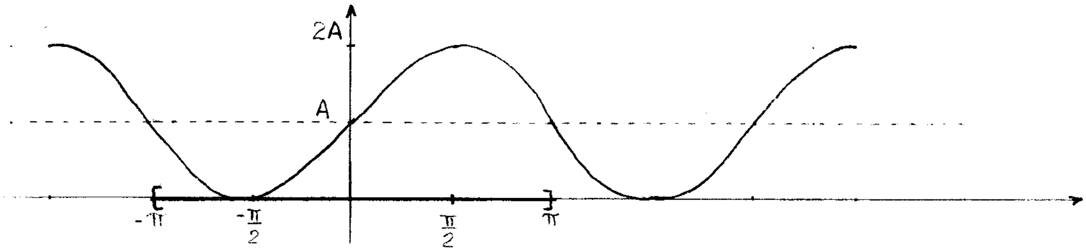
① Desarrollar en Serie de FOURIER las siguientes funciones periódicas

a - $f_1(x) = A + A\,\text{sen}\,x = A(1+\text{sen}\,x)$. $\quad A \in \mathbb{R}^+$

$-\pi \leq x \leq \pi$



$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} A(1+\text{sen}\,x)\,dx = \frac{A}{\pi}\left[\int_{-\pi}^{\pi} dx + \int_{-\pi}^{\pi}\text{sen}\,x\,dx\right]$$

$$a_0 = \frac{A}{\pi}\left[\Big|\, x \Big|_{-\pi}^{\pi} + \Big|-\cos x\Big|_{-\pi}^{\pi}\right] = \frac{A}{\pi}\left[(\pi-(-\pi)) + (\cancel{1}-\cancel{1})\right] = \frac{A}{\cancel{\pi}}\cdot 2\cancel{\pi}$$

$$\therefore \quad a_0 = 2A$$

Nota: La función $f(x) = A(1+\text{sen}\,x)$, es una función impar, lo cual indica que los coeficientes a_k son nulos. Verifiquemoslo.

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} A(1+\text{sen}\,x)\cos kx\,dx = \frac{A}{\pi}\left[\underbrace{\int_{-\pi}^{\pi}\cos kx\,dx}_{=0} + \underbrace{\int_{-\pi}^{\pi}\text{sen}\,x.\cos kx\,dx}_{=0}\right]$$

$$\therefore a_k = 0$$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} A(1+\text{sen}\,x)\,\text{sen}\,kx\,dx = \frac{A}{\pi}\left[\underbrace{\int_{-\pi}^{\pi}\text{sen}\,kx\,dx}_{=0} + \int_{-\pi}^{\pi}\text{sen}\,x\,.\,\text{sen}\,kx\,dx\right]$$

Nota: Recordemos de la teoria

$$\int_{-\pi}^{\pi}\text{sen}(mx).\text{sen}(kx)\,dx = \begin{cases} 0 & \text{si} \quad k \neq m \\ \pi & \text{si} \quad k = m \end{cases}$$

En nuestro caso $m = 1$ entonces para $k = m = 1$ tenemos

$$b_1 = \frac{A}{\pi}\int_{-\pi}^{\pi} sen^2 x \, dx = \frac{A}{\pi}\int_{-\pi}^{\pi} \frac{1 - \cos 2x}{2}\, dx = \frac{A}{2\pi}\left[\int_{-\pi}^{\pi} dx - \int_{-\pi}^{\pi} \cos 2x \, dx\right]$$

$$b_1 = \frac{A}{2\pi}\left[\left(\pi - (-\pi)\right) - \left(\frac{sen 2(\pi) - sen 2(-\pi)}{2}\right)\right] = \frac{A}{2\pi}\left(2\pi - 0\right) = A$$

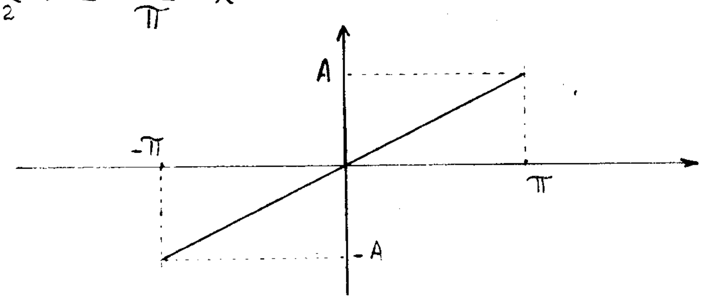
$$\therefore \; b_1 = A$$

Nota: Si $k \neq 1$; $(k = 2, 3, \ldots, n)$; entonces $b_2 = b_3 = \cdots = b_n = 0$

