Programming Language Concepts

Higher Order Functions

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Outline

Lambda Calculus

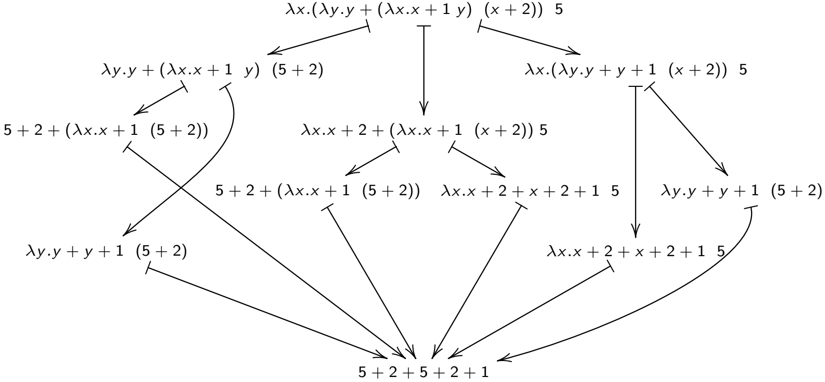
- 1930's by Alonso Church and Stephen Cole Kleene
- Mathematical foundation for computatibility and recursion
- Simplest functional paradigm language
- $\lambda var.expr$
  defines an anonymous function. Also called lambda abstraction
- *expr* can be any expression with other lambda abstractions and applications. Applications are one at a time.
- $(\lambda x.\lambda y.x + y)\ 3\ 4$

- In '$\lambda var.expr$ ' all free occurences of *var* is bound by the λvar.
- Free variables of expression $FV(expr)$
  - $FV(name) = \{name\}$ if *name* is a variable
  - $FV(\lambda name.expr) = FV(expr) - \{name\}$
  - $FV(M\ N) = FV(M) \cup FV(N)$
- α conversion: expressions with all bound names changed to another name are equivalent:
  $\lambda f.f\ x \equiv_\alpha \lambda y.y\ x \equiv_\alpha \lambda z.z\ x$
  $\lambda x.x + (\lambda x.x + y) \equiv_\alpha \lambda t.t + (\lambda x.x + y) \equiv_\alpha \lambda t.t + (\lambda u.u + y)$
  $\lambda x.x + (\lambda x.x + y) \not\equiv_\alpha \lambda x.x + (\lambda x.x + t)$

β Reduction

- Basic computation step, function application in λ-calculus
- Based on substitution. All bound occurences of λ variable parameter is substituted by the actual parameter
- $(\lambda x.M)N \mapsto_{\beta} M[x/N]$ (all x's once bound by lambda are substituted with N).
- $(\lambda x.(\lambda y.y + (\lambda x.x + 1)\ y)(x + 2))\ 5$
- If no further β reduction is possible, it is called a normal form.
- There can be different reduction strategies but should reduce to same normal form. (Church Rosser property)

All possible reductions of a λ-expression. All reduce to the same normal form.



Introduction

- Mathematics:
  $(f \circ g)(x) = f(g(x))$, $(g \circ f)(x) = g(f(x))$
- "$\circ$" : Gets two unary functions and composes a new function.
  A function getting two functions and returning a new function.
- in Haskell:

```
f x = x+x
g x = x*x
compose func1 func2 x = func1 (func2 x)
t = compose f g
u = compose g f
```

- t 3 = (3*3)+(3*3) = 18
  u 3 = (3+3)*(3+3) = 36
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- "`compose`" function is a function getting two functions as parameters and returning a new function.
- Functions getting one or more functions as parameters are called Higher Order Functions.
- Many operations on functional languages are repetition of a basic task on data structures.
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Functions/Curry

- Cartesian form vs curried form:
  $\alpha \times \beta \to \gamma$ vs $\alpha \to \beta \to \gamma$
- Curry function gets a
  binary function in cartesian form and converts it to curried form.

```
curry f x y = f(x,y)
add (x,y) = x+y
increment = curry add 1
---
increment 5
6
```

- curry: $(\alpha \times \beta \to \gamma) \to \alpha \to \beta \to \gamma$
- Haskell library includes it as `curry`.

Functions/Map

```
square x = x*x
day no = case no of 1 -> "mon" ; 2 -> "tue" ; 3 -> "wed";
        4 -> "thu" ; 5 -> "fri" ; 6 -> "sat" ; 7 -> "sun"
map func [] = []
map func (el:rest) = (func el):(map func rest)
----
map square [1,3,4,6]
[1,9,6,36]
map day [1,3,4,6]
["mon","wed","thu","sat"]
```

- map:$(\alpha \rightarrow \beta) \rightarrow [\alpha] \rightarrow [\beta]$
- Gets a function and a list. Applies the function to all elements and returns a new list of results.
- Haskell library includes it as `map`.

Functions/Filter

```
iseven x = if mod x 2 == 0 then True else False
isgreater x = x>5

filter func [] = []
filter func (el:rest) = if func el then
                                   el:(filter func rest)
                              else (filter func rest)
----
filter iseven [1,2,3,4,5,6,7]
[2,4,6]
filter isgreater [1,2,3,4,5,6,7]
[6,7]
```

- `filter`:$(\alpha \rightarrow Bool) \rightarrow [\alpha] \rightarrow [\alpha]$
- Gets a boolean function and a list. Returns a list with only members evaluated to `True` by the boolean function.
- Haskell library includes it as `filter`.

