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A SYSTEMS MODEL OF STREAM FLOW AND WATER QUALITY IN THE BEDFORD-OUSE RIVER—1. STREAM FLOW MODELLING

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Abstract—This paper, the first of a two part description of the modelling activities associated with the Bedford-Ouse River Study, concentrates on streamflow characterisation. The streamflow models are of a stochastic-dynamic type in which a simple lumped parameter differential equation model for mainstream flow is enhanced by stochastic time-series descriptions of rainfall-runoff behaviour. The models have been developed using a new systematic approach to the modelling of badly defined dynamic systems which is centred around the exploitation of recursive methods of parameter estimation and time-series analysis. At the same time, the models have quite strong links with more conventional models used previously in hydrological systems analysis and the implications of the modelling results can easily be interpreted in conventional hydrologic terms. An important aspect of the modelling exercises described in the paper is that they are objective orientated and, in the Bedford-Ouse Study, the models are developed specifically with operational control and management applications in mind.

1. INTRODUCTION

The Bedford-Ouse study was initiated in 1972 by the Great Ouse River Division of the Anglian Water Authority and the Department of the Environment in association with the Control and Management Systems Division of the Engineering Department, University of Cambridge. The study is concerned with the existing utilisation and future potential of the Ouse. In particular, the models are designed to be of specific use in assessing the possible impact on river water quality of the new city of Milton Keynes, which is intended to have a population of 250,000 by the turn of the century and is situated some 55 kilometres upstream of the Bedford City Water Board Abstraction Plant at Clapham (see Fig. 1).

Since water quality is clearly dependent on flow, the present description of the Bedford-Ouse study is divided into two major parts: part I is devoted to the stochastic-dynamic modelling of flow; while part II concentrates on the theme of water quality.

Most conventional approaches to stream flow simulation modelling tend to be based upon compartmental hydrological methods (e.g. Linsley & Crawford, 1963); models that are, of necessity, fairly detailed and complex, and so difficult to justify in the context of the present study. For this reason the approach adumbrated by Jamieson *et al.* (1971) has been utilised here: a simple deterministic model is used to relate flow variations at different points in the river system to input variations at the system boundary and this is then enhanced by stochastic time series models that represent the flow variations due to rainfall and runoff effects.

2. THE FLOW ROUTING PROBLEM IN THE MAIN STREAM

The stretch of the Bedford-Ouse of particular interest to the dynamic modelling part of the Bedford-Ouse study is between the proposed effluent discharge point for the new city of Milton Keynes, situated at Tickford Abbey, and the Bedford Water Board abstraction point, 55 km downstream at Clapham. The principle features of the area, such as major tributaries, weirs and towns, which discharge localised effluent, are shown in Fig. 1. The flow gauging stations at Newport Pagnell and Bedford define the upstream and downstream boundaries of the system. The objective of the main streamflow model of this region is to translate the "input" flow from the upstream system boundaries, defined as the flow gauging stations at Newport Pagnell, to the downstream system boundary at Bedford. This model should incorporate geographical features because of the effects on the redistribution of flow of factors such as change in river shape and size, the location of weirs, effluent discharge and tributaries. Clearly, the combination of these features, plus the effects of dispersion in the flowing media, introduce considerable complications into any mathematical description of the system.

The flow routing procedure used to model the system shown in Fig. 1 is based on a multi-reach structure, in which each reach is characterised by a number of compartments. The model for flow variations in each compartment is based on an analogy with the lumped parameter equations for the variations in the concentration of a conservative pollu-

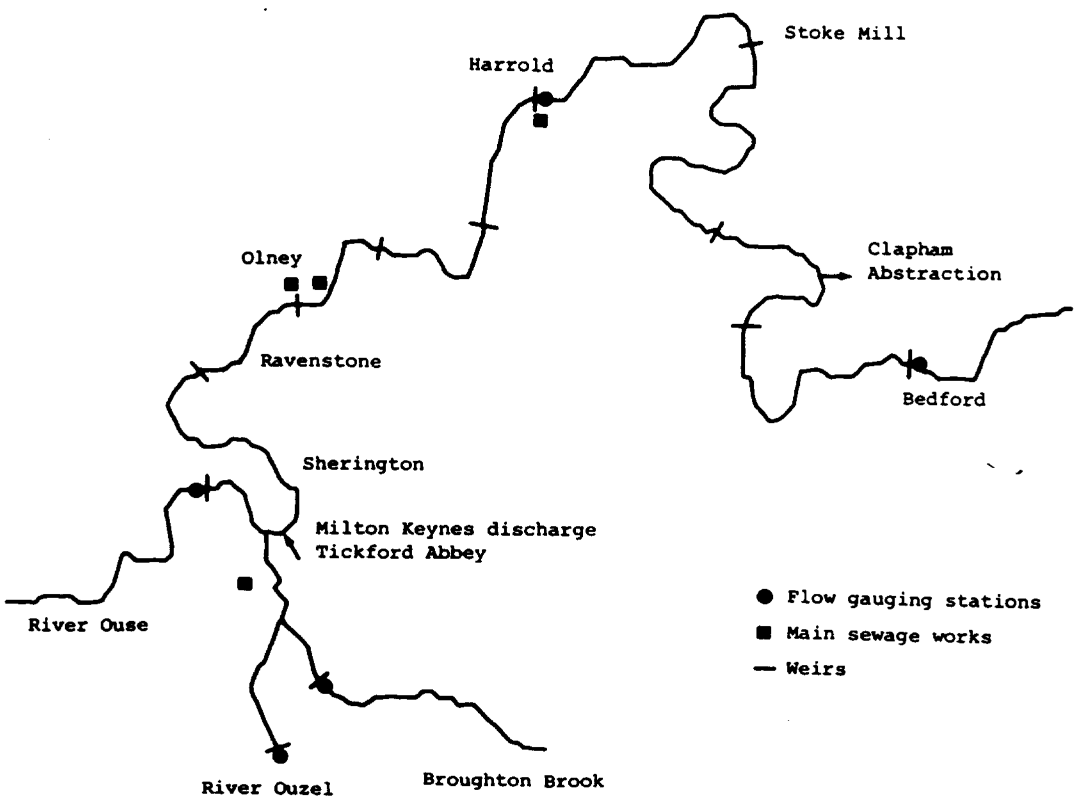

Fig. 1. Map of the Bedford-Ouse system.

tant, under the assumption of uniform mixing over the compartment. Here the concentration c_j at the jth compartment is given by the well known mass conservation equation

$$\frac{\mathrm{d}(Vc_j)}{\mathrm{d}t} = Q(c_{j-1} - c_j) \tag{1}$$

where $Q = Q(t)$ is the flow rate and $V = V(t)$ is the compartmental volume. Using a probabilistic argument, it is possible to draw an analogy between the variations or "perturbations" in flow δQ_j about some mean or reference level and the changes in concentration of the conservative pollutant (Himmelblau & Yates, 1968). The resulting equation is similar to (1) and takes the form

$$\frac{\mathrm{d}(\bar{V}_j \delta Q_j)}{\mathrm{d}t} = \bar{Q}_j(\delta Q_{j-1} - \delta Q_j) \tag{2}$$

where $\bar{Q}_j$ and $\bar{V}_j$ are, respectively, the reference flow and volume in the jth compartment. In effect, and to be somewhat simplistic, the perturbation flow is being considered as a measure of the "concentration" of equi-velocity particles in the stream.

Since $\bar{Q}_j$ and $\bar{V}_j$ in (2) are constant (or slowly variable) $\bar{V}_j$ can be taken outside the differentiation operator to yield

$$\tau_j \frac{\mathrm{d}\delta Q_j}{\mathrm{d}t} = \delta Q_{j-1} - \delta Q_j \tag{3}$$

where $\tau_j = \bar{V}_j/\bar{Q}_j$. This equation represents the flow processes in a single compartment using a single dynamic lag (i.e. a system of first order with unity gain and time constant τ_j). To model both the advective and dispersive properties of flow in a whole and possibly complex reach, it is desirable to allow for a number of lags (e.g. Whitehead, Young & Michell, 1978; Himmelblau & Yates, 1968). The multi-lag model for each reach can then be written in Laplace transfer function terms as

$$F(s) = \frac{\delta Q_j(s)}{\delta Q_{j-n}(s)} = \frac{1}{(1 + \tau_j s)^n} \tag{4}$$

where s is the Laplace operator and n is the chosen number of lags.

