Ejercicios Series de Fourier

1) Calcular la Serie de Fourier asociada a $f(x) = x$

su derivada es $f'(x) = 1$

f y f' son seccionalmente continuas

f es una función impar $\Rightarrow a_k = 0$

$$b_k = \frac{2}{\pi}\int_0^{\pi} x \cdot \operatorname{sen} kx \cdot dx$$

para integrar por partes hacemos $\begin{cases} u = x \Rightarrow du = 1 \\ dv = \operatorname{sen} kx \Rightarrow v = \dfrac{-\cos kx}{k} \end{cases}$

$$b_k = \frac{2}{\pi}\left[\left(-x \cdot \cos kx\right)_0^{\pi} + \int_0^{\pi} \frac{\cos kx}{k}\, dx\right] =$$

$$b_k = \frac{2}{\pi}\left[\left(\frac{-\pi \cdot \cos k\pi}{k}\right) + \underbrace{\cancel{\left(\frac{1}{k}\cdot\frac{1}{k}\operatorname{sen} kx\right)_0^{\pi}}}_{=0}\right] = \frac{-2}{\pi}\cdot\frac{\pi \cos k\pi}{k} =$$

$$b_k = \frac{-2}{k}(-1)^k = \frac{2}{k}(-1)^{k+1}$$

Por lo tanto la serie nos queda:

$$f(x) \simeq 2 \cdot \sum_{k=1}^{\infty} (-1)^{k+1} \frac{\operatorname{sen} kx}{k} = 2\Big(\underbrace{\operatorname{sen} x}_{b_1 = 1} \underbrace{- \tfrac{1}{2}\operatorname{sen} 2x}_{b_2 = -\frac{1}{2}} \underbrace{+ \tfrac{1}{3}\operatorname{sen} 3x}_{b_3 = \frac{1}{3}} + \dots\Big)$$

Verificación para algunos valores

si $x=0 \longrightarrow f(0) = 2\left(\operatorname{sen} 0 - \frac{1}{2}\operatorname{sen} 0 + \frac{1}{3}\operatorname{sen} 0 - \frac{1}{4}\operatorname{sen} 0\right) = 0$

si $x = \frac{\pi}{2}$ ✓

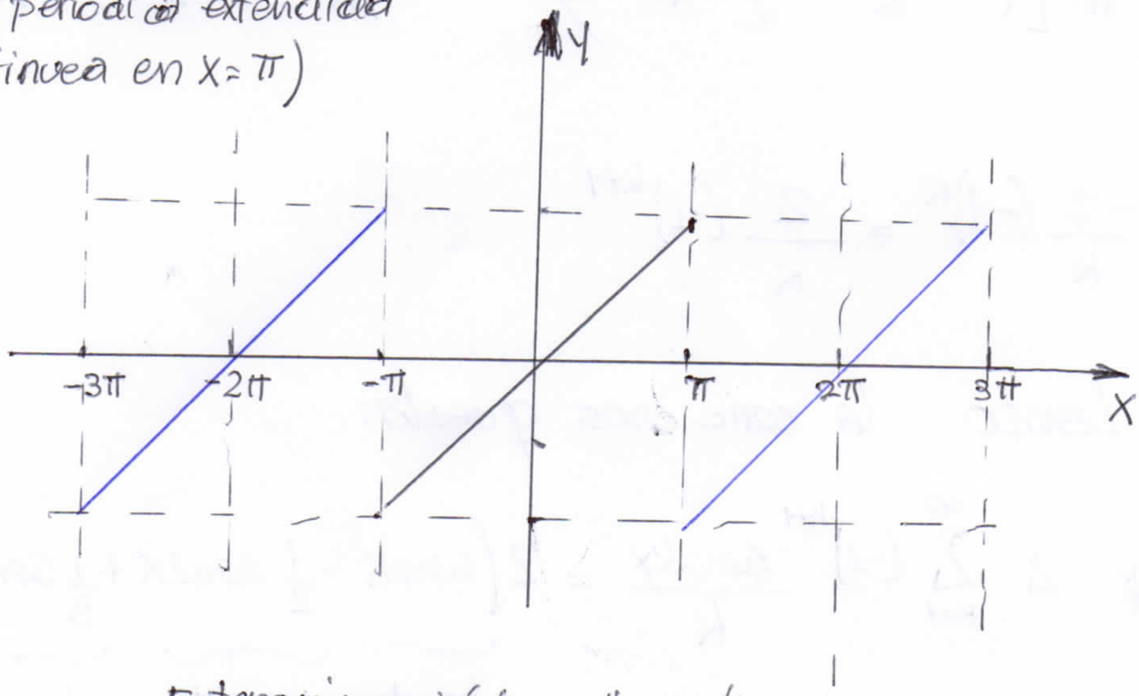
$$f\left(\frac{\pi}{2}\right) = 2\left(\operatorname{sen}\frac{\pi}{2} - \frac{1}{2}\operatorname{sen} 2\frac{\pi}{2} + \frac{1}{3}\operatorname{sen}\frac{3}{2}\pi - \frac{1}{4}\operatorname{sen}\frac{4\pi}{2} + \frac{1}{5}\operatorname{sen}\frac{5}{2}\pi + \dots\right)$$

