Math 341 / 650 Spring 2017
Midterm Examination One

Solutions

Professor Adam Kapelner

Monday, March 5, 2018

Full Name ____________________

Code of Academic Integrity

Since the college is an academic community, its fundamental purpose is the pursuit of knowledge. Essential to the success of this educational mission is a commitment to the principles of academic integrity. Every member of the college community is responsible for upholding the highest standards of honesty at all times. Students, as members of the community, are also responsible for adhering to the principles and spirit of the following Code of Academic Integrity.

Activities that have the effect or intention of interfering with education, pursuit of knowledge, or fair evaluation of a student's performance are prohibited. Examples of such activities include but are not limited to the following definitions:

Cheating Using or attempting to use unauthorized assistance, material, or study aids in examinations or other academic work or preventing, or attempting to prevent, another from using authorized assistance, material, or study aids. Example: using an unauthorized cheat sheet in a quiz or exam, altering a graded exam and resubmitting it for a better grade, etc.

I acknowledge and agree to uphold this Code of Academic Integrity.

____________________ signature ____________ date

Instructions

This exam is seventy five minutes and closed-book. You are allowed **one** page (front and back) of a "cheat sheet." You may use a graphing calculator of your choice. Please read the questions carefully. If the question reads "compute," this means the solution will be a number otherwise you can leave the answer in *any* widely accepted mathematical notation which could be resolved to an exact or approximate number with the use of a computer. I advise you to skip problems marked "[Extra Credit]" until you have finished the other questions on the exam, then loop back and plug in all the holes. I also advise you to use pencil. The exam is 100 points total plus extra credit. Partial credit will be granted for incomplete answers on most of the questions. $\boxed{\text{Box}}$ in your final answers. Good luck!

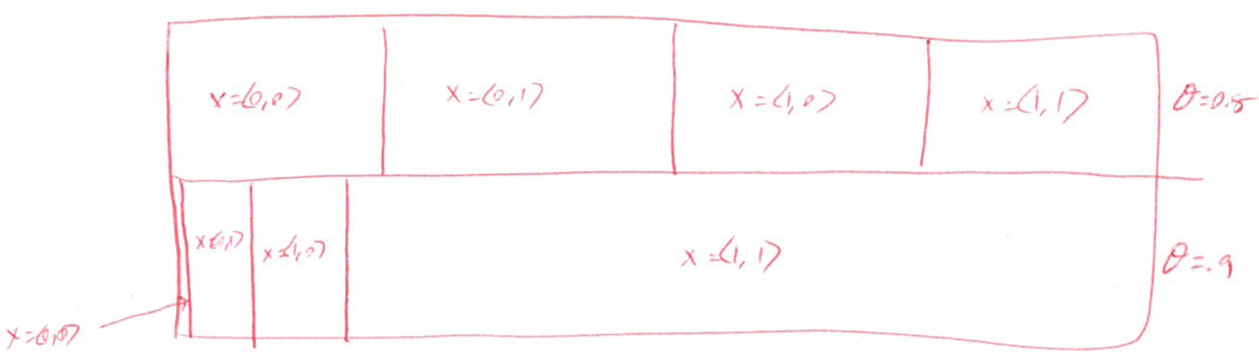
Distribution of r.v.	Quantile Function	PMF / PDF function	CDF function	Sampling Function
beta	qbeta(p, α, β)	d-(x, α, β)	p-(x, α, β)	r-(α, β)
betabinomial	qbetabinom(p, n, α, β)	d-(x, n, α, β)	p-(x, n, α, β)	r-(n, α, β)
betanegativebinomial	qbeta_nbinom(p, r, α, β)	d-(x, r, α, β)	p-(x, r, α, β)	r-(r, α, β)
binomial	qbinom(p, n, θ)	d-(x, n, θ)	p-(x, n, θ)	r-(n, θ)
exponential	qexp(p, θ)	d-(x, θ)	p-(x, θ)	r-(θ)
gamma	qgamma(p, α, β)	d-(x, α, β)	p-(x, α, β)	r-(α, β)
geometric	qgeom(p, θ)	d-(x, θ)	p-(x, θ)	r-(θ)
inversegamma	qinvgamma(p, α, β)	d-(x, α, β)	p-(x, α, β)	r-(α, β)
negative-binomial	qnbinom(p, r, θ)	d-(x, r, θ)	p-(x, r, θ)	r-(r, θ)
normal (univariate)	qnorm(p, θ, σ)	d-(x, θ, σ)	p-(x, θ, σ)	r-(θ, σ)
poisson	qpois(p, θ)	d-(x, θ)	p-(x, θ)	r-(θ)
T (standard)	qt(p, ν)	d-(x, ν)	p-(x, ν)	r-(ν)
uniform	qunif(p, a, b)	d-(x, a, b)	p-(x, a, b)	r-(a, b)

Table 1: Functions from R (in alphabetical order) that can be used on this exam. The hyphen in colums 3, 4 and 5 is shorthand notation for the full text of the r.v. which can be found in column 2.

