Notes on Gravitational Waves and Memory Effect

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1 Gravitaional Waves

Ripples in space-time cause by enegetic gravitational processes in the Universe like colliding black holes, merging neutron stars, exploding stars, and possibly even the birth of the Universe itself. Albert Einstein predicted the existence of GWs in 1916 in his GTR. Einstein's mathematics showed that massive accelerating objects(such as neutron stars or black holes orbiting each other) would disrupt space-time in such a way that 'waves' undulating space-time would propagate in all directions away from the source. The cosmic ripples would travel at the speed of light, carrying with them information about their origins, as well as clues to the nature of gravity itself. In 1993, astrophysicists Russell Hulse and Joseph Taylor received the Nobel Prize in Physics for their 1974 discovery of a binary pair of neutron stars 21,000 light years from Earth.

Finally more than 40 years later, on Sep 14, 2015, GWs were directly detected by LIGO's interferometers which are generated by two colliding black holes 1.3 billion light-years away. LIGO can sense the whispers of GWs through the imprint of that radiation on laser light. It can detect a change in arm length of about $10^{-19}m$.

- GWS are ripples in the curvature of spacetime that propagate with the speed of light. Hence graviton is massless.
- Act as time varying tidal forces.
- These can be described by linearised theory in the far zone. [or wave zone region (away from source), first order in metric perturbations those are plane waves that travel at the speed of light.]

Now the question is, why do we need to do the weak field approach to gravitational radiation? The answer is

- Any observable gravitational radiation is likely to be of very low intensity.
- It is only possible to attach a precise meaning to the concept of an elementary particle when it is far away from all other particles, and far gravitons this corresponds to a weak field solution of the field equation.

GWs d=4 have two polarization state (in case of linearly polarized wave: $h_+, h_\times$)

GWs are unrelated to EM radiation. Here we need to solve multidimensional, non-linear coupled partial differential equations of spacetimes. Gravitational wave components of the spacetime metric usually constitute small fractions of the smooth background metric. Moreover to extract the waves from the background in a simulation requires that one prove the numerical spacetime in the far-field, or radiation zone which is typically at large distance from the strong-field central source [1]. This will be an unprecedented direct test of GR, especially in the highly dynamical and non-linear strong-field regime.

1.1 Gravitational Waves in Einstein Linearised Theory

Here we study the weak-field approximation of the Gravitational radiation.

- In Linearised theory;
  Metric: $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ $\quad |h_{\mu\nu}| << 1$
  Connection: $\Gamma^{\mu}_{\alpha\beta} = \frac{1}{2}\eta^{\mu\nu}(h_{\alpha\nu,\beta} + h_{\beta\nu,\alpha} - h_{\alpha\beta,\nu})$
  Ricci tensor: $R_{\mu\nu} = \frac{1}{2}({h_\mu}^{\alpha}{}_{,\nu\alpha} + {h_\nu}^{\alpha}{}_{,\mu\alpha} - {h_{\mu\nu,\alpha}}^{\alpha} - h_{,\mu\nu})$
- Introducing the *Trace-reversed potential*: $\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h$
  The name makes sense, since $\bar{h}^{\mu}_{\mu} = -h^{\mu}_{\mu}$ [The Einstein tensor is simply the trace-reversed Ricci tensor.]

Einstein's field equation is given by

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi T_{\mu\nu} \tag{1.1}$$

Hence the linearlised field equation in terms of the redefined trace reversed potential is given by

$$-\bar{h}_{\mu\nu,\alpha}{}^{\alpha} - \eta_{\mu\nu}\bar{h}_{\alpha\beta,}{}^{\alpha\beta} + \bar{h}_{\mu\alpha,}{}^{\alpha}{}_{\nu} + \bar{h}_{\nu\alpha,}{}^{\alpha}{}_{\mu} = 16\pi T_{\mu\nu} \tag{1.2}$$

