Calculus

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Introduction

Calculus is the mathematical study of change. Traditionally it has two major branches, **differential calculus** and **integral calculus**. These two branches are related to each other by the **fundamental theorem of calculus**.

Functions

What is a Function?

A **function** is a relation between a set of **inputs**, called **variables** or **arguments**, and a set of permissible **outputs**. Each set of inputs must map to exactly one output. **(research: what about functions? containing a random part?)**

Functions can be defined by a mathematical formula or an algorithm. However that doesn't need to be the case. A fuction can also be defined by plain text or some mix these two. E.g. this **piecewise**-defined function:

$$f(x) = \begin{cases} the\, number\, of\, even\, digits\, in\, x & if\, x\, is\, a\, whole\, number \\ 0 & otherwise \end{cases}$$

Two functions are the same if they yield the same output for the same input. E.g. $f(x) = (1+x)^2$ and $g(x) = x^2 + 2x + 1$ are the same. However $f(x) = \frac{x^2}{x}$ and $g(x) = x$ are not the same, because 0 is in the domain of $f(x)$, but not in the domain of $g(x)$.

Domain & Codomain

A function has a **domain**, the set of input values for which the function is defined, and a **codomain**, a set of values into which all the outputs of the function fall. The set of input-output pairs is called the **graph**. E.g. $f(x) = \frac{1}{x}$ is a function with the domain of all real numbers excluding 0.

The domain of a complex, multi-part function is simply the domain of all its parts. E.g. for $f(x) = \sqrt{(}1-x) + \sqrt{(}1+x)$, the domain for the first part is $(-\infty, 1]$ and the domain for the second part is $[-1, \infty)$. Therefore the domain for the entire function is $[-1, 1]$. In this notation the round bracket means the value is excluded, while the squared bracket indicates that it is included.

Different Functions:

Square Root Function

The domain of the square root function, $f(x) = \sqrt{(}x)$, are all non-negative numbers, or $[0, \infty)$.

From this it follows that $\sqrt{(}x^2) = |x|$ and **NOT** $\sqrt{(}x^2) = x$. The square root of 0 is 0.

Limits

Definition

The **limit** is the output value that a function “approaches” as the input value gets closer to a certain value. E.g. the following means that the function $f(x)$ gets very close to 9 when it’s input value x gets very close to 3.

$$\lim_{x \to 3} f(x) = 9$$

Rules

The **algebraic limit theorem** states that the following rules hold true, provided the limits on the right sides of the equations exist and the denominator in the last equation is not zero.

$$\lim_{x \to p} (f(x) + g(x)) = \lim_{x \to p} f(x) + \lim_{x \to p} g(x)$$
$$\lim_{x \to p} (f(x) - g(x)) = \lim_{x \to p} f(x) - \lim_{x \to p} g(x)$$
$$\lim_{x \to p} (f(x) * g(x)) = \lim_{x \to p} f(x) * \lim_{x \to p} g(x)$$
$$\lim_{x \to p} \left(\frac{f(x)}{g(x)}\right) = \frac{\lim_{x \to p} f(x)}{\lim_{x \to p} g(x)}$$

Not Existing Limits

Some limits simply do not exist. E.g. the following function oscilliates so much that it doesn’t really approach any particular value.

$$\lim_{x \to 0} sin\left(\frac{1}{x}\right)$$

Squeeze Theorem

For some functions that are not defined at a certain point a limit exists nevertheless.

The **squeeze theorem** states that for every input value x not equal to a we have:

$$g(x) \leq f(x) \leq h(x)$$

and:

$$\lim_{x \to a} g(x) = \lim_{x \to a} h(x) = L$$

Then it follows that

$$\lim_{x \to a} f(x) = L$$

An example of this is $f(x) = \left(\frac{sin(x)}{x}\right)$ at the point 0. Because $\lim_{x \to 0} cos(x) = 1$, the application of the squeeze theorem prooves that $\lim_{x \to 0} \left(\frac{sin(x)}{x}\right) = 1$:

$$cos(x) < \left(\frac{sin(x)}{x}\right) < 1$$

Differential Calculus

Overview

Differential calculus is the study of the definition, properties, and applications of the derivative of a function. It is concerned with the study of the rates at which quantities change. Primary study objects are a functions **derivates**, which describe the rate of change of a function near an input value. The process of finding a derivative is called differentiation.

The derivative is a **linear operator (research this term)**, which inputs a function and outputs another function. It is often denoted with a prime: The derivative of f is f' **(turn this into LaTeX)**. The derivate (f') represents the change with respect to the input of the original function (f). E.g. if f takes time as an input and returns the position of an object at that time, f' represents the change in position of the object with respect to time (velocity).