Equations (3) and (4) represent a "kinematic" method for flow routing (see e.g., Weinmann & Laurenson, 1977) and the restriction to small perturbations about a mean flow is theoretically necessary to proscribe effects of "dynamic" waves. In practice, however, this restriction may not be limiting (see later).

It is interesting to note the similarity between the above multiple-lag model and the conventional Muskingum–Cunge (M–C) method of flow routing (McCarthy, 1938; Cunge, 1969). In the M–C method the equations for mass balance over the jth compartment of a reach relate the "storage" S_j to the inflow Q_{j-1} and outflow Q_j i.e.,

$$\frac{\mathrm{d}S_j}{\mathrm{d}t} = Q_{j-1} - Q_j \tag{5}$$

Here, the storage is defined by the rather arbitrary relationship

$$S_j = \tau_j(\epsilon Q_{j-1} + (1-\epsilon)Q_j)$$

where τ_j is the residence time (known as the storage coefficient, k, in the hydrological literature; see Laurenson, 1975) and ϵ is a "tuning" parameter which, together with the choice of the number of compartments n in the reach, can be manipulated to provide a good model fit to measured data.

If ϵ is set to zero in equation (5) then $S_j = \tau_j Q_j$ and the compartmental equation becomes

$$\tau_j \frac{dQ_j}{dt} = Q_{j-1} - Q_j \tag{6}$$

which can be compared directly with equation (3).

Cunge (1969) has argued that an n compartment reach model based on an equation such as (5) for each compartment provides a good approximation to the convection–diffusion equation

$$\frac{\partial Q}{\partial t} = -u\frac{\partial Q}{\partial x} + D\frac{\partial^2 Q}{\partial x^2} \tag{7}$$

where

x = distance in the longitudinal direction,
t = time,
Q = the flow,
u = velocity (strictly wave celerity) and
D = a coefficient of dispersion.

The dispersion coefficient in (7) is related to ϵ in the M–C model and n the number of lags in series, by the relationship

$$D = (\tfrac{1}{2} - \epsilon)\frac{ul}{n} \tag{8}$$

in which l is the length of the reach. In an extensive comparative study of flow routing methods (Flood Studies Report, 1975) it has been shown that there is little difference in practice between a multi-lag linear reservoir model based on equation (5) and the linear diffusion model. Indeed it is concluded that the multi-lag model is preferable to linear and non-linear diffusion methods because it does not suffer from computational difficulties with tributaries.

If ϵ is not set to zero, it is straightforward to show that the reach model for the M–C method can be written in the following form

$$F_{mc}(s) = \frac{Q_j(s)}{Q_{j-n}(s)} = \frac{(1-\epsilon\tau_j s)^n}{(1+(1-\epsilon)\tau_j s)^n}. \tag{9}$$

Comparing this transfer function with the equivalent transfer function for the presently proposed method, as given in equation (4), we see that the general M–C model is a more complicated representation with a transfer function which has multiple zeros (i.e. the roots associated with the transfer function numerator polynomial) in the right half of the complex plane. In systems terms, such a transfer function is said to possess "non-minimum phase characteristics" (see Truxal, 1958) and it is well known (Nash, 1959) that such a model can generate *reductions* in flow (and even negative flows) at the compartmental and reach outputs in response to sharp *increases* in flow at the reach input.†

These obviously undesirable model response characteristics are a direct result of a non-zero choice for the factor ϵ in equations (5) and (9) and are not encountered when $\epsilon = 0$. This is one reason why we prefer the "low pass filter" multiple lag model (4) rather than (9)‡. Another is that the simpler representation (4) appears adequate for many practical flow routing applications, in the sense that the selection of n, together with the definition of $\tau_j = Q_j/V_j$, seem sufficient to reproduce satisfactorily the advection and dispersion characteristics of the reach.

For the above reasons, the multiple lag model (4) was taken as the basis for the streamflow model used in the Bedford-Ouse Study. It was found by experiment, however, that the small perturbation restriction was not necessary in the Bedford-Ouse situation and the time constant (storage coefficient) was defined directly in terms of the actual variations of flow and volume, i.e.

$$\tau = \frac{V(t)}{Q(t)} \tag{10}$$

while δQ_j and δQ_{j-1} in (2) were replaced by Q_j and Q_{j-1} respectively.

The expression (10) was evaluated by reference to the reach data available for the river§: the volume changes computed from the expression

$$V = Al \tag{11(i)}$$

where l is the reach length and A is the cross sectional area. A was obtained from the following simple expression derived from hydrographic data supplied by the Great Ouse River Division (see Table 1).

$$A = h_t b + \frac{h_t^2}{s} \tag{11(ii)}$$

where h_t is the total reach depth given by

$$h_t = h_w + h_m\left(\frac{Q - 1.31}{Q_m - 1.31}\right) \tag{11(iii)}$$

in which h_w is the minimum depth (due to weirs), h_m is a flow dependent depth and Q_m is the maximum reach flow before flooding. h_m is computed under the assumption that the reach has uniform trapezoidal cross-section although, as we shall see, this assumption is not limiting since the overall model is even-

† The transfer function for a single compartment is, in fact, similar to the first order Pade approximation for a pure transportation time delay (Truxal, 1958). It could be considered, therefore, as an attempt to model the various transportation time delay effects of each compartment.

‡ Clearly such *a priori* information is not always available and the general utilisation of this model may well require additional experimentation to estimate the relationship between V and Q, for example via dye tracer studies as in Whitehead, Young & Mitchell (1978).

§ Although we recognise that, for long stretches of river, it may well be necessary to introduce pure transportation time delays, Beck & Young (1975).

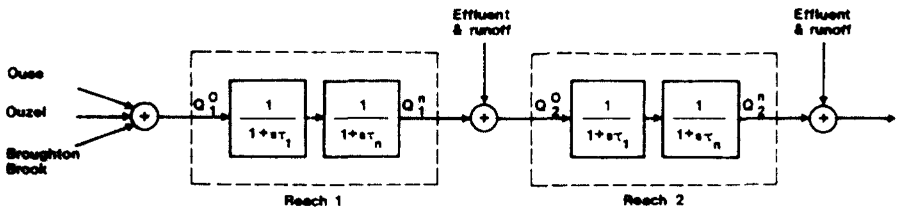

Fig. 2. Multi-compartment reach model. Q_j^i subscript j refers to reach j, superscript i refers to compartment i.

tually made stochastic and this, in effect, injects an additional degree of freedom and so allows for some uncertainty in the definition of these relationships.

The combination of simple flow-volume relationships such as (11) and the multiple lag structure of equation (4) provides a convenient description of the river behaviour which can be illustrated diagrammatically as shown in Fig. 2. Spatial characteristics are incorporated into the river model by the reach structure, with each reach boundary based on the location of weirs, effluent discharges and tributaries. The selected reach length depends to a large extent on the intensity of phenomena affecting quality and flow and the reach details are tabulated in Table 1.

The one parameter in the flow routing model that has not been defined at this point is the number of lags in each reach (n in equation (4)). The choice of n affects the relative importance of advection and dispersion of a flood wave and choice of appropriate values is a vexing problem (Laurenson, 1962; Whitehead, Hornberger & Black, 1980).

The analogy between a convection-diffusion, partial differential equation representation of the flow routing problem and a multiple lag model was pointed out above. Bearing in mind equation (1), a similar analogy can be made for the transport of a pollutant in a river (e.g. Zirvin & Shinar, 1976) and we maintain that tracer experiments can, therefore, be used initially to identify the structure of a multiple-lag model. That is, a value of n chosen to model adequately the advective–dispersive properties of a river with respect to tracer concentration can also be used in a flow model. Of course, the effectiveness of this procedure will have to be judged on the basis of the results but it seems a logical step especially since tracers have been used successfully to study runoff processes on watersheds (Pilgrim, 1977).

