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C. E. Lemke,

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BIMATRIX EQUILIBRIUM POINTS AND MATHEMATICAL PROGRAMMING*†

C. E. LEMKE

Rensselaer Polytechnic Institute, Troy, New York

Some simple constructive proofs are given of solutions to the matric system $Mz - w = q$; $z \geqq 0$; $w \geqq 0$; and $z^T w = 0$, for various kinds of data M, q, which embrace the quadratic programming problem and the problem of finding equilibrium points of bimatrix games.

The general scheme is, assuming non-degeneracy, to generate an adjacent extreme point path leading to a solution. The scheme does not require that some functional be reduced.

A. Introduction

In this paper, simple constructive proofs are given of the existence of solutions for certain systems of the form: $Mz - w = q$; $z \geqq 0$; $w \geqq 0$, when such exist. The quadratic programming problem and the problem of finding equilibrium points for bimatrix games may be posed in the given form, and thus a general algorithm is given for these problems.

The element of proof adapts the techniques used in the constructive proof of the existence of equilibrium points for bimatrix games [7] to a wider class of problems. The main characteristic of the technique, combining the familiar concepts of non-degeneracy and extreme-point path, is the generation of an adjacent extreme-point path (which is not based upon a successive-approximation scheme) which terminates in an equilibrium point, when such exists. In somewhat more geometrical detail, visualizing the convex polyhedron in z-space of points satisfying:

$$Mz \geqq q; \quad z \geqq 0,$$

the path of points generated consists wholly of points for which $z^T w = (z)_s (w)_s$; that is, points for which the sum has at most one (non-negative) summand $(z)_s (w)_s$ positive, for fixed s. It is arranged that the path start on an unbounded edge, which thereafter uniquely defines the path to be traversed, and for particular kinds of data M and q the path will end in an equilibrium point.

By way of background, the quadratic programming problem (which includes the linear programming problem) in the well-known "Kuhn-Tucker format" takes the above form. Indeed, the majority of published solution techniques may be described in terms of the formulation. The bimatrix (or non-zero-sum two-person matrix) game may be cast in the given form, as may other "quadratic-like" types of problems (see, for example, those discussed in [6]). The results

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given below extend somewhat the class of problems for which an adjacent extreme point path scheme will lead to a solution. Also, additional light (Theorem 5) is shed on the results of Charnes, Cooper, and Thompson [3] touching on the boundedness of the constraint sets of linear programming problems. In passing, the results generalize those contained in a report [5] of Dantzig and Cottle, and a similar scheme of Gomory and Balinski developed for the assignment and transportation problems, [1].

From the computational point of view, it is supposed that adjacent extreme point algorithms are sufficiently well-known,[1] so that details of computation may be omitted.

B. Existence Proofs

We shall consider sets of the form:

$$(1) \qquad Z = \{z: Mz - w = q; \quad z \geqq 0; \quad w \geqq 0\} \subset R_n,$$

where M is a square matrix of order n, and q, z, and w are columns. R_n is the space of columns of numbers with n components. The notation $A > 0$ for a matrix means that all components are positive; and $A \geqq 0$ means that all components are non-negative.

We shall use the notation A^T to denote the transpose of A. $(A)_i$ denotes the i^{th} column of matrix A; and $(a)_i$ denotes the i^{th} component of column a. We shall use the columns e_i and e defined by: $(e)_i = 1$, for all i, and $(e_i)_i = 1$; $(e_i)_j = 0$, for $j \neq i$.

Referring to system (1), given z, $w = Mz - q$ serves always to define w.

Def. 1. A point of Z which satisfies:

$$(2) \qquad z^T w = (z)_1(w)_1 + (z)_2(w)_2 + \cdots + (z)_n(w)_n = 0$$

is called an *equilibrium point.*

Since z and w are non-negative, (2) is equivalent to:

$$(3) \qquad (z)_i(w)_i = 0, \qquad \text{for } 1 \leqq i \leqq n.$$

For each i, the pair $(z)_i$, $(w)_i$ is a *complementary* pair, and each is the *complement* of the other.

We shall summarize the relevant well-known facts concerning Z, and consequences of the assumption of non-degeneracy.

