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A LINEAR TWO-LEVEL PROGRAMMING PROBLEM*†

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Abstract—A stackleberg game in which a first player can affect the resources available to a second player is defined. For given first move the second player will maximize a linear program, subject to the resources available. The first player attempts to maximize his own linear objective function which contains variables under the control of the second player. Even though all functions are linear, it is shown that local optima can exist. An implicit search is described which uses necessary conditions for a better solution, to limit the extent of the search.

1. INTRODUCTION

In a mixed economy, policy problems involve policymakers in indirectly influencing the decisions of private individuals and companies.

If x_1 refers to the decision variables of private individuals, and x_2 to the variables under control of policymakers, we can state the linear two-level programming problem as:

*P*1. Find $x = (x_1, x_2)$ such that:

$$x_2 \text{ solves: } \underset{x_2}{\text{maximize}}\ c_2 x \tag{1.1}$$

subject to

$$H_1 x_1 + H_2 x_2 = b \tag{1.2}$$

$$x \geq 0 \tag{1.3}$$

and where

$$x_1 \text{ solves } \max_{x_1} [c_1 x | x_2] \tag{1.4}$$

where

$$H_1 \text{ is } (m \times n_1) \text{ and } H_2 \text{ is } (m \times n_2).$$

We may note, immediately, that for given $x_2 = \bar{x}_2$ (i.e. for a given setting of the policy variables), the choice of x_1 reduces to a linear programming problem:

*P*2. For any given (lth) set of non-negative values of the policy variables, $x_2 = \bar{x}^{(l)} \geq 0$, find the value of x_1, such that:

$$f_1 = \max_{x_1} [c_1(x_1, \bar{x}_2^{(l)})] \tag{2.1}$$

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subject to

$$H_1x_1 = b - H_2\bar{x}_2^{(l)} \tag{2.2}$$

$$x_1 \geq 0. \tag{2.3}$$

Note that for arbitrary $\bar{x}_2^{(l)} \geq 0$, there is no guarantee that $P2$ will have a feasible solution. We refer to $P2$ as the "behavioral" or "inner" L.P. problem.

Despite all functions in $P1$ being linear, multiple local optima can exist. Viewed as a mathematical program from the policymakers' standpoint, the problem treated herein is that of maximizing a (generally non-concave) piecewise linear function over a polyhedron. The example treated in Section 7 has two isolated local solutions.

This problem $P1$, is in marked contrast to the usual definition of mathematical programming, where a single decision maker has direct control over all variables.

The algorithm presented in this paper is concerned only with policies which affect the right hand side of $P2$, "the resource control problem"[1], though the same general problem structure would include situations where the outer decision maker (the policy maker) also affects the parameters of the inner (behavioral) objective function, e.g. by product or input prices subsidies or taxes; or, indeed, factor-product transformation rates, via research or extension. This more general formulation is referred to as *multi-level* programming.

Examples of the range of policy problems to which this general structure is applicable, include:

(i) *Energy policy.* With the desire to reduce dependence on imported oil, the Government can impose import levies, retail sales tax, import quotas, and rationing. Individual consumers and companies will then decide their consumption patterns in the light of the resulting availabilities and prices, with consequent effects on the levels of imports, government revenue, and the general price level.

(ii) *Drug abuse.* With the desire to enforce the existing law, the government may establish enforcement efforts with given probabilities of success, and levels of punishment. More generally, Congress can change the relative levels of punishment and/or taxation. In the light of these costs (illegal) importers and producers can change the focus and level of their activity; and final consumers can decide whether the "high" is worth the price.

(iii) *Economic development.* A government, alone or in consultation with donor agencies, can decide on an investment plan to provide transport and communications infrastructure and processing capacity, together with price levels. Individual farmers and traders then make their production and marketing decisions which either do, or do not, result in higher GNP per capita, changed income distribution and surplus or deficit on the balance of payments.