The first term in these linearized equations is the usual flat-space d'Alermbertian, and the other terms serve merely to keep the equations "gauge-invariant". Without loss of generality, one can impose the "gauge conditions".
Here we are imposing the *Harmonic gauge* ($\bar{h}^{\mu\alpha}{}_{,\alpha} = 0$) or Lorentz gauge [2] (Tensor analogue of Lorentz gauge $A^{\alpha}{}_{\alpha} = 0$ in electromagnetic theory) Einstein's full equation is,

$$\Box\bar{h}_{\mu\nu} = -16\pi T_{\mu\nu} \tag{1.3}$$

2 Transverse Traceless Gauge

Now we find the degrees of freedom of the Gravitational waves by constraining the wave equations [3]. The simplest solution to vacuum equation are,

$$\bar{h}_{\mu\nu} = Re[A_{\mu\nu}exp(ik_{\alpha}x^{\alpha})]$$

Using wave equation;

(i) In Lorentz gauge the wave amplitude components must be orthogonal to the wave vector k^{α}. Hence the condition, $A_{\mu\alpha}k^{\alpha} = 0, \quad [\because \bar{h}^{\mu\alpha}{}_{,\alpha} = 0]$

(ii) Traceless condition : $A^{\alpha}_{\alpha} = 0,$

(iii) On further restriction by using our remaining gauge freedom, we can change the gauge while remaining within the Lorentz class of gauges using any vector solving the wave equation,

$$A_{\alpha\beta}U^{\beta} = 0 \quad (U^{\beta} \equiv \text{timelike 4-velocity vector})$$

where U is some fixed 4-velocity, that is, any constant timelike unit vector we wish to choose.
[under $x^{\mu} \to x^{\mu} + \xi^{\mu}$, using remaining gauge freedom]

These are called **transverse-traceless(TT)gauge**.
Now we have used all our gauge freedom, so any remaining independent components

of $A_{\alpha\beta}$ are physically important.
Suppose the plane wave is travelling in the $+z-$direction,

$$\bar{h}_{\mu\nu}^{(TT)} = A_{\mu\nu}^{(TT)} cos[w(t-z)]$$

Let us go to a Lorentz frame for the background Minkowski spacetime that is to make a background Lorentz transformation in which the vector $U^\beta = \delta_0^\beta$. Our TT gauge is based on this time like basis vector. Hence

$$A_{\alpha 0} = 0 \quad \forall\alpha.$$

In this frame our spatial coordinate axes are oriented such that the wave is travelling in the $z-$direction $k^\alpha = (\sharp, 0, 0, \sharp)$. This implies that

$$A_{\alpha z} = 0 \quad \forall\alpha.$$

These two conditions constrain the amplitude tensor $A_{\alpha\beta}$ $(A_{xx} = -A_{yy}, A_{xy} = A_{yx})$. Hence there are only two independent degrees of freedom A_{xx} and A_{xy}.

Now one can see that, the Gravitational wave has 'no' effect on a particle at rest. This means it will be at constant coordinate position after the passing of the GWs. Here the interpretation is such that we have found a coordinate system that stays attached to the particle. Hence we can't find the effect of the GWs as such. Therefore we need to study the effect on the relative separation of the two test particles [3, Chapter 9].

3 Production of GWs

Disturbances in the gravitational field at (t, x) are calculated in terms of the events on the past light cone. We have in Eq. (1.3) coupled to matter using Green's function formalism we have the solution to EFE,

$$\bar{h}_{\mu\nu}(t, \mathbf{x}) = 4\int \frac{1}{|\mathbf{x}-\mathbf{y}|} T_{\mu\nu}(t_r, \mathbf{y}) d^3y$$

where, $t = x^0$, $t_r = t - |\mathbf{x}-\mathbf{y}| \equiv$ Retarded time
and the Retarded Green's function,
$G(x^\alpha - y^\alpha) = -\dfrac{1}{4\pi|\mathbf{x}-\mathbf{y}|}\ \delta(|\mathbf{x}-\mathbf{y}| - (x^0 - y^0))\ \Theta(x^0 - y^0).$