$$= 2\Big(\underbrace{1 - 0 + \frac{1}{3} - 0 + \frac{1}{5} - 0 + \frac{1}{7} - 0 + \frac{1}{9} - 0 + \dots}_{\text{Esta serie converge a } \frac{\pi}{4}}\Big) = 2\left(\frac{\pi}{4}\right) = \frac{\pi}{2} \quad ✓$$

si $x = \pi$

$$f(\pi) = 2\left(\operatorname{sen}\pi - \frac{1}{2}\operatorname{sen} 2\pi + \frac{1}{3}\operatorname{sen} 3\pi - \dots\right) = 0$$

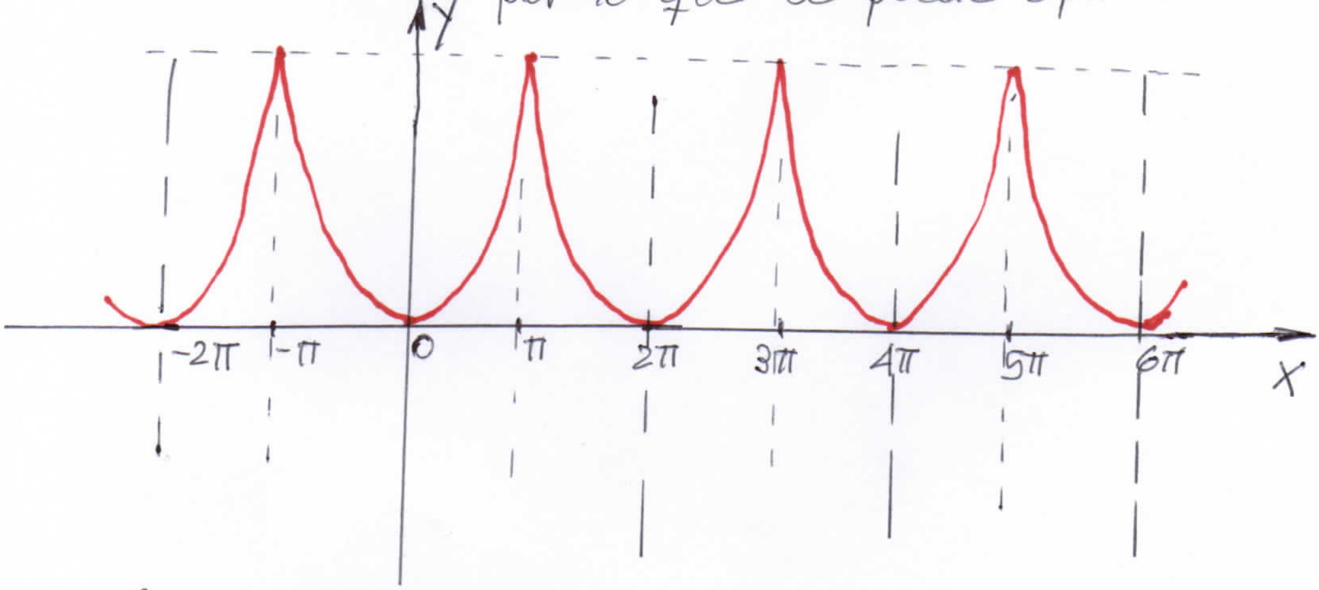
La serie converge a la semisuma de los límites laterales: $\frac{-\pi+\pi}{2} = 0$
(la función periódica extendida
no es continua en $x=\pi$)