Functions/Reduce (Fold Right)

```
sum x y = x+y
product x y = x*y

reduce func s [] = s
reduce func s (el:rest) = func el (reduce func s rest)
----
reduce sum 0 [1,2,3,4]
10                              // 1+2+3+4+0
reduce product 1 [1,2,3,4]
24                              // 1*2*3*4*1
```

- reduce:$(\alpha \to \beta \to \beta) \to \beta \to [\alpha] \to \beta$
- Gets a binary function, a list and a seed element. Applies function to all elements right to left with a single value. $reduce\ f\ s\ [a_1, a_2, ..., a_n] = f\ a_1\ (f\ a_2\ (....\ (f\ a_n\ s)))$
- Haskell library includes it as `foldr`.

- Sum of a numbers in a list:
  `listsum = reduce sum 0`
- Product of a numbers in a list:
  `listproduct = reduce product 1`
- Sum of squares of a list:
  `squaresum x = reduce sum 0 (map square x)`

Functions/Fold Left

```
subtract x y = x - y
foldl func s [] = s
foldl func s (el:rest) =
          foldl func (func s el) rest
----
reduce subtract 0 [1,2,3,4]
-2                              // 1-(2-(3-(4-0)))
foldl subtract 0 [1,2,3,4]
-10                             // ((((0-1)-2)-3)-4)
```

- `foldl`:$(\alpha \to \beta \to \alpha) \to \alpha \to [\beta] \to \alpha$
- Reduce operation, left associative.:
  $reduce\ f\ s\ [a_1, a_2, ..., a_n] = f\ (f\ (f\ ...(f\ s\ a_1)\ a_2\ ...))\ a_n$
- Haskell library includes it as `foldl`.

Functions/Iterate

```
twice x = 2*x

iterate func s 0 = s
iterate func s n = func (iterate func s (n-1))
----
iterate twice 1 4
16                        // twice (twice ( twice (twice 1))
iterate square 3 3
6561                      // square (square (square 3))
```

- `iterate`: $(\alpha \to \alpha) \to \alpha \to int \to \alpha$
- Applies same function for given number of times, starting with the initial seed value. $iterate\ f\ s\ n = f^n\ s = \underbrace{f\ (f\ (f\ ...(f\ s))}_{n}$

Functions/Value Iteration (for)

```
for func s m n =
      if m>n then s
      else for func (func s m) (m+1) n

----
for sum 0 1 4
10          // sum (sum (sum (sum 0 1) 2) 3) 4
for product 1 1 4
24          // product (product (product (product 1 1) 2) 3) 4
```

- for:$(\alpha \rightarrow int \rightarrow \alpha) \rightarrow \alpha \rightarrow int \rightarrow int \rightarrow \alpha$
- Applies a binary integer function to a range of integers in order.
  $for\ f\ s\ m\ n = f(f\ (f\ (f\ (f\ s\ m)\ (m+1))\ (m+2))\ ...)\ n$

- multiply (with summation):
  `multiply x = iterate (sum x) x`
- integer power operation (Haskell '^'):
  `power x = iterate (product x) x`
- sum of values in range 1 to n:
  `seriessum = for sum 0 1`
- Factorial operation:
  `factorial = for product 1 1`

Higher Order Functions in C

C allows similar definitions based on function pointers. Example: bsearch() and qsort() funtions in C library.

```
typedef struct Person { char name[30]; int no;} person;
int cmpnmbs(void *a, void *b) {
    person *ka=(person *)a; person *kb=(person *)b;
    return ka->no - kb->no;
}
int cmpnames(void *a, void *b) {
    person *ka=(person *)a; person *kb=(person *)b;
    return strncmp(ka->name,kb->name,30);
}
int main() {        int i;
    person list[]={{"veli",4},{"ali",12},{"ayse",8},
                   {"osman",6},{"fatma",1},{"mehmet",3}};
    qsort(list,6,sizeof(person),cmpnmbs);
    ...
    qsort(list,6,sizeof(person),cmpnames);
    ...
}
```

Fibonacci

Fibonacci series: 1 1 2 3 5 8 13 21 ..

$fib(0) = 1 \;\; ; \;\; fib(1) = 1 \;\; ; \;\; fib(n) = fib(n-1) + fib(n-2)$

```
fib n = let f (x,y) = (y,x+y)
            (a,b) = iterate f (0,1) n
        in b
----
fib 5       // f(f(f(f(0,1))))
8           //(0,1)->(1,1)->(1,2)->(2,3)->(3,5)->(5,8)
```

Sorting

Quicksort:

1. First element of the list is x and rest is xs
2. select smaller elements of xs from x, sort them and put before x.
3. select greater elements of xs from x, sort them and put after x.

```
notfunc f x y = not (f x y)

sort _ [] = []
sort func (x:xs) = (sort func (filter (func x) xs)) ++
                (x: (sort func (filter ((notfunc func) x) xs)))
---
sort (>) [5,3,7,8,9,3,2,6,1]
[1,2,3,3,5,6,7,8,9]
sort (<) [5,3,7,8,9,3,2,6,1]
[9,8,7,6,5,3,3,2,1]
```

List Reverse

- Taking the reverse
  - First element is x rest is xs
  - Reverse the xs, append x at the end

  Loose time for appending x at the end at each step (N times append of size N).
- Fast version, extra parameter (initially empty list) added:
  - Take the first element, insert at the beginning of the extra parameter.
  - Recurse rest of the list with the new extra parameter.
  - When recursion at the deepest, return the extra parameter.

  Inserts to the beginning of the list at each step. Faster (N times insertion)

```
reverse1 [] = []
reverse1 (x:xs) = (reverse1 xs) ++ [x]

reverse2 x = reverse2' x []  where
        reverse2' [] x = x
        reverse2' (x:xs) y = reverse2' xs (x:y)
---
reverse1 [1..10000]          // slow
reverse2 [1..10000]          // fast
```

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