In a 50 row by 300 column of agricultural production in a region of Mexico, Candler and Norton[3], have shown that the value of (1.1) if (1.4) is ignored, can be twice as large as if the profit maximizing behavior of individuals is recognized as part of the problem. That is, if the private objectives (1.4) could be changed to exactly coincide with the policymakers' objectives (1.1), then the level of the policymakers' objectives could be doubled. Thus, the errors involved in solving planning models as if all variables were under central control, can effect by an order of magnitude, the estimated benefits of the plan. See also [4].

Enough has been said to indicate that multi-level programming is not only an interesting mathematical problem, but also a problem of major policy significance. Related problem structures have also been investigated in connection with armed conflict[2]. The problem structure can be tied both to game theory and mathematical programming[4].

The problem addressed here is related to the linear max–min problem studied by Falk[6]. Indeed, this latter problem is a special case of the two-level problem of this paper in which the behavioral objective function is in direct opposition to the policy objective function. As a result, Falk's problem is equivalent to maximizing a piecewise linear convex function over a polyhedron, while the implied objective function of this paper need not be convex.

The problem considered can have local optima, hence there is little hope of a monotonic improvement in the objective function. Rather some form of implicit enumeration has to be relied on, using the global (but necessary condition) information generated in the search for locally better solutions.

By paying attention to all necessary condition information generated to date, we can avoid returning to any previously explored basis. This provides the rationale for the proposed algorithm which is, in essence, simply an implicit search.

The next section of this paper sets the two-level linear programming problem in the wider context of multi-level programming, and defines several associated linear programs used in the solution. A third section discusses a high point, which can be found for a given basic solution to the behavioral problem $P2$; shows how some monotonic improvement may be possible using local information; and, shows how the high point can be used to generate global necessary information for better bases to $P2$. Section 4 then considers global optimality and additional necessary conditions for better bases. Section 5 discusses algorithmic principles, and Section 6 presents the algorithm itself. A seventh section provides a numerical example and commentary, and a final section provides brief concluding remarks.

2. PROBLEM DEFINITION

The linear two-level problem, $P1$, is but one example of much larger, *multi-level programming*, structure.

For the k-level programming problem we have a $(1 \times n)$ solution vector x partitioned into k sub-vectors:

$$x = (x_1, x_2, \ldots, x_k)$$

Define

$$\tilde{x}_j = (x_{j+1}, x_{j+2}, \ldots, x_k)$$

We can now state the k-level programming problem:

$P3$. Find x where

$$x_k \text{ solves: } \underset{x_k}{\text{maximize}}\ g_k(x) \tag{3.1}$$

$$\text{subject to } h_i(x) = 0 \quad i = 1, \ldots, m \tag{3.2}$$

$$x \geqslant 0 \tag{3.3}$$

and where

$$x_j \text{ solve } \max_{x_j} [g_j(x)|\tilde{x}_j] \quad j = k-1, \ldots, 1 \tag{3.4}$$

If $k = 1$, this reduces to the normal statement of the mathematical programming problem.

If $k = 2$, all functions are linear, and x_2 only affects the constraints of the inner problem, the case for which an algorithm is offered, the problem can be written as $P1$.

In problem $P1$, x_2 must be set so that: (i) $H_1x_1 + H_2x_2 = b$

$$x_1, x_2 \geqslant 0$$

is feasible for some x_1, and, for all such x_2: (ii) x_2 yields the highest value of c_2x, with the understanding that x_1 will be set optimally in a different and subsequent optimization.

This paper assumes that for given x_2, there will be a *unique* solution x_1. The importance of this assumption can be seen from the following example:

$PE1$. Find $x = (x_{11}, x_{12}; x_2)$ such that

$$f_2 = \max_{x_2} (-x_2 + x_{11} - x_{12})$$

subject to $0 \leq x_2 \leq 1$, where x_{11}, x_{12} solve

$$f_1 = \max_{x_{11}, x_{12}} (x_{11} + x_{12})$$

subject to

$$x_{11} + x_{12} = 1 - x_2$$

$$x_{11}, x_{12} \geq 0.$$

In this case the solution to the inner problem, in x_{11} and x_{12}, is not unique; any value of x_{11} and x_{12} which satisfies the restraint yields the same value of f_1. For the outer problem the choice of x_{11} or x_{12} is crucial. This type of degeneracy is not treated in this paper.