This is the $1/r$, coulomb-type gravitational field generated by the four-momenta of the source's various independent pieces. Here we don't have to compute all of the components of $h_{\mu\nu}$, since the Lorentz gauge condition reduces the number of independent components or use the TT gauge to find the solution easily. The disturbance in the Gravitational field at (t, x) is a sum of the influences from the energy

and momentum sources at the point $(t_r, x-y)$ on the past light cone.Here I am not going to the details of the derivation as it's not required for this note.
The memory of the burst what I am going to introduce is basically equal to the change, from before the burst to afterwards, in the TT part of the "$1/r$, coulomb type" gravitational field generated by the four-momenta of the source's various independent pieces.

3.1 GWs from a Weak-field, slow velocity source

Now we can find the Quadrupole moment tensor of the energy density of the source, which is basically a constant tensor on each surface of constant time. In the Weak-field(Newtonian Potential) is defined as $\Phi << 1$ and slow velocity $v << 1$ [1]. In the wave zone at large distance r from the source we may expand the retarded integral in powers of $\frac{x'^i}{r}$,

$$\bar{h}_{ij}(t, x^k) = \frac{4}{r}\int d^3x' T_{ij}(t-r, x'^k)$$

Using the *Tensor-virial theorem* we can write,

$$\int d^3x' T_{ij} = \frac{1}{2}\frac{d^2}{dt^2}\int d^3x' T_{tt} x'^i x'^j$$

Again we have,

$$I^{ij}(t) \equiv \int d^3x' \rho_0(t, x'^k) x'^i x'^j \equiv \text{Second moment of mass distribution}$$

Hence

$$\boxed{\bar{h}_{ij}(t, x^k) = \frac{2}{r}\ddot{I}_{ij}(t-r)}$$

To write the above equation in the TT gauge let us define the *reduced Quadrupole moment*:

$$\mathcal{I}_{ij} = I_{ij} - \frac{1}{3}\eta_{ij} I$$

Hence we have the quadrupole approximation,

$$\boxed{\bar{h}_{ij}{}^{TT}(t, x^k) = \frac{2}{r}\ddot{\mathcal{I}}^{TT}_{ij}(t-r)}$$

Here we have used the projection operator $P_i^j \equiv \eta_i^j - n_i n^j$.
where, $n_i = \frac{x_i}{r}$ is the unit vector pointing in wave's local direction of propagation.

$$\mathcal{I}^{TT}_{ij} = (P_i^k P_j^l - \frac{1}{2}P_{ij}P^{kl})\mathcal{I}_{kl}$$

and

$$\mathcal{I}^{TT}_{ij} = I^{TT}_{ij}$$

These are the stuff carried away by the GWs:

- Averaging the quadratic first order perturbations h_{ij} in the vacuum Einstein equation behaves as a *source term* for the background curvature, T_{ab}^{GW}.
- In nearly Minkowskian frame, averaging over the several wavelength the stress energy tensor reduces to,

$$T_{ab}^{GW} = \frac{1}{32\pi}\langle \partial_a h_{ij}^{TT} \partial_b h^{ij\,TT} \rangle$$

Listing stuffs carried away by the GWs:

- Outgoing energy flux in radial direction: $\mathbf{T^{GW}_{tr}}$
- GW luminosity: Equal and opposite of the energy change of the source,

$$\begin{aligned} L_{GW} \equiv -\frac{dE}{dt} &= -\lim_{r\to\infty} \int T_{tr}^{GW} r^2 d\Omega. \\ &= \lim_{r\to\infty} \frac{r^2}{16\pi} \int \langle \dot{h}_+^2 + \dot{h}_\times^2 \rangle d\Omega \end{aligned}$$

  where, $\partial_r h_{ij}^{TT} = -\partial_t h_{ij}^{TT}$ for out going radiation.