In order to assess the dispersive properties of the river in this manner, a number of experiments were conducted in conjunction with the Great Ouse River Division of the Anglian Water Authority. The experiments involved the release of a known quantity of potassium iodide solution into the river at an upstream point and the continuous monitoring of the dye concentration 55 km downstream at Clapham. A typical response of the tracer at Clapham, as shown in Fig. 3, indicates a pure time delay of some 64 h followed by an asymmetrical curve indicating the degree of mixing and dispersion within the river system. Also shown in Fig. 3 is the output of a multi-lag model with reach representations similar to equation (4) but with δQ_j and δQ_{j-n} replaced by the concentrations c_j and c_{j-n} respectively. Here, the 14 reaches constituting the model were each characterised by n compartments and it was found that the response shown in Fig. 3 could be obtained with $n = 3$. A trial and error method was used to obtain this "best" value for n but it would clearly be possible to use a more systematic procedure such as that suggested by Himmelblau & Yates (1968).

Having obtained a satisfactory value of n from the tracer experiment data, this same value of n was used to define the structure of the flow routing model (4). A typical example of the performance of the model

Table 1. Reach data

Reach Number	Reach length l m	Base width, b m	Bank slope s	Minimum depth, h_w m	Additional depth, h_m m	Maximum (Banks full) flowrate, Q_m m^3 s^{-1}
1	1800	20.1	2	1.52	1.37	88.9
2	3400	20.1	2	1.52	1.37	88.9
3	3000	20.1	2	3.04	0.52	88.9
4	4800	20.1	1	1.37	1.82	88.9
5	4000	20.1	1.5	1.58	1.00	66.6
6	4800	17.6	2	1.76	1.37	75.6
7	5300	19.9	2	1.37	1.67	83.8
8	8040	20.7	2	1.52	1.52	105.4
9	8850	17.0	2	0.91	1.52	73.1
10	6260	17.3	2	2.31	1.75	120.1
11	3370	17.3	1.5	1.52	1.52	65.7
12	3220	17.3	1.5	2.73	0.61	65.7
13	4000	17.3	1.5	1.52	1.21	65.2
14	5150	17.3	1.75	2.13	0.91	65.2

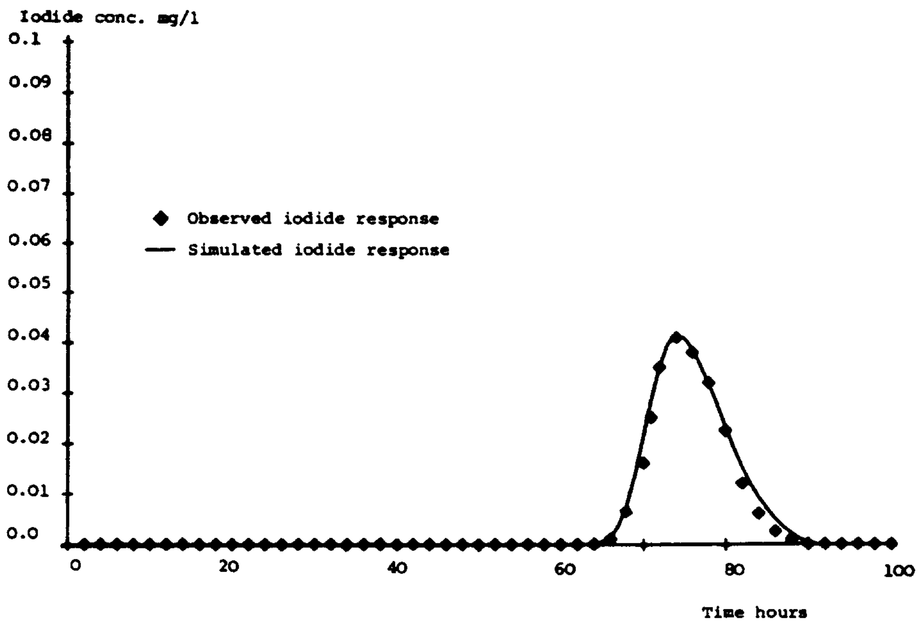

Fig. 3. Observed and simulated iodide response.

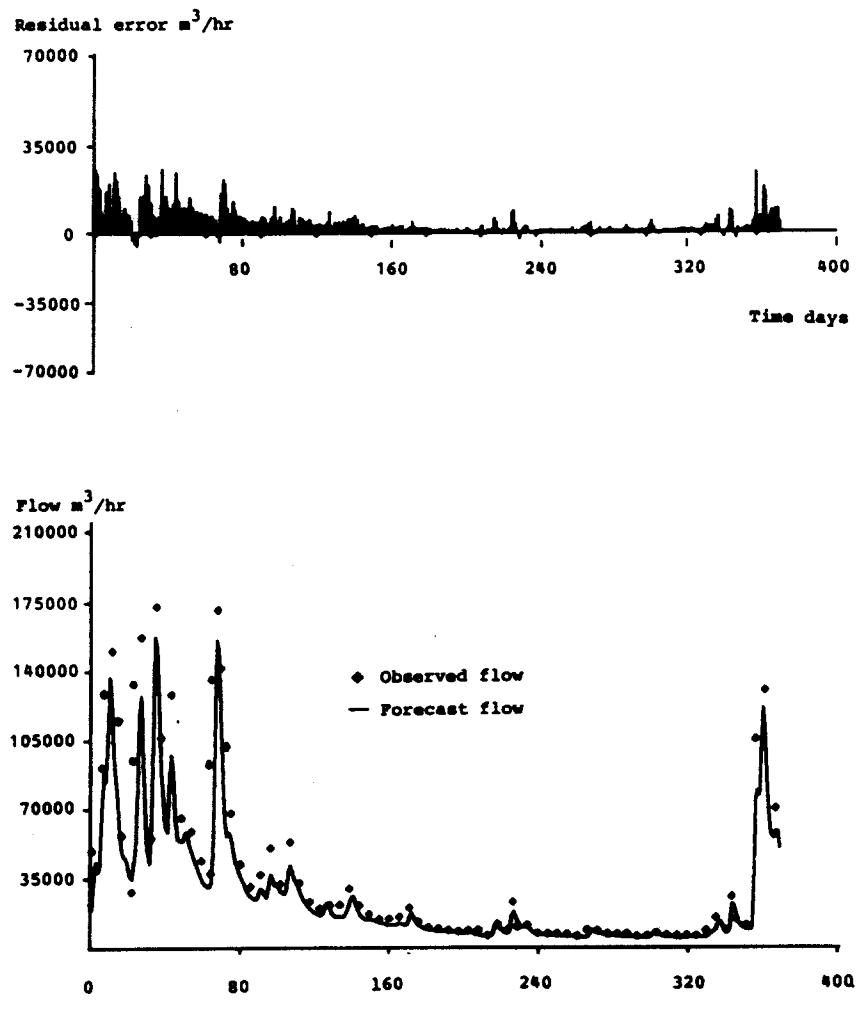

Fig. 4. Deterministic flow forecast at Harrold 1972.