Entonces $F_1(x) = \frac{a_0}{2} + b_1 . sen\, x = A + A\, sen x$

Esta función está dada de tal manera que su desarrollo en serie de FOURIER tiene esa misma expresión.

b. $f_2(x) = \frac{A}{\pi} \cdot x$ $\quad -\pi \leq x \leq \pi$ $\quad A \in \mathbb{R}^+$



$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{A}{\pi} . x \, dx = \frac{A}{\pi^2}\left|\frac{x^2}{2}\right|_{-\pi}^{\pi} = \frac{A}{\pi^2}\left(\frac{\pi^2}{2} - \frac{(-\pi)^2}{2}\right) = 0$$

Nota: Como $f(x)$ es una función impar los coeficientes a_k son nulos

$$\therefore \; a_k = 0$$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{A}{\pi} . x \, sen\, kx \, dx = \frac{2A}{\pi^2}\int_{0}^{\pi} x . sen\, kx \, dx$$

Integramos por partes

$$u = x \quad \rightarrow \quad du = dx$$

$$dv = \operatorname{sen} kx \, dx \quad \rightarrow \quad v = -\frac{\cos kx}{k}$$

$$b_k = \frac{2A}{\pi^2}\left[-\frac{x\cos kx}{k} + \int_0^{\pi} \frac{\cos kx}{k}\,dx\right] = \frac{2A}{\pi^2}\left[\left|\frac{-x\cos kx}{k}\right|_0^{\pi} + \left|\frac{\operatorname{sen} kx}{k^2}\right|_0^{\pi}\right]$$

$$b_k = \frac{2A}{\pi^2}\left[\left(\frac{-\pi\cdot\cos k\pi}{\cancel{k}} - 0\right) + \frac{\operatorname{sen} k\pi}{k^2} - 0\right] = -\frac{2A}{\pi\cdot k}\cdot\cos k\pi$$

Entonces :

- si k es impar $\quad b_k > 0$
- si k es par $\quad b_k < 0$

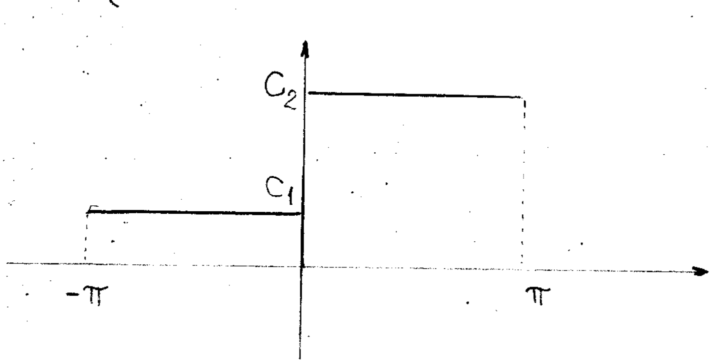
$$\therefore \quad b_k = \frac{2A}{\pi k}\cdot(-1)^{k+1}$$

Desarrollando la serie nos queda

$$F_2(x) = \frac{A}{\pi}x = \frac{2A}{\pi}\left[\operatorname{sen} x - \frac{\operatorname{sen} 2x}{2} + \frac{\operatorname{sen} 3x}{3} + \ldots + \frac{\operatorname{sen} 2nx}{2n} + \operatorname{sen}(2n+1)x\right]$$

C.-

$$f_3(x) = \begin{cases} C_1 & \text{si} \quad -\pi \leq x < 0 \\ C_2 & \text{si} \quad 0 \leq x \leq \pi \end{cases}$$



Calculo los coeficientes de FOURIER

$$a_0 = \frac{1}{\pi}\left[C_1\int_{-\pi}^{0} dx + C_2\int_{0}^{\pi} dx\right] = \frac{1}{\pi}\left[C_1\left(0-(-\pi)\right) + C_2\left(\pi - 0\right)\right] = \frac{1}{\pi}\left(C_1\pi + C_2\pi\right)$$

$$\therefore\ a_0 = C_1 + C_2$$

$$a_k = \frac{1}{\pi}\left[C_1\int_{-\pi}^{0}\cos kx\, dx + C_2\int_{0}^{\pi}\cos kx\, dx\right] = \frac{1}{\pi}\left[C_1 \underbrace{\left|\frac{\operatorname{sen} kx}{k}\right|_{-\pi}^{0}}_{=0} + C_2 \underbrace{\left|\frac{\operatorname{sen} kx}{k}\right|_{0}^{\pi}}_{=0}\right]$$

$$\therefore\ a_k = 0$$

Nota: El resultado de los coeficientes "a_k" anteriormente obtenido es real ya que se trata de una función impar desplazada "A" unidades sobre el eje de las ordenadas, donde $A = (C_1 + C_2)/2$, y recordemos que para dichas funciones los coeficientes a_k son nulos.