Problem 1 Consider the $\overset{iid}{\sim}$ Bernoulli model if θ known, $n = 2$, $\Theta_0 = \{0.5, 0.9\}$ and the prior of indifference. Note that $\boldsymbol{X}$ refers the vector of the $n = 2$ random variables.

(a) [10 pt / 10 pts] Illustrate the space $\boldsymbol{X} \times \Theta$ to scale and properly denoted as we did in class.

$\mathcal{X}$



(M)

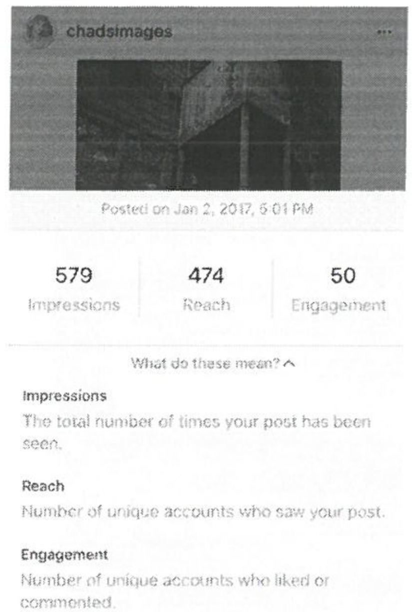
$P(X=\langle 0,0\rangle|\theta=.9) = .1^2 = .01$

$P(X=\langle 0,1\rangle|\theta=.9) = .1\cdot .9 = .09$

$P(X=\langle 1,0\rangle|\theta=.9) = \text{''} = .09$

$P(X=\langle 1,1\rangle|\theta=.9) = .9^2 = .81$

Problem 2 Imagine you are a celebrity who has a instagram page where you post photos. This is important to you so you own a business account. Here is typically what you see:



In case you can't read the text in the image, the "reach" is how many unique people viewed your photo post and the "engagement" is the number of people who liked it or commented on it.

The proportion of engagement : reach is a critical number — it demonstrates how hooked your audience is. We'll call this θ going forward and we'll try to estimate it. As of now, we do not have any reason to exclude any values from Θ.

We'll assume each unique person who visits the post of the photo as an independent Bernoulli trial where 1 represents they engaged with the post and 0 represents that they did not engage.

(a) [4 pt / 14 pts] What would the uninformative prior be for θ?

$$\theta \sim U(0,1) = Beta(1,1)$$

(b) [4 pt / 18 pts] Assume the uninformative prior. Your post just got 10 views but no engagement. What is your best guess of θ right now using the Bayesian formulation?

$$\hat{\theta}_{MMSE} = \frac{\alpha+x}{\alpha+\beta+n} = \frac{1+0}{1+1+10} = \boxed{\frac{1}{12}}$$

(c) [4 pt / 22 pts] You just got another 10 views but no engagement. What is your best guess of θ right now (at $n = 20$) using the Bayesian formulation?

$$\hat{\theta}_{MMSE} = \frac{1+0}{1+1+20} = \boxed{\frac{1}{22}}$$

(d) [4 pt / 26 pts] What is your best guess of θ right now (at $n = 20$) using the Frequentist formulation?

$$\hat{\theta}_{MLE} = \frac{x}{n} = \frac{0}{20} = 0$$

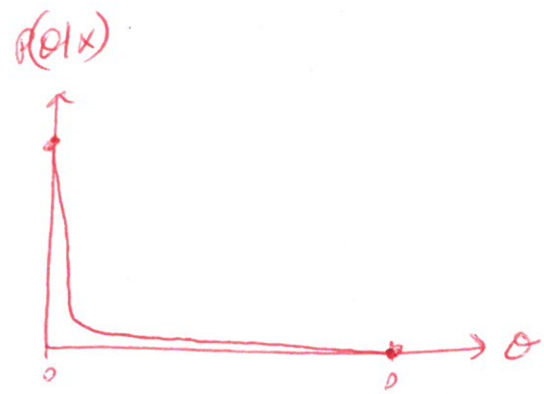
(e) [4 pt / 30 pts] Why is the Frequentist formulation not realistic?

Estimating $\theta = 0$ means engagement is absolutely impossible which is too presumptuous especially after only 20 observations

(f) [4 pt / 34 pts] After these 20 observations, what is the posterior distribution? Notate it and its distribution correctly below.

$$\theta \mid X \sim \text{Beta}(\alpha + x, \beta + n - x) = \text{Beta}(1, 21)$$

(g) [6 pt / 40 pts] Plot the PDF of the posterior distribution as best as you can. Include important tick marks on both the x and y axes as well as labels for both the x and y axes.



