1) Find first 3 terms of p.s. to

$F = y''(x) + \varepsilon y(x)^2 = 0 \qquad y(0)=0,\ y(1)=1$

y_0: $\varepsilon = 0$

$$y''(x) = 0$$

$$\frac{dy}{dx} = c_1$$

$y_0 = c_1 x + c_2$; Apply BC's:

$y_0(0) = 0 = c_2 \qquad c_2 = 0$

$y_0(1) = 1 = c_1 + c_2 \quad c_1 = 1$

$y_0 = x$

$\times$ all $y_k(0) = y_k(1)$ for $k \geq 1$

$$y = y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + O(\varepsilon^3)$$

$$y'' = \underset{0}{y_0''} + \varepsilon y_1'' + \varepsilon^2 y_2'' + O(\varepsilon^3)$$

$$\varepsilon y^2 = \varepsilon^5 y_2^2 + 2\varepsilon^4 y_1 y_2 + 2\varepsilon^3 y_2 y_0 + \varepsilon^3 y_1^2 + 2\varepsilon^2 y_1 y_0 + \varepsilon y_0^2$$

$$y'' = \varepsilon y_1'' + \varepsilon^2 y_2'' + O(\varepsilon^3)$$

$$\varepsilon y_0^2 + \varepsilon^2 (y_1'' + 2y_0 y_1) + \varepsilon^3 (y_1^2 + 2y_0 y_2 + y_2'') + \ldots$$