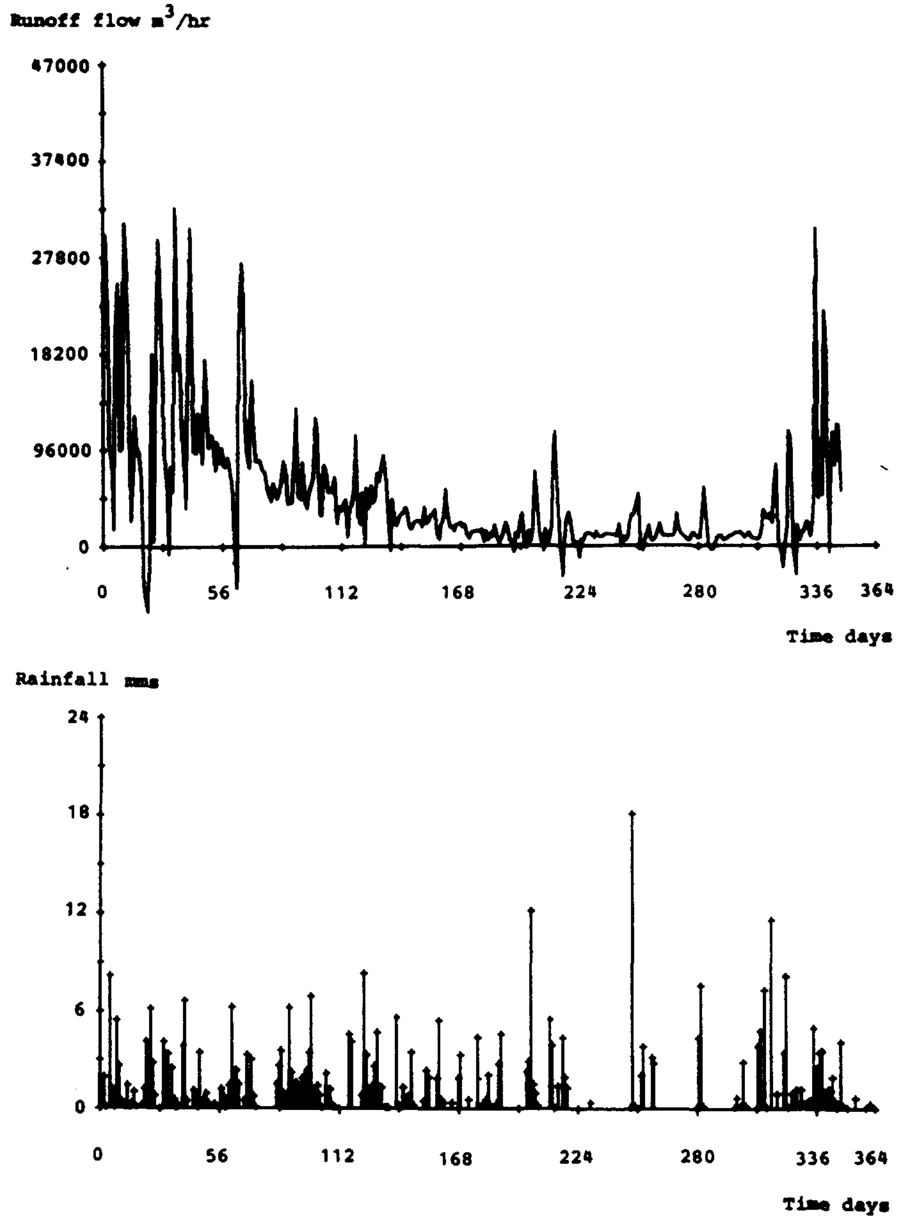

Fig. 5. Rainfall and runoff data 1972.

obtained in this manner is shown in Fig. 4, which illustrates the model forecast flow at Harrold over a one year period based on the input data over the same period collected at the upstream flow gauging station. Although the fit of the deterministic model is not perfect, the main patterns of flow variation are faithfully reproduced in the predicted flows and this can be taken as an indication of the adequacy of the multi-lag representation.

The fact that the deterministic model fit is not perfect is hardly surprising since the downstream flow forecast is totally dependent upon upstream flow and the model does not allow for rainfall–runoff effects along the river. We would expect, therefore, that the error between the forecasted and actual flow would show some patterns of variation that might be attributed, at least in part, to rainfall–runoff phenomena. This error sequence is also shown in Fig. 4 and it is clear that it exhibits a definite long term "seasonal" variation with additional short term effects. A comparison of the error series with concurrent rainfall variations, as shown in Fig. 5, demonstrates that these short term variations are strongly, but not linearly correlated with rainfall. It is clear, therefore, that a large part of the error series may be considered as an indication of runoff or lateral flow into the river.

3. RAINFALL–RUNOFF CHARACTERISATIONS IN TIME-SERIES TERMS

As we have pointed out previously, the conventional approach to modelling rainfall–runoff effects is based on an attempt to simulate deterministically the

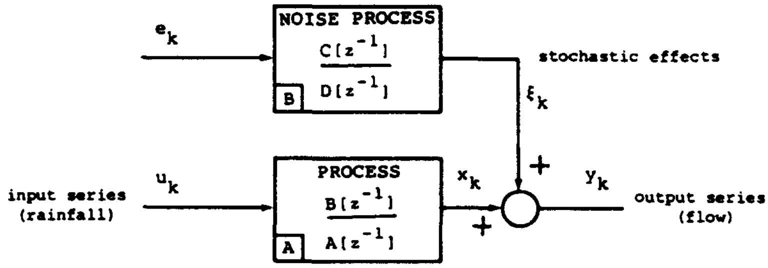

Fig. 6. Rainfall–runoff time-series model.

entire land phase hydrology in detailed terms and this seems inappropriate in the present circumstances. In the Bedford-Ouse study, therefore, a rather simple stochastic input–output approach to rainfall–runoff characterisation is employed in which the dynamic behaviour between the rainfall and the estimated runoff is inferred directly from the available data. These data, which are shown in Fig. 5, represent the daily values of rainfall averaged from three rain gauges in the Bedford-Ouse area over 1972, together with estimated daily runoff flow values for the same period, obtained as the residual error from the deterministic flow simulation discussed in the previous section. For obvious reasons, such data are often termed *time-series data* and, the analysis of the data is, in consequence, referred to as *time-series analysis*.

It is difficult to say when time-series analysis was first used as a tool for the modelling of rainfall–flow behaviour, but Eagelson *et al.* (1966) and Clarke (1974) both made initial contributions in this area. Since 1971, the time-series approach has received increasing attention and useful contributions have been made by Kashyap & Rao (1973), Norton (1975) Natale & Todini (1974) and Szollosi-Nagy (1976). In the latter two cases, an approach based upon the impulse response or weighting sequence model is utilised. It is well known that this model, which represents an attempt to model the unit hydrograph directly, suffers from a number of estimation problems: in particular, because the impulse response (or unit hydrograph) is nominally infinite dimensional, the model is often parametrically rather inefficient and the estimates can have relatively high estimation error variance (see later section 3.1).

The approach to time-series analysis used in the present study avoids the problems associated with the weighting sequence model by employing a parametrically more efficient representation in the form of a finite dimensional transfer function between rainfall and run-off.|| The parametric efficiency of the transfer function model is well known and has been discussed in some detail, for example, by Box & Jenkins.

In the present approach, the parameters characterising the time-series model between the "input" rainfall and the "output" runoff are estimated by a *recursive* instrumental variable (IV) procedure in which the estimates of parameters characterising the transfer function are updated a sample at a time while working serially through the data. The advantages of such a recursive approach over the more conventional "block data" least squares analysis discussed in most statistical texts will become obvious as we proceed. For the moment, however, it will suffice to say that it adds flexibility to the analysis by allowing explicitly for the estimation of possible parametric change over the observation interval.

3.1 *The form of the time-series model*

The form of the model which provides the starting point for the time-series analysis used here is the discrete-time z (or pulse) transfer function representative of a linear stochastic dynamic system, as shown in Fig. 6. This model, which is also the basis of the time-series analysis technique suggested by both Box & Jenkins (1970) and Bohlin (1970), relates the output of the system, y_k to two inputs: the first, u_k, is considered completely deterministic and measurable, while the second, e_k, is a zero mean, purely stochastic "white noise" variable with variance σ^2 which is uncorrelated both in time and with u_k. In the present hydrological context, y_k is the measured or inferred runoff flow while u_k represents the measured rainfall input. The stochastic input e_k is included to account for unavoidable uncertainties in the relationship, such as those arising from additional disturbances and measurement noise.

It is the presence of the noise model which most differentiates the time-series approach to hydrological modelling from the more conventional approach used heretofore. The explicit inclusion of the stochastic influence is an admission that it is not possible to explain the real world in completely deterministic terms; that, try as we might, there will remain an element of uncertainty in any analysis based on real data.

