

$$n=2 \quad \theta_{MLE} = 0.5$$

$$X=1$$

$$CI_{\theta, 1-\alpha} := \left[ \hat{\theta}_{MLE} \pm z_{\frac{\alpha}{2}} SE[\hat{\theta}_{MLE}] \right]$$

$$SE[\hat{\theta}_{MLE}] = \sqrt{\frac{\theta(1-\theta)}{n}} \approx \sqrt{\frac{0.5(1-0.5)}{2}}$$

my PhD class has been wrong about this

$$CI_{\theta, 95\%} := \left[ 0.5 \pm 2 \sqrt{\frac{0.5(1-0.5)}{2}} \right]$$

$$= [0.5 \pm 0.707] = [-0.21, 1.21]$$

absurd!  $\theta \in (0, 1)$

At best completely useless!!!

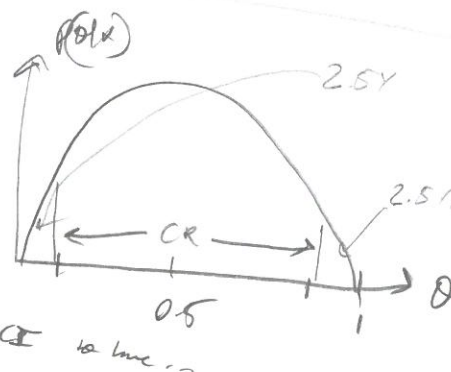
Also... can't get  $P(\theta \in CI) = 95\%$  !!

Can we do better with Bayes?

$$\theta \sim \text{Bern}(\alpha, \beta) \Rightarrow \theta|x \sim \text{Bern}(\alpha+x, \beta+1-x)$$

as in exple  $\alpha = \beta = 1$  (uniform prior)

$$\Rightarrow \theta|x \sim \text{Bern}(1+1, 1+(2-1)) = \text{Bern}(2, 2)$$



Now we have a credible region (CR) i.e. the Bayesian CI

$$P(\theta \in CR) = 1 - \alpha$$

How to make CR?? General formula (applies for all Bayesian formulations)

$$CR_{\theta, 1-\alpha} := [\text{qunile}[\theta|x, \frac{\alpha}{2}], \text{qunile}[\theta|x, 1-\frac{\alpha}{2}]]$$

In our exple,

$$CR_{\theta, 95\%} = [\text{qunile}[\text{Bern}(2, 2), 2.5\%], \text{qunile}[\text{Bern}(2, 2), 97.5\%]]$$

$$= [\text{qbern}(0.025, 2, 2), \text{qbern}(0.975, 2, 2)]$$

$$= [0.094, 0.906]$$

much better!!!

Just so you understand...

Why is a computer needed here?

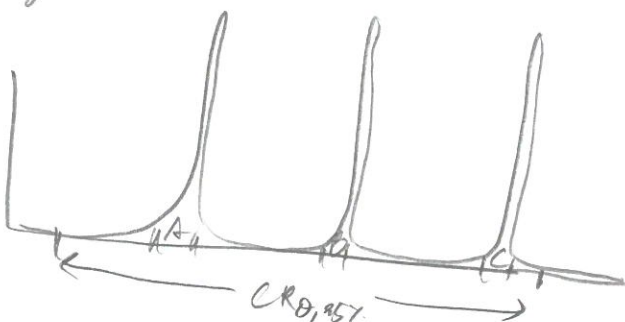
12

$$Q_{rank}[\text{Beta}(\alpha, \beta), 25\%] = x \text{ s.t. } \frac{1}{B(\alpha, \beta)} \int_0^x \theta^{\alpha-1} (1-\theta)^{\beta-1} d\theta = .025$$

must be done numerically except  
in special cases of  $\alpha, \beta$ .

Can we make a smaller CR? Yes...

Figure



What about AUBUC. This has the smallest total length of the interval.  
It is called the highest density region (HDR) approach.

- ① ~~First~~ having a non-contiguous CR... doesn't sit right with my people...
- ② ~~Very computationally intensive to get it correct.~~
- ③ ~~Other definitions for contiguous CR, he will be doing~~

Disadvantages

- ① Not preferable to have a non-contiguous interval  
e.g.  $CR_{0.25\%} = [0.1, 0.2] \cup [0.8, 0.9]$  i.e. "low or high"  
makes little sense
- ② Very computationally intensive to find this region  
 $\Rightarrow$  he will use the contiguous CR default base

What we've been discussing are 2-sided CR's.

We can also do 1-sided CR's

$$\begin{aligned}
 CR_{L, \theta, 1-\alpha_0} &:= (-\infty, Q_{\text{unif}}[\theta | x, 1-\alpha_0]) \\
 \uparrow \\
 \text{left sided} &= [0, q_{\text{beta}}(1-\alpha_0, \alpha, \beta)] \quad \text{in our beta-binomial model} \\
 &= [Q_{\text{unif}}[\theta | x, \alpha_0], \infty) \\
 CR_{R, \theta, 1-\alpha_0} &= [q_{\text{beta}}(\alpha_0, \alpha, \beta), 1] \quad \dots \dots
 \end{aligned}$$

