SYDE 556/750

Simulating Neurobiological Systems
Lecture 6: Recurrent Dynamics

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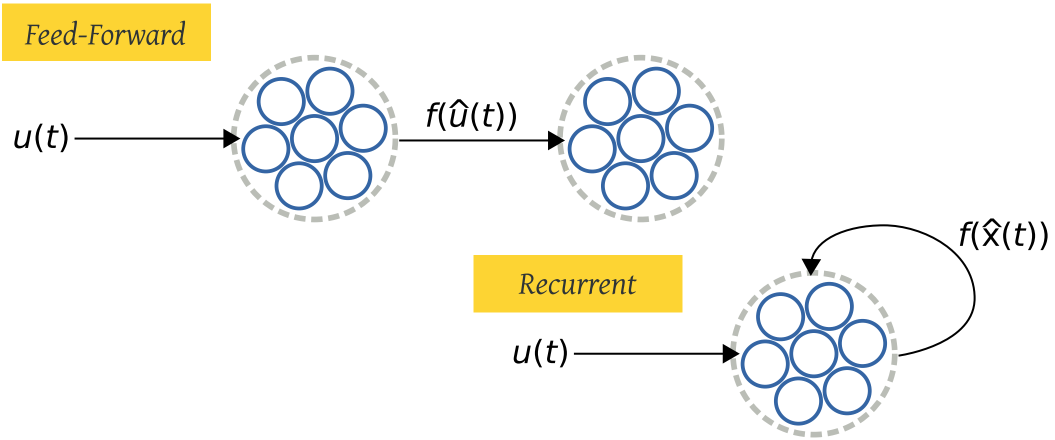
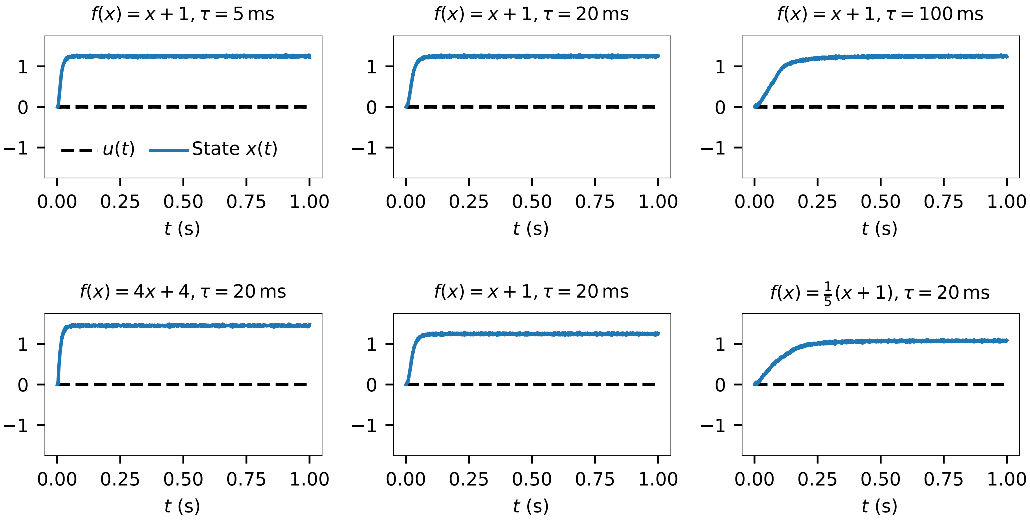
NEF Principle 3: Dynamics

NEF Principle 3 – Dynamics
Neural dynamics are characterized by considering neural representations as control theoretic state variables. We can use control theory (and dynamical systems theory) to analyse and construct these systems.