- Loss of angular momentum:

$$\frac{dJ_z}{dt} = \lim_{r\to\infty} \frac{r^2}{16\pi} \int \langle \partial_t h_+ \partial_\phi h_\times + \partial_t h_\times \partial_\phi h_\times \rangle d\Omega.$$

- Loss of linear momentum:

$$\frac{dP_i}{dt} = \lim_{r\to\infty} -\frac{r^2}{16\pi} \int \frac{x^i}{r} \langle \dot{h}_+^2 + \dot{h}_\times^2 \rangle d\Omega$$

In weak-field, slow velocity source, we can express the radiated energy in terms of source's reduced quadrupole moment,

$$L_{GW} \equiv -\frac{dE}{dt} = \frac{1}{5}\langle \dddot{\mathcal{I}}_{ij} \dddot{\mathcal{I}}^{ij} \rangle$$

Similary,

$$\frac{dJ_i}{dt} = -\frac{2}{5}\epsilon_{ijk}\langle \ddot{\mathcal{I}}^{jm} \dddot{\mathcal{I}}^{k}{}_{m} \rangle.$$

These are called the *quadrupole formula* which can be used to calculate the loss of energy and angular momentum from any weak-field, slow-velocity source.

4 Geodesic

In flat space, we can parallel transport a vector by simply keeping its Cartesian components constant. Parallel transport in sphere or on a curved manifold, the result of parallel transport can depend on the path taken.

- These are the shortest-distance paths in curved spacetime or affine-parametrized path of the freely falling particle
- They parallel transport their own tangent vectors. i.e. there is no change to the tangent vector along the path.
- Geodesics are defined by parallel transport or covariant differentiation.

Suppose u is a tangent vector to a curve and this curve is a geodesic parallel transports its own tangent vector along itself or we can say the directional covariant derivative is equal to zero along the curve x^μ parameterized by λ. One can test whether any curve is a geodesic or not if $\nabla_u u = 0$. This implies the curve with tangent vector u, which is parametrized by the proper time τ parallel transports its own tangent vector u. That curve is a geodesic.

On the other hand, the vector field v is parallel transported along the vector $u = \frac{d}{d\tau}$, if $\frac{dv}{d\lambda} = \nabla_u v = 0$. This is the equation of parallel transport.

$$\nabla_\mu u^\nu = \partial_\mu u^\nu + \Gamma^\nu_{\mu\lambda} u^\lambda = 0 \Rightarrow \frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\lambda}(x)\frac{dx^\nu}{d\tau}\frac{dx^\lambda}{d\tau} = 0.$$

This is the Geodesic equation (parallel transport of the tangent vector).

4.1 Geodesic Deviation

Geodesic deviation is defined as the relative separation of two infinitesimally close geodesics. Let us consider two nearby freely falling particles 1 and 2 moving along two nearby geodesics $x^\mu(\tau)$ and $x^\mu(\tau) + D^\mu(\tau)$ with equal values of affine parameter τ the proper time. Affine parameter is defined as clock time along free-fall trajectory.

$$\begin{aligned} \frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\lambda}(x)\frac{dx^\nu}{d\tau}\frac{dx^\lambda}{d\tau} &= 0 \\ \frac{d^2(x^\mu + D^\mu)}{d\tau^2} + \Gamma^\mu_{\nu\lambda}(x+D)\frac{d(x^\nu + D^\nu)}{d\tau}\frac{d(x^\lambda + D^\lambda)}{d\tau} &= 0 \end{aligned} \tag{4.1}$$

Evaluating the difference to first order in D^μ, the Geodesic Deviation equation;

$$\boxed{u^\rho \nabla_\rho (u^\sigma \nabla_\sigma D^\mu) = -R^\mu_{\alpha\beta\gamma} u^\alpha D^\beta u^\gamma}$$

where, $u^\alpha = \frac{dx^\alpha}{d\tau}$. Here the LHS is the acceleration of nearby points relative to each other. u^α vector components along the geodesic D^μ is the separation 4-vector/deviation vector/connecting vector.