$$b_k = \frac{1}{\pi}\left[C_1\int_{-\pi}^{0}\operatorname{sen} kx\, dx + C_2\int_{0}^{\pi}\operatorname{sen} kx\, dx\right] = \frac{1}{\pi}\left[C_1\left|\frac{-\cos kx}{k}\right|_{-\pi}^{0} + C_2\left|\frac{-\cos kx}{k}\right|_{0}^{\pi}\right]$$

$$b_k = -\frac{1}{\pi}\left[C_1\left(\frac{1-(-1)^k}{k}\right) + C_2\left(\frac{(-1)^k - 1}{k}\right)\right]$$

Entonces:

- $b_k = 0 \quad \forall\ k$ par $\quad (k = 2, 4, \ldots, 2n)$
- $k = 1 \quad \Rightarrow \quad b_1 = \dfrac{2\,(C_2 - C_1)}{\pi}$
- $k = 3 \quad \Rightarrow \quad b_3 = \dfrac{2}{3}\cdot\dfrac{(C_2 - C_1)}{\pi}$
- $k = 2n+1 \quad \Rightarrow \quad b_{2n+1} = \dfrac{2\,(C_2 - C_1)}{(2n+1)\,\pi}$

Entonces la funcion desarrollada nos queda :

$$F_3(x) = \frac{a_0}{2} + \sum_{1}^{\infty} b_k \operatorname{sen} kx = \frac{C_1+C_2}{2} + \frac{2(C_2-C_1)}{\pi}\left[\operatorname{sen} x + \frac{\operatorname{sen} 3x}{3} + \frac{\operatorname{sen} 5x}{5} + \dots\right]$$

Nota : Observemos dos casos particulares de esta función

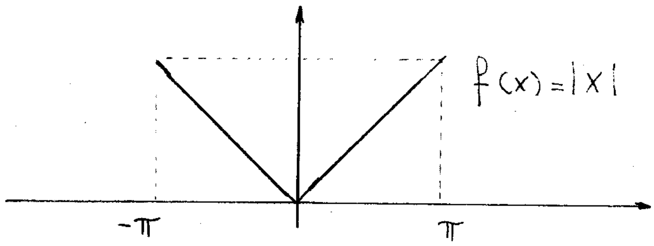
1° - $C_1 = -C_2 = \gamma$ (Onda cuadrada)

$$F_3(x) = \frac{-4\gamma}{\pi}\left(\operatorname{sen} x + \frac{\operatorname{sen} 3x}{3} + \dots\dots + \frac{\operatorname{sen}(2n+1)x}{(2n+1)}\right)$$

2° $C_1 = 0$, $C_2 = \gamma$ (Pulso - Cero ; Onda binaria)

$$F_3(x) = \frac{\gamma}{2} + \frac{2\gamma}{\pi}\left(\operatorname{sen} x + \frac{\operatorname{sen} 3x}{3} + \dots + \frac{\operatorname{sen}(2n+1)x}{(2n+1)}\right)$$

d. $f_4(x) = |x|$ $\qquad -\pi \leq x \leq \pi$



Nota : Esta función es par por lo tanto $b_k = 0$

$$a_0 = \frac{1}{\pi}\left[-\int_{-\pi}^{0} x\,dx + \int_{0}^{\pi} x\,dx\right] = \frac{1}{\pi}\left[-\left|\frac{x^2}{2}\right|_{-\pi}^{0} + \left|\frac{x^2}{2}\right|_{0}^{\pi}\right]$$

$$a_0 = \frac{1}{\pi}\left[-\left(0 - \frac{\pi^2}{2}\right) + \left(\frac{\pi^2}{2} - 0\right)\right] = \frac{1}{\pi}\left[\frac{\pi^2}{2} + \frac{\pi^2}{2}\right] = \pi$$

$$\therefore \qquad a_0 = \pi$$

$$a_k = \frac{1}{\pi}\left[-\int_{-\pi}^{0} x.\cos kx\,dx + \int_{0}^{\pi} x.\cos kx\,dx\right] = \frac{1}{\pi}\left[-\left(\frac{x\,\mathrm{sen}\,kx}{k} - \int \frac{\mathrm{sen}\,kx}{k}\,dx\right)\Bigg|_{-\pi}^{0} + \left(\frac{x\,\mathrm{sen}\,kx}{k} - \int \frac{\mathrm{sen}\,kx}{k}\,dx\right)\Bigg|_{0}^{\pi}\right]$$

