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PARAMETER CONSTRAINTS IN A STREAM ECOSYSTEM MODEL: INCORPORATION OF A PRIORI INFORMATION IN MONTE CARLO ERROR ANALYSIS *

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ABSTRACT

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Using a model of mass and nutrient dynamics in a stream, we evaluated the use of a priori information (e.g., mass balance constraints, limits for the state variables, etc.) to increase the accuracy and efficiency of Monte Carlo error analysis. The results indicated that a priori information can effectively eliminate physically unrealistic simulations and reduce uncertainty about the predicted values. Some types of a priori information proved more helpful than increased accuracy in measuring model parameters.

INTRODUCTION

The field of ecological simulation modeling has expanded rapidly over the past two decades, and it is clear that simulation models are both feasible and useful (Shugart and O'Neill, 1979). However, significant questions remain about the reliability and accuracy of these models (Mankin et al., 1974). In a series of recent papers (Gardner and O'Neill, 1982; Gardner et al., 1980a, 1980b, 1981; O'Neill and Gardner, 1979; O'Neill et al., 1980), we have focused on prediction uncertainty resulting from errors associated with parameter values in the model. This question has particular relevance to

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ecological problems and has been addressed by a number of investigators (Tiwari et al., 1978; Tiwari and Hobbie, 1976; Wood, 1976; Furness, 1979; Lettenmaier and Richey, 1979). Our approach begins with a deterministic model that is assumed to be correct. However, exact parameter values are unknown and must be estimated by some sampling procedure. Sampling results are used as the basis of Monte Carlo simulations that produce means and variances for model output.

This approach is different from stochastic modeling of a system. In a stochastic model, parameter distributions represent the actual frequency distributions of these rate processes in the system and the output is regarded as the population of potential behaviors. In error analysis, distributions represent sampling error in estimating parameter values and the frequency distribution of model output reflects uncertainty associated with predictions. The variances of parameters are not estimates of the 'true' distribution, but are expressions of estimation errors.

In our approach, we assume that parameter values are measured in independent experiments in the laboratory or field, a common procedure in ecosystem modeling (e.g., Shugart et al., 1974; Innes, 1978). This approach is different from estimating model parameters from a time-series measurement of state variables. For time-series problems some form of nonlinear least squares or Extended Kalman Filter (Beck, 1982) is used to estimate all parameters simultaneously. The uncertainty associated with parameters is based on estimation error (goodness of fit) and is unique for that set of time series measurements. Variances are typically smaller than estimates from independent measurements over a wider range of conditions.

Sensitivity methods may complement error analysis, but the two techniques are not the same. Sensitivity measures the change in a prediction resulting from a change in a parameter, assuming that parameter errors are small and that each parameter contributes independently to prediction error. In earlier studies (Gardner et al., 1980b), we found that sensitivity analysis tells more about the structure of the model than about prediction error. When there are large errors in all parameters, nonlinear and higher order effects (e.g., interactions between parameters) dominate the outcome, and sensitivity analysis is a poor indicator of the relationship between prediction uncertainty and errors on individual parameters (Gardner et al., 1981).

Monte Carlo methods are ideally suited to the investigation of parameter uncertainty. They can account for all of the information available to the modeler and provide a straightforward estimation of resulting errors. A serious drawback of the method has been that it includes all possible behavior of the model within the range of parameters used. Because it is impractical to inspect each iteration of the model, absurd as well as reasonable results are used to estimate means and variances of predicted values.

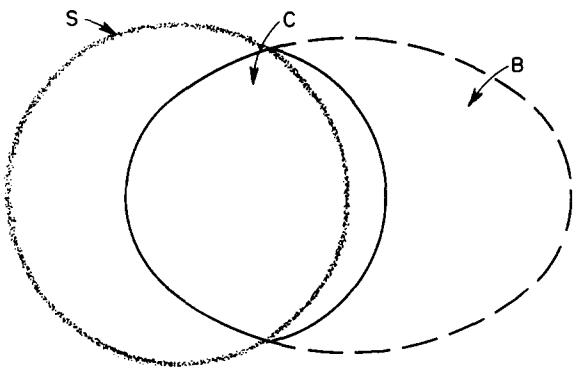

Fig. 1. A Venn diagram comparison of the region of model behavior with and without constraints (B and C, respectively) with the actual behavior of the system, S. S is fuzzy because all possible behaviors of S are not known.

