HW 1

Due 23:59 January 11, 2019

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Instructions:

1. Do not change the problem statements we are giving you. Simply add your solutions by editing this latex document.
2. Take as much space as you need for each problem. You'll tell us where your solutions are when you submit your paper to gradescope.
3. Export the completed assignment as a PDF file for upload to gradescope.
4. On gradescope, upload only **one** copy per partnership. (Instructions for uploading to gradescope will be posted on the HW1 page of the course website.)

Academic Conduct: I certify that my assignment follows the academic conduct rules for of CPSC 221 as outlined on the course website. As part of those rules, when collaborating with anyone outside my group, (1) I and my collaborators took no record but names away, and (2) after a suitable break, my group created the assignment I am submitting without help from anyone other than the course staff.

1. (2 points) Using 140 characters or less, post a synopsis of your favorite movie to the course piazza space under the "HW1 tell me something!" notice, so that your post is visible to everyone in the class, and tagged by #HW1num1. Also, use Piazza's code-formatting tools to write a *private* post to course staff that includes at least 5 lines of code. It can be code of your own or from a favorite project—it doesn't even have to be syntactically correct—but it must be formatted as a code block in your post, and also include the tag #HW1num1. (Hint: Check http://support.piazza.com/customer/portal/articles/1774756-code-blocking). Finally, please write the 2 post numbers corresponding to your posts here:

Favorite Movie Post (Public) number:	59
Formatted Code Post (Private) number:	229

2. (16 points) In this problem, you will be a math detective! Some of the symbols and functions may not be familiar to you, but with a little digging, reading, and observing, you'll be able to figure them out. Your task is to simplify each of the following expressions as much as possible, **without using an calculator (either hardware or software)**. Do not approximate. Express all rational numbers as improper fractions. You may assume that n is an integer greater than 1. Show your work in the space provided, and write your final answer in the box.

(a) i. $\sum_{k=1}^{n} k2^{k+1} - \sum_{k=1}^{n} k2^{k}$

Answer for (a.i):	$(n-1)2^{n+1} + 2$

$= \sum_{k=1}^{n} (k2^{k+1} - k2^{k})$

$= \sum_{k=1}^{n} (2^{k+1} - 2^{k})k = \sum_{k=1}^{n} k2^{k}(2-1) = \sum_{k=1}^{n} k2^{k} := y = 1\cdot2 + 2\cdot2^2 + 3\cdot2^3 + \dots + (n-1)2^{n-1} + n2^n$

let $y = \sum_{k=1}^{n} k2^{k}$, thus $2y = \sum_{k=1}^{n} k2^{k+1} = 1\cdot2^2 + 2\cdot2^3 + \dots \quad + (n-1)2^n + n2^{n+1}$

$2y - y = n\cdot2^{n+1} - \sum_{k=1}^{n} 2^k = n\cdot2^{n+1} - \sum_{k=0}^{n} 2^k + 1 = n\cdot2^{n+1} - \left(\frac{(2^{n+1}-1)}{2-1}\right) + 1$

$= n\cdot2^{n+1} - 2^{n+1} + 1 + 1 = (n-1)2^{n+1} + 2$

ii. $\prod_{i=1}^{n-1} \frac{2i}{n-i}$

Answer for (a.ii):	2^{n-1}

$$\prod_{i=1}^{n-1} \frac{2i}{n-i} = \frac{2(\cancel{1}) \cdot 2(\cancel{2}) \cdot 2(\cancel{3}) \cdots 2(\cancel{n-3}) \cdot 2(\cancel{n-2}) \cdot 2(\cancel{n-1})}{(\cancel{n-1}) \cdot (\cancel{n-2}) \cdot (\cancel{n-3}) \cdots \cancel{3} \cdot \cancel{2} \cdot \cancel{1}}$$

$$= \underbrace{2 \cdot 2 \cdot 2 \cdots 2}_{n-1 \text{ times}}$$

$$= 2^{n-1}$$

(b) i. $15^{333} \bmod 2$

Answer for (b.i):	1

Any number mod 2 is 1 iff the number is odd.
Since the product of odd numbers is odd, any power of 5 is od