$O(1)$: $y_0 = x$

$O(\varepsilon)$: $\quad y_1'' + y_0^2 = 0 \qquad$ B.C.'s: $y_1(0) = y_1(1) = 0$

$$\frac{d^2 y_1}{dx^2} = -x^2$$

$$\int d^2 y_1 = -\int x^2 dx^2$$

$$\int dy_1 = \int -\left(\frac{1}{3}x^3 + c\right) dx$$

$$y_1 = -\frac{1}{12}x^4 + c_1 x + c_2$$

$y_1(0) = c_2 = 0$

$y_1(1) = -\frac{1}{12} + c_1 = 0$

$c_1 = \frac{1}{12}$

$$y_1 = \frac{x - x^4}{12}$$

$O(\varepsilon^2)$: $y_2'' + 2y_0y_1 = 0$

$$\frac{d^2y_2}{dx^2} + \frac{x^2-x^5}{6} = 0$$

$$\frac{d^2y_2}{dx^2} = \frac{x^5-x^2}{6} = \frac{1}{6}(x^5-x^2)$$

$$d^2y_2 = \int \frac{1}{6}(x^5-x^2)\,dx^2$$

$$\int dy_2 = \int \left(\frac{1}{36}x^6 - \frac{1}{18}x^3 + C_1\right)dx$$

$$y_2 = \frac{1}{252}x^7 - \frac{1}{72}x^4 + C_1x + C_2$$

$y_2(0) = y_2(1) = 0$

$$0 = C_2$$

$$0 = \frac{1}{252} - \frac{1}{72} + C_1$$

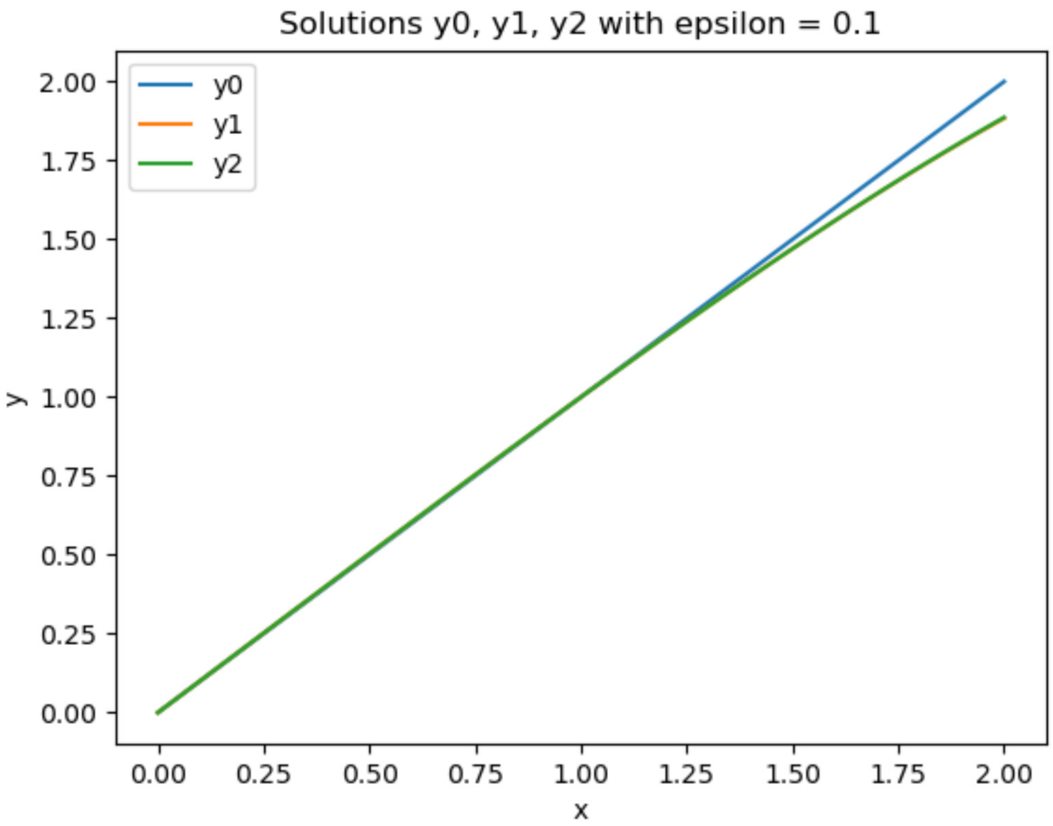
$$C_1 = 5/504$$

$$y_2 = \frac{1}{252}x^7 - \frac{1}{72}x^4 + \frac{5}{504}x$$

So:
1st 3 terms are

$$y = \underbrace{x}_{1^{st}} + \underbrace{\varepsilon\left(\frac{x-x^4}{12}\right)}_{2^{nd}} + \underbrace{\varepsilon^2\left(\frac{1}{252}x^7 - \frac{1}{72}x^4 + \frac{5}{504}x\right)}_{3^{rd}}$$

Plots:


Solutions y0, y1, y2 with epsilon = 0.1
y0
y1
y2
y
x
2.00
1.75
1.50
1.25
1.00
0.75
0.50
0.25
0.00
0.00
0.25
0.50
0.75
1.00
1.25
1.50
1.75
2.00

2) $\varepsilon y'' - y' = 1 \qquad y(0)=0 \quad y(1)=0$

$\Rightarrow$ Singular problem because $\varepsilon = 0$ reduces the order, causing us to lose a solution

Set $\varepsilon = 0$: (Outer soln)

$$-y_0' = 1 \qquad\qquad y_0(0) = 0$$

$$\frac{dy_0}{dx} = 1 \qquad\qquad \Rightarrow 0 = C$$

$$dy_0 = -1 \qquad\qquad y_0(x) = -x$$

$$y_0 = -x + C \qquad\qquad y_0(1) = -1 \neq 0 \Rightarrow \text{indicates existence of BL near } x = 1$$

Inner Soln:

define $X = \dfrac{x-1}{\varepsilon}$ as rescale parameter

s.t. $x = 1 + \varepsilon X$

$$\frac{d}{dx} = \frac{1}{\varepsilon}\frac{d}{dX} \qquad \frac{d^2}{dx^2} = \frac{1}{\varepsilon^2}\frac{d^2}{dX^2}$$

so

$$\varepsilon y'' - y' = 1$$

$$\Rightarrow \frac{\varepsilon}{\varepsilon^2}\frac{d^2y}{dX^2} - \frac{1}{\varepsilon}\frac{dy}{dX} = 1$$

$$\frac{1}{\varepsilon}\frac{d^2y}{dX^2} - \frac{1}{\varepsilon}\frac{dy}{dX} = 1$$