The model shown in Fig. 6 is characterised by the two "transfer functions" relating x_k to u_k and ξ_k to e_k. In the first case, the hypothetical noise-free runoff x_k at the kth observational instant is related to past values $x_{k-1}, x_{k-2}, \ldots, x_{k-n}$ as well as to present and past values of the rainfall input u_k by the discrete time model

$$x_k + a_1 x_{k-1} + \ldots a_n x_{k-n} = b_0 u_k + \ldots b_n u_{k-n} \quad (12)$$

|| This approach has been outlined briefly in previous publications; Whitehead & Young (1975).

Similarly the noise variable ξ_k is related to e_k by a discrete-time auto-regressive moving average (ARMA) model of the form

$$\xi_k + c_1\xi_{k-1} + \ldots c_n\xi_{k-n} = e_k + d_1 e_{k-1} + \ldots d_n e_{k-n} \quad (13)$$

The overall model is completed by the output equation which merely defines y_k as the sum of x_k and ξ_k, that is

$$y_k = x_k + \xi_k \quad (14)$$

Since equation (13) relates stochastic variables it remains to define the statistical properties of the input "white noise" e_k; in particular, we require e_k to be zero mean, serially uncorrelated with constant variance σ^2 and, finally, to be uncorrelated with u_k, i.e.,

$$E\{e_k\} = 0;\ E\{e_k e_j\} = \begin{cases}\sigma^2 & k = j\\ 0 & k \neq j\end{cases};\ E\{e_k u_j\} = 0 \quad \text{for all } k, j$$

where E is the expected value operator.

Equation (12) relates the hypothetical noise-free runoff x_k to the rainfall u_k. By substituting from equation (14) into equation (12) it is possible to obtain the discrete-time relationship between the *observed* (in this case estimated) runoff y_k and u_k as

$$y_k + a_1 y_{k-1} + \ldots a_n y_{k-n} = b_1 u_k + \ldots b_1 u_{k-n} + \eta_k \quad (15)$$

where η_k is a stochastic residual defined as

$$\eta_k = \xi_k + a_1\,\xi_{k-1} + \ldots a_n\,\xi_{k-n} \quad (16)$$

In general, η_k will be both serially correlated in time and correlated with y_i, $i = k, \ldots, k-n$. Equation (16) can now be expressed conveniently in vector form as

$$y_k = \mathbf{z}_k^T\mathbf{a} + \eta_k \quad (17)$$

where $\mathbf{z}_k^T = [-y_{k-1}, \ldots, -y_{k-n}, u_k, \ldots, u_{k-n}]$ and $\mathbf{a} = [a_1, \ldots, a_n, b_0, \ldots b_n]^T$

Equations (12) and (13) can also be expressed in a rather convenient operational notation form by the introduction of the backward shift operation z^{-1}, where

$$z^{-1}x_k = x_{k-1}$$

In this way, equation (12) can be written as

$$A[z^{-1}]x_k = B[z^{-1}]u_k \quad \text{or} \quad x_k = \frac{B[z^{-1}]}{A[z^{-1}]}u_k \quad (18)$$

while equation (13) becomes

$$C[z^{-1}]\xi_k = D[z^{-1}]e_k \quad \text{or} \quad \xi_k = \frac{D[z^{-1}]}{C[z^{-1}]}e_k \quad (19)$$

where

$$A[z^{-1}] = 1 + a_1 z^{-1} + \cdots + a_n z^{-n}$$
$$B[z^{-1}] = b_0 + b_1 z^{-1} + \cdots + b_n z^{-n}$$
$$C[z^{-1}] = 1 + c_1 z^{-1} + \cdots + c_n z^{-n}$$
$$D[z^{-1}] = 1 + d_1 z^{-1} + \cdots + d_n z^{-n}$$

An alternative expression for y_k is then given simply by

$$y_k = \frac{B[z^{-1}]}{A[z^{-1}]}u_k + \frac{D[z^{-1}]}{C[z^{-1}]}e_k \quad (20)$$

and we see that the transfer functions characterising the "system model" and "noise model" blocks of Fig. 6 are given by

$$\frac{B[z^{-1}]}{A[z^{-1}]} \quad \text{and} \quad \frac{D[z^{-1}]}{C[z^{-1}]}, \text{ respectively.}$$

It is clear from equation (20) that, if we divide the polynomial $B[z^{-1}]$ by $A[z^{-1}]$, we obtain a general model of the form

$$y_k = G[z^{-1}]u_k + \xi_k \quad (21)$$

where the polynomial $G[z^{-1}]$ is nominally infinite dimensional and can be defined as

$$G[z^{-1}] = g_0 + g_1 z^{-1} + \cdots + g_m z^{-m} + \cdots + g_r z^{-r} \quad (22)$$

in other words,

$$y_k = g_0 u_k + g_1 u_{k-1} + \cdots + g_r u_{k-r} + \xi_k. \quad (23)$$

This is the impulse response or weighting sequence model referred to earlier, in which the output flow of time k is nominally given by the weighted sum of *all* past values of rainfall and the stochastic term ξ_k. Its name derives from the fact that, in the deterministic case, (i.e. $\xi_k = 0$) the response y_k to the unit impulse u_k $\left(\text{i.e. } u_k = \begin{Bmatrix}1, k = 0\\ 0, k \neq 0\end{Bmatrix}\right)$ is given by $y_k = g_k$, $k = 0, 1, \ldots, \infty$. The impulse response model (23) is also interesting in the hydrological context because it is directly equivalent to the *unit hydrograph* representation used in hydrological systems analysis for some considerable time. This is because, in the present situation, the unit impulse of u_k is equivalent to a unit storm disturbance.

3.2 *The approach to time series analysis*

The problem of time-series analysis posed by the model (20) involves the estimation of the parameters characterising the polynomials A, B, C and D on the basis of the time series data y_k, u_k obtained over some observation interval $k = 1, 2, \ldots, T$. The approach to this problem utilised in the present study is based on the three phases of model building: namely, model structure identification; parameter estimation and, finally, model validation (see Young, 1977).

Identification and parameter estimation. The initial identification studies make use of the correlation approach of Box & Jenkins (1970) but, in the present rather special situation, this analysis is supplemented by a new approach based on the use of recursive methods of parameter estimation. (see also Beck & Young, 1976).

It is not intended to describe the recursive methods of parameter estimation in detail here since this has been done elsewhere (e.g. Young, 1974). Suffice it to say that they are based on recursive instrumental variable-approximate maximum likelihood (IV-AML) algorithms which are used to estimate the elements of the parameter vectors $\mathbf{a}$ and $\mathbf{c}$, where $\mathbf{a}$ is the vector of unknown parameters in the A and B polynomials, while $\mathbf{c}$ is a similar vector for the C and D polynomials, i.e.,

$$\mathbf{a} = [a_1, a_2, \ldots, a_n, b_0, b_1, \ldots, b_n]^T \quad (24)$$
$$\mathbf{c} = [c_1, c_2, \ldots, c_n, d_1, d_2, \ldots, d_n]^T$$

In these recursive algorithms, an estimate of the unknown parameter vector is updated recursively while working serially through the data. In the case of the vector $\mathbf{a}$, for example, the estimate $\hat{\mathbf{a}}_k$ of $\mathbf{a}$ at the kth point is given by an algorithm of the general form

$$\hat{\mathbf{a}}_k = \hat{\mathbf{a}}_{k-1} + \mathbf{g}_k \{y_k - \hat{y}_k\} \quad (25)$$

where the second term on the right hand side of the equation is, in simplistic terms, a correction factor based on the difference between the latest runoff flow measurement y_k and the "estimate" $\hat{y}_k$ of this flow based on the estimated model coefficients obtained at the previous $(k-1)$th instant; in other words, $\hat{y}_k$ is generated by the equation

$$\hat{y}_k = \mathbf{z}_k^T \hat{\mathbf{a}}_{k-1} \quad (26)$$

This equation follows directly from equation (17).

The advantage of the recursive formulation of the estimation problem is that it can be easily modified to allow for the possibility of parametric variation over the observation interval. There is insufficient space in the present paper to discuss this but it is described fully by Young *et al.* (1971), Young (1974), Whitehead & Young (1977) and Young (1978).