$\therefore \quad 15^{333} \bmod 2 \equiv 1$

ii. $32^{333} \bmod 15$

Answer for (b.ii):	2

$32^{333} \bmod 15 \equiv 1 \equiv (32^{332} \cdot 32) \bmod 15$

$\equiv 1 \cdot 2 \bmod 15$

$\equiv 2$

(c) i. $\sum_{r=0}^{n} \left(\frac{1}{n}\right)^r$

Answer for (c.i):	$\frac{n}{n-1} - \frac{1}{(n-1)n^n}$

$$= \frac{1-\left(\frac{1}{n}\right)^{n+1}}{1-\left(\frac{1}{n}\right)} = \frac{1-\left(\frac{1}{n}\right)^{n+1}}{\left(\frac{n-1}{n}\right)}$$

$$= \left(\frac{n}{n-1}\right)\left(1-\left(\frac{1}{n}\right)^{n+1}\right)$$

$$= \left(\frac{n}{n-1}\right)\left(1-\frac{1}{n\cdot n^n}\right)$$

$$= \frac{n}{n-1} - \frac{1}{(n-1)n^n}, \quad n>1$$

ii. $\sum_{r=3}^{\infty} \left(\frac{2}{3}\right)^r$

Answer for (c.ii):	$\frac{8}{9}$

$$= \sum_{r=0}^{\infty}\left(\frac{2}{3}\right)^r - \sum_{r=0}^{2}\left(\frac{2}{3}\right)^r$$

① $\sum_{r=0}^{\infty}\left(\frac{2}{3}\right)^r = \frac{1}{1-\frac{2}{3}} = 3$

② $\sum_{r=0}^{2}\left(\frac{2}{3}\right)^r = 1+\frac{2}{3}+\frac{4}{9} = \frac{19}{9}$

① − ② $= 3 - \frac{19}{9} = \frac{8}{9}$

(d) i. $4^{(\log_2 n)/2}$

Answer for (d.i):	n

$$4^{\frac{\log_2 n}{2}} = (2^2)^{\frac{\log_2 n}{2}} = 2^{\log_2 n} = n$$

ii. $\frac{\log_7 1024}{\log_7 32} - \frac{\log_3 1024}{\log_3 32}$

Answer for (d.ii):	0

(applying log rule: $\frac{\log_c(b)}{\log_c(a)} = \log_a(b)$)

$= \log_{32} 1024 - \log_{32} 1024$

$= 0$

iii. $\frac{\log_2 n}{\log_{32} n}$

Answer for (d.iii):	5

$$\frac{\log_2 n}{\log_{32} n} = \frac{\log_2 n}{\log_{2^5} n} = \frac{\log_2 n}{\frac{1}{5}\log_2 n} = \frac{5}{1} = 5$$

3. (12 points)

(a) (3 points) Find a closed form expression for $f(n) = \sum_{j=1}^{n}(2j-1), \forall n \geq$ ____ (fill in the blank with the smallest reasonable value!), and use induction to prove the formula is correct.

Formula:	n^2

$f(n) = \sum_{j=1}^{n}(2j-1) = 2\sum_{j=1}^{n} j - \sum_{j=1}^{n} 1 = 2 \cdot \frac{n(n+1)}{2} - n$

$= n^2 + n - n = n^2$

i) Base case, when $n=1$

$f(1) = \sum_{j=1}^{1}(2j-1) = 2-1 = 1 = 1^2$

ii) Induction case

IH: $f(n) = \sum_{j=1}^{n}(2j-1) = n^2$ for $\forall n \geq 1$

and we need to prove $n+1$ case, that $f(n+1) = (n+1)^2$

$f(n+1) = \sum_{j=1}^{n+1}(2j-1) = \sum_{j=1}^{n}(2j-1) + (2(n+1)-1)$ (by IH)

$= n^2+2n+1 = (n+1)^2$

Thus, formula holds when $n \geq 1$. QED

(b) (2 points) Find a closed form expression for

$$\frac{1+3+5+\ldots+(2k-1)}{(2k+1)+(2k+3)+\ldots+(4k-1)}.$$

Formula:	$\frac{1}{3}$

(c) Give a rigorous, direct proof that your solution to part (b) is correct.

i. (2 points) Express the fraction using a new variable that quantifies the number of terms in the numerator and in the denominator. Be sure to bound that variable.

Formula:	$\dfrac{\sum_{n=1}^{k}(2n-1)}{\sum_{n=k+1}^{2k}(2n-1)}$, for $k \geq 1$

ii. (5 points) Complete the proof. A concise solution will apply the result from part (a) three times.

$$\frac{\sum_{n=1}^{k}(2n-1)}{\sum_{n=k+1}^{2k}(2n-1)} = \frac{\sum_{n=1}^{k}(2n-1)}{\sum_{n=1}^{2k}(2n-1) - \sum_{n=1}^{k}(2n-1)} = \frac{2\cdot\frac{k(k+1)}{2} - k}{2\cdot\frac{2k(2k+1)}{2} - 2k - 2\cdot\frac{k(k+1)}{2} + k}$$

$$= \frac{k^2}{4k^2+2k-2k-k^2-k+k} = \frac{k^2}{3k^2} = \frac{1}{3} \quad \# \text{ QED}$$

4. (8 points) Prove that $f(n) \in O(g(n))$ if and only if $g(n) \in \Omega(f(n))$.

Part1) Consider an arbitrary integer n and arbitrary function f and g.