We will denote a typical column in H_1 by H_{1j}, and a typical column vector in H_2 by H_{2q}. For convenience we assume rank $(H_1) = m$.

Thus we have:

$$\begin{aligned} x &= (x_{11}, \ldots, x_{1j}, \ldots, x_{1n_1}, x_{21}, \ldots, x_{2q}, \ldots, x_{2n_2}) \\ &= (\text{behavioral variables, policy variables}) \end{aligned}$$

A similar nonconvex programming problem, (formulated as a max–min problem with a single objective function $f(x_1, x_2)$), has been described by Falk[6]. The algorithm described here can be used to solve the Falk's problem; and a trivial generalization of his algorithm would allow solution of the general linear two-level problem.

We now present in addition to $P2$, two further (primal) L.P. problems that are associated with the two-level programming problem, $P1$.

$P4$. Given a basis set B_l from H_1, that is optimal with respect to the behavioral L.P. problem, $P2$, find values of $(x_1^{(l)}, x_2)$ such that:

$$f_2 = \max_{x_1^{(l)}, x_2} [c_2(x_1^{(l)}, x_2)] \tag{4.1}$$

subject to

$$B_l x_1^{(l)} + H_2 x_2 = b \tag{4.2}$$

$$x_1^{(l)}, x_2 \geq 0. \tag{4.3}$$

Note that this "policy" or "outer" L.P. problem is defined only for behavioral variables (activities) that are members of B_l.

$P5$. Find values of x_1 and x_2 such that:

$$f_2 = \max_{x_1, x_2} c_2 x \tag{5.1}$$

subject to

$$H_1 x_1 + H_2 x_2 = b \tag{5.2}$$

$$x_1, x_2 \geq 0. \tag{5.3}$$

Clearly, the solution to $P5$ places an upper bound on the solution to $P1$. As shown in [3, 4], this bound may be several times the optimal value of $P1$.

In Falk's problem the optimal solution has to be both a vertex of the constraints of $P5$, as well as a vertex of the projection of this set onto the policy space. In the present paper, an optimal setting for the policy variables could be *interior* to this projection.

We now define two classes of basis which may be selected from H_1.

BOB. Any basis set B_l, from H_1, that satisfies

$$c_{1l}B_l^{-1}H_{1j} - c_{1j} \geqslant 0 \text{ for all } j = 1, \ldots, n_1$$

is a *behaviorally optimal basis*, or BOB.

In particular if $P2$ has a behaviorally optimal solution, which may or may not be feasible, the associated basis B_l is BOB.

Note that behavioral optimality is unaffected by the settings for the policy variables x_2. The values of the policy variables only affect the *feasibility* of a behavioral basis.

FBOB. Any BOB basis B_l, for which $P4$ has a feasible solution is a *feasible behaviorally optimal basis*, or FBOB.

Bases, from H_1, which give a feasible solution to $P2$, or $P4$, but which are not BOB, are of no interest. Since such bases would *never* be chosen by the behavioral decision makers, regardless of the setting of x_2.

If problem $P2$ has an optimal feasible solution $x_1^{(l)}$, then $x^{(l)} = (x_1^{(l)}, \bar{x}_2^{(l)})$ is a feasible solution to the two-level linear programming problem, $P1$.

If there is no feasible solution to $P2$, then $x_2 = \bar{x}_2^{(l)}$ is not a feasible setting for the policy variables in $P1$.