3.3 *The time series analysis in practice*

In the present example, initial identification studies based on Box–Jenkins methods of correlation analysis (Box & Jenkins, 1970) provide a rough indication of the impulse response characteristics (i.e. the unit hydrograph) between rainfall and runoff flow.¶ On the basis of this initial analysis, a candidate model for rainfall-flow is suggested of the form

$$y_k = \frac{b_1 z^{-1} + b_2 z^{-2}}{1 + a_1 z^{-1} + a_2 z^{-2}} u_k + \xi_k. \quad (27)$$

The recursive estimates of the coefficients in this model obtained using the recursive IV-AML algorithm display signs of temporal variability: in particular, the estimates of b_1 and b_2 appear to vary considerably over the observation interval even when the algorithm is informed that they are constant. To investigate this further, the b parameters are estimated using the IV-AML algorithm modified under the assumption that the b coefficients may vary over the observation interval, as mentioned in the previous section.

The estimation results obtained in this manner are shown in Fig. 7(a). It is clear that both b parameters show distinct variations over the year: in addition to a long term variation which seems indicative of seasonal influences, there are also short term effects which appear to occur predominantly when storms follow periods of dry weather. From the hydrological point of view these changes are most probably due to evapotranspiration effects in the long term and soil moisture in the short term. This soil moisture effect is particularly apparent on day 250, for example, where b_1 falls from 1000 to 120 following the end of a dry period.

The concept of an "effective rainfall" measure has been developed by hydrologists in order to allow for factors such as evapotranspiration and soil moisture. Penman (1950) and Grindley (1967) have proposed methods for estimating runoff for rural areas where the evaporation and transpiration effects are calculated as a direct loss depending upon temperature, relative humidity and vegetation types, and where the soil moisture effect is calculated as the cumulative balance of inflow and outflow from the soil storage zones.

In the Bedford-Ouse study, two simple relationships are utilised to allow for evapotranspiration and soil moisture effects.

First, in order to allow for the predominantly temperature dependent evapotranspiration effects a modified rainfall measure r_k^* is formed by modulating the basic rainfall r_k by a temperature dependent factor proportional to the difference between the prevailing mean monthly air temperature T_i and the overall maximum temperature T_m, i.e.

$$r_k^* = k(T_m - T_i)r_k; \quad k = \text{prop. constant.} \quad (28)$$

The soil moisture compensation is motivated by a simple analysis which has much in common with the Antecedent Precipitation Index (API) approach, but which uses a much simpler and parametrically more efficient approach to exponential weighting into the past. A measure of soil moisture content is obtained by filtering r_k^* by means of a discrete first order filter of the form

$$s_k = s_{k-1} + \frac{1}{T_s}(r_k^* - s_{k-1}) \quad (29)$$

where s_k is the output of the filter and T_s is a time constant to be chosen empirically. The final "effective" rainfall measure u_k is then obtained by modulating the modified rainfall r_k^* by s_k, i.e.

$$u_k = s_k r_k^* \quad (30)$$

In this manner, a prolonged period of dry weather will lead to a low value of s_k which, because of the

¶ Superior methods of identification are now available (Young *et al.*, 1978) which avoid some of the subjectivity inherent in the Box-Jenkins methods.

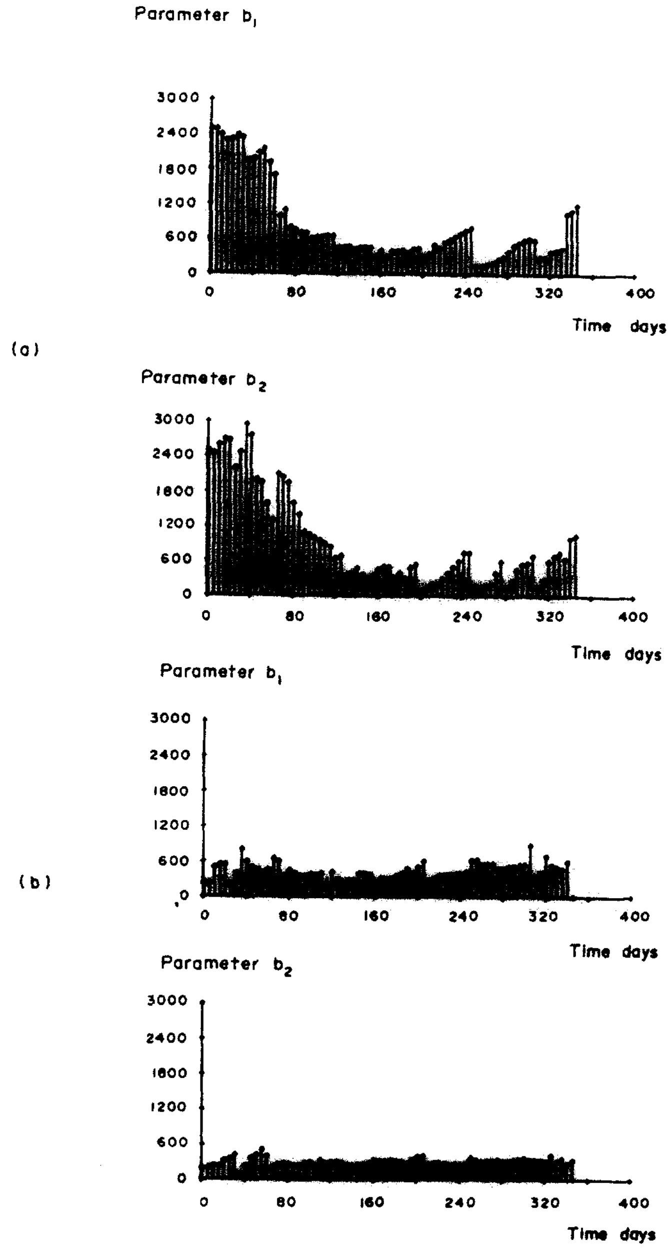

Fig. 7. Recursive estimation of parameters; (a) using actual rainfall-runoff data. (b) using effective rainfall-runoff data.

exponential weighting into the past imposed by the filter mechanisms (29), will not change dramatically after a single storm and will achieve higher sustained values only after continued rainfall. Thus, u_k should provide an indication of the effective rainfall, i.e., the rainfall which results in runoff. The reader will note the similarity of the parametrically efficient procedure used here with that used in the basic rainfall-runoff model (20) compared with model (21).

The proportionality constant k and the filter time constant T_s are both selected by a trial and error process with k chosen largely as a scaling factor. u_k generated from equation (30) utilising these values of k and T_s, is then used, together with the run-off flow estimate y_k, as the basis for the recursive estimation of the time-series model parameters. This procedure is repeated with different values of T_s *until relatively time-invariant estimates of the b_1 and b_2 parameters are obtained*, as shown in Fig. 7(b). Here T_s is set at 25 days which is, presumably, indicative of the slow dynamics associated with soil wetting and drying processes in the Bedford-Ouse catchment.†

At this point, the time-series model between the "effective" rainfall u_k and runoff flow y_k is character-

† In contrast, we have found recently that in the Australia Capital Territory, T_s values of only 5 days seem to pertain.

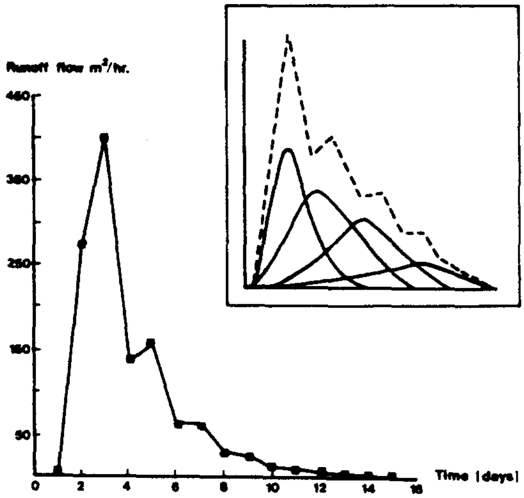

Fig. 8. Impulse response (unit hydrograph) of the rainfall–runoff system.

ised by coefficients which appear predominantly constant over the observational interval. Thus we have "identified" a candidate time-invariant parameter model structure which can now form the basis of more statistically efficient parameter estimation *under the assumption of parametric invariance.* Such estimation was carried out using an interative version of the IV-AML algorithm which yields relatively efficient estimates of the model parameter (i.e. estimates with low estimation error variance).§ The final estimates obtained in this manner together with their estimated approximate standard errors (in parentheses) are as follows:

$$a_1 = -0.09\,(0.017)$$
$$a_2 = -0.36\,(0.02)$$
$$b_1 = 275\,(1.01)$$
$$b_2 = 383\,(4.40)$$

In other words, the transfer function model between y_k and u_k is:

$$y_k = \frac{275z^{-1} + 383z^{-2}}{1 - 0.09z^{-1} - 0.36z^{-2}} u_k + \xi_k. \quad (31)$$

Correlation analysis applied to the estimated noise term ξ_k in this equation indicates that it is a serially uncorrelated series of random variables, thus eliminating the need for a more complicated ARMA noise model in this case.

The nature of the unit hydrograph (impulse response) for this model is shown in Fig. 8. Note the aparametric efficiency of the model: a normal unit hydrograph model would probably have required 8–10 coefficients for adequate characterisation, rather than the 4 required in (31). Note also the slight oscillatory nature of the response: this is probably due to the existence of different lagged pathways between rainfall and its effect on flow. In other words, the unit hydrograph obtained in Fig. 8 is probably an approximation to a "true" hydrograph which is the sum of a number of non-oscillatory unit hydrographs with different lag effects (see inset to Fig. 8). This demonstrates how the procedure proposed here can be a useful prelude to more detailed hydrological analysis, if this is demanded by the objectives of the study.

§ A more sophisticated version of this analysis which yield asymptotically efficient (i.e. minimum variance) estimates has been suggested by Young (1976), Young & Jakeman (1978) and Jakeman & Young (1978). In the present situation, statistical efficiency is not warranted.

4. THE COMPLETE STREAM FLOW MODEL

Having identified and estimated the rainfall-runoff model characteristics, it is now possible to combine this model with the basic stream flow model to produce the complete characterisation of the hydrological system as shown in Fig. 9. Figure 10 compares the river flow estimated from this overall model with the observed flow used in the estimation of the model. It is clear from the residual error plot that the estimation is fairly good; the mean percentage error is 8.6% which is well within the accuracy of the flow gauging station, which is estimated at 10% by the river authority. In addition, the model explains 99% of the variance of the original flow series and the errors are within 10% of the observed flows for 70% of the time.

The final and continuing stage in model building is validation; here the model's forecasting ability is evaluated on data *other than that used in the identification and estimation studies.* If the model continues to forecast well over this test data interval, it is assumed that it is *conditionally* acceptable, in the sense that, *as far as it is possible to test,* the model seems satisfactory. Figure 11 compares the observed stream flow measured at Harrold during 1973 with the flow generated by the model estimated previously over 1972. As might be expected, the fit is not as good as in 1972 but, except for the large storm on day 175, the forecasted flow behaviour is satisfactory with good general agreement between observed and simulated flows.

Flow data were also available for 1973 at Bedford and provided an additional check on the model validity. The *same* rainfall–runoff model to the river section between Harrold and Bedford as applied between Tickford Abbey and Harrold provides a satisfactory flow forecast at Bedford, as shown in Fig. 12. The validation exercise is, of course, a continuing procedure since the model will need to be reassessed in the light of future developments and additional data. However, the models as at present validated, provide what appears to be a reasonable characterisation of the flow behaviour in the study section of the Bedford–Ouse; a characterisation which is certainly good enough for the initial water quality modelling studies which are the subject of part II of this paper.

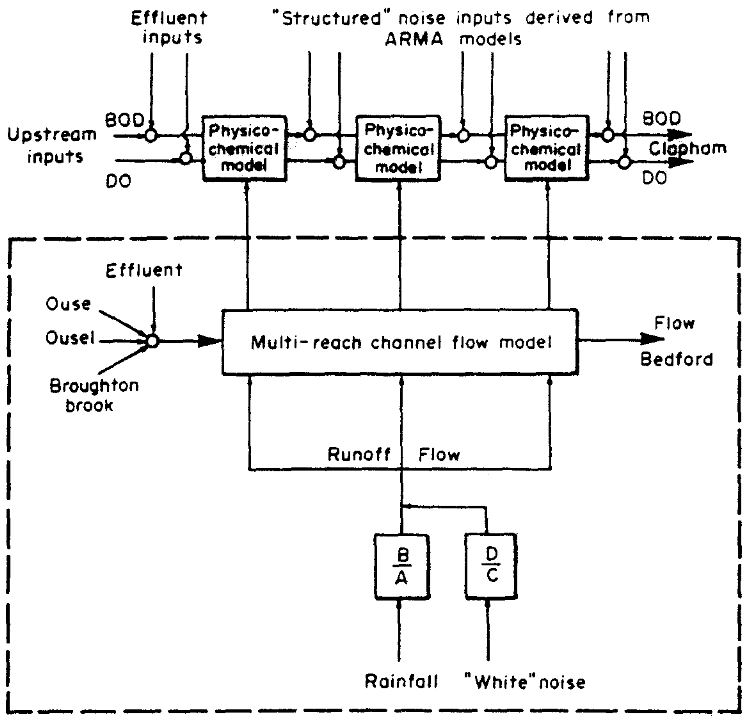

Fig. 9. Combined rainfall-runoff, river flow (within dotted box) and water quality model.

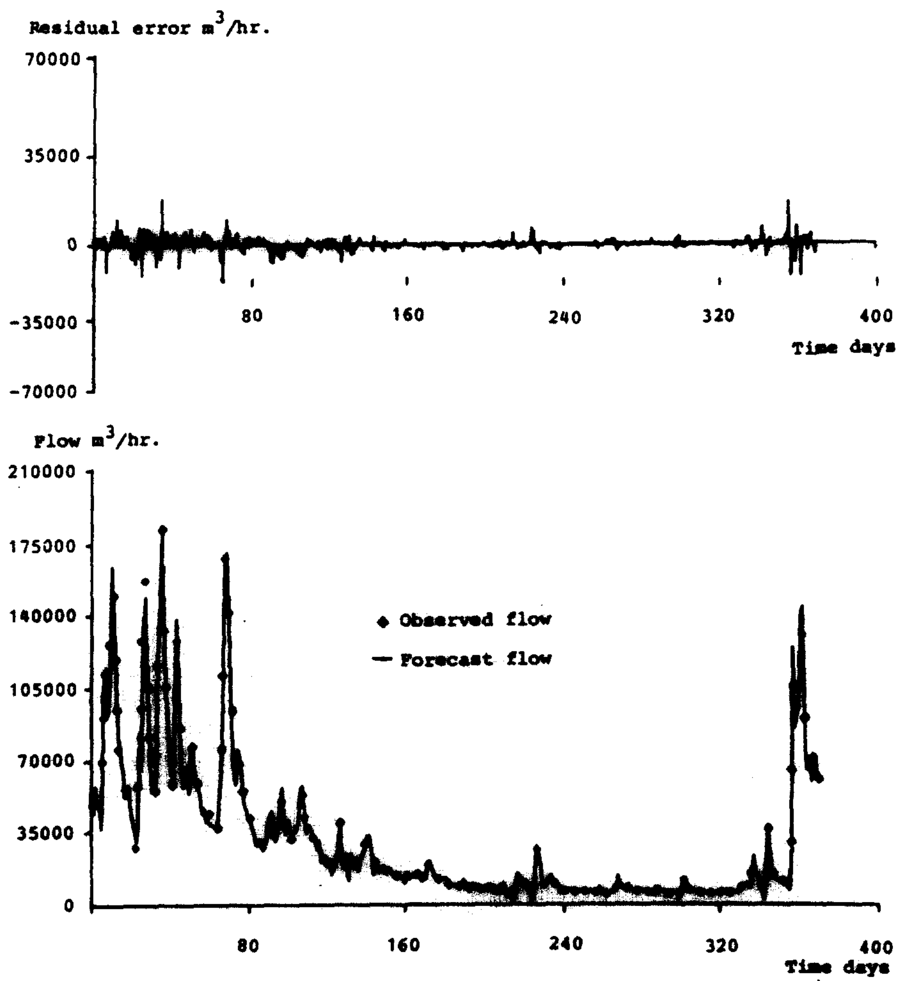

Fig. 10. Final flow forecast at Harrold 1972.