$$a_k = \frac{1}{\pi}\left[-\left(\underbrace{\frac{x.\mathrm{sen}\,kx}{k}}_{=0} + \frac{\cos kx}{k^2}\right)\Bigg|_{-\pi}^{0} + \left(\underbrace{\frac{x.\,\mathrm{sen}\,kx}{k}}_{=0} + \frac{\cos kx}{k^2}\right)\Bigg|_{0}^{\pi}\right]$$

$$a_k = \frac{1}{\pi}\left[-\left(\frac{1-(-1)^k}{k^2}\right) + \left(\frac{(-1)^k - 1}{k^2}\right)\right] = \frac{2}{\pi}\left(\frac{(-1)^k - 1}{k^2}\right)$$

$$\therefore \qquad a_k = \frac{2}{\pi k^2}\cdot(-1^k - 1)$$

Entonces :

- $\forall$ k par $\Rightarrow$ $a_2 = a_4 = \dots = a_{2n} \equiv 0$
- $k = 1$ $\Rightarrow$ $a_1 = -\frac{4}{\pi}$
- $k = 3$ $\Rightarrow$ $a_3 = \frac{-4}{3^2\pi} = \frac{-4}{9\pi}$
- $k = 2n+1$ $\Rightarrow$ $a_{2n+1} = \frac{-4}{(2n+1)^2\pi}$

Por lo tanto la función desarrollada nos queda :

$$F_4(x) = \frac{a_0}{2} + \sum_{k=1}^{n} a_k\cdot\cos kx$$

$$F_4(x) = \frac{\pi}{2} - \frac{4}{\pi}\left(\cos x + \frac{\cos 3x}{9} + \frac{\cos 5x}{25} + \dots + \frac{\cos(2n+1)x}{(2n+1)^2}\right) \equiv |x|$$

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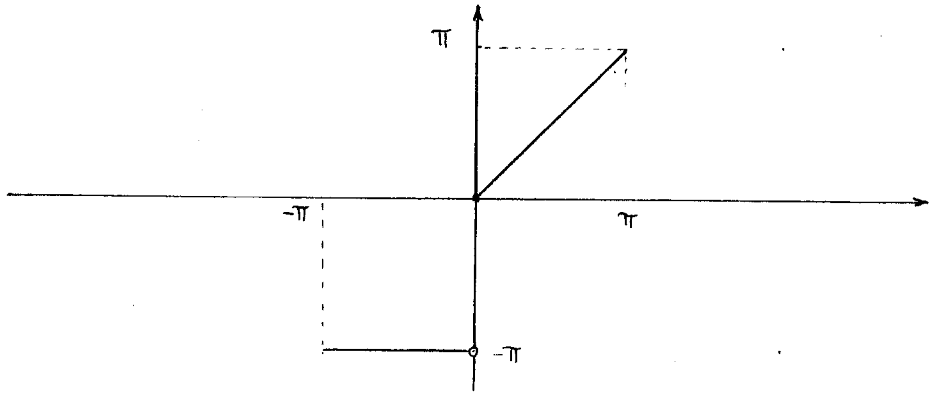
$$f_5(x) = \begin{cases} -\pi & \text{si} \quad -\pi \leqslant x < 0 \\ x & \text{si} \quad 0 \leqslant x \leqslant \pi \end{cases}$$



Nota: Observemos que esta función no es par ni impar en el intervalo $[-\pi;\pi]$, Veamos que valores toman los coeficientes de FOURIER.

$$a_0 = \frac{1}{\pi}\left[-\pi\int_{-\pi}^{0} dx + \int_0^{\pi} x\,dx\right] = \frac{1}{\pi}\left[-\pi(x)\Big|_{-\pi}^{0} + \left(\frac{x^2}{2}\right)\Big|_0^{\pi}\right]$$

$$a_0 = \frac{1}{\pi}\left(-\pi(+\pi) + \frac{\pi^2}{2}\right) = \frac{1}{\pi}\left(-\pi^2 + \frac{\pi^2}{2}\right) = -\frac{\pi}{2}$$