Thus, a *feasible* solution to $P1$ is defined as any $x_2 = \bar{x}_2^{(l)}$ satisfying (1.2) and (1.3), and where problem $P2$ has an optimal feasible solution for $x_2 = \bar{x}_2^{(l)} \geqslant 0$, and $x_1^{(l)}$ is the solution. If $x_1 = \bar{x}_1^{(l)}$ is a feasible, but non-optimal solution to $P2$ given $x_2 = \bar{x}_2^{(l)}$, then $x = (\bar{x}_1^{(l)}, \bar{x}_2^{(l)})$ is *not* a feasible solution to $P1$.

The (feasible) solution space for $P1$, is then $\{x_2 \geqslant 0 | B_l^{-1}b - B_l^{-1}H_2x_2 \geqslant 0,\ c_{1l}B_l^{-1}H_{1j} - c_{1j} \geqslant 0$ for all $j = 1, \ldots, n_1; B_l$ a basis from $H_1\}$. This is a convex polyhedron.

Now, if B_l is a behavioral optimum basis (BOB), then from (2.2), B_l will be a feasible behavioral optimum basis (FBOB) if there exist values of the policy variables such that:

$$B_l^{-1}H_2x_2 \leq B_l^{-1}b \tag{6}$$

since,

$$x_1^{(l)} = B_l^{-1}b - B_l^{-1}H_2x_2. \tag{7}$$

Thus for values of $x_2 \geq 0$ satisfying (6), the corresponding solution to the behavioral L.P. problem, $P2$, is given by (7).

Therefore, given (BOB): B_l, values for $(x_1^{(l)}, x_2)$ satisfying the constraint set for the policy L.P. problem $P4$, i.e. satisfying (4.2) and (4.3), are feasible solutions to $P1$.†

We now turn to the consideration of optimal feasible solutions. Let $x_2 = x_2^*$ be an optimal feasible setting for the policy variables in the two-level programming problem, $P1$. Then, from the definition of $P1$, there exists a basic optimal feasible solution to the behavioral L.P. problem $P2$:

$$\begin{aligned} x_1^* &= B_*^{-1}b - B_*^{-1}H_2x_2^* \geq 0 \text{ for } (x_{1j}^* | H_{1j} \in (\text{BOB}): B_*) \\ &= 0 \text{ otherwise,} \end{aligned} \tag{8}$$

such that $f_2^* = c_2(x_1^*, x_2^*)$ is the optimal value for the policy (outer) objective function (1.1).‡

Consider now the policy L.P. problem $P4$ associated with (BOB): B_*. Clearly (x_1^*, x_2^*) is a feasible solution to this L.P. problem. Let W_* denote an optimal feasible basis for $P4$, and let f_2^{**} denote the corresponding value of the policy objective function. Then $f_2^{**} = f_2^*$, since if

†If the intersection of the sets of values for x_2 given by (6) and the non-negativity condition, is empty, then there is no set of values for the policy variables such that this particular (BOB): B_l gives a feasible solution to the behavioral L.P. problem, $P2$. In this case B_l does not contribute to a feasible solution to the two-level programming problem, $P1$.

‡As noted earlier, in order to avoid complexities associated with indeterminate solutions to the multi-level programming problem $P1$, we assume that the optimal solution to the behavioural L.P. problem, $P2$, (given $x_2 = x_2^*$), is unique. Strictly it is only necessary to assume that the value of f_2, or any feasible solution to $P2$, with $x_2 = x_2^*$, is unique.

$f_2^{\dagger**} > f_2^{\dagger}$, $f_2^{\dagger}$ is not the optimal value for the policy objective function for the two-level programming problem $P1$.§ This result proves Theorem 1.

Theorem 1

Given there exists an optimal feasible solution to the two-level programming problem P1, there exists a (BOB): B_* such that the corresponding basic optimal feasible solution to the policy L.P. problem $P4$ is an optimal feasible solution to $P1$.

As a corollary to Theorem 1 we have:

Corollary

An optimal feasible solution to the linear two-level programming