1) $A\underline{x} = \underline{b}$ $\quad A = \begin{pmatrix} 1 & 0 & 1 & 4 \\ 1 & 0 & 2 & -1 \\ 2 & 1 & 3 & -2 \end{pmatrix}$

$M = 3 \quad N = 4$

$\underline{b} = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}$

$M < N \Rightarrow$ Underconstrained

$A^T A = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 0 & 1 \\ 1 & 2 & 3 \\ 4 & -1 & -2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 & 4 \\ 1 & 0 & 2 & -1 \\ 2 & 1 & 3 & -2 \end{pmatrix}$

$A^T A = \begin{pmatrix} 6 & 2 & 9 & -1 \\ 2 & 1 & 3 & -2 \\ 9 & 3 & 14 & -4 \\ -1 & -2 & -4 & 21 \end{pmatrix}$

$A^T A + \lambda \underline{I} = \begin{pmatrix} 6+\lambda & 2 & & \\ \vdots & 1+\lambda & & \\ \vdots & & \ddots & 21+\lambda \end{pmatrix}$

Take inverse

$(A^T A + \lambda I)^{-1} (A^T \underline{b})$

$$= \frac{-1}{\lambda^3 + 42\lambda^2 + 430\lambda + 132} \begin{pmatrix} 3\lambda^2 + 44\lambda + 162 \\ 3\lambda^2 + 76\lambda + 482 \\ 5\lambda^2 + 52\lambda - 178 \\ 15\lambda^2 + 244\lambda + 62 \end{pmatrix}$$

Take $\lim_{\lambda \to 0}$ of each component

$$\underline{x} = \begin{pmatrix} -27/22 \\ -241/66 \\ 89/66 \\ 81/66 \end{pmatrix}$$

2) Same idea; take $y = a + bx + cx^2$

$$X = \begin{pmatrix} 1 \\ x_i \\ x_i^2 \end{pmatrix}$$ and solve $A = (X^T X)(X^T y)$

$\Rightarrow$ MATLAB code appended yields

$a = 2.2515 \quad b = 0.8638 \quad c = 3.4024$

For $y = a + bx + cx^2 + dx^3$

$\Rightarrow \quad a = 2.2515 \quad b = 0.8638 \quad c = 3.4024$

$d = 0.0025$

Cubic term is almost negligible and rest of solution barely changed

Average residual for quadratic $r1 \approx 1.126$

for cubic $r2 \approx 1.129$

Barely improves by adding d

3) $A = \begin{pmatrix} 0.1 & 0.5 \\ 0.01 & 0.02 \end{pmatrix}$

Actual:

$$\underline{I} - A = \begin{pmatrix} 0.9 & 0.5 \\ 0.01 & 0.98 \end{pmatrix}$$

$$\det(\underline{I} - A) = 0.877$$

$$(\underline{I} - A)^{-1} = \begin{pmatrix} 0.98 & -0.5 \\ -0.01 & 0.9 \end{pmatrix} \left(\frac{1}{0.877} \right)$$

Neumann Series:

1-term:

$$S_0 = \underline{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

2-term:

$$S_1 = \underline{I} + A = \begin{pmatrix} 1.1 & 0.5 \\ 0.01 & 1.02 \end{pmatrix}$$

3-term:

$$S_2 = \underline{I} + A + A^2 = \begin{pmatrix} 1.115 & 0.56 \\ 0.0112 & 1.0254 \end{pmatrix}$$

For $A = \begin{pmatrix} 0.1i & 0.5(1+i) \\ 0.01i & 0.02i \end{pmatrix}$,

1 term:

$$S_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

2 term:

$$S_1 = \begin{pmatrix} 1+0.1i & 0.5+0.5i \\ 0.01i & 1+0.02i \end{pmatrix}$$

3-term:

$$S_2 = \begin{pmatrix} 0.985+0.105i & 0.44+0.56i \\ -0.0012+0.01i & 0.9946+0.025i \end{pmatrix}$$

3) cont

$$(I-A) = \begin{pmatrix} 1-0.1i & 0.5+0.5i \\ 0.9i & 1-0.02i \end{pmatrix}$$

$$(I-A)^{-1} = \begin{pmatrix} 0.9846+0.1027i & 0.1297+0.992i \\ -0.0017+0.0098i & 0.9940+0.2418i \end{pmatrix}$$