(h) [6 pt / 46 pts] Create a 98% credible region for θ. Use notation from Table 1.

$$CR_{\theta, 98\%} = [\,\text{qbeta}(.01, 1, 21),\ \text{qbeta}(.99, 1, 21)\,]$$

(i) [2 pt / 48 pts] Is the interval you just created also the 98% HDR? Yes/No.

(j) [6 pt / 54 pts] What is the unconditional probability of those 20 observations? Answer exactly.

$$P(X) = \int_{\Theta} P(X|\theta)\,P(\theta)\,d\theta = \int_0^1 (1-\theta)^{20}(1)\,d\theta = \left[-\frac{(1-\theta)^{21}}{21}\right]_0^1 = -\left(0 - \frac{1}{21}\right) = \boxed{\frac{1}{21}}$$

(k) [4 pt / 58 pts] What is the probability that the engagement proportion is greater than 0.5? You do not need to evaluate the integral.

$$P(\theta > 0.5 | x) = \int_{0.5}^{1} \text{Beta}(1, 21)\,d\theta = \frac{1}{B(1,21)} \int_{0.5}^{1} \theta^{1-1}(1-\theta)^{21-1}\,d\theta = \frac{1}{B(1,21)} \int_{0.5}^{1} (1-\theta)^{20}\,d\theta$$

(l) [5 pt / 63 pts] What is the probability that the next viewer will not engage? Use the posterior predictive distribution to answer.

$$P(x^*|x) = \text{Bern}\left(\frac{\alpha + x}{\alpha+\beta+n}\right) = \text{Bern}\left(\frac{1}{22}\right) \Rightarrow P(x^* = 0 | x) = \boxed{\frac{21}{22}}$$

We now use previous posts to build a better prior. If we look at previous θ estimates, we can fit a $\alpha = 17$ and $\beta = 245$ beta distribution. Use this as the prior from now on.

(m) [4 pt / 67 pts] This prior is equivalent to how many pseudo-engagements on the post?

17

(n) [4 pt / 71 pts] What is the prior expectation?

$$E(\theta) = \frac{17}{17+245} = \boxed{\frac{17}{262}}$$

(o) [6 pt / 77 pts] Find $\hat{\theta}_{\text{MMSE}}$, $\hat{\theta}_{\text{MAE}}$ and $\hat{\theta}_{\text{MAP}}$ considering the $n = 20$ non-engagements from part (d).

$$\hat{\theta}_{MMSE} = \frac{\alpha + x}{\alpha+\beta+n} = \frac{17}{17+245+20} = \boxed{\frac{17}{282}}, \quad \hat{\theta}_{MAE} = \text{qbeta}(0.5, 17, 265), \quad \hat{\theta}_{MAP} = \frac{\alpha + x - 1}{\alpha+\beta+n-2} = \boxed{\frac{16}{280}}$$

(p) [4 pt / 81 pts] If we are using $\hat{\theta}_{\text{MMSE}}$ to estimate θ, calculate ρ, the shrinkage proportion. Round to the nearest percent.

$$\rho = \frac{\alpha+\beta}{n+\alpha+\beta} = \frac{17+245}{20+17+245} = \frac{262}{282} \approx \boxed{93\%}$$

(q) [4 pt / 85 pts] Does the shrinkage indicate a *strong* prior? Yes / no.

(r) [6 pt / 91 pts] Create a 95% credible region for the probability of *not engaging*.

$$\theta \mid X \sim \text{Beta}(\alpha + x, \beta + n - x)$$

$$1-\theta \mid X \sim \text{Beta}(\beta + n - x, \alpha + x) = \text{Beta}(265, 17)$$

$$\Rightarrow CR_{1-\theta, 95\%} = \left[\text{qbeta}(.025, 265, 17), \text{qbeta}(.975, 265, 17)\right]$$

Problem 3 These are purely theoretical exercises.

(a) [5 pt / 96 pts] Prove $B(1,1) = 1$ from first principles.

$$B(1,1) = \frac{\Gamma(1)\Gamma(1)}{\Gamma(2)} = \frac{\Gamma(1)^{\cancel{2}}}{1\,\cancel{\Gamma(1)}} = \Gamma(1) = \boxed{1}$$

$$\Gamma(1) := \int_0^\infty t^{1-1} e^{-t}\,dt = \int_0^\infty e^{-t}\,dt = -\left[e^{-t}\right]_0^\infty = -(0-1) = 1$$

(b) [4 pt / 100 pts] If $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, compute $\Gamma(\frac{7}{2})$ to the nearest 2 digits.

$$\Gamma\left(\frac{7}{2}\right) = \frac{5}{2}\Gamma\left(\frac{5}{2}\right) = \frac{5}{2}\cdot\frac{3}{2}\Gamma\left(\frac{3}{2}\right) = \frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\underbrace{\Gamma\left(\frac{1}{2}\right)}_{=\sqrt{\pi}} = \boxed{3.32}$$