$$\therefore \quad a_0 = -\frac{\pi}{2}$$

$$a_K = \frac{1}{\pi}\left[-\pi\int_{-\pi}^{0}\cos Kx\,dx + \int_0^{\pi} x\cos Kx\,dx\right] = \frac{1}{\pi}\left[\left(-\pi.\frac{\operatorname{sen} Kx}{K}\right)\Big|_{-\pi}^{0} + \left(\frac{x.\operatorname{sen} Kx}{K} - \int\frac{\operatorname{sen} Kx}{K}dx\right)\Big|_0^{\pi}\right]$$

$$a_K = \frac{1}{\pi}\left[\underbrace{\left(\frac{-\pi\operatorname{sen} Kx}{K}\right)\Big|_{-\pi}^{0}}_{=0} + \left(\underbrace{\frac{x.\operatorname{sen} Kx}{K}}_{=0} + \frac{\cos Kx}{K^2}\right)\Big|_0^{\pi}\right] = \frac{1}{\pi}\left(\frac{(-1)^K - 1}{K^2}\right)$$

Entonces:

- $\forall$ K par $\Rightarrow$ $a_2 = a_4 = \dots = a_{2n} = 0$

- $K = 1 \qquad \Rightarrow \qquad a_1 = -\frac{2}{\pi}$
- $K = 3 \qquad \Rightarrow \qquad a_3 = -\frac{2}{3^2\pi} = -\frac{2}{9\pi}$
- $K = 2n+1 \qquad \Rightarrow \qquad a_{2n+1} = -\frac{2}{(2n+1)^2\pi}$

$$\therefore \quad a_K = \frac{(-1)^K - 1}{\pi \cdot K^2}$$

$$b_K = \frac{1}{\pi}\left[-\pi\int_{-\pi}^{0} \operatorname{sen} Kx\,dx + \int_{0}^{\pi} x.\operatorname{sen} x\,dx\right] = \frac{1}{\pi}\left[\left(\frac{\pi\cos Kx}{K}\right)\Big|_{-\pi}^{0} + \left(-\frac{x.\cos Kx}{K} + \int \frac{\cos Kx}{K}\,dx\right)\Big|_{0}^{\pi}\right]$$

$$b_K = \frac{1}{\pi}\left[\left(\frac{\pi.\cos Kx}{K}\right)\Big|_{-\pi}^{0} - \left(\frac{x.\cos Kx}{K} - \underbrace{\frac{\operatorname{sen} Kx}{K^2}}_{=0}\right)\Big|_{0}^{\pi}\right] = \frac{1}{\pi}\left[\pi\left(\frac{1-(-1)^K}{K}\right) - \frac{\pi(-1)^K}{K}\right] = \left(\frac{1-2(-1)^K}{K}\right)$$

Entonces :

- $\forall \; K$ impar $\qquad \Rightarrow \qquad b_K = \frac{3}{K}$
- $\forall \; K$ par $\qquad \Rightarrow \qquad b_K = -\frac{1}{K}$

$$\therefore \quad b_K = \frac{1 - 2.(-1)^K}{K}$$

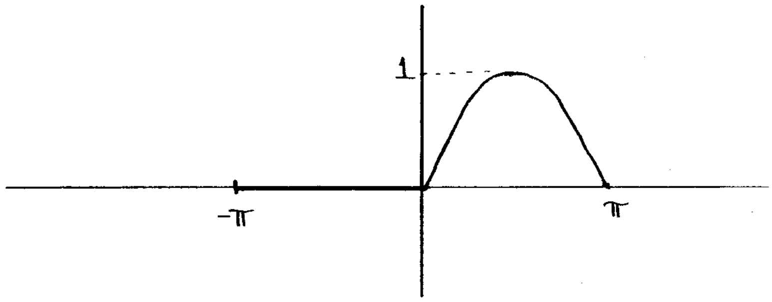
Por lo tanto la función desarrollada en serie nos queda

$$F_5(x) = \frac{a_0}{2} + \sum_{K=1}^{\infty}\left(a_K \cos Kx + b_K \operatorname{sen} Kx\right)$$

$$F_5(x) = -\frac{\pi}{4} - \frac{2}{\pi}\left(\cos x + \frac{\cos 3x}{9} + \frac{\cos 5x}{25} + \cdots\right) + \left(3\operatorname{sen} x - \frac{\operatorname{sen} 2x}{2} + 3\operatorname{sen} 3x - \frac{\operatorname{sen} 4x}{4} + \cdots\right)$$

f .