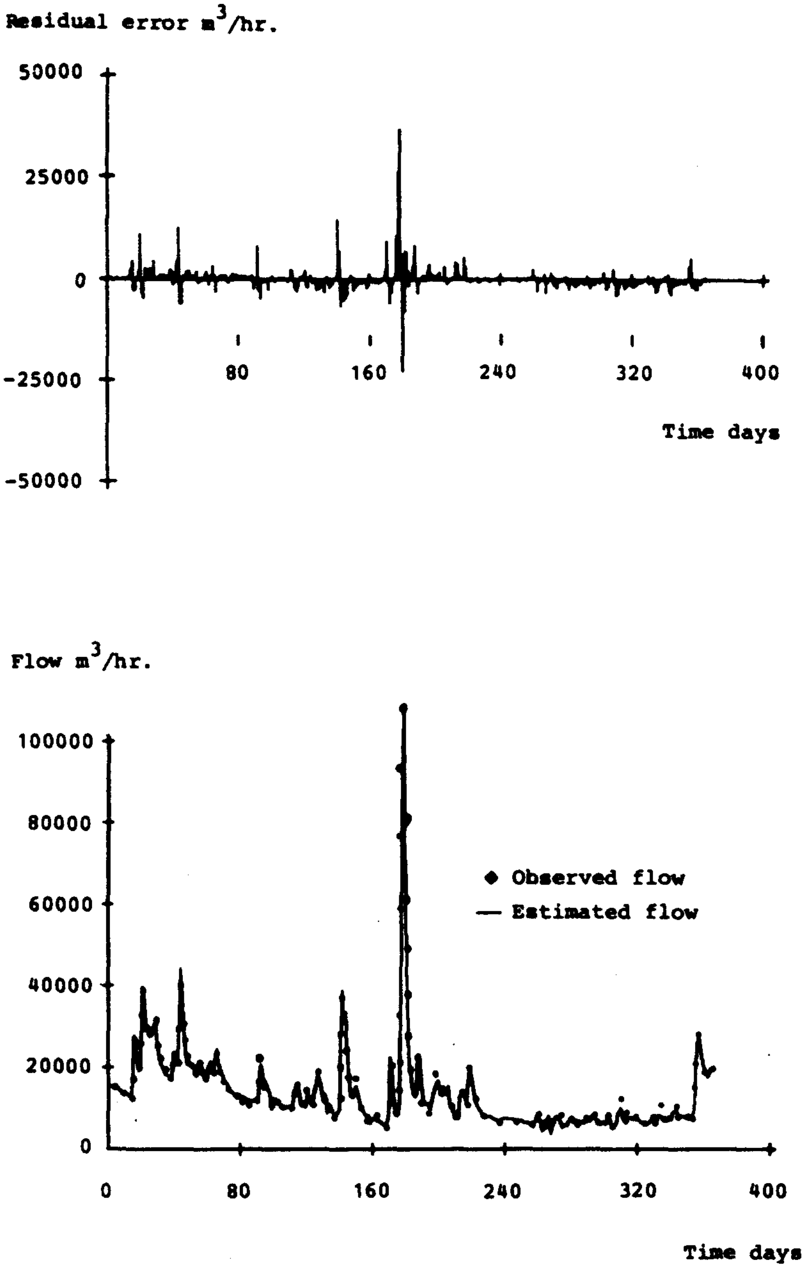

Fig. 11. Flow forecast at Harrold 1973.

5. CONCLUSIONS

In the first part of this paper we have concentrated on streamflow modelling. The approach described here has yielded models that are inherently stochastic, describe the "dominant mode" behaviour of the system and have been chosen carefully to suit the nature of the problem at hand. In addition a systematic methodology is proposed using the methods of recursive model identification, parameter estimation and validation.

In the present context the stream flow models are designed principally for use in the later water quality modelling exercises which are the subject of part II of this paper. But they can also be useful in other applications. For example, the stream flow characterisation is at a level of detail sufficient to be useful in hydrological system management and planning; for example, in flow and flood forecasting, dam control and design, flood damage assessment etc. In this latter context, it is interesting to note that the IV-AML estimation programs have been acquired by the Institute of Hydrology (Venn & Day, 1978) for on-going hydrological studies.

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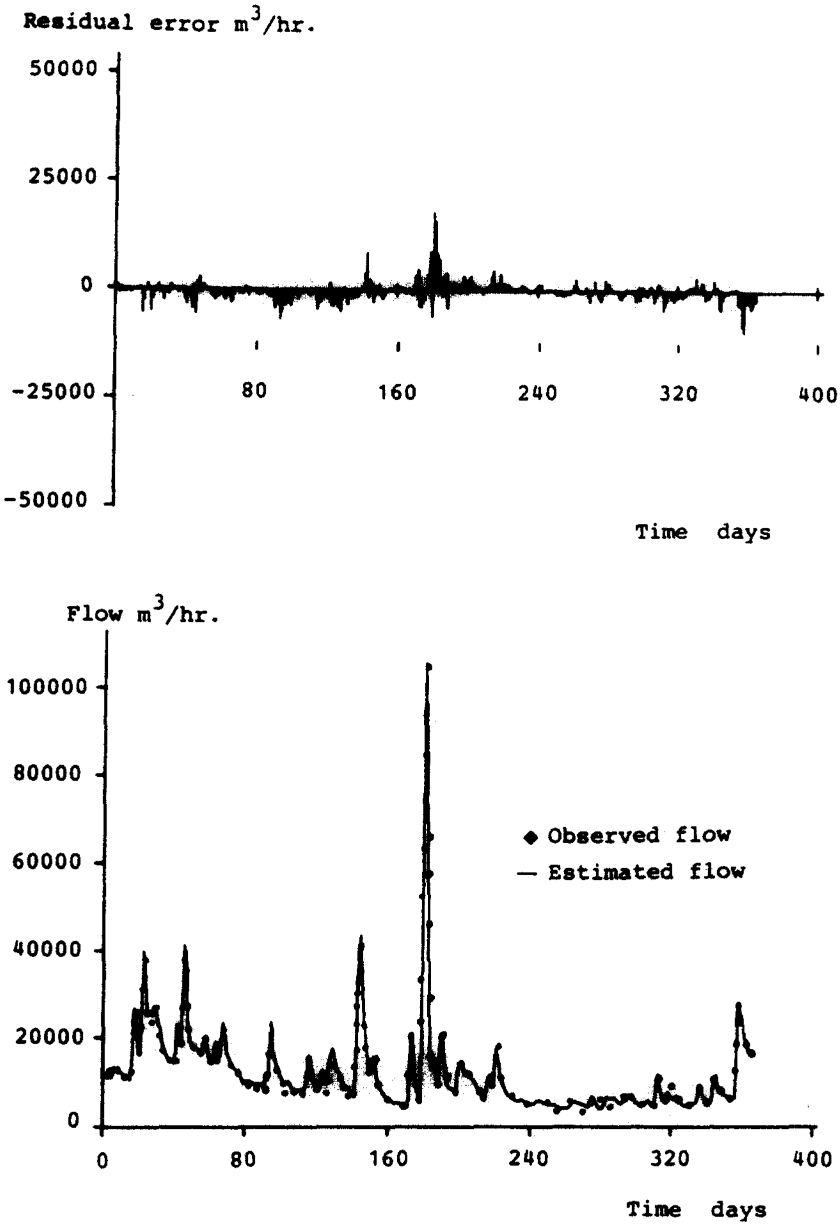

Fig. 12. Flow forecast at Bedford 1973.

ment and the Anglian Water Authority (formerly the Great Ouse River Authority). The authors are extremely grateful for this support and for the help received from many people in the Water Authority including Neville Taylor, Alan Fawcett and Roy Billington. The opinions presented here are, of course, the responsibility of the authors and do not necessarily represent the view of others concerned in the Bedford-Ouse study.

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