4.2 Action of Tidal forces on particles

In Local Inertial Frame (LIF), coordinate distances are proper distances. In a coordinate system closely associated with measurements, the LIF at the point of the first geodesic where D originates. That means in these coordinates the components of D do indeed correspond to measurable proper distances if the geodesics are near enough to one another. So,

$$\frac{d^2 D^\mu}{d\tau^2} = -R^\mu{}_{\alpha\beta\gamma} u^\alpha D^\beta u^\gamma$$

Taking, $\mathbf{u} \to (1,0,0,0)$ and initially, $\mathbf{D} \to (0,\epsilon,0,0)$.
Then at $O(h_{\mu\nu})$,

$$\frac{d^2 D^\mu}{d\tau^2} = -\epsilon R^\mu{}_{0x0}.$$

This shows that Riemann tensor is locally measurable by simply watching the proper distance changes between nearby geodesics. Now the Riemann tensor is itself gauge invariant, so its components do not depend on the choice we made between a LIF and the TT coordinates, since Riemann components are simple in TT gauge, for the wave travelling in the z-direction, we can write all the components.

In TT gauge, for example, that two particles initially separated in the x-direction have a separation vector D whose component's proper length obey,

$$\frac{\partial^2 D^x}{\partial t^2} = \frac{1}{2}\epsilon \frac{\partial^2 h^{TT}_{xx}}{\partial t^2}, \quad \frac{\partial^2 D^y}{\partial t^2} = \frac{1}{2}\epsilon \frac{\partial^2 h^{TT}_{xy}}{\partial t^2}.$$

In this local frame, the wave acts like a **tidal force**. By tidal we mean differential force. Gravity stretches and squeezes the mass. The tidal force is a force that stretches a body towards and away from the center of mass of another body due to a gradient in gravitational field from the other body and this force depends on the separation vector.

5 Gravitaional Wave Memory Effect

Memory effects is a permanent change in the relative displacement of test particles. Classical observable effects in the low-energy region of gravity and gauge theories. All gravitational wave sources possess some form of gravitational-wave memory. GWs without memory causes oscillatory deformations but eventually returns the detector to its initial state. But on the other hand when a GW with memory passes through an *idealized* detector, it causes permanent deformation, which we call 'Memory' of the wave.

There are two kinds of Memory effects:

- Linear Memory: arises in systems with unbounded components (two body scattering, matter or neutrinos ejected from a supernova or gamma-ray burst jets.
- Non-linear Memory: arises from the contribution of the emitted GWs to the changing quadrupole and higher mass moments.

5.1 Detection of Memory effect

Experimenters divide the GWs into 4 classes (which can be detected in future):

- **bursts:** in which the wave field h_{ij}^{TT} rises from zero, oscillates for only a few cycles and then returns to zero.
- periodic waves
- stochastic waves
- **bursts with memory(BWM):** in which h_{ij}^{TT} rises from zero, oscillates for a few cycles, and then after a burst of duration Δt settles down into a non-zero final value δh_{ij}^{TT}. For any kind of detector the best way to search for a BWM is to integrate up the signal for an integration time $\frac{1}{f_{opt}}$, where f_{opt} is the frequency at which the detector has optimal amplitude sensitivity to ordinary bursts one cycle long with frequency f_{opt}. It is possible, though not highly probable, that BWM will be amond the earliest kinds of GWs detected; therefore experimenters should take them into account when planning their search strategies and data analyses.
- Memory effects are not included in most numerical relativity models and are hence typically not incorporated in gravitational waveforms of compact coalescences. Because memory appears in the $m = 0$ modes of the waveform which are difficult to resolve with numerical relativity simulations. The slow build-up of memory during compact binary coalescences causes low-frequency contributions to gravitational-wave signal. Recent advances in modelling and gravitational-wave signal analysis have made it possible to coherently search for the presence of memory in an ensemble of GW signals. A milestone which is likely to occur during the LIGO A+/Virgo+ era. [4, 5]
- continuous gravitational waves are almost monochromatic signals generated typically by rotating, non-axisymmetric neutron stars [5].