The implications of this problem are illustrated by the Venn diagram in Fig. 1. The set S represents the known behaviors of the real system. The margins of S are indistinct because precise information on the total region of S is seldom available. The set B represents the possible behaviors of the model. We maintain that the situation represented in the figure is characteristic of all models. Some behaviors of the model match the known behaviors of the system, but the model is also capable of unrealistic behaviors. All elements in B are typically included in Monte Carlo simulations, and the estimated uncertainty is proportional to the area of B.

To obtain an unbiased estimate of prediction uncertainty, we must restrict the Monte Carlo simulations to the physical limits of the real system. This subset, C, of the total model behavior, M, is shown by the solid line in Fig. 1. The purpose of this study is to explore possible ways in which a priori information can be used to avoid physically impossible or unrealistic combinations of parameter values. We will illustrate this process with a nonlinear model of a stream ecosystem, developed in a previous study (O'Neill et al., 1979).

MODEL AND METHODS

The model considers mass and nutrient dynamics in a longitudinal reach of a stream. The state variables of the model are the mass, M; the average particle size, S, of organic matter; the biomass, H, of heterotrophs; and the mass of nitrogen, N, contained in the organic matter. Particle size is included to account for physical transport and microbial decomposition. The model is based on conservation of mass and considers inputs of organic matter and

TABLE I

Equations for the lumped stream model. The first column is the left hand side of the equations, and the remaining columns are terms for the right hand side.

	Litter fall	Through-fall and emergence	Physical transport	Heterotrophic feeding	Micro-bial respira-tion	Excretion	Assimi-lation	Predation and emergence
Mass of organic matter								
$dM/dt =$	10L		$-0.1aMS^{-1/3}$	$-2gNT/C_h(M+H)$	$-dS^{-2/3}M$			
Hetero-trophic biomass								
$dH/dt =$			$-0.01bH^2$	$gNH/C_h(M+H)$				$-0.1pH^2$
Nitrogen								
$dN/dt =$	10L C_1	10T	$-0.1aNS^{-1/3}$				$\frac{-gNH}{M+H}$	
Particle size								
$dS/dT =$	$\frac{10LS}{MS_1}(S_1-S)$			$\left(1-\frac{2N}{C_hM}\right)\frac{gHS}{M+H}$	$-dS^{1/3}$	$-\left(1-\frac{2N}{C_hM}\right)\frac{H_SgHS}{M+H}$		

nitrogen, hydrologic transport of all components and biotic processes such as feeding, respiration, excretion and predation. The terms of the model are shown in Table I.

TABLE II

Parameter definitions and values defining the uniform, triangular, and normal distributions for each parameter of the lumped stream model

Parameter * name	Parameter description	Uniform ** maximum	Triangular *** mode	Normal **** C.V.
L	Litter fall	5.0	1.5	50
a	Organic matter discharge coefficient	0.02	0.01	20
g	Heterotrophic feeding rate with no food limitation	2.5	0.3	330
C_h	Nitrogen (N) concentration in heterotrophs	0.13	0.11	10
d	Microbial respiration coefficient	0.01	0.002	50
b	Heterotrophic drift coefficient	0.01	0.001	50
p	Heterotrophic predation coefficient	0.1	0.02	50
C_l	N concentration of litter fall	0.02	0.01	20
T	N input in process other than litter fall	0.02	0.0001	1000
S_l	Average leaf size	3.0	0.5	50
H_s	Number of particles excreted per particle consumed	200.0	3.08	1000

* Functions used to calculate d, p and H_s. M_0, H_0, N_0 and S_0 are functions of the initial values of M, H, N and S, respectively ($M_0 = 200$; $H_0 = 2$; $N_0 = 3$; $S_0 = 0.2$).

$$D = S_0^{2/3}\left[0.0075 - \left((2gN_0H_0)/(M_0C_h(M_0 + H_0))\right)\right]$$

$$P = gN_0/(H_0C_h(M_0 + H_0)) - b$$

$$H_s = 1.0 + \left[\left((L/S_l)(S_l - S_0)\right)/\left(1 - \left((2N_0)/(C_hM_0)\right)\left((gH_0M_0)/(H_0 + M_0)\right)\right)\right]$$

** Minimum values for the uniform and triangular are 0.0 except for S_l and H_s which were 0.2 and 1.0, respectively.

*** The minimum and maximum values for the triangular distribution are the same as those for the uniform. The mean for the normal is the same as the mode for the triangular. Values randomly selected from the normal distribution which were below the minimum for the uniform were rejected and new values were chosen.