$$y'' - y' = \varepsilon$$

Set $\varepsilon = 0$

$$\frac{d^2y}{dX^2} - \frac{dy}{dX} = 0 \qquad r^2 - r = 0$$

$$\Rightarrow y(X) = C_1 + C_2 e^{x}$$

2) cont $\quad$ B.C.'s: $y(\bar{X}=0) = y(x=1) = 0$

$$y(0) = c_1 + c_2 e^0 = 0$$

$$c_2 + c_1 = 0$$

$$y(\bar{X}) = c_2\left(e^{\bar{X}} - 1\right)$$

As $\bar{X} \to -\infty$, we move from B.L back to domain

$$x = 1, \quad y(\bar{X} \to -\infty) = -c_2$$

$$y_0(x=1) = y(\bar{X} \to -\infty)$$

$$-1 = -c_2 \qquad c_2 = 1$$

$$\Rightarrow y_0(\bar{X}) = e^{\bar{X}} - 1$$

$$y(x) = e^{\frac{x-1}{\varepsilon}} - 1$$

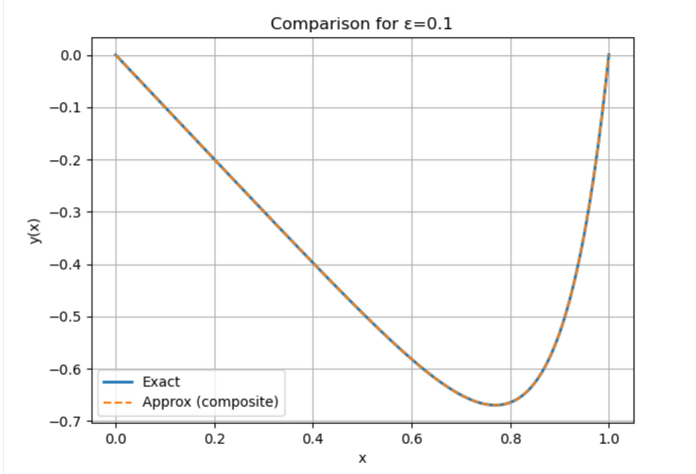
Composite soln:

$$y_c(x) = \underbrace{y_{outer}}_{-x} + \underbrace{y_{inner}}_{e^{\frac{x-1}{\varepsilon}} - 1} - \underbrace{y_{overlap}}_{\lim_{x \to 1^-} = -1}$$

$$y_c(x) = -x + e^{\frac{x-1}{\varepsilon}} - 1 + 1$$

$$y_0(x) = -x + e^{\frac{x-1}{\varepsilon}} + O(\varepsilon^2)$$

Plots with $\varepsilon = 0.1$:



3) $\ddot{x} + \sin x = 0 \qquad x(0) = \varepsilon \quad \dot{x}(0) = 0$

let $x = \varepsilon y$ s.t.

$$y = x/\varepsilon$$

$$y(0) = x(0)/\varepsilon = 1 \quad \checkmark$$

$$\dot{y}(0) = \dot{x}(0)/\varepsilon = 0 \quad \checkmark$$

$$\Rightarrow \quad \dot{x} = \varepsilon \dot{y} \quad \ddot{x} = \varepsilon \ddot{y}$$

$$\varepsilon \ddot{y} + \sin(\varepsilon y) = 0$$

Expand sin as taylor series:

$$\sin(\varepsilon y) = \varepsilon y - \frac{1}{6}\varepsilon^3 y^3 = 0$$

$$\Rightarrow \ddot{y} + y - \frac{1}{6}\varepsilon^2 y^3 = 0$$

Strained coordinates: $\qquad y_n(\tau) = y_n\left(\frac{t}{\sum_{n=0} c_n \varepsilon^n}\right)$

$$t = \tau \sum_{n=0}^{\rho} c_n \varepsilon^n$$

$$\dot{y} = \left(1 - c_1\varepsilon + (c_1^2 - c_2^2)\varepsilon^2\right)\frac{dy}{d\tau}$$

$$\ddot{y} = \left(1 - 2c_1\varepsilon + (3c_1^2 - 2c_2)\varepsilon^2\right)\frac{d^2y}{d\tau^2}$$

$\tau = \frac{t}{\sum_{n=0}^{\rho} c_n \varepsilon^n}$ so $\tau(t=0) = 0$ so IC's are same

$$\Rightarrow \left(1 - 2c_1\varepsilon + (3c_1^2 - 2c_2)\varepsilon^2\right)\left(\frac{\partial^2 y_0}{\partial \tau^2} + \varepsilon\frac{\partial^2 y_1}{\partial \tau^2} + \varepsilon^2\frac{\partial^2 y_2}{\partial \tau^2}\right) + \left(y_0 + \varepsilon y_1 + \varepsilon^2 y_2\right)$$

$$- \frac{1}{6}\varepsilon^2\left(y_0 + \varepsilon y_1 + \dots\right)^3 = 0$$

Group $O(1)$ terms:

$$\frac{\partial^2 y_0}{\partial \tau^2} + y_0 = 0$$

$$r^2 + 1 = 0$$

$$\Rightarrow y_0(\tau) = A\cos(\tau) + B\sin(\tau)$$

$$y_0(0) = A = 1$$

$$\dot{y}_0(0) = 0 = B$$

$$y_0(\tau) = \cos(\tau)$$

$O(\varepsilon)$:

$$0 = -2c_1 \frac{\partial^2 y_0}{\partial \tau^2}$$

$\frac{\partial^2 y_0}{\partial \tau^2}$ is of form $\cos\tau$, so we have to set $c_1 = 0$ to kill resonant $\tau\cos\tau$ term.