Extensión periódica Normalizada

2) Calcular la serie de Fourier asociada a $f(x) = x^2$

$f'(x) = x$ f y f' son seccionalmente lisos por lo que se puede aplicar Fourier



f es par $\Longrightarrow b_k = 0$

$$a_k = \frac{2}{\pi}\int_0^{\pi} x^2 \cdot \cos kx \, dx = \frac{2}{\pi}\left[\left[\frac{2x}{k^2}\cos kx\right]_0^{\pi} + \left[\frac{x^2}{k} - \underbrace{\cancel{\frac{2}{k^3}}}_{=0}\right]\operatorname{sen} kx\Big|_0^{\pi}\right] =$$

$$a_k = \frac{2}{\pi}\left[\frac{2\pi}{k^2}\cos k\pi - 0\right] = \frac{4}{k^2}(-1)^k$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x^2 = \frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi} = \frac{2}{3}\pi^2 \Longrightarrow \frac{a_0}{2} = \frac{\pi^2}{3}$$

La serie (de cosenos) de Fourier asociada a $f(x) = x^2$ es:

$$f(x) \cong \frac{\pi^2}{3} + \sum_{k=1}^{\infty} \frac{4}{k^2} \cdot (-1)^k \cdot \cos kx$$

$$f(x) \cong \frac{\pi^2}{3} + 4\left[-\cos x + \frac{1}{2^2}\cos 2x - \frac{1}{3^2}\cos 3x + \frac{1}{4^2}\cos 4x - \frac{1}{5^2}\cos 5x + \dots\right]$$

$$f(x) = \frac{\pi^2}{3} + 4\left[-\cos x + \frac{1}{4}\cos 2x - \frac{1}{9}\cos 3x + \frac{1}{16}\cos 4x - \frac{1}{25}\cos 5x + \dots\right]$$

Verificación para algunos valores

$$f(0) = \frac{\pi^2}{3} + 4\left(-\cos 0 + \frac{1}{4}\cos 0 - \frac{1}{9}\cos 0 + \frac{1}{16}\cos 0 - \frac{1}{25}\cos 0 + \dots\right.$$

$$= \frac{\pi^2}{3} + 4\left(-1 + \frac{1}{4} - \frac{1}{9} + \frac{1}{16} - \frac{1}{25} + \dots\right) = -0{,}065 \cong 0$$

$$f\left(\frac{\pi}{2}\right) \cong \frac{\pi^2}{3} + 4\left(-\cos\frac{\pi}{2} + \frac{1}{4}\cos\pi - \frac{1}{9}\cos\frac{3}{2}\pi + \frac{1}{16}\cos 2\pi - \frac{1}{25}\cos\frac{5}{2}\pi + \dots\right)$$

$$f\left(\frac{\pi}{2}\right) \cong \frac{\pi^2}{3} + 4\left(0 - \frac{1}{4} - 0 + \frac{1}{16} - 0 + \frac{1}{36} - 0 + \dots\right) = \frac{\pi^2}{3} + 4\left(-0{,}1597\overline{2}\right) =$$

$$f\left(\frac{\pi}{2}\right) \cong \frac{\pi^2}{3} - 0{,}\overline{638} \cong \left(\frac{\pi}{2}\right)^2$$

$$f(\pi) \cong \frac{\pi^2}{3} + 4\left(-\cos\pi + \frac{1}{4}\cos 2\pi - \frac{1}{9}\cos 3\pi + \frac{1}{16}\cos 4\pi - \frac{1}{25}\cos 5\pi + \dots\right)$$

$$= \frac{\pi^2}{3} + 4\left(1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \frac{1}{36} + \dots\right) =$$

$$= \frac{\pi^2}{3} + 4\left(1{,}4913\overline{8}\right) = 9{,}2554 \cong \pi^2$$

En este caso la suma de los límites laterales coincide con el valor de la función (la función periódica extendida es continua en $x = \pi$