Laser interferometers respond to the proper-distance separations between their (ideally) free mirrors. If they are truly free, then they can store the signal forever. They register the integral of h along the path between the mirrors. Therefore, if their mirrors were truly free, they would experience a truly permanent deformation when a BWM passes; in principle they can store the signal forever.

(i) Ground-based interferometers (Advanced LIGO) are sensitive to the memory, from most sources because they (the LIGO test masses) are not truly free. The principle sources for the interferometric detectors(LIGO, Virgo, LISA) are orbiting bodies, for which the dominant signal is oscillatory. However, it has long been known that GWs will also contain non-oscillatory features, resulting from a net change in time derivatives of the multiple moments of the system [6, 7].

(ii) LISA is capable of memory detection considering **two facts**;

  - Its proof-masses are truly freely-floating
  - The good sensitivity in the low-frequency band

  Firstly its proof-masses are truly freely-floating as a result it can maintain a permanent displacement and secondly the good sensitivity in the low-frequency band, where the memory sources are stronger.

5.2 Generic form of Linear Memory

More specifically, before the burst is emitted and after it is finished, the source will consist of a set of freely moving systems that are gravitationally unbound from each other. For example, two stars flying toward each other or apart, or a neutron-star binary, which is to be regarded as a single system rather than two independent stars, since the two stars are bound to each other gravitationally. Let us consider N freely moving systems labeling by the index $A = 1, 2, 3...N$. Then the Memory expression due to the motion of the source expressed in the geometrized units ($G = c = 1$) is given by [7, 8],

$$\triangle\bar{h}^{TT}_{jk} = \triangle\Big[\sum_{A=1}^{N}\frac{4P^A_j P^A_k}{\mathbf{k}\,.\,\mathbf{P^A}}\Big]^{TT} \tag{5.1}$$

by writing it explicitly we have,

$$\triangle\bar{h}^{TT}_{jk} = \triangle\sum_{A=1}^{N}\frac{4M_A}{r\sqrt{1-v_A^2}}\Big[\frac{v^A_j v^A_k}{1-v_A cos\theta_A}\Big]^{TT} \tag{5.2}$$

where $A \equiv$ a freely moving system, $M_A \equiv$ mass of system A and $v_A \equiv$ velocity of CM of A. This is the expression of the **Linear Memory effect**.

Here on the right-hand side $\triangle$ means the final value of the summation(after the burst) minus the initial value (before the burst), M_A is the mass of the system A, and the other symbols denote the following quantities as measured in the rest frame of the detector: r is the distance from source to detector, v^j_A is the velocity of the CM of the system A, and θ is the angle between v^j_A and the direction from the source to the detector. TT means "in the rest frame of the source project out the piece that is transverse to the line between source and detector and remove that piece's trace", that is, take the transverse, traceless or TT part. The RHS of Eq. (5.2) is precisely the TT part of the change in the source's $1/r$, coulomb-type gravitational field.

Sources of BWM:

In order to produce a BWM, an astrophysical system, either before the emission of the burst or afterwards or both, must consist of two or more freely moving masses. These masses could be, for example, stars, black holes, gravitationally bound clusters of stars or black holes, or bursts of neutrinos; the only constraint is that each of the masses must move freely and therefore be characterized by a single, constant 4-momentum. Thus the source must consist either of the collision of two or more initially free masses, or an explosion of an initial single mass into several freely and independently moving masses. In either case, so long as the source is not at a cosmologically large distance, the permanent change in the gravitational-wave field(the burst memory) δh_{ij}^{TT} is equal to the 'transverse traceless (TT)'part of the time-independent, coulomb-type, $1/r$ field of the final system minus that of the initial system[1]. These examples with the limiting sensitivities of proposed detectors, suggest that **BWM could be among the first sources detected.**

- Linear Memory[7] arises in systems with unbound components: a binary on a hyperbolic orbit(two-body scattering), matter or neutrinos ejected from a supernova or gamma-ray burst jets.
- This memory effect tells us about non-periodic sources (binary scattering, supernovae, GRB jets etc.)