$$f_6(x) = \begin{cases} 0 & \text{si} \quad -\pi \leqslant x < 0 \\ \operatorname{sen} x & \text{si} \quad 0 \leqslant x \leqslant \pi \end{cases}$$



$$a_0 = \frac{1}{\pi}\int_0^{\pi} \operatorname{sen} x \, dx = \frac{1}{\pi}(-\cos x)\Big|_0^{\pi} = -\frac{1}{\pi}\Big((-1) - 1\Big) = \frac{2}{\pi}.$$

$$\therefore \; a_0 = \frac{2}{\pi}$$

$$a_k = \frac{1}{\pi}\left[\int_{-\pi}^{0}(0).\cos kx \, dx + \int_0^{\pi} \operatorname{sen} x \cos kx \, dx\right]$$

Nota: Aplicando, en este caso, la siguiente identidad trigonométrica tendremos

$$\operatorname{sen} x . \cos kx = \frac{1}{2}\left(\operatorname{sen}(k+1)x - \operatorname{sen}(k-1)x\right)$$

$$a_k = \frac{1}{\pi}\int_0^{\pi} \operatorname{sen} x \cos kx \, dx = \frac{1}{2\pi}\left[\int_0^{\pi} \operatorname{sen}(k+1)x \, dx - \int_0^{\pi} \operatorname{sen}(k-1)x \, dx\right]$$

$$u = (k+1)x \rightarrow du = (k+1)dx \qquad \text{si} \qquad x = \pi \Rightarrow u = (k+1)\pi$$

$$v = (k-1)x \rightarrow dv = (k-1)dx \qquad \text{si} \qquad x = \pi \Rightarrow v = (k-1)\pi$$

$$a_k = \frac{1}{2\pi}\left[\left(\frac{-\cos u}{k+1}\right)\Bigg|_0^{(k+1)\pi} - \left(\frac{-\cos v}{k-1}\right)\Bigg|_0^{(k-1)\pi}\right] = \frac{1}{2\pi}\left[-\left(\frac{(-1)^{k+1} - 1}{k+1}\right) + \left(\frac{(-1)^{k-1} - 1}{k-1}\right)\right]$$

- $K = 1 \quad \Rightarrow \quad a_1 = -\frac{2}{\pi}$
- $K = 3 \quad \Rightarrow \quad a_3 = -\frac{2}{3^2\pi} = -\frac{2}{9\pi}$
- $K = 2n+1 \quad \Rightarrow \quad a_{2n+1} = -\frac{2}{(2n+1)^2\pi}$

$$\therefore \quad a_K = \frac{(-1)^K - 1}{\pi \cdot K^2}$$

$$b_K = \frac{1}{\pi}\left[-\pi\int_{-\pi}^{0} \operatorname{sen} Kx\,dx + \int_{0}^{\pi} x.\operatorname{sen} x\,dx\right] = \frac{1}{\pi}\left[\left(\frac{\pi \cos Kx}{K}\right)\Bigg|_{-\pi}^{0} + \left(-\frac{x.\cos Kx}{K} + \int \frac{\cos Kx}{K}\,dx\right)\Bigg|_{0}^{\pi}\right]$$

$$b_K = \frac{1}{\pi}\left[\left(\frac{\pi.\cos Kx}{K}\right)\Bigg|_{-\pi}^{0} - \left(\frac{x.\cos Kx}{K} - \underbrace{\frac{\operatorname{sen} Kx}{K^2}}_{=0}\right)\Bigg|_{0}^{\pi}\right] = \frac{1}{\pi}\left[\pi\left(\frac{1-(-1)^K}{K}\right) - \frac{\pi(-1)^K}{K}\right] = \left(\frac{1-2(-1)^K}{K}\right)$$

Entonces :

- $\forall$ K impar $\Rightarrow$ $b_K = \frac{3}{K}$
- $\forall$ K par $\Rightarrow$ $b_K = -\frac{1}{K}$

$$\therefore \quad b_K = \frac{1 - 2.(-1)^K}{K}$$

Por lo tanto la función desarrollada en serie nos queda

$$F_5(x) = \frac{a_0}{2} + \sum_{K=1}^{\infty}\left(a_K \cos Kx + b_K \operatorname{sen} Kx\right)$$

$$F_5(x) = -\frac{\pi}{4} - \frac{2}{\pi}\left(\cos x + \frac{\cos 3x}{9} + \frac{\cos 5x}{25} + \cdots\right) + \left(3\operatorname{sen} x - \frac{\operatorname{sen} 2x}{2} + 3\operatorname{sen} 3x - \frac{\operatorname{sen} 4x}{4} + \cdots\right)$$

$$\therefore \quad a_k = \frac{(k+1)[(-1)^{k-1}-1] - (k-1)[(-1)^{k+1}-1]}{2\pi\,(k-1)(k+1)}$$