$O(\varepsilon^2)$:

$$\frac{\partial^2 y_2}{\partial \tau^2} + y_2 = \frac{1}{6} y_0^3 - 2c_2 \ddot{y}_0 \quad (c_1 \text{ term died})$$

$$\frac{\partial^2 y_2}{\partial \tau^2} + y_2 = \frac{1}{6} (\cos\tau)^3 - 2c_2\cos\tau$$

expand $(\cos\tau)^3$

$$= \frac{3}{4}\cos\tau + \frac{1}{4}\cos(3\tau)$$

$$\frac{\partial^2 y_2}{\partial \tau^2} + y_2 = \left(\frac{1}{8} - 2c_2\right)\cos\tau + \frac{1}{24}\cos(3\tau)$$

kill c_2 term to prevent resonance

$$2c_2 = \frac{1}{8}$$

$$c_2 = 1/16$$

$$\frac{\partial^2 y_2}{\partial \xi^2} + y_2 = \frac{1}{24}\cos(3\xi)$$

y_H: $r^2 + 1 = 0$

$$y_{2,H} = A\cos\xi + B\sin\xi$$

y_P: $y_{2,P} = A\cos(3\xi) + B\sin(3\xi)$

$$y'_{2,P} = -3A\sin(3\xi) + 3B\cos(3\xi)$$

$$\ddot{y}_{2,P} = -9A\cos(3\xi) - 9B\sin(3\xi)$$

$\Rightarrow$ $-8A\cos(3\xi) - 8B\sin(3\xi) = \frac{1}{24}\cos(3\xi)$

$$B = 0$$

$$-8A = 1/24 \quad A = \frac{-1}{192}$$

$\Rightarrow$ $y_2 = A\cos\xi + B\sin\xi - \frac{1}{192}\cos(3\xi)$

IC's: $y_2(0) = y_0(0) + \varepsilon^2 y_2(0) = 1$

$$\Rightarrow \quad 1 + \varepsilon^2 y_2(0) = 1$$

$$y_2(0) = 0$$

$$y'_2(0) = y'_0(0) + \varepsilon^2 y'_2(0) = 0$$

$$0 + \varepsilon^2 y'_2(0) = 0$$

$$y'_2(0) = 0$$

$\Rightarrow$ $y_2(0) = A - \frac{1}{192} = 0$

$$A = 1/192$$

$$y'_2(0) = B = 0$$

$$y_2(\xi) = \frac{1}{192}(\cos\xi - \cos 3\xi)$$

$$\Rightarrow y(\tau) = \cos\tau + \frac{\varepsilon^2}{192}\left(\cos\tau - \cos 3\tau\right)$$

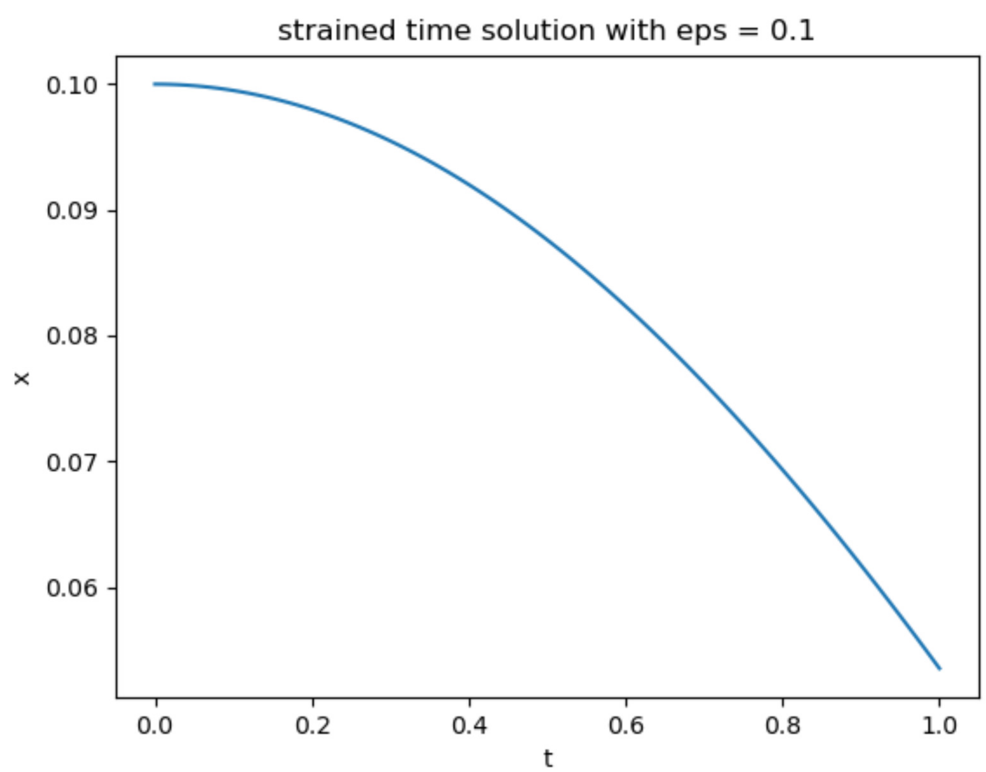
$$t = \tau\left(1 + \frac{1}{16}\varepsilon^2\right)$$