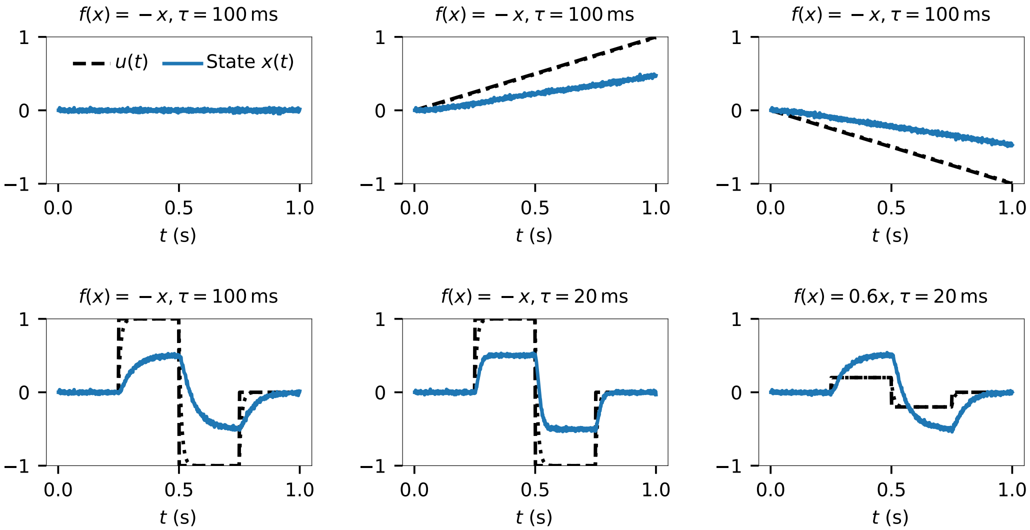
Feed Forward vs. Recurrent Connections



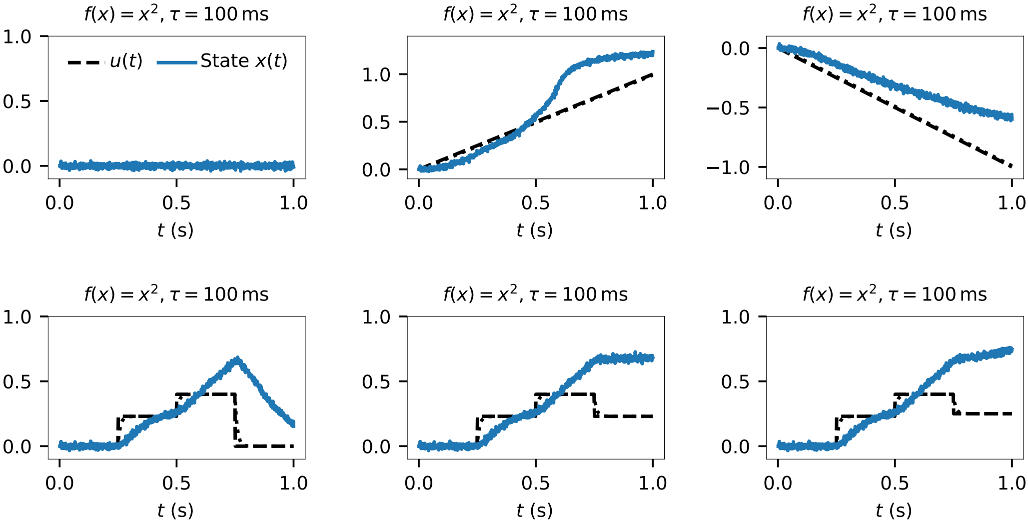
Recurrence Experiments (I)