**** C.V. is the coefficient of variation (standard deviation/mean). Initial values for the state values, M, H, N and S, are 2000, 20, 30, and 0.2, respectively.

A detailed description of the model derivation is available elsewhere (O'Neill et al., 1979), so we will only briefly state the assumptions of the model. The physical transport of organic matter, mass and nitrogen are assumed to be inversely proportional to the mean diameter of particles. The feeding rate is considered to be a function of the available food and the biomass of consumers. As food becomes abundant, feeding is a function of consumer biomass. As food becomes scarce, feeding is a function of the available food supply. Heterotrophic growth rate is limited by food quality (i.e., nutrient concentration) and respiration is assumed to be 50% of assimilated energy. Heterotrophic drift and predation losses are assumed to be density dependent. Microbial decomposition is modeled as an enzymatic process limited by available surface area of the particles.

Monte Carlo simulations consider each parameter an independent random variable. Each Monte Carlo iteration began with the selection of model parameters from independent uniform, triangular or normal distributions. The uniform distribution generates all values within an interval (defined by minima and maxima, Table II) with equal probability (Hahn and Shapiro, 1968). The minimum values were set as the lowest physically possible limit. The maxima were chosen by estimating the largest realistic value that the actual stream system could be expected to take.

The triangular distribution requires a mode or most likely value, in addition to the maximum and minimum possible values. We have developed a simple algorithm which calculates the probabilities from trigonometric relationships without assuming a symmetrical distribution.

The normal distribution has often been used, when sufficient information is available to estimate means and variances, because the distribution of means of individual observations from any distribution will be normally distributed (Hahn and Shapiro, 1968). We have used algorithms for the independent normal distribution (McGrath et al., 1975) and a multivariate normal (IMSL, 1979) to generate parameters with means and variances as given in Table II. To prevent physically unreasonable values from being generated, the limits of the uniform distribution (Table II) were used to truncate values selected from the normal distribution.

Five hundred iterations of the Monte Carlo process were performed for each case considered. The parameter values and predicted state values at day 100 were recorded and statistical properties were analyzed using SAS (Barr et al., 1979).

PARAMETER CONSTRAINTS IN MASS-BALANCE MODELS

In applying the stream model, initial conditions must be measured for all four state variables and, in addition, some combination of field and labora-

tory experiments must be devised to estimate each of the eleven parameters. Ordinarily, the investigator will make optimal use of the state variable estimates to reduce the number of parameters that must be experimentally determined. This may be done by assuming that the state variables are at steady-state. Under this assumption, some parameters can be calculated by setting the derivatives to zero. It has been our experience that whatever set of assumptions is applied the investigator can avoid the expense of estimating all parameters. A careful reading of the original documentation for the stream model (O'Neill et al., 1979) indicates that three parameters of the model, p, d and H_s, were calculated from mass-balance considerations. In each case, an assumption of steady-state was made, an equation was set to zero, and values of p, d, and H_s were calculated. Thus, these three parameters are not independent random variables but are functions of other parameters and state variable values (Table II). In our Monte Carlo simulations, the other eight parameters were independently varied.

PARAMETER ESTIMATES AND ASSUMPTIONS ABOUT DISTRIBUTIONS

In Monte Carlo error analysis, the basic assumptions concern the statistical distributions used to describe parameters. If all that is known are the bounds of the distribution, then the least biased distribution is the uniform (Tiwari and Hobbie, 1976) because no information about the shape of the distribution is required. If, in addition to the range, the investigator also has an estimate of central tendency, e.g., a mean or mode, the triangular distribution is the least biased estimator (Tiwari and Hobbie, 1976). This additional information is particularly valuable since it insures that fewer of the random parameter values will be drawn from the extremes of the range. To use a normal distribution, the mean and variance of the parameters must be available.

We performed three sets of Monte Carlo simulations of the stream model by assuming that all parameters were either uniformly, triangularly, or normally distributed. Coefficients of variation for the normal distribution are based on estimates of the realistic ability to measure these parameters in the field and are adopted here from Gardner et al. (1981). Table III shows that all three distributions yield reasonable values for the expected state variables. Comparing the predictions from the three distributions, the normal made the least biased estimate of M but the most biased estimate of N. The triangular was most accurate in predicting N and S. On the basis of the mean values, there is little reason to choose between the distributions in the prediction of H.

