HW 3

Due: 23:59, Monday, March 4, 2018

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Instructions:

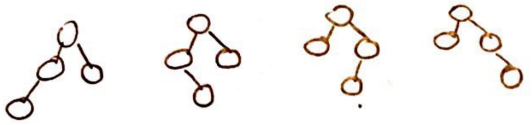
1. Do not change the problem statements we are giving you. Simply add your solutions by editing this latex document.
2. Take as much space as you need for each problem. You'll tell us where your solutions are when you submit your paper to gradescope.
3. Export the completed assignment as a PDF file for upload to gradescope.
4. On gradescope, upload only **one** copy per partnership. (Instructions for uploading to gradescope will be posted on the HW3 page of the course website.)

1. **Lanky AVL Trees [30 points].**

In class we showed that there are 2^{n-1} binary trees of maximal height given n nodes, if we put no constraints on the shape of the tree. In this homework assignment we will explore AVL trees of maximal height. To that end, let $L(h)$ be the *number* of maximal height AVL trees of height h . We call such trees "Lanky Trees."

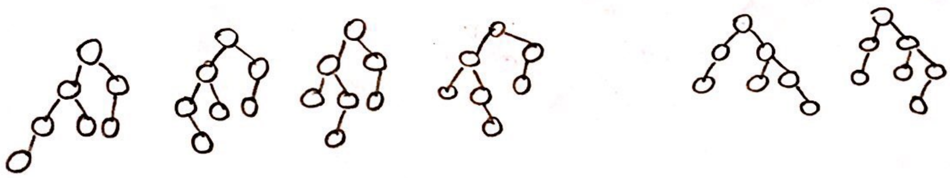
A "Lanky Tree" of height h is a smallest (i.e., fewest nodes) AVL tree of height h.

(a) Draw all Lanky Trees of height 2:



$\boxed{L(2) = 4}$

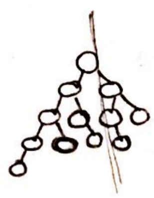
(b) Draw at least six different Lanky Trees of height 3:



$\boxed{L(3) = 16}$

(c) Give an expression for $L(4)$ in terms of $L(3)$ and $L(2)$.

$\boxed{L(4) = 2 \cdot L(3) \cdot L(2)}$



leftside right

4x4 x2

x4

(d) Write a recurrence for $L(h)$.

Base cases: $\boxed{L(0) = 1 \quad , L(1) = 2}$

Recursive case: $\boxed{L(h) = 2 \cdot L(h-1) \cdot L(h-2)}$

(e) The recurrence from part (d) probably looks quite strange to you. We will solve $L(h)$ by first transforming it into something more familiar. Define $B(h) = \log_2 L(h), \forall h \geq 0$. Write $B(h)$ as a recurrence below.

Scratch work: $B(0) = \log_2 L(0) = \log_2 1 = 0$

$B(1) = \log_2 L(1) = \log_2 2 = 1$

$\vdots$

$B(h) = \log_2 L(h) = \log_2[2 \cdot L(h-1) \cdot L(h-2)]$

$= \log_2 2 + \log_2 L(h-1) + \log_2 L(h-2)$

$= 1 + \log_2 L(h-1) + \log_2 L(h-2)$

$= \log_2 (h-1) + \log_2 (h-2) + 1 = B(h-1) + B(h-2) + 1$

(by I.H)

Base cases: $\boxed{B(0) = 0 \quad , B(1) = 1}$

Recursive case: $\boxed{B(h) = B(h-1) + B(h-2) + 1}$

(f) While this recurrence is getting closer to something familiar, it still contains a pesky additive term! This can be eliminated by defining yet another function to be:

$$\boxed{P(h) = B(h) + \underline{\ 1\ }, \forall h \geq 0}$$

(g) Write the recurrence for $P(h)$ below.

Scratch work: $P(0) = B(0) + 1 = 0 + 1 = 1$

$P(1) = B(1) + 1 = 1 + 1 = 2$

$\vdots$

$P(h) = B(h) + 1$, from 1(f)

$= B(h-1) + B(h-2) + 1 + 1$, from 1(e)

$P(h-1) = B(h-1) + 1$ & $P(h-2) = B(h-2) + 1$, By I.H

$\therefore P(h) = P(h-1) + B(h-2) + 1$

$= P(h-1) + P(h-2)$

Base cases: $P(0) = 1$, $P(1) = 2$

Recursive case: $P(h) = P(h-1) + P(h-2)$

(h) Write the first 10 terms of the sequence defined by $P(h)$:

1, 2, 3, 5, 8, 13, 21, 34, 55, 89

Write the first 10 terms of the Fibonacci sequence, $F(h), \forall h \geq 0$.

$0, 1, \ldots$ 1, 2, 3, 5, 8, 13, 21, 34

(i) Describe the function $P(h)$ in terms of the elements of the Fibonacci sequence:

$P(h) = F(h+2)$

(j) Use the work from the previous parts and known results about the Fibonacci sequence, to write down a closed form solution for $L(h)$, the number of Lanky Trees of height $h \geq 0$.

Scratch work:

$P(h) = F(h+2) = B(h) + 1$, from 1(i) & 1(f)

$F(h+2) = \log_2 L(h) + 1$, def$\underline{n}$ of $B(h)$ in 1(e)

$\log_2 L(h) = F(h+2) - 1$, by alg.

$L(h) = 2^{F(h+2)-1}$

closed form of Fib-Sq $\Rightarrow$ $F(h) = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^h - \left(\frac{1-\sqrt{5}}{2}\right)^h\right)$

$F(h+2)-1 = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{h+2} - \left(\frac{1-\sqrt{5}}{2}\right)^{h+2}\right) - 1$

$$\boxed{L(h) = 2^{F(h+2)-1} = 2^{\left[\frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{h+2} - \left(\frac{1-\sqrt{5}}{2}\right)^{h+2}\right)-1\right]}}$$

(That's a lot of trees!)

2. **More Lanky AVL Trees [20 points].**

In this problem we will further explore the structure Lanky AVL trees. In each part, you should use the strategies you learned in problem 1 to create concise solutions. In some cases you'll start by writing down a recurrence, in others you'll observe something about the structure of the tree. We will be grading your scratch work, so please be as clear as you can in describing your derivations.

(a) All Lanky Trees of height h have the fewest nodes of any AVL tree of height h. How many nodes do they have? Express your answer using the Fibonacci function $F()$.

Scratch work:

$N(0) = 1$, $N(1) = 2$

$N(2) = 4 = 2+1 + 1 = N(1)+N(0)+1$

$N(3) = 7 = 4+2 + 1 = N(2)+N(1)+1$

$\vdots$

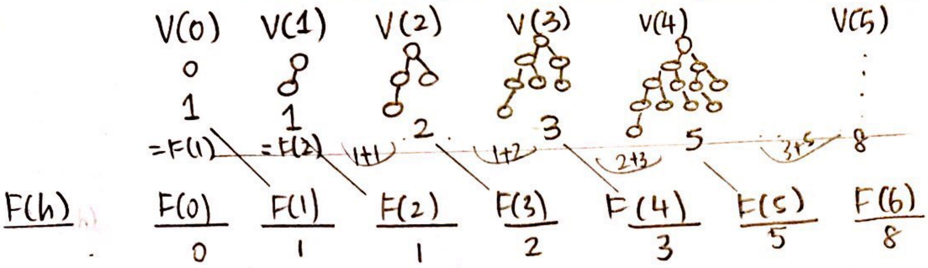
$\Rightarrow N(h) = N(h-1)+N(h-2)+1$

h	F(h)	N(h)
0	0	1 = F(0+3) - 1
1	1	2 = F(1+3) - 1
2	1	4 = F(2+3) - 1
3	2	7 = F(3+3) - 1
4	3	12
5	5	20
6	8	33
7	13	54
8	21	88
h	F(h)	N(h) = F(h+3) - 1

The number of nodes in a Lanky Tree of height h is:

$$\boxed{N(h) = F(h+3) - 1}$$

(b) How many leaves does a Lanky AVL Tree of height h have? Use everything you've learned so far to give us a concise description of this value.



$V(h)$

$= V(h-1)+V(h-2)$

$= F(h+1)$

$V(0) = 1 = F(1)$
$V(1) = 1 = F(2)$
$V(2) = 2 = F(3)$ .... $V(h) = F(h+1)$

$$\boxed{V(h) = V(h-1)+V(h-2) = F(h+1)}$$

(c) In any tree, a non-leaf node is referred to as an *internal* node. How many internal nodes does a Lanky AVL Tree of height h have? Be as concise as you can.

$N(h) = N(h-1) + N(h-2) + 1 = F(h+3) - 1$, from 2(a)
$V(h) = V(h-1) + V(h-2) = F(h+1)$, from 2(b)

$I(h) = N(h) - V(h)$
$= \underline{F(h+3)} - 1 - F(h+1)$
$= \underline{F(h+2) + F(h+1)} - 1 - F(h+1)$, by def of Fib sq.
$= F(h+2) - 1$

$$\boxed{I(h) = F(h+2) - 1}$$

(d) Another way to count the number of Lanky Trees is to notice that the only difference between Lanky Trees is the order of the children at the internal nodes. Since the two children of an internal node have different heights in a Lanky Tree, swapping the order of the children produces a different Lanky Tree of the same height.

How many Lanky Trees of height h are there **as a function of** $I(h)$ defined above?

$$\boxed{2^{I(h)}}$$

$L(h) = 2^{F(h+2)-1}$, from 1j
$I(h) = F(h+2) - 1$, from 2c
$\therefore L(h) = 2^{I(h)}$, by alg.