5.3 Non-linear or Christodoulou Memory

It arises from the GW stress-energy tensor (GWs produced by GWs)

$$\Box \bar{h}_{\mu\nu} = -16\pi\tau_{\mu\nu} \quad ; \tau_{\mu\nu} = T_{\mu\nu} + t_{\mu\nu} \tag{5.3}$$

Here τ depends on the matter stress-energy tensor T, the Landau-Lifshitz pseudotensor t, and other terms quadratic in h. Of the many nonlinear terms in t, there is a piece that is proportional to the stree-energy tensor for GWs:

$$t_{\mu\nu} \propto T_{jk}^{gw} = \frac{1}{r^2}\frac{dE^{gw}}{dtd\Omega} n_j n_k \tag{5.4}$$

where $\frac{dE^{gw}}{dtd\Omega} \equiv$ GW energy flux and $n_j \equiv$ unit radial vector. Eq.(5.4) is the stress-energy tensor for the gravitational waves.

[1] For some sources the magnitude of the components of δh_{ij}^{TT} can be estimated from the equation, $\delta h \to \frac{E_{CM}^{kin}}{r}$.

The correction to the GW field [9], when applying the Green's function $\Box^{-1}$ to the RHS of the EFE, will be;

$$\Delta \bar{h}^{TT}_{jk} = \frac{4}{r}\int_{-\infty}^{t_r} dt^{'} \Big[\int \frac{dE^{gw}}{dt' d\Omega'} \frac{n_j^{'} n_k^{'}}{(1 - \mathbf{n}'.\mathbf{N})} d\Omega^{'} \Big]^{TT} \tag{5.5}$$

where $\mathbf{N}$ points from source to observer, Δ means difference between late and early time values. This is the Non-linear memory effect. The time integral in the above equation is what gives the memory its hereditary nature: the memory piece of the GW field for any value of t_r depends on the entire past history of the source.

- discovered in the 1990's independently by Blanchet & Damour and Christodoulou.
- Arises from the GW stress-energy tensor (GWs produced by GWs)
- A nonlinear contribution also results from the interaction of the waves with themselves. Nonlinear memory[10] is sourced by the emitted GWs. The non-oscillatory nature of the memory suggests that it will not be a dominant feature in interferometric detectors, for which drifts in the background metric are factored out. However, the step induced during merger may be observable in pulsar timing measurements, as has been noted in a recent set of papers [11]
- non-oscillatory and visually distinctive in the waveform
- affects the signal amplitude starting at leading (Newtonian-quadrupole) order
- like GW "tails", the nonlinear memory is hereditary.
- arises from a contribution of the emitted GWs to the changing quadrupole and higher mass moments to the radiative mass multipole moments, that is sourced by the energy-flux of the radiative GWs.
- As discussed by Thorne, the nonlinear memory can be described in terms of a linear memory in which the unbound masses are the individual radiated gravitons. This implies that nearly all GW sources are sources with memory(even if the component objects remain bound).

The memory (linear and nonlinear) should not be mistaken as a change in the monopolar-piece of the 1/R expansion of the metric. Rather it is a change in the quadrupolar (and higher-order) pieces of the $1/R$-spatial-part of the TT-projection of the metric. It is a purely GW effect, and is not directly connected with the change in the 'Coulomb part' of the metric or the 'mass loss of the source' (spherically symmetric mass loss produces no GWs!) except indirectly through the changing mass's effect on the quadrupole and higher-order multipole moments.