In general, the model yields expected values and coefficients of variation which are insensitive to the assumed distributions. There appears to be a

TABLE III

Means (upper value) and coefficients of variation (lower value) of 500 predicted state values when 8 of 11 parameters were randomly varied from the uniform, triangular, and normal distributions (Table II). Parameters d, p, and H_s were calculated by functions shown in Table II. The deterministic solution was obtained by setting all parameters equal to their mean value (Table II).

State variables	Deterministic solution	Distributions		
		Uniform	Triangular	Normal
M	1635.0	2264.9	1991.8	1722.8
		44.4	38.1	32.9
H	22.9	24.9	23.7	18.5
		49.9	37.4	34.7
N	29.2	35.2	31.7	23.1
		52.1	42.0	40.4
S	0.1045	0.1167	0.1089	0.1217
		47.8	29.2	27.3

slight reduction in the coefficients of variation as one proceeds from the uniform to the triangular and then to the normal distribution, but the differences are generally small.

The expected values and coefficients of variation for the uniform distribution are very close to the normal. Randomly generated values from the uniform distribution result in more frequent extreme values; therefore, one would expect greater variability in the predictions. In the stream model, these extremes were compensated for by the functions used to calculate p, d, and H_s. The additional information required by the triangular (mode) and normal distribution (mean and variance) resulted in a greater central tendency of randomly generated parameter values, but this effect did not significantly alter the distribution of predictions (Table III).

EXAMINING THE INFLUENCE OF THE PARAMETER CONSTRAINTS

The most important factor causing this insensitivity to distributions is the constraint on the system's behavior imposed by the functions (Table II). If the other parameter values tend to force the system away from its expected behavior, the values calculated for d, p, and H_s will compensate for this tendency.

To get some impression of the influence of these functions, we can compare the results with a case in which the functions were not used (Table

TABLE IV

Means (upper value) and coefficients of variation (lower value) of predicted state values for three different methods of varying parameters from the normal distribution

State variables	Normal with functions *	Normal alone **	Multivariate normal ***
M	1722.8	402.2	1504.8
	32.9	152.9	50.6
H	18.5	26.6	24.7
	34.7	73.1	45.9
N	23.1	7.0	22.9
	40.4	120.8	61.8
S	0.1217	0.0683	0.0826
	27.3	356.9	81.9

* Normal with functions: 8 parameters were independently varied from the normal distribution and d, p and H_s calculated from a priori constraints (Table II).

** Normal alone: only information for the normal distribution (Table II) was used to independently vary the 11 stream model parameters.

*** Multivariate normal: covariances and maximum values for d, p and H_s were specified to approximate the effect of the functions.

IV). Without the functions, the predictions of state variables show tremendous bias. Although the prediction of H is reasonable, the other values are off by factors ranging from 2–4. The coefficients of variation are increased by as much as ten times.

Two factors were noticed when the results from the parameter constraint case were examined. The calculated values of p, d and H_s are truncated from the limits shown in Table II, and there are numerous correlations among the parameters. Therefore, it appears possible to duplicate the effect of the functions by choosing the parameters from a multivariate normal distribution which incorporates the variance–covariance relationships between the parameters. Previous results (Gardner et al., 1980a, 1980b; Gardner and O'Neill, 1982) indicate that this multivariate information can reduce predicted uncertainties by as much as four times. Five hundred iterations of this case were performed, and the results are shown in column 3 of Table IV. The multivariate case approximates the results of cases using the functions, but the coefficients of variation are always larger.

THE EFFICIENCY OF MONTE CARLO SIMULATIONS WITH CONSTRAINTS

Monte Carlo error analysis may be inefficient (McGrath et al., 1975) if a large number of iterations are required for reasonable estimates of means