For each $z \,\varepsilon\, R_n$ one obtains a unique matrix $N(z)$ obtained from the matrix (M^T, I) by deleting, for each i, $(M^T)_i$ if and only if $(w)_i \neq 0$, and $(I)_i$ if and only if $(z)_i \neq 0$. (Possibly $N(z)$ has no columns.)

Def. 2. A point $z \,\varepsilon\, R_n$ is an *extreme point* of Z if and only if $z \,\varepsilon\, Z$, and rank $N(z) = n$. A point z lies on an *open edge* of Z if and only if $z \,\varepsilon\, Z$ and rank $N(z) = n - 1$.

Def. 3. Z is *non-degenerate* if and only if whenever A is a matrix obtained from (M^T, I) by deleting some (but not all) columns, and there is a $z \,\varepsilon\, R_n$ such that $A = N(z)$, then the number of columns of A equals its rank.

[1] For a description of adjacent extreme point algorithms see [2], pp. 269–348, Vol. I.

Concerning the existence of equilibrium points, it may be shown (see, for example, [7]) that Z may be perturbed to a set $Z' \supset Z$, such that Z' is non-degenerate and in such a way that if Z' has an equilibrium point then Z does. In any case, non-degeneracy will be explicitly assumed where required.

If Z is non-degenerate, the following holds: if z is an extreme point of Z, then $N(z)$ is non-singular, and if z is a point on an open edge of Z, then $N(z)$ has $n - 1$ columns. It further follows that an extreme point is an end-point of exactly n edges of Z. More precisely, if $\bar{z}$ is an extreme point of Z, and $N = N(\bar{z})$, let N_i be obtained from N by deleting its i^{th} column. Then the set of points $z \varepsilon Z$ for which $N_i = N(z)$ is a (non-empty) open edge of Z having $\bar{z}$ as end-point. Otherwise put, in terms of the components of w and z, moving from $\bar{z}$ along an edge of Z *exactly one* of those n of the $2n$ variables $(z)_i$ and $(w)_i$ which are zero at $\bar{z}$ is increased from zero (the other $n - 1$ remaining at zero value), and that edge may be associated with that variable.

Finally, if z is an equilibrium point, by (3) $N(z)$ has at least n columns; hence, by non-degeneracy, has just n columns, and hence is an extreme point. Hence, if Z is non-degenerate it has a finite number of equilibrium points.

An edge of Z having two end-points is bounded. Its two end-points are *adjacent* extreme points. (If Z is non-degenerate, two extreme points z_1 and z_2 of Z are adjacent if and only if $N(z_1)$ and $N(z_2)$ have just $n - 1$ columns in common). An edge having just one end-point is unbounded, and will be called a *ray* of Z. Since $z \geqq 0$ (i) Z cannot contain an entire line, and (ii) if Z is non-empty it has an extreme point.

Def. 4. A non-empty connected set consisting of a non-empty class of closed edges of Z such that no three edges of the class intersect is called an *adjacent extreme-point path* or more briefly an *adjacency path.*

Thus, an extreme point contained in an adjacency path of Z meets just one or two edges of the path. If such a point meets just one such edge it will be called an *end-point* of the path. Thus, an adjacency path has 0, 1, or 2 *end-points*, and contains 0, 1, or 2 rays of Z. It has 0 end-points if and only if it contains either 0 rays (a closed path) or 2 rays; it has 2 end-points if and only if it has 0 rays; and has 1 end-point if and only if it has 1 ray.

Def. 5. For fixed i, *the set* Z_i is the set:

$$Z_i = \{z: z \varepsilon Z \quad \text{and} \quad z^T w = (z)_i (w)_i\}. \tag{4}$$

Hence, Z_i is the set of points of Z for which $(z)_j (w)_j = 0$, for $j \neq i$; and hence the set S of all equilibrium points of Z is contained in the set Z_i for each i, and is in fact the intersect over i of these sets.

Theorem 1. For fixed s, if Z is non-degenerate, Z_s is either empty or is the union of disjoint adjacency paths of Z. The set S of equilibrium points of Z is precisely the set of end-points of the adjacency paths comprising Z_s.

Proof. If $\bar{z} \varepsilon Z_s$, then $(\bar{z})_i (\bar{w})_i = 0$; for $i \neq s$. Hence $N(\bar{z})$ has n or $n - 1$ columns. Hence $\bar{z}$ is either an extreme point of Z or lies on an open edge of Z.

If $\bar{z}$ is on an open edge of Z, then $N(\bar{z}) = N(z)$ for all points z of that edge, and hence the entire edge is contained in Z_s. Since at an end-point of such an edge, just one additional variable is made 0, such end-point is also a point of Z_s. Hence, if Z_s is non-empty, it contains at least one extreme point of Z.

If $\bar{z}$ is an extreme point of Z, then either (Case I) $(\bar{z})_s(\bar{w})_s = 0$, or (Case II) $(\bar{z})_s(\bar{w})_s > 0$. In Case I $\bar{z}$ is an equilibrium point. In this case, for each i, precisely one of the pair $(\bar{z})_i$, $(\bar{w})_i$ is equal to 0. Hence, that edge along which, from $\bar{z}$, the zero member of the pair $(\bar{z})_s$, $(\bar{w})_s$ is increased from 0, is the one and only edge from $\bar{z}$ contained in Z_s. In Case II, for just one value of i, say $i = r \neq s$, $N(\bar{z})$ contains as columns both $(M^T)_r$ and $(I)_r$; that is: $\bar{w}_r = \bar{z}_r = 0$. An edge of Z (with end-point $\bar{z}$) along which just one of these variables is increased from 0 is contained in Z_s. The two such edges are the only edges of Z with end-point $\bar{z}$ contained in Z_s.

It remains to point out that an extreme point $\bar{z}$ in Z_s lies on one and only one adjacency path of Z of points of Z_s.

If an extreme point $\bar{z}$ in Z_s is incident with just one edge of points of Z_s, that edge is either a ray (in which case it constitutes the desired adjacency path), or is not. If not the other end-point is either an equilibrium point (in which case the edge constitutes the desired path) or is not. If not, there is just one other edge of points of Z_s along which one may continue. Continuing in this manner the process terminates either in a ray, or at an equilibrium point, yielding the desired path.

If an extreme point $\bar{z}$ in Z_s is incident with two edges of points of Z_s, selecting one of these edges to start, a path is described as in the preceding paragraph, except that the path may return to $\bar{z}$, in which case it terminates. If not, a similar portion of the desired adjacency path is swept out starting from the other edge coincident with $\bar{z}$, and the two portions constitute the desired path. This concludes the proof.

Existence of Equilibrium Points

The technique which furnished a constructive proof of the existence of equilibrium points of non-zero sum matrix games [7] will be adapted, in what follows, to certain other types of sets Z. A main result is contained in Theorem 4.

In the case of the game example, the resultant Z was clearly non-empty. An example similar to this case will illustrate the technique there applied. In other cases, one needs to take account of the possibility that Z is empty. We shall use the more obvious half of the following well-known result (see, for example, [2]):

Lemma 2. Z is empty if and only if there exists a $u \geqq 0$ satisfying:

$$M^T u \leqq 0; \quad \text{and} \quad u^T q > 0. \tag{5}$$

We shall also use the following property of the sets Z_s, which is an immediate corollary of Theorem 1:

Theorem 2. Let Z be non-degenerate and have the property that for some s, Z_s contains precisely one ray of Z. Then Z has an odd number of equilibrium points.

Proof. Label as E_0 the single ray of Z_s. The adjacency path of Z of points of Z_s which contains E_0 must terminate in an equilibrium point.

If an adjacency path contained in Z_s does not contain E_0 it is either a closed path (containing no equilibrium points) or has two end-points (which are distinct equilibrium points). This concludes the proof.

As an example, similar to the game case:

Corollary. Let Z be non-degenerate. If $q = e$, and $M > 0$, then Z has an odd number of equilibrium points.

Proof. For fixed s, we need merely point out that Z_s contains only one ray of Z. Consider the non-negative orthant of points satisfying $z \geqq 0$. Z is obtained from it by intersecting it with n half-spaces of the form $a^T z \geqq 1$, where $a > 0$ represents any row of $M > 0$. The hyperplane $a^T z = 1$ therefore cuts each coordinate axis, and points of the form:

$$z = ke_i, \qquad k > 0, \tag{6}$$

are in Z for k large enough, and in fact, for some $k_0 > 0$, points (6) are in Z for $k \geqq k_0$, and not in Z for $k < k_0$. The ray of points for which $k \geqq k_0$ lies in Z_i (for each i) and not in Z_j for $j \neq i$. Since that part of Z satisfying $a^T z = 1$ is bounded these are the only rays, and in particular, for fixed s, Z_s has one and only one ray, completing the proof.

Regarding the number of equilibrium points, there is a general class for which, if Z is non-degenerate, an equilibrium point is unique.

Theorem 3. If Z is non-degenerate and M satisfies $z^T M z \geqq 0$ (that is, if M is non-negative definite) there is at most one equilibrium point.

Proof. Let z_1 and z_2 be equilibrium points. Set $w_i = Mz_i - q$, so that $w_i^T z_i = 0$, for $i = 1, 2$. Then:

$$0 \leqq (z_2 - z_1)^T M(z_2 - z_1) = (z_2 - z_1)^T (w_2 - w_1) = -(z_2^T w_1 + z_1^T w_2). \tag{7}$$

Since all variables are non-negative, this implies:

$$z_2^T w_1 = 0 = z_1^T w_2 = z_1^T w_1 = z_2^T w_2 . \tag{8}$$

Since z_i is an equilibrium point, by non-degeneracy each pair (z_1, w_1) and (z_2, w_2) has precisely n zero components. Now the pair $(z_1 + z_2, w_1 + w_2)$ has *at least* n positive components. But (8) implies: $(z_1 + z_2)^T (w_1 + w_2) = 0$. Hence the pair has *at most* n positive components. Hence, precisely n positive components. Hence the pairs (z_1, w_1) and (z_2, w_2) have the *same* components positive. Hence $N(z_1) = N(z_2)$. Hence $z_1 = z_2$, completing the proof.

To attack a wider class of problems, consider the following device. Let $z^* = \begin{pmatrix} z \\ z_0 \end{pmatrix}$ where z_0 is a scalar variable. Define the set:

$$Z^* = \{z^* : Mz + z_0 e - w = q; \quad w, z \geqq 0; \quad z_0 \geqq 0\} \subset R_{n+1} . \tag{9}$$

Define the set:

$$Z_0^* = \{z^* \text{ in } Z^* : z^T w = 0\}. \tag{10}$$

Note that Z^* is non-empty. In fact, the set E_0^* of points satisfying:

(11) $z = 0; \quad z_0 > \text{Max}_i\,(q)_i\,; \quad \text{and} \quad w = z_0 e - q,$

is a ray of Z^* which is further contained in Z_0^*.

To apply Theorem 2, we shall add a single constraint. Let Z^{**} be the set of points of Z^* satisfying:

(12) $$-e^T z - w_0 = -k; \qquad w_0 \geqq 0,$$

where k is taken large enough so that any extreme point of Z^* satisfies $e^T z < k$ (that is, $w_0 > 0$). The equality constraints for Z^{**} in block form become:

(13) $$\begin{pmatrix} M & e \\ -e^T & 0 \end{pmatrix} z^* - w^* = \begin{pmatrix} q \\ -k \end{pmatrix}; \quad \text{where} \quad w^* = \begin{pmatrix} w \\ w_0 \end{pmatrix};$$

so that Z^{**} has the form (1). Let Z_0^{**} be the set of points of Z_0^* which are contained in Z^{**}. Letting w_0 play the role of complement of z_0, we are concerned with equilibrium points of Z^{**}. Z_0^{**} is then the set of points of Z^{**} satisfying:

(14) $$z^{*T} w^* = z_0 w_0\,.$$

If it is assumed that Z^* is non-degenerate, the choice of k ensures that Z^{**} is non-degenerate. It may be remarked that if Z^* is non-degenerate, then Z is, and that if Z is non-degenerate, Z^* may be perturbed so that Z^* is non-degenerate.

From the computational point of view, the constraint (12) is artificial and unnecessary.

Note that E_0^* is still a ray of Z^{**}. The additional constraint (12) ensures that it is the only ray of Z^{**} contained in Z_0^{**}. To see this, we proceed as follows:

Points on any ray of Z^* or of Z^{**} will have the form:

(15) $$z^* = \bar{z}^* + \theta u^*; \quad \text{for } \theta \geqq 0; \quad \text{where} \quad \bar{z}^* = \begin{pmatrix} \bar{z} \\ \bar{z}_0 \end{pmatrix}; \qquad u^* = \begin{pmatrix} u \\ u_0 \end{pmatrix};$$

and where $\bar{z}^*$ is an extreme point of Z^* or of Z^{**} (whichever is being discussed); $u^* \neq 0$; and θ and u_0 are scalar quantities. Setting $\bar{w} = M\bar{z} + \bar{z}_0 e - q$; and $v = Mu + u_0 e$, we may write:

(16) $$M(\bar{z} + \theta u) + (\bar{z}_0 + \theta u_0)e - (\bar{w} + \theta v) = q.$$

The conditions $\bar{z} + \theta u \geqq 0$; $\bar{z}_0 + \theta u_0 \geqq 0$; and $\bar{w} + \theta v \geqq 0$ for all $\theta \geqq 0$ require that:

(17) $$u \geqq 0; \qquad u_0 \geqq 0; \quad \text{and} \quad v \geqq 0.$$

Then, if the ray is a ray of Z^{**}, the condition that $e^T z < k$ requires that $u = 0$. Then $u^* \neq 0$ requires that $u_0 > 0$. Hence, for θ large enough, one must have that $w = \bar{w} + \theta v > 0$. Then the condition $z^T w = \bar{z}^T w = 0$ for Z_0^{**} requires that $\bar{z} = 0$, and hence that the ray is E_0^*.

Hence:

Lemma 2. Z^{**} has an equilibrium point.

Proof. Z^{**} satisfies the requirements of Theorem 2.

Next, consider the path of points of Z_0^{**} which terminates in an equilibrium point z^*. Then $z_0 w_0 = 0$. If the path were to end in $w_0 = 0$ (hence in $z_0 > 0$), the choice of k would ensure that the equilibrium point lies on a ray of Z^*. We seek (Theorem 4) conditions on M which will ensure that if this occurs it must be that Z is empty. We shall therefore, from now on, disregard the artificial constraint (12) and examine the possibility of rays of Z^* other than E_0^* contained in Z_0^* (i. e., satisfying $z^T w = 0$).

Any ray of Z^* is a set of points satisfying (15), (16), and (17), where now $\bar{z}^*$ is an extreme point of Z^*. We have:

$$Mu + u_0 e - v = 0. \tag{18}$$

If it is next supposed that the ray is in Z_0^*, then $(\bar{z} + \theta u)^T(\bar{w} + \theta v) = 0$. Since all quantities are non-negative, this is equivalent to:

$$\bar{z}^T \bar{w} = \bar{z}^T v = u^T \bar{w} = u^T v = 0. \tag{19}$$

There are two cases: Case I: $u = 0$ and Case II: $u \neq 0$.

If $u = 0$, then $u_0 > 0$. As before, we may conclude that the ray is E_0^*.

Hence, (Case II) we shall suppose that $u \neq 0$. We may then take u so that $e^T u = 1$. Then (18) and (19) yield:

Lemma 3. A ray of Z^* contained in Z_0^* which is not the ray E_0^* satisfies:

$$u^T M u + u_0 = 0. \tag{20}$$

We next place conditions on M:

Theorem 4. Let Z be non-degenerate. Let M have the property that if $u \geqq 0$, then:

(i) $\quad u^T M u \geqq 0,$

(ii) $\quad u^T M u = 0 \quad$ implies that:

$$Mu + M^T u = 0. \tag{21}$$

Then, if Z is non-empty, it has an equilibrium point.

Proof. We need only show that, with the conditions of the theorem, the conditions (20) imply that Z is empty, unless $\bar{z}_0 = 0$.