In both cases of A, more terms encourage better agreement with exact solution, indicating convergence eventually

4) $y''(x) + k^2 y(x) = 0$ $\quad$ $y'(0) = 0$

$y(1) + \frac{1}{2} y'(1) = 0$

$r^2 + k^2 = 0$

$r^2 = -k^2$

$r = \pm ik$

$y(x) = c_1 e^{ikx} + c_2 e^{-ikx}$

so

$y(x) = A\cos(kx) + B\sin(kx)$ by Euler's formula

$y'(x) = -Ak\sin(kx) + Bk\cos(kx)$

$y'(x=0) = 0 = Bk \quad \Rightarrow B = 0$

$y(x=1) = A\cos(k)$

$\frac{1}{2}y'(x=1) = -\frac{1}{2}Ak\sin(k)$

$A\cos(k) - \frac{1}{2}Ak\sin(k) = 0$

For nontriviality, $A \neq 0$

$\cos(k) = \frac{1}{2}k\sin(k)$

$\tan k = 2/k$

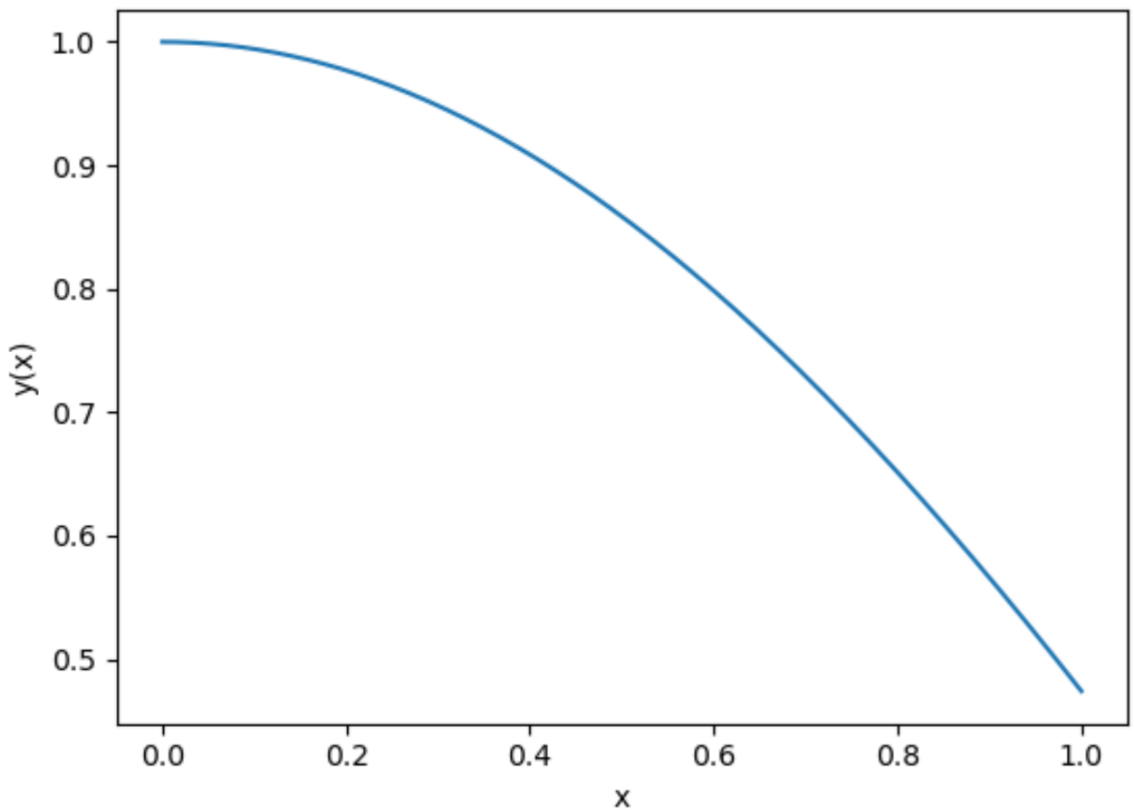
$\tan(k) - \frac{2}{k} = 0$

Code outputs $\underline{k = 1.076}$

A can be anything: let it be $A = 1$ for simplicity

$y(x) = \cos(1.076x)$

Plot:



5) $\dot{\underline{x}} = A\underline{x}(t)$

so $\underline{x}(t) = e^{tA}\underline{x}_0$

see appended code for work; plots shown here