e.g.  $n=2, x=1$

$$CR_{L, \theta, 95\%} = [0, q_{\text{beta}}(0.95, 2, 2)] = [0, .86]$$

$$CR_{R, \theta, 95\%} = [q_{\text{beta}}(0.05, 2, 2), 1] = [.14, 1]$$

midpoint  $\uparrow$   


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midpoint  $\downarrow$

# Param Hypothesis Testing (3<sup>rd</sup> goal of inference)

4

## → Theory Testing

theory:  $H_1$  ... which I want to prove

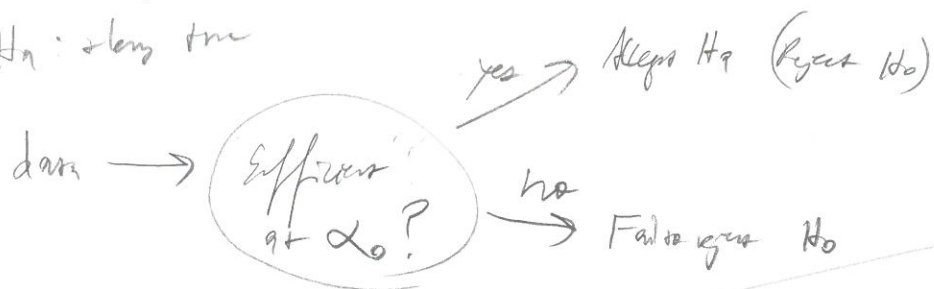
but I want to be sure I'm right.

So I assume the <sup>(H<sub>0</sub>)</sup> ~~opposite~~ and only accept my theory if there is overwhelming evidence.

Evidence is determined by  $\alpha$ .

$H_0$ : theory not true

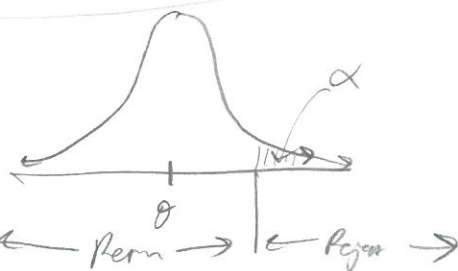
$H_1$ : theory true



Review / Frequency Hyp. testing

$$H_0: \theta \leq 0.5 = \theta_0$$

$$H_1: \theta > 0.5 = \theta_0$$



one sample  
one-sided /  
right-sided  
test of proportion

$$\hat{p} \sim N\left(\theta_0, \sqrt{\frac{\theta_0(1-\theta_0)}{n}}\right) = N\left(0.5, \sqrt{\frac{0.5(1-0.5)}{100}}\right)$$

if  $n=100$

$$\text{Rej Region} = \left[0, \theta_0 + z_{\alpha} \sqrt{\frac{\theta_0(1-\theta_0)}{n}}\right] = [0, 0.5 + 1.64 \cdot 0.05] = [0, 0.58]$$

if  $\hat{\theta} \in \text{Rej Region} \Rightarrow \text{F.T.R. } H_0$

if  $\hat{\theta} \notin \text{''} \Rightarrow \text{Reject } H_0$

$$p\text{-val} := P(\text{seeing data or more extreme} \mid H_0 \text{ true}) = P(\hat{p} > \theta \mid \hat{p} \sim N\left(\theta_0, \sqrt{\frac{\theta_0(1-\theta_0)}{n}}\right)) = \arg\theta_0 \{ \theta \in \text{Rej Region} \}$$

$$H_0: \theta = 0.5 = \theta_0$$

$$H_1: \theta < 0.5 = \theta_0$$



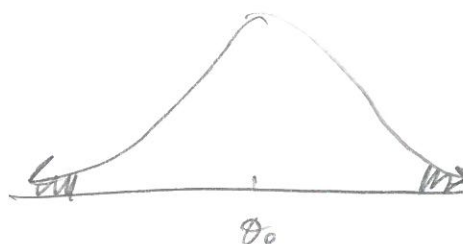
$$\text{Ret. Region} = \left[ \theta_0 - z_{\alpha} \sqrt{\frac{\theta_0(1-\theta_0)}{n}}, 1 \right]$$

$$H_0: \theta = 0.5 = \theta_0$$

$$H_1: \theta \neq 0.5 = \theta_0$$

$$\theta > 0.5 \text{ or } \theta < 0.5$$

two ways to reject



$$\text{Ret. Region} = \left[ \theta_0 \pm z_{\frac{\alpha}{2}} \sqrt{\frac{\theta_0(1-\theta_0)}{n}} \right]$$

$$p_{\text{val}} = \inf_{\theta_0} \{ \theta \in \text{Ret. Region} \}$$

$$p_{\text{val}} = P(\text{reject } H_0 \mid H_0 \text{ true})$$

$$\left. \begin{aligned} &\neq P(H_0) \\ &\neq P(H_0 | X) \\ &\neq P(H_1) \\ &\neq P(H_1 | X) \end{aligned} \right\} \text{these are things you want!}$$

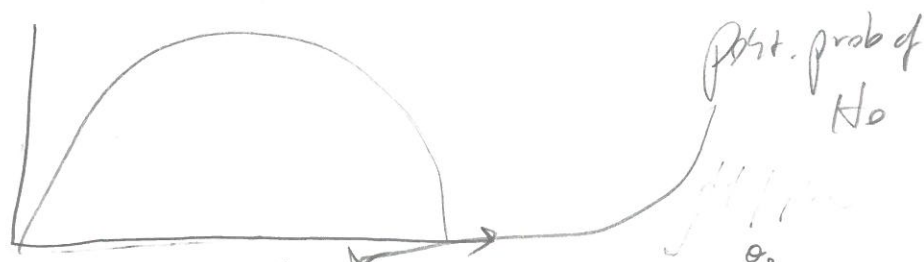
Bayesian Hyp. Testing

$$H_0: \theta \leq 0.5 = \theta_0$$

$$H_1: \theta > 0.5 = \theta_0$$

with data

we know  $P(\theta | X)$



$$\text{we can calc } p_{\text{val}} = P(H_0 | X) = P(\theta \leq \theta_0 | X) = \int_{\theta_0}^1 \frac{1}{P(\theta_0, b, a, c)} \theta^{a-1} (1-\theta)^{b-1} d\theta$$

at low.. reject

$= p_{\text{best}}(\theta_0, X, a, b, c)$