5.4 Non-linear Memory is Interesting!

- It has a large contribution to the time-domain waveform amplitude
- It is *hereditary* effect -the memory amplitude at any retarded time depends on the entire past motion of the source (and not just on the source's instantaneous retarded-time configuraiton)
- A unique nonlinear effect because of its non-oscillatory nature, makes it distinctly visible in the waveform.
- unique among the many other nonlinear effects present in the GW signal. Unlike other post-Newtonian corrections, the memory affects the waveform at leading (Newtonian) order. Its "hereditary" nature allows a small effect to build-up to a large value over time.
- it is observable and could serve as a test of general relativity.
- can also think of it as a nonlinear correction to the multipoles: $T^{gw}_{\alpha\beta} \sim O(h^2)$
- affects the waveform at leading-order.
- imparts a unique and visually apparent signature to the waveform.

5.5 Detection of Non-linear Memory

- Considering the memory's detectability with GW interferometer, we compute the sky-averaged rms signal-to-noise ratio(SNR)for a detector.
- detectable with the pulsar timing arrays.

5.6 Memory in Asymptotic flat spacetime

Expression for Memory in asymptotically flat spacetime[12]

$$\boxed{\Delta D_\mu = \frac{1}{2}\Delta h^{TT}_{\mu\nu} D^\nu} \quad \rightarrow \quad \textit{Non-trivial memory}$$

where,
$D^\nu \equiv$ Geodesic deviation of the pair of test particles
$\Delta h^{TT}_{\mu\nu} \equiv$ Net change in metric perturbation in TT gauge

Memory Effect vanishes to leading $O(\frac{1}{r})$ in all asymptotically flat spacetimes of even dimension[12] $d > 4$, that are stationary near spatial infinity and- fter a burst of gravitational radiation-become(nearly) stationary again at late times at null infinity.

Simple example;[13]

Memory for a point particle scattering in linearized gravity:

Change in displacement of the two test particles; in GICS in the presence of scattering process

6 Rough Work

Our Universe is curved not flat!!

Concerns:

- ΔD_μ involves integration over history of motion of the particles unlike flat spacetime.
- D_μ get contributions from both background curvature as well as the GWs.
- The retarded propagator of metric perturbation contains a *tail term*[2].

Net Memory[14] $= \sum$ All tail terms from (reflected waves+original waves)

7 Review: Electromagnetic Memory

Review of [?]:

Any gravitational memory effect corresponds to an asymptotic charge generated by a particular asymptotic symmetry. For example, the displacement memory is related to the supertranslation charges. Given the particular kind of memory effects and various extensions of the asymptotic symmentry group, the better ways are to relate these various memory effect to the various asymptotic charges. The best way to do that is to use the tetrad formalism [? ? ? ?]. The precession of a freely falling gyroscope in the asymptotic region of spacetime caused by gravitational waves and a net rotation of the gyroscope after the passage of gravitational waves is called the 'gyroscopic memory effect'. This memory is related to the dual higher derivative or Gauss-Bonnet charge, that is generated by internal Lorentz transformations [?]. Also the Pontryagin charge generated by an internal Lorentz transformation is related to a new subleading memory effect, which is called as the radial kick memory. GW Memory effect arise from the non-oscillatory components of the gravitational wave signals. These effects are the predictions of GR in the non-relativistic regime

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and this has the close connections to the asymptotic properties of the isolated gravitating systems [**?**].
There are many types of memory effects. Displacement memory, spin memory and velocity memory effects.

- Displacement memory is a change in the relative separation of two initially comoving observers due to a burst of GWs
- Spin memory is a portion of the change in relative separation of observers with initial relative velocity.

As these effects are very small, we can only detect these from the events which are much louder than those that have been detected so far. So combining data from multiple events which could be detected in a population of binary mergers.
So the question is how long current and future detectors will need to operate in order to measure these effects from populations of binary BH systems that are consistent with the populations inferred from the detections from LIGO, VIRGO and KAGRA sensitivity for 1.5years

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