle} g_0$

$$\langle f_1, g_0\rangle = \int_{-\infty}^{\infty} te^{-t^2} dt$$

te^{-t^2} is an odd function, so

$= 0$

$g_1 = f_1$

$$\|g_1\|^2 = \langle f_1, f_1\rangle = \int_{-\infty}^{\infty} t^2 e^{-t^2} dt$$

$$\int t^2 e^{-t^2} dt = \sqrt{\pi}/2$$

3. $g_2 = f_2 - \dfrac{\langle f_2, g_0\rangle}{\langle g_0, g_0\rangle} g_0 - \dfrac{\langle f_2, g_1\rangle}{\langle g_1, g_1\rangle} g_1$

$\langle f_2, g_1\rangle = \int t^3 e^{-t^2} dt = 0$, odd function

$$\langle f_2, g_0\rangle = \int_{-\infty}^{\infty} t^2 e^{-t^2} dt = \sqrt{\pi}/2$$

$$\langle g_0, g_0\rangle = \sqrt{\pi}$$

$$g_2 = f_2 - \frac{\sqrt{\pi}/2}{\sqrt{\pi}} g_0 = t^2 e^{-t^2/2} - \frac{1}{2} e^{-t^2/2} = e^{-t^2/2}\left(t^2 - \frac{1}{2}\right)$$

$$\|g_2\|^2 = \langle g_2, g_2\rangle = \int \left(t^2 - \frac{1}{2}\right)^2 e^{-t^2} dt$$

1) cont $\|g_2\|^2 = \int t^4 e^{-t^2} dt - \int t^2 e^{-t^2} dt + \frac{1}{4}\int e^{-t^2} dt$

$$= \frac{3\sqrt{\pi}}{4} - \frac{\sqrt{\pi}}{2} + \frac{\sqrt{\pi}}{4} = \sqrt{\pi}/2$$

so an orthogonal set is

$$e^{-t^2/2}, \quad t e^{-t^2/2}, \quad (t^2 - \tfrac{1}{2}) e^{-t^2/2}$$

2) $x(t) = \sin(4t) \quad L_2[0,1]$

$\langle f(t), g(t) \rangle = \int_0^1 f(t) g(t) dt$ in this space

2.1) $\text{span}\{t, t^2\}$

$$\begin{pmatrix} \langle t,t \rangle & \langle t,t^2 \rangle \\ \langle t^2,t \rangle & \langle t^2,t^2 \rangle \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} \langle x,t \rangle \\ \langle x,t^2 \rangle \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} 1/3 & 1/4 \\ 1/4 & 1/5 \end{pmatrix}}_{A} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} \int_0^1 t\sin(4t)dt \\ \int_0^1 t^2\sin(4t)dt \end{pmatrix}$$

$$\int_0^1 t\sin(4t)dt = -\frac{1}{4}t\cos(4t) + \frac{1}{16}\sin(4t) \Big|_0^1$$

$$= -\frac{1}{4}\cos(4) + \frac{1}{16}\sin(4)$$

$$\int_0^1 t^2\sin(4t)dt = \left(\frac{2}{4^3} - t^2/4\right)\cos(4t) + \frac{2t}{4^2}\sin(4t) \Big|_0^1$$

$$= \left(\frac{2}{64} - \frac{1}{4}\right)\cos(4) + \frac{2}{16}\sin(4)$$

$$- \left(\frac{2}{4^3}\right)$$

$$= -7/32\cos 4 + \frac{1}{8}\sin(4) - 1/32$$

$$A = \begin{pmatrix} 1/3 & 1/4 \\ 1/4 & 1/5 \end{pmatrix} \quad \det(A) = \frac{1}{15} - 1/16 = 1/240$$

$$A^{-1} = 240 \begin{pmatrix} 1/5 & -1/4 \\ -1/4 & 1/3 \end{pmatrix}$$

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = 240 \begin{pmatrix} 1/5 & -1/4 \\ -1/4 & 1/3 \end{pmatrix} \begin{pmatrix} -\frac{1}{4}\cos 4 + \frac{1}{16}\sin 4 \\ -\frac{7}{32}\cos 4 + \frac{1}{8}\sin 4 - \frac{1}{32} \end{pmatrix}$$

$c_1 = \frac{9}{8}\cos 4 - \frac{9}{2}\sin 4 + \frac{15}{8} \approx 4.5448$

$c_2 = -\frac{5}{2}\cos 4 + \frac{25}{4}\sin 4 - \frac{5}{2} \approx -5.5959$

projection:

$$p(t) = 4.5448\,t - 5.5959\,t^2$$

2.2: $t,\ t^2,\ \tan t$

$\langle t, t\rangle = 1/3 \quad \langle t, t^2\rangle = 1/4 \quad \langle t^2, t^2\rangle = 1/5$

$\langle t, \tan t\rangle = \int_0^1 t\tan(t)\,dt \quad \langle t^2, \tan(t)\rangle = \int_0^1 t^2\tan(t)\,dt$