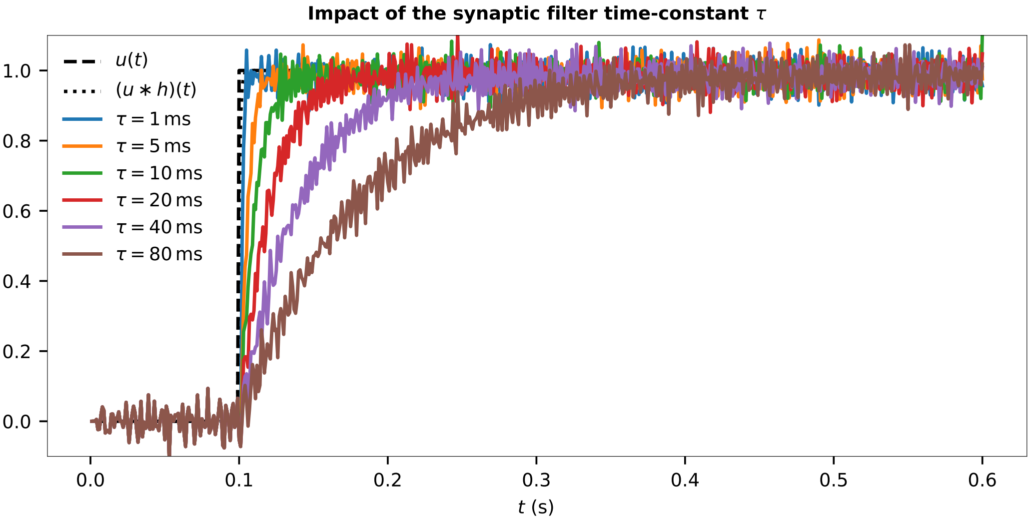
Recurrence Experiments (II)



Recurrence Experiments (III)



Making Sense of Dynamics



Behaviour of a Recurrent Connection

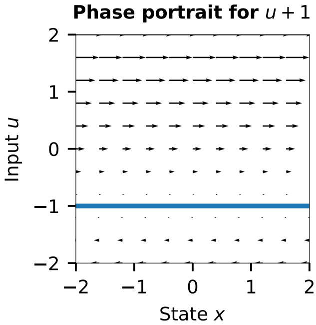
Feed-forward

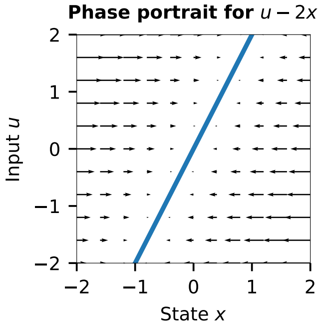
$$x(t) = g(u(t)) * h(t)$$

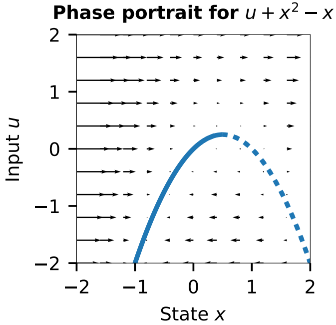
Recurrent

$$\begin{aligned}
x(t) &= (g(u(t)) + f(x(t))) * h(t) \\
X(s) &= (G(s) + F(s))H(s) \\
X(s) &= (G(s) + F(s))\frac{1}{1 + s\tau} \\
X(s) + s\tau X(s) &= G(s) + F(s) \\
s\tau X(s) &= G(s) + F(s) - X(s) \\
sX(s) &= \frac{F(s) - X(s)}{\tau} + \frac{G(s)}{\tau} \\
\frac{dx}{dt} &= \frac{f(x(t)) - x(t)}{\tau} + \frac{g(u(t))}{\tau}
\end{aligned}$$

Revisiting Recurrence Experiments







Behaviour of a Recurrent Connection

$$\frac{dx}{dt} = \frac{f(x(t)) - x(t)}{\tau} + \frac{g(u(t))}{\tau}$$

If we want this

$$\frac{dx}{dt} = a(x) + b(u)$$

Then we set

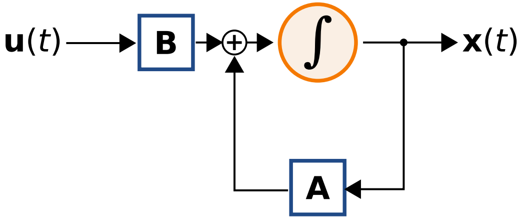
$$f(x) = \tau a(x) + x$$
$$g(u) = \tau b(u)$$

And we get

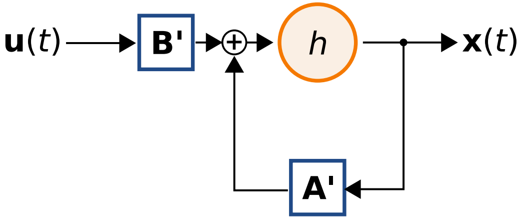
$$\frac{dx}{dt} = \frac{(\tau a(x) + x) - x}{\tau} + \frac{(\tau b(u))}{\tau}$$
$$\frac{dx}{dt} = a(x) + b(u)$$

Implementing Dynamics using a Neural Ensemble

Evaluating an LTI
using an Integrator



Evaluating an LTI
using a Synaptic Filter



Implementing Dynamical Systems as a Neural Ensemble

LTI System

$$\begin{aligned}\varphi(\mathbf{u},\mathbf{x}) &= \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}\\ \varphi'(\mathbf{u},\mathbf{x}) &= \mathbf{A}'\mathbf{x} + \mathbf{B}'\mathbf{u}\\ \mathbf{A}' &= \tau\mathbf{A} + \mathbf{I}\\ \mathbf{B}' &= \tau\mathbf{B}\,.\end{aligned}$$

Additive Time-Invariant System

$$\begin{aligned}\varphi(\mathbf{u},\mathbf{x}) &= f(\mathbf{x}) + g(\mathbf{u})\\ \varphi'(\mathbf{u},\mathbf{x}) &= f'(\mathbf{x}) + g'(\mathbf{u})\\ f'(\mathbf{x}) &= \tau f(\mathbf{x}) + \mathbf{x}\\ g'(\mathbf{u}) &= \tau g(\mathbf{u})\end{aligned}$$

“General” Recipe

Scale the original dynamics by τ, add feedback $\mathbf{x}$

NEF Principle 3: Dynamics

Time-Invariant Dynamical System

$$\frac{\mathrm{d}\mathbf{x}(t)}{\mathrm{d}t} = f(\mathbf{x}(t), \mathbf{u}(t))$$

Linear Time-Invariant (LTI) Dynamical System

$$\frac{\mathrm{d}\mathbf{x}(t)}{\mathrm{d}t} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$

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Low-pass Filter Example

Desired behaviour

$$\frac{dx}{dt} = \frac{u - x}{\tau_{desired}}$$

Feed-forward input

$$g(u) = \tau_{synapse}(\frac{u}{\tau_{desired}})$$

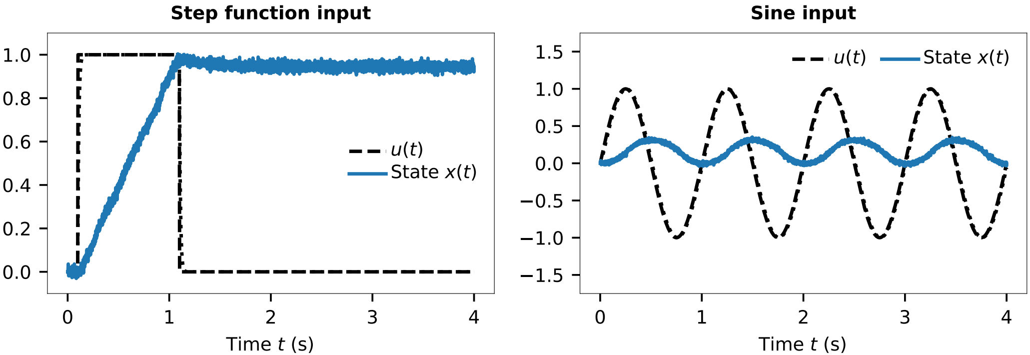
$$g(u) = \frac{\tau_{synapse}}{\tau_{desired}} u$$