Entonces :

- $\forall$ k impar $\Rightarrow$ $a_k = 0$
- $k = 2$ $\Rightarrow$ $\dfrac{3(-2) - 1(-2)}{2\pi\,(1)(3)} = \dfrac{-2}{3\pi}$
- $k = 4$ $\Rightarrow$ $\dfrac{5(-2) - 3(-2)}{2\pi\,(3)(5)} = \dfrac{-2}{3.5.\pi}$

$$b_k = \frac{1}{\pi}\left[\int_{-\pi}^{0}(0)\,\text{sen}\,kx\,dx + \int_{0}^{\pi}\text{sen}x.\text{sen}\,kx\,dx\right] = \frac{1}{\pi}\int_0^{\pi}\text{sen}\,x.\text{sen}\,kx\,dx$$

Nota : Como hemos demostrado en la teoria la integral anterior tiene las siguientes soluciones

$$\int_0^{\pi}\text{sen}\,mx.\text{sen}\,kx\,dx = \begin{cases} 0 & \text{si} \quad k \neq m \\ \dfrac{\pi}{2} & \text{si} \quad k = m \end{cases}$$

Entonces para nuestro caso $m = 1$ tendremos $k = 1$

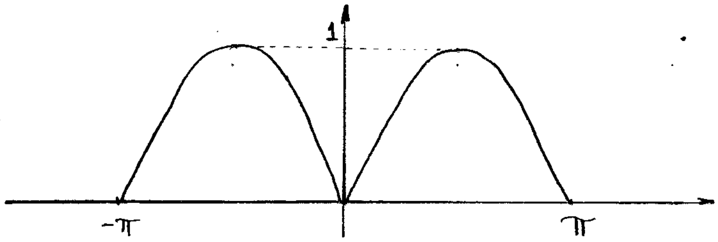
$$b_1 = \frac{1}{\pi}\int_0^{\pi}\text{sen}\,x.\text{sen}\,x\,dx = \frac{1}{\pi}\int_0^{\pi}\text{sen}^2x\,dx = \frac{1}{\cancel{\pi}}.\frac{\cancel{\pi}}{2} = \frac{1}{2}$$

$$\therefore \quad b_1 = \frac{1}{2} \qquad \wedge \quad b_k = 0 \qquad (k = 2,3,\ldots,n)$$

Por lo tanto la función desarrollada nos queda así

$$F_6(x) = \frac{1}{\pi} - \frac{2}{\pi}\left(\frac{\cos 2x}{3} + \frac{\cos 4x}{3.5} + \frac{\cos 6x}{5.7} + \cdots\right) + \frac{\text{sen}\,x}{2}$$

9- $\quad f_7(x) = |\operatorname{sen} x| \qquad -\pi \leqslant x \leqslant \pi$



Nota: Ésta función es par por lo tanto $b_K = 0$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} |\operatorname{sen} x|\, dx = \frac{2}{\pi}\int_0^{\pi} \operatorname{sen} x\, dx = \frac{2}{\pi}\cdot(-\cos x)\Big|_0^{\pi} = -\frac{2}{\pi}\big((-1) - 1\big) = \frac{4}{\pi}$$

$$\therefore \quad a_0 = \frac{4}{\pi}$$

$$a_K = \frac{1}{\pi}\int_{-\pi}^{\pi} |\operatorname{sen} x| \cos Kx\, dx = \frac{2}{\pi}\int_0^{\pi} \operatorname{sen} x \cdot \cos Kx\, dx$$

Aplicamos la siguiente identidad trigonométrica para resolver esta integral.

$$\operatorname{sen} x \cdot \cos Kx = \frac{1}{2}\Big(\operatorname{sen}(K+1)x - \operatorname{sen}(K-1)x\Big)$$

Entonces

$$a_K = \frac{2}{\pi}\int_0^{\pi} \operatorname{sen} x \cdot \cos Kx\, dx = \frac{1}{\pi}\left[\int_0^{\pi} \operatorname{sen}(K+1)x\, dx - \int_0^{\pi} \operatorname{sen}(K-1)x\, dx\right]$$

$$u = (K \pm 1)x \qquad \rightarrow \qquad du = (K \pm 1)dx$$

$$\text{si} \begin{cases} x = 0 & \Rightarrow \quad u = 0 \\ x = \pi & \rightarrow \quad u = (K \pm 1)\pi \end{cases}$$

$$\therefore \quad a_k = \frac{(k+1)\left[(-1)^{k-1}-1\right] - (k-1)\left[(-1)^{k+1}-1\right]}{2\pi\,(k-1)(k+1)}$$

Entonces :

- $\forall$ k impar $\Rightarrow$ $a_k = 0$
- $k = 2$ $\Rightarrow$ $\dfrac{3(-2) - 1(-2)}{2\pi\,(1)(3)} = \dfrac{-2}{3\pi}$
- $k = 4$ $\Rightarrow$ $\dfrac{5(-2) - 3(-2)}{2\pi\,(3)(5)} = \dfrac{-2}{3.5.\pi}$

$$b_k = \frac{1}{\pi}\left[\int_{-\pi}^{0}(0)\,\text{sen}\,kx\,dx + \int_{0}^{\pi}\text{sen}x.\text{sen}\,kx\,dx\right] = \frac{1}{\pi}\int_{0}^{\pi}\text{sen}\,x.\text{sen}\,kx\,dx$$

Nota : Como hemos demostrado en la teoria la integral anterior tiene las siguientes soluciones

$$\int_{0}^{\pi}\text{sen}\,mx.\text{sen}\,kx\,dx = \begin{cases} 0 & \text{si} \quad k \neq m \\ \dfrac{\pi}{2} & \text{si} \quad k = m \end{cases}$$

Entonces para nuestro caso $m = 1$ tendremos $k = 1$

$$b_1 = \frac{1}{\pi}\int_{0}^{\pi}\text{sen}\,x.\text{sen}\,x\,dx = \frac{1}{\pi}\int_{0}^{\pi}\text{sen}^2 x\,dx = \frac{1}{\cancel{\pi}}.\frac{\cancel{\pi}}{2} = \frac{1}{2}$$

$$\therefore \quad b_1 = \frac{1}{2} \quad \wedge \quad b_k = 0 \qquad (k = 2, 3, \ldots, n)$$

Por lo tanto la función desarrollada nos queda asi

$$F_6(x) = \frac{1}{\pi} - \frac{2}{\pi}\left(\frac{\cos 2x}{3} + \frac{\cos 4x}{3.5} + \frac{\cos 6x}{5.7} + \cdots\right) + \frac{\text{sen}\,x}{2}$$

$$a_K = \frac{1}{\pi}\left[\int_0^{(K+1)\pi} \frac{\operatorname{sen} u}{(K+1)}\, du - \int_0^{(K-1)\pi} \frac{\operatorname{sen} u}{(K-1)}\, du\right] = \frac{1}{\pi}\left[\left(\frac{-\cos u}{(K+1)}\right)\Bigg|_0^{(K+1)\pi} - \left(\frac{-\cos u}{(K-1)}\right)\Bigg|_0^{(K-1)\pi}\right]$$

$$a_K = -\frac{1}{\pi}\left[\left(\frac{(-1)^{K+1} - 1}{(K+1)}\right) - \left(\frac{(-1)^{K-1} - 1}{(K-1)}\right)\right] = \frac{1}{\pi}\left(\frac{(-1)^{K-1} - 1}{K-1} - \frac{(-1)^{K+1} - 1}{K+1}\right)$$

$$\therefore \quad a_K = \frac{(K+1)\left[(-1)^{K-1} - 1\right] - (K-1)\left[(-1)^{K+1} - 1\right]}{\pi \ (K^2 - 1)}$$

Entonces :

- $\forall \quad K$ impar $\quad \rightarrow \quad a_K = 0$
- $K = 2 \quad \rightarrow \quad a_2 = \dfrac{(2+1)(-2) - (2-1)(-2)}{\pi\ (2^2 - 1)} = \dfrac{-4}{3\pi}$
- $K = 4 \quad \rightarrow \quad a_4 = \dfrac{(4+1)(-2) - (4-1)(-2)}{\pi\ (4^2 - 1)} = \dfrac{-4}{15\pi}$
- $K = 6 \quad \rightarrow \quad a_6 = \dfrac{(6+1)(-2) - (6-1)(-2)}{\pi\ (6^2 - 1)} = \dfrac{-4}{35\pi}$

Por lo tanto la función desarrollada en Serie nos queda

$$F_7(x) = \frac{2}{\pi} - \frac{4}{\pi}\left(\frac{\cos 2x}{3} + \frac{\cos 4x}{15} + \frac{\cos 6x}{35} + \cdots\cdots\right) \equiv |\operatorname{sen} x|$$

② Desarrollar las siguientes funciones en una serie de FOURIER expresada en términos del seno y coseno.

a $\qquad f(x) = x$

$$0 \leqslant x \leqslant \pi$$