By (i) and (20):

$$u_0 = u^T M u = 0. \tag{22}$$

Then, by (ii) and (18):

$$Mu = v \geqq 0 \quad \text{and} \quad M^T u = -Mu \leqq 0. \tag{23}$$

Next, by (19):

$$\begin{aligned} 0 &= \bar{z}^T v = \bar{z}^T M u, \\ 0 &= u^T \bar{w} = u^T(M\bar{z} + z_0 e - q) = \bar{z}^T M^T u + \bar{z}_0 - u^T q. \end{aligned} \tag{24}$$

Adding, and using (23):

$$\bar{z}^T(M^T u + Mu) + \bar{z}_0 - u^T q = \bar{z}_0 - u^T q = 0. \tag{25}$$

Now, if $\bar{z}_0 = 0$, then $\bar{z}$ is already an equilibrium point of Z. If $\bar{z}_0 > 0$, then $u^T q > 0$. In this case, a u has been found satisfying the conditions of Lemma 1, and Z is empty. This concludes the proof.

It may be observed that it has not been shown that the conditions (i) and (ii) on M ensure that E_0^* is the only ray of Z^* contained in Z_0^* (as in the case of the game example). But the conditions do ensure that, starting from the ray E_0^*; one terminates an adjacency path in $z_0 = 0$; that is, in an equilibrium point of Z. This latter statement embodies the suggested computational scheme.

C. Discussion

With regard to Theorem 4, we note that the case for which M satisfies $z^T M z \geqq 0$ for *all* z, most recently considered by Dantzig and Cottle, is included. To show that this condition implies (ii) of the statement of Theorem 4, we need only observe that $z^T M z = \frac{1}{2} z^T (M + M^T) z$; that $M + M^T$ is therefore symmetric and non-negative definite; and hence that $z^T (M + M^T) z = 0$ implies $(M + M^T) z = 0$. We have extended this result to, for example, the case for which $M > 0$, which evidently satisfies (i) and (ii) of Theorem 4. The bi-matrix game case does not satisfy (ii) of the theorem.

Regarding the computational aspects, we have implied a computational scheme by the specification of a definite adjacency path, and have given no consideration to the question of how this might duplicate previous results. We shall compare our formulation briefly with that of Dantzig and Cottle [5], which we take the liberty of describing in our terms. Our constraints read:

$$Mz + z_0 e - w = q; \qquad w \geqq 0.$$

Let $w' = w - z_0 e$. Dantzig and Cottle apply themselves to the form:

$$Mz - w' = q; \quad \text{and} \quad w' \geqq -z_0 e,$$

where z_0 is taken fixed and $z_0 > \max_i (q)_i$.

Starting with $z = 0$ and $w' = -q$, they proceed to describe an adjacency path retaining $z \geqq 0$ and $w'^T z = 0$, and aim at successively reducing the number of negative components of w' to zero.

In conclusion, we may observe the following result:

Theorem 5. Let Z be non-empty and non-degenerate, and let M be non-negative definite. Then Z has at least n rays.

Proof. By Theorem 2, Z has a unique equilibrium point. Hence that point is the intersect of the sets Z_i. For fixed i, that adjacency path of the set Z_i which contains the equilibrium point must end in a ray belonging to Z_i and not to Z_j for $j \neq i$. Since this holds for each i, there are at least n rays.

As an example of Theorem 5, consider the case of linear programming. The matrix A of order m by r is given. Then Z is the set of points $z = \begin{pmatrix} x \\ y \end{pmatrix}$ satisfying:

$$Ax \geqq a; \qquad x \geqq 0, \quad \text{and} \quad -A^T y \geqq -b; \qquad y \geqq 0.$$

The assertion of the theorem is that when both of these sets are non-empty, then the number of rays, each of which is a ray either of one set or of the other, is at least $m + r$. In particular, if one of the sets, say $Ax \geqq a$; $x \geqq 0$, is bounded, then the dual set $A^T y \leqq b$; $y \geqq 0$ has at least $m + r$ rays. This extends the results of Clark [4] and of Charnes, Cooper and Thompson.[2]

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