$$\Rightarrow y(t) = \cos\left(\left(1+\frac{1}{16}\varepsilon^2\right)t\right) + \frac{\varepsilon^2}{192}\left[\cos\left(\left(1+\frac{1}{16}\varepsilon^2\right)t\right) - \cos\left(\left(3+\frac{3}{16}\varepsilon^2\right)t\right)\right]$$

$$\Rightarrow x(t) = y(t)\cdot\varepsilon$$

$$= \varepsilon\left\{\cos\left(\left(1+\frac{1}{16}\varepsilon^2\right)t\right) + \frac{\varepsilon^2}{192}\left[\cos\left(\left(1+\frac{1}{16}\varepsilon^2\right)t\right) - \cos\left(\left(3+\frac{3}{16}\varepsilon^2\right)t\right)\right]\right\}$$

Plots:



4) WKBJ to

$$\varepsilon^2 y''(x) = (x^2+1)^2 y(x) \quad y(0)=0 \quad y(1)=1$$

$x_0 = 0$, $x \in (0,1)$ plot

$$y(x) = \frac{1}{f(x)^{1/4}}\left[\hat{C}_1 \exp\left(\frac{1}{\varepsilon}\int_{x_0}^{x}\sqrt{f(s)}\,ds\right) + \hat{C}_2 \exp\left(\frac{-1}{\varepsilon}\int_{x_0}^{x}\sqrt{f(s)}\,ds\right)\right]$$

$$f(x) = (x^2+1)^2$$

$$\Rightarrow S_0(x) = (x^2+1)$$

$$y(x) = \frac{1}{(x^2+1)^{1/2}}\left(\hat{C}_1 \exp\left(\frac{1}{\varepsilon}\int_0^x s^2+1\,ds\right) + \hat{C}_2 \exp\left(\frac{-1}{\varepsilon}\int_0^x s^2+1\,ds\right)\right)$$

$$y(x) = \frac{1}{(x^2+1)^{1/2}}\left(\hat{C}_1 e^{\frac{1}{\varepsilon}\left(\frac{1}{3}x^3+x\right)} + \hat{C}_2 e^{-\frac{1}{\varepsilon}\left(\frac{1}{3}x^3+x\right)}\right)$$

$$y(0) = 0 = 1\,(\hat{C}_1 + \hat{C}_2)$$

$$\hat{C}_1 = -\hat{C}_2$$

$$y(x) = \frac{1}{(x^2+1)^{1/2}}\left(\hat{C}_1\left(e^{1/\varepsilon\left(\frac{1}{3}x^3+x\right)} - e^{-1/\varepsilon\left(\frac{1}{3}x^3+x\right)}\right)\right)$$

$$y(1) = 1 = \frac{1}{\sqrt{2}}\left(\hat{C}_1\left(e^{4/3\varepsilon} - e^{-\frac{4}{3\varepsilon}}\right)\right)$$

$$\frac{\sqrt{2}}{e^{4/3\varepsilon} - e^{-4/3\varepsilon}} = \hat{C}_1$$

$$\Rightarrow \quad y(x) = \frac{\sqrt{2}}{(x^2+1)^{1/2}} \, \frac{1}{\left(e^{4/3\epsilon} - e^{-4/3\epsilon}\right)} \left(e^{\frac{1}{\epsilon}\left(\frac{1}{3}x^3 + x\right)} - e^{-\frac{1}{\epsilon}\left(\frac{1}{3}x^3+x\right)} \right)$$

or sinh form:

$$y(x) = \frac{\sqrt{2}}{(x^2+1)^{1/2}} \, \frac{\sinh\left(\frac{x + x^3/3}{\epsilon}\right)}{\sinh\left(\frac{4}{3\epsilon}\right)}$$

Plot it: