In [21], the recurrence relations of the proposition 2.1 were called *rectangular* because their r.h.s. contains monodromy matrix elements from a rectangular domain of the monodromy matrix, see section 8.

The second property can be viewed as a dual version of the previous one since it can be used to prove the recurrence relations.

Proposition 2.2. *Let $\mathbb{B}(\bar t)$ be an off-shell Bethe vector. Then, the action of the monodromy matrix entries reads*

$$T_{i,j}(z)\cdot\mathbb{B}(\bar t)=\lambda_n(z)\mu^n_{n'}(z;\bar t)\sum_{part\ \bar w^s}\mathbb{B}(\bar w_{\mathrm{II}})\frac{\Gamma_i(\bar w_{\mathrm{I}})}{\psi_{n'}(z;\bar w_{\mathrm{I}})}\frac{\bar\Gamma_j(\bar w_{\mathrm{III}})}{\phi_n(z;\bar w_{\mathrm{III}})}\,\Omega(\bar w_{\mathrm{I}},\bar w_{\mathrm{II}},\bar w_{\mathrm{III}})\alpha(\bar w_{\mathrm{III}}),$$

where $\Gamma_i(\bar w_{\mathrm{I}}),\psi_{n'}(z;\bar w_{\mathrm{I}}),\bar\Gamma_j(\bar w_{\mathrm{III}})$ and $\phi_n(z;\bar w_{\mathrm{III}})$ are rational fanctions, see propositions 4.1 and 6.1

Corollary 2.1. *The off-shell Bethe vectors have definite colors:*

$$\mathfrak{q}_a\mathbb{B}(\bar t)=|\,\bar t^a\,|\,\mathbb{B}(\bar t),\quad a\in J_{\mathfrak{g}}\,.$$

The next property justifies the name off-shell Bethe vectors we use in the definition 2.1:

Proposition 2.3. *Let $\mathbb{B}(\bar t)$ be an off-shell Bethe vector. If the Bethe equations*

$$\alpha_a(\bar t^a_{\mathrm{I}})=\frac{\Omega^R_a(\bar t_{\mathrm{I}},\bar t_{\mathrm{II}})}{\Omega^L_a(\bar t_{\mathrm{II}},\bar t_{\mathrm{I}})},\qquad a\in J_{\mathfrak{g}}$$

are obeyed, $\mathbb{B}(\bar t)$ is an eigenvector of the transfer matrix $\mathcal{T}_{\mathfrak{g}}(z)=\mathrm{tr}T(z)$:

$$\mathcal{T}_{\mathfrak{g}}(z)\mathbb{B}(\bar t)=\tau_{\mathfrak{g}}(z;\bar t)\mathbb{B}(\bar t)\,.$$

Moreover, the on-shell Bethe vectors are also highest weight vectors for the $\mathfrak{g}$ algebra:

$$\mathfrak{T}_a\mathbb{B}(\bar t)=0,\quad a\in J_{\mathfrak{g}}\,.$$

The explicit form of the eigenvalues are given in (4.27), (7.10), (7.12) and (7.13).

The next proposition allows to build Bethe vectors for composite models. It is usually called the coproduct formula for off-shell Bethe vectors.

Proposition 2.4. *Let $\mathbb{B}^{[2]}(\bar t)$ and $\mathbb{B}^{[1]}(\bar t)$ be two off-shell Bethe vectors obeying the definition 2.1 for two different commuting monodromy matrices $T^{[2]}(u)$ and $T^{[1]}(u)$ respectively. Then, the composed Bethe vector*

$$\mathbb{B}^{[2,1]}(\bar t)=\sum_{\mathrm{part}}\Omega(\bar t_{\mathrm{I}},\bar t_{\mathrm{II}})\mathbb{B}^{[2]}(\bar t_{\mathrm{I}})\mathbb{B}^{[1]}(\bar t_{\mathrm{II}})\prod_{s\in J_{\mathfrak{g}}}\alpha^{[2]}_s(\bar t^s_{\mathrm{II}})$$

also obeys the definition 2.1 for the composite model defined by the monodromy matrix

$$T^{[2,1]}(z)=T^{[2]}(u)T^{[1]}(u),\qquad i.e.\quad T^{[2,1]}_{i,j}(z)=\sum_{k=n'}^{n}T^{[2]}_{i,k}(u)T^{[1]}_{k,j}(u)\,.$$

The rational function Ω is defined in (9.2).