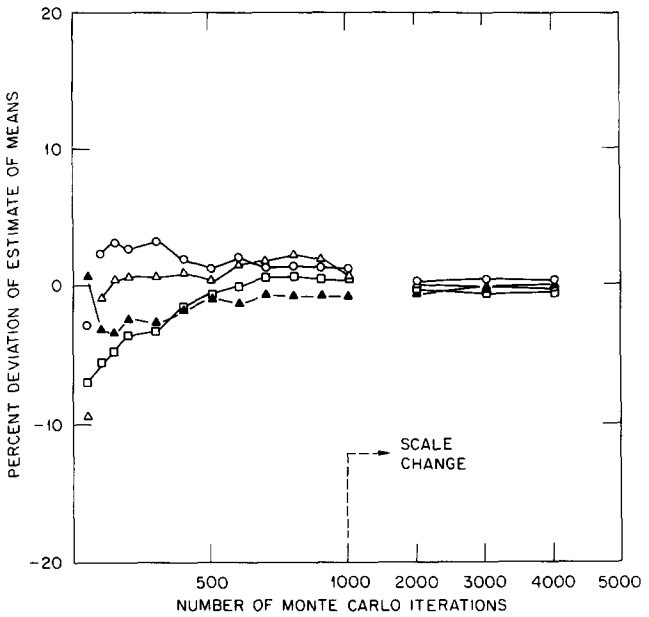

Fig. 2. The estimate of the mean of state values of the stream model for successively increasing numbers of Monte Carlo iterations. Estimates are expressed as a percent deviation from the calculated values at 5000 iterations. State values: M □——□; H ○——○; N △——△; O ▲——▲.

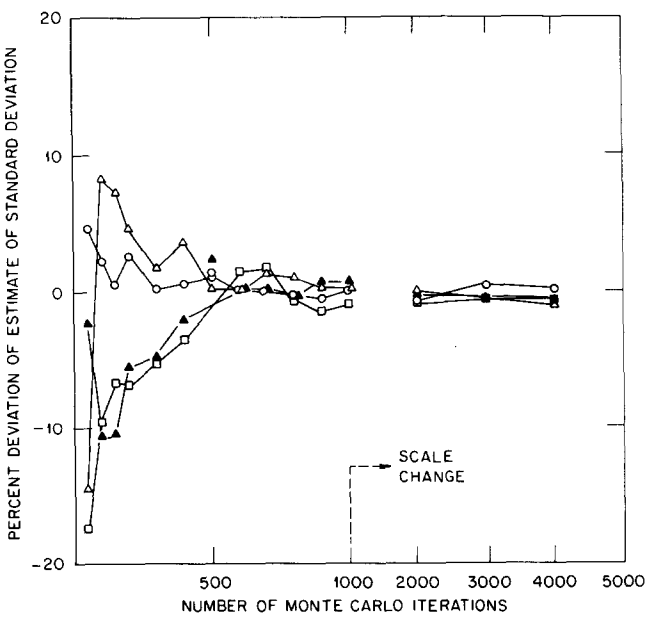

Fig. 3. The estimate of the standard deviation of state values of the stream model for successively increasing numbers of Monte Carlo iterations. Estimates are expressed as a percent deviation from the calculated values at 5000 iterations. State values, as in Fig. 2.

and variances. Unless a large number of iterations are obtained, a few extreme values will dominate the results. In our approach, extreme values have largely been eliminated by a priori constraints. As a result, the Monte Carlo process becomes much more efficient.

Figure 2 shows the percentage deviation between the mean and standard deviation for 5000 Monte Carlo iterations, and the values calculated as a running average. The results have converged at 500 iterations to be within 3% of the final calculation. Thus, the constraints in this analysis produce more reasonable error bounds and more accurate estimates of the means, and also improve the efficiency of the Monte Carlo procedure.

CONSTRAINTS BASED ON STATE VARIABLES

Another approach to constraining the stream model is to set limits on the state variables. For instance, all of the processes operating in the stream tend to decrease the average particle size, S. As a result, the value of S can never be larger than S_1, the size of leaves being deposited into the stream system. This information can be used as a check during the Monte Carlo simulation. If S is larger than S_1, that iteration, as well as the parameter set associated with it, is rejected and a new iteration begun.

This approach improves the predictions of the state variables compared to the normal distribution without any constraints (Table V). The variances are smaller, with the exception of H which contributes little to the prediction of

TABLE V

Comparison of the means and coefficients of variation of 500 predicted state values of the lumped stream model when all 11 parameters were randomly varied from the normal distribution (Table II); all 11 parameters were varied from the normal distribution, but simulations where S was less than S_1 were rejected; and 8 of 11 parameters were varied from the normal distribution with d, p, and H_s calculated by functions

State variables	Normal alone	Normal S not $< S_1$	Normal with functions
M	402.2	1482.8	1722.8
	152.9	56.1	32.9
H	26.6	22.2	18.5
	73.1	103.2	34.7
N	7.0	29.1	23.1
	120.8	45.1	40.4
S	0.0683	0.0803	0.1217
	356.9	85.2	27.3

particle size, S. Because the constraint on S involves a single state variable, the reduction in the coefficients of variation are consistently less than the functions which involve constraints on p, d, and H_s. Remember also that we have only set an upper bound on the value of S, while the functional constraints (Table II) bound both upper and lower values.