Recurrent

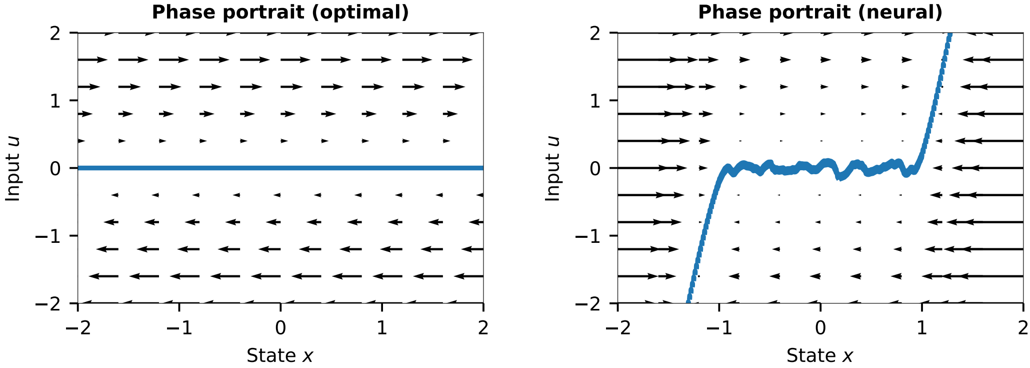
$$f(x) = \tau_{synapse}(\frac{-x}{\tau_{desired}}) + x$$

$$f(x) = (1 - \frac{\tau_{synapse}}{\tau_{desired}})x$$

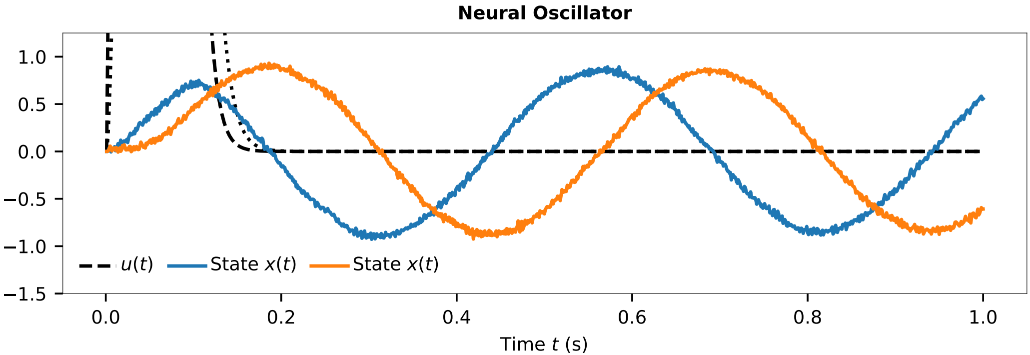
Integrator Example (I)



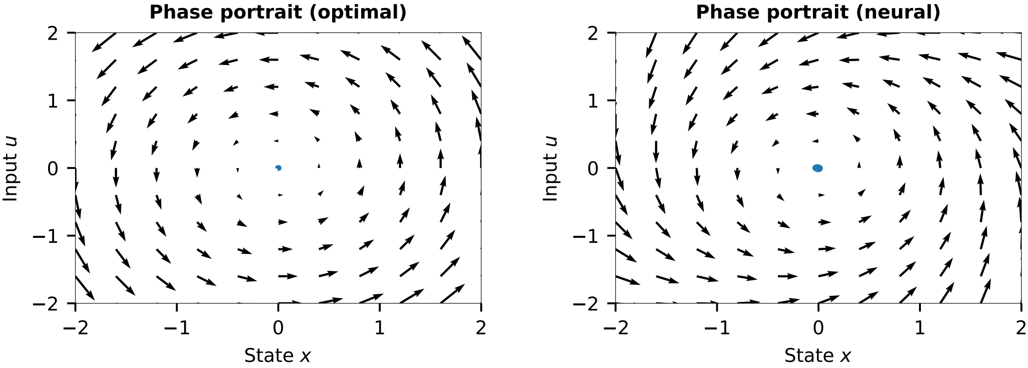
Integrator Example (II)



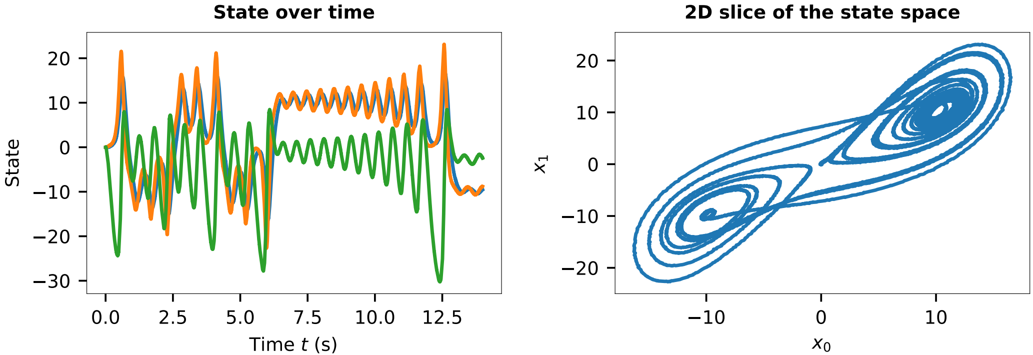
Oscillator Example (I)



Oscillator Example (II)

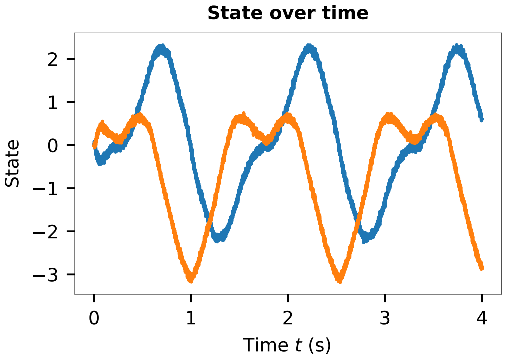


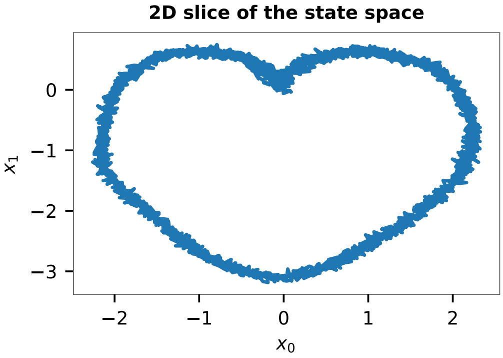
Lorentz Attractor



$$\frac{\mathrm{d}\mathbf{x}(t)}{\mathrm{d}t} = \begin{pmatrix} 10x_2(t) - 10x_1(t) \\ -x_1(t)x_3(t) - x_2(t) \\ x_1(t)x_2(t) - \frac{8}{3}(x_3(t) + 28) - 28 \end{pmatrix}$$

Heart Shape


State over time
2
1
0
−1
−2
−3
State
0
1
2
3
4
Time t (s)


2D slice of the state space
0
−1
−2
−3
x_1
−2
−1
0
1
2
x_0

Horizontal Eye Control

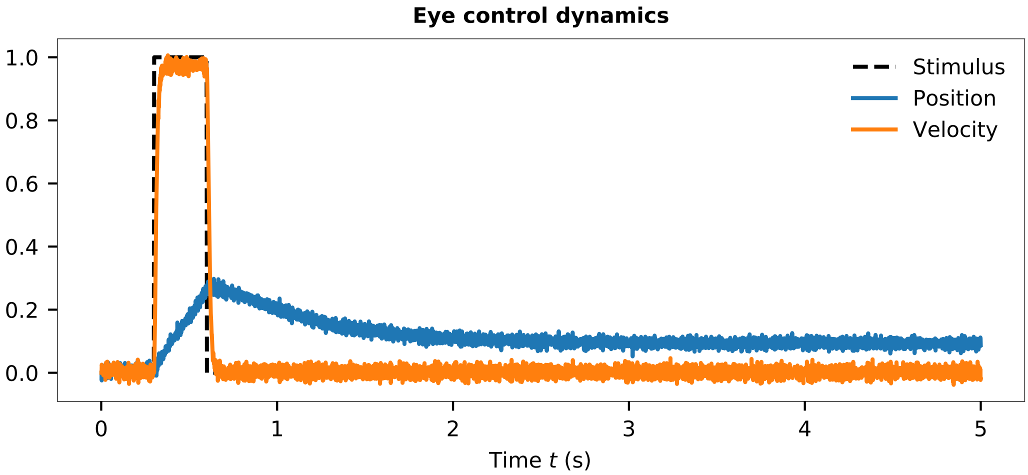


Image sources

Title slide
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