WLOG, assume $f(n) \in O(g(n))$ and we want to prove $g(n) \in \Omega(f(n))$

by showing $\exists$ c_1 and n_1 such that $g(n) \geq c_1 f(n)$ for all $n \geq n_1$ (Definition of Ω)

By assumption, $\exists$ c_2 and n_2 such that

$f(n) \leq c_2 g(n)$ for $\forall n \geq n_2$ where $c_2 \geq 0$.

By dividing by c_2, $\frac{1}{c_2} f(n) \leq g(n)$, $\forall n \geq n_2$ and by choosing $c_1 = \frac{1}{c_2}$ and

$n_1 = n_2$, $c_1 f(n) \leq g(n)$, $\forall n \geq n_1 = n_2$.

Therefore by the definition, $g(n) \in \Omega f(n)$

Part2) Consider an arbitrary integer n and arbitrary function f and g and WLOG.

assume $g(n) \in \Omega(f(n))$ and we want to prove $f(n) \in O(g(n))$

by assumption. $\exists$ c_1 and n_1 such that $g(n) \geq c_1 f(n)$ for $\forall n \geq n_1$

where $c_1 \geq 0$.

By dividing by c_1, $\frac{1}{c_1} g(n) \geq f(n)$, $\forall n \geq n_1$

Therefore, by choosing $c_2 = \frac{1}{c_1}$ and $n_2 = n_1$, $c_2 g(n) \geq f(n)$, $\forall n \geq n_2$

Therefore by the definition of Big O, $f(n) \in O(g(n))$ where $c_2 = \frac{1}{c_1}$ and $n_2 = n_1$

Thus, $f(n) \in O(g(n))$ if and only if $g(n) \in \Omega(f(n))$. QED

5. (8 points) Indicate for each of the following pairs of expressions $(f(n), g(n))$, whether $f(n)$ is O, Ω, or Θ of $g(n)$. Prove your answers. Note, if you choose O or Ω, you must also show that the relationship is not Θ. You may use the result from problem 4, if you find it useful.

(a) $f(n) = n\log(n^2+1) + n^2\log n$, and $g(n) = n^2$.

Answer for (a):	$f(n) \in \Omega(g(n))$

Consider an arbitrary integer n, and arbitrary functions f and g

WLOG,
i) Suppose $f(n) \in O(g(n))$ then $\exists\, c_1$ and n_0 such that

$n\lg(n^2+1) + n^2\lg n \le cn^2$ for $\forall n \ge n_0$ and by isolating C,

$\frac{n\lg(n^2+1) + n^2\lg n}{n^2} \le c$ and there is no constant that can bound

$\frac{n\log(n^2+1) + n^2\log n}{n^2}$ as n increase, because $\frac{n\lg(n^2+1) + n^2\lg n}{n^2} \to \infty$ as $n \to \infty$

Therefore, by contradiction $f(n) \notin O(g(n))$

WLOG,
ii) Suppose $f(n) \in \Omega(g(n))$ Then $\exists\, c_2$ and n_0 s.t $n\lg(n^2+1) + n^2\lg n \ge C \cdot n^2$ for $\forall n \ge n_0$

We know that $n^2\lg n \ge n^2$ when $n \ge 1$ because $\lg n$ is an increasing function

and we also know that $n\lg(n^2+1) \ge 0$ when $n \ge 1$ since $n\lg(n^2+1) \ge \lg 2$

as $\lg n$ is increasing function.

Therefore by choosing $c_2 = 1$ and $n_0 = 1$, $n\lg(n^2+1) + n^2\lg n \ge n^2$ for $\forall n \ge 1$,

thus $f(n) \in \Omega(g(n))$

Furthermore, $f(n) \in \Theta(g(n))$ iff $f(n) \in O(g(n))$ and $f(n) \in \Omega(g(n))$

However, $f(n) \notin O(g(n))$. therefore $f(n) \notin \Theta(g(n))$. QED

(b) $f(n) = n^2$, and $g(n) = \frac{1}{2}n^3 - 2n^2 - 3n - 17$.

Answer for (b):	$f(n) \in O(g(n))$

Consider an arbitrary integer n and,

i) Suppose $f(n) \in g(n))$. Then by def of big O, $\exists$ c_0 and n_0 such that

$n^2 \leq c_0(\frac{1}{2}n^3 - 2n^2 - 3n - 17)$ for $\forall n \geq n_0$ and WLOG we chose $n_0 = 6$ and by isolating c, we get

$$\frac{n^2}{(\frac{1}{2}n^3 - 2n^2 - 3n - 17)} \leq \frac{6^2}{1} = 36 \leq c \text{ for } \forall n \geq n_0 = 6$$

Since, $\frac{1}{2}n^3 - 2n^2 - 3n - 17 \geq 0$ and it is an increasing function when $n \geq 6$, and $n^2 \geq 0$ and it is also an increasing function. Thus when $n = 6$ the value of $\frac{n^2}{(\frac{1}{2}n^3 - 2n^2 - 3n - 17)}$ is max.

Therefore by chosing $c_0 = 36$ and $n_0 = 6$, $f(n) \in O(g(n))$

ii) Suppose $f(n) \in \Omega(g(n))$ and by def of Ω, $\exists$ c_1 and n_1 such that

$n^2 \geq c_1(\frac{1}{2}n^3 - 2n^2 - 3n - 17)$ for $\forall n \geq n_1$ and by isolating c, we get

$$\frac{n^2}{\frac{1}{2}n^3 - 2n^2 - 3n - 17} \geq c. \text{ However } \frac{n^2}{\frac{1}{2}n^3 - 2n^2 - 3n - 17} \to 0 \text{ as } n \to \infty.$$

Therefore, there is no positive constants c. Contradiction!

Thus, $f(n) \notin \Omega(g(n))$

Furthermore, $f(n) \in \Theta(g(n))$ iff $f(n) \in O(g(n))$ and $f(n) \in \Omega(g(n))$

but $f(n) \notin \Omega(g(n))$. Therefore, $f(n) \notin \Theta(g(n))$

$n \geq c$

6. (16 points) For each C++ function below, give the tightest asymptotic upper bound that you can determine for the function's **runtime**, in terms of the input parameter. Where prompted, also compute the return value of the function. For all function calls, assume that the input parameter $n \geq 2$.

(a)

```
int raspberryscone(int n) {
   int s = 0;
   for (int q = 0; q < n; q++)
      s = s + q;
   for (int r = s; r > 2; r--)
      s--;
   return r * s;
}
```

The 1st for loop: q increments n times at constant time.

$S = \sum_{q=0}^{n} q = \frac{n(n+1)}{2}$

2nd for loop: r&s will always be 2 when $n>2$ or 1 when $n=2$.

$\therefore$ return value $= 1 \times 1$ when $n=2$ or 2×2 when $n>2$

Return value for (a):	$\begin{cases} 1, n=2 \\ 4, n>2 \end{cases}$

Running time of (a):	$O(n^2)$

The return value of 1st loop $\frac{n(n+1)}{2} = \frac{1}{2}n^2 + \frac{1}{2}$ dominates the running time.

(b)

```
int peanutbuttercheesecake(int n) {
   int i = 0;
   int j = 4 * n;
   for (int k = j * j; k > 1; k = k / 2) {
      i++;
   }
   return i * i / 4;
}
```

$i = \lfloor \log_2(16n^2) \rfloor$ Since each divides by 2.

$= 4 + \lfloor \log_2 n^2 \rfloor$

$= 4 + 2\lfloor \log_2 n \rfloor$

return value $= \frac{(4 + 2\lfloor \log_2 n \rfloor)^2}{4}$

$= \left(\frac{4 + 2\lfloor \log_2 n \rfloor}{2}\right)^2$

$= (2 + \lfloor \log_2 n \rfloor)^2$

Return value for (b):	$(2 + \lfloor \log_2 n \rfloor)^2$

Running time of (b):	$O(\log_2 n)$

for loop running time is $4 + 2\log_2 n = O(\log_2 n)$, which dominates as other lines are constant time.

(c)

```
int almondcaramelbrittle(int n) {
   int c = 0;
   int s;
   for (s = 1; s < n * n * n * n; s = s * n) {
        c++;
   }
   return c * s;
}
```

$C = n * n * n * n$
$= n^4$

Return value for (c):	$4n^4$

regardless of n, for loop will always run at constant time of 4 times. $\therefore O(1)$

Running time of (c):	$O(1)$

(d)

$O(n^2)$ / 1 or 4 $\quad$ $O(1)$ / $4n^4$

```
void vanillacustardblueberrytorte(int n) {
   int j = 1;

   for (int k = 0; k < raspberryscone(n/2) * almondcaramelbrittle(n); k++)
        j = 2 * j;

   for (int m = j; m > 1; m = m / 2)
        cout << "I am having so much fun with asymptotics!" << endl;
}
```

$\log n$

The 1st for loop is quartic in n, $\therefore O(n^6) = 4n^4 \times n^2 = 4n^6$ dominates running time.

Running time of (d):	$O(n^6)$

(e)

```
int pirouline(int n) {
    int j = 0;
    for (int k = 0; k < raspberryscone(n) * n; k++)
        j = j + 2;
   return peanutbuttercheesecake(almondcaramelbrittle(j));
}
```

4

The for loop's running time of $O(n^2) \cdot n = O(n^3)$ dominates running time, compared

Running time of (e):	$O(n^3)$