Although some improvement in error bounds is achieved by state variable constraints (Table V), this approach requires that simulations be performed in order to determine whether or not the constraints are violated. For example, using the normal distribution, 2315 additional simulations of the model were needed before 500 satisfactory realizations were achieved. When we used the uniform distribution, 23308 additional simulations were required. Therefore, constraints on state variables are significantly less efficient than parameter constraints.

INCORPORATING A PRIORI INFORMATION

Our results indicate a logical sequence for incorporating a priori information into Monte Carlo simulations. The following is a prioritized list of procedures based on effectiveness in constraining the system, ease of implementation, and effect on the simulation efficiency.

Parameter truncation

The simplest way to constrain the problem is by truncating each parameter at physically realistic upper and lower bounds. If values beyond these limits are drawn randomly, they are discarded and a new value is selected.

Mass-balance constraints

The second step is to examine possible constraints on the derivative of the state equations. This is the approach we used to develop the functions in Table II. If steady-state is appropriate for the problem, then the number of independent measured parameters can be reduced.

If steady state is too severe an assumption, other constraints may still be reasonable. These constraints might involve the maximum and minimum values which the derivatives can take. If a range can be reasonably estimated, the equation could be set equal to each of the extreme values. This yields an upper and lower bound on some parameters. A uniform distribution can then be used to draw values between these limits.

A number of additional approaches suggest themselves. Perhaps some derivatives can only be negative when the remainder of the state variables take on certain values. Perhaps one derivative can never be greater than

another derivative. Any of these approaches specify constraints on the parameters, with the results that the Monte Carlo analysis can become quite efficient. Investing the effort to uncover these relationships is strongly recommended.

State-variable constraints

Constraints on state variables, such as S less then S_1 (Table V), are of lower priority because they involve many additional simulations of the model. Often, if proper constraints can be placed on the derivatives, this step is unnecessary. When our model is constrained by the functions in Table II, the additional constraint, S less than S_1, does not eliminate any further parameter sets.

The possibilities of applying this class of constraints is only limited by the creativity of the analyst. Upper and lower bounds might be placed on the state variables. In our model, the quantity N/M, which represents the concentration of nutrient in the organic matter, might be constrained to remain below some reasonable value. Again, the criterion must be that the investigator would clearly eliminate a model run if this criterion was violated. The normal, sensible checks which the modeler would apply are simply incorporated into the analysis.

Correlations between parameters

In unusual circumstances, the investigator may have data that demonstrate a significant correlation between parameters. This information is incorporated by drawing these parameters from a multivariate normal distribution which incorporated this information.

In previous studies (Gardner et al., 1980a; O'Neill et al., 1980) we have estimated, a posteriori, covariances between parameters. This was done by comparing Monte Carlo results to data and rejecting unreasonable iterations. The parameter sets which remain (those which generated 'reasonable' results) are then statistically analyzed for changes in means, variances, and correlations between parameters. The model is re-run using the a posteriori information to generate a more reasonable estimate of the uncertainty of the model. This approach, which is akin to parameter fitting, is obviously inefficient, requiring a second set of Monte Carlo simulations.

In many cases, the investigator will not have measured correlations between parameters. However, there may still be considerable a priori information. For example, an independent study may have regressed one parameter against another. This second parameter could then be calculated from the regression equation, and then varied about this estimate based on the error associated with the regression.

DISCUSSION

This study has attempted to use all the information available about a system to estimate the uncertainty of model predictions. The results indicate that great improvements can be made in estimating means and variances of predictions by placing realistic limits on model behavior. Such information is often available at little expense and, in our example, proved more useful in reducing variances and bias of estimators than did increased information about the frequency distributions of parameters.

There is an important restriction on the use of a priori information that must be considered. The constraints must correspond to physical or 'real' limits on the system. Only constraints that would produce clearly erroneous results should be admitted. If a particular behavior is possible even though unlikely, the possibility must remain in the analysis. Only limits based on objective criteria can be included. The intent of the constraint is to limit the system to physically realistic behaviors so that means and variances reflect the true uncertainty.

The most surprising result of the study is the insensitivity of prediction uncertainty to parameter distributions (Table III). It is not possible, on the basis of this one study, to evaluate the potential generality of this result. The stream model is very well-behaved and the parameter restrictions were based on a steady-state assumption. Therefore, three of the parameters were always being calculated to ensure that the model would approach a steady state. The peculiarities of the model we used or the ability to set precise limits on the parameters may have caused the insensitivity. The results must now be checked against additional models.

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