$\langle \tan(t), \tan(t)\rangle = \int_0^1 \tan^2(t)\,dt = \tan(1) - 1$

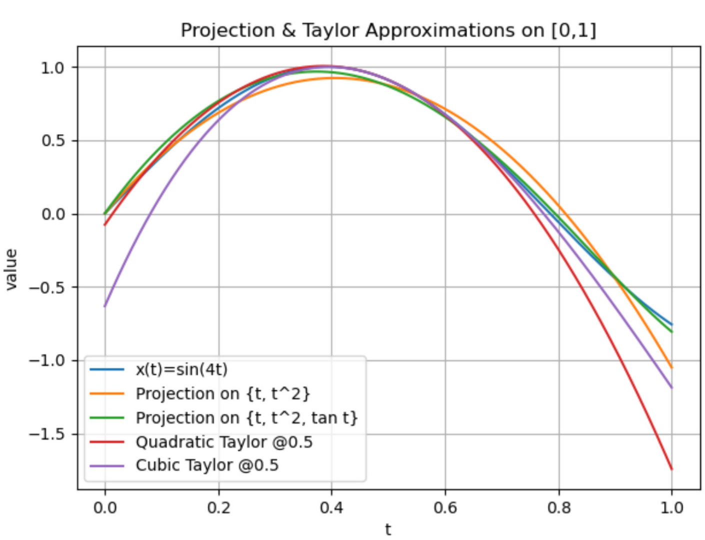
: harder integrals that have to be solved with a fourier series

↳ also, taking inverse of 3x3 will be worse, meriting a move to solving using python

→ See appended code:

$$p(t) = c_1 t + c_2 t^2 + c_3 \tan(t)$$

$$p(t) = 2.489\,t - 7.73\,t^2 + 2.845\tan(t)$$



3) $Ly = y'' = \lambda y \quad y(0) = y(1) = 0$ on $[0,1]$

3 sign cases:

1. $\lambda = 0$

$y'' = 0 \quad y' = A \quad y = Ax + B$

$0 = B$

$0 = A$ } trivial solution

$y = 0$, no eigenfunction

2. $\lambda > 0$

$\lambda = \alpha^2$

$y = Ae^{\alpha x} + Be^{-\alpha x}$

$0 = A + B \quad A = -B$

$0 = Ae^{\alpha} - Ae^{-\alpha} \Rightarrow A = 0, B = 0$

Again $y = 0$, no eigenfunction

3. $\lambda < 0$

$\lambda = -k^2, \; k > 0,$

$y = A\sin kx + B\cos kx$

$0 = B$

$0 = A\sin k$ either $A = 0$ or $\sin k = 0$ ⇒ trivial

$\sin k = 0$

$k = 0, \pi, 2\pi \ldots$

$k = n\pi \quad n = 1, 2, \ldots$

$\boxed{\lambda_n = -(n\pi)^2 \quad y_n(x) = \sin(n\pi x) \quad n = 1, 2, \ldots}$

↳ eigenvalues ↳ eigenfunctions

4) $LT := -k\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{\partial^2 T}{\partial z^2}\right) = 0$

$\Rightarrow -kT_r(R,z) = \beta T(R,z)$

Galerkin:

$\hat{T}(r,z) = T_1(z)\phi_1(r) + T_2(z)\phi_2(r)$ $\quad \phi_1 = 1$

$\langle u, v\rangle = \int_0^R ruv\,dr$ $\quad \phi_2 = 2\frac{r^2}{R^2} - 1$

$\psi_n = \phi_m (m=1,2)$

$\langle \phi_m, L\hat{T}\rangle = 0 \quad m=1,2$ (enforce this condition) $*$

T_1, T_2 dependent on just z:

$\hat{T}_r = T_2(z)\phi_2'(r) \qquad \hat{T}_{zz} = T_1''(z)\phi_1(r) + T_2''(z)\phi_2(r)$

$\langle \phi_m, -k\frac{1}{r}\partial_r(r\hat{T}_r)\rangle = -k\int_0^R \phi_m \partial_r(r\hat{T}_r)\,dr$

IBP:

$= -k\left[\phi_m r\hat{T}_r\right]_0^R + k\int_0^R \phi_m' r\hat{T}_r\,dr$

At $r=R$, we have $-k\hat{T}_r = \beta\hat{T}$ and $\phi_1(R) = \phi_2(R) \Rightarrow$ from $*$

$\hat{T}(R,z) = T_1 + T_2$

$\Rightarrow \quad -k\left[\phi_m r\hat{T}_r\right]_0^R = \beta R\phi_m(R)(T_1+T_2)$

$= \beta R(T_1+T_2)$

$\hat{T}_r = T_2\phi_2' \quad \Rightarrow$

$k\int_0^R \phi_m' r\hat{T}_r\,dr = kT_2\int_0^R r\phi_m'\phi_2'\,dr$

From $\hat{T}_{zz}$, we have

$\langle \phi_m, -k\hat{T}_{zz}\rangle = -kT_1''\int_0^R r\phi_m\phi_1\,dr - kT_2''\int_0^R r\phi_m\phi_2\,dr$

4) cont. $M_{11} = \int_0^R r\phi_1\phi_1 dr = R^2/2$

$M_{12} = M_{21} = \int_0^R r\phi_1\phi_2 dr = 0$ (orthogonal ϕ_1, ϕ_2)

$M_{22} = \int_0^R r\phi_2^2 dr = R^2/6$

$A_1 = \int_0^R r\phi'_1\phi'_2 dr = 0$

$A_2 = \int_0^R r\phi'_2\phi'_2 dr = \int_0^R r\left(\frac{4r}{R^2}\right)^2 dr = 4$

Galerkin ODE's:

$m=1$

$\beta R(T_1+T_2) + kT_2A_1 - kT_1''M_{11} - kT_2''M_{12}$

$\Rightarrow -\frac{kR^2}{2}T_1'' + \beta R(T_1+T_2) = 0$

$m=2$

$\beta R(T_1+T_2) + kT_2A_2 - kT_1''M_{21} - kT_2''M_{22} = 0$

$-\frac{kR^2}{6}T_2'' + 4kT_2 + \beta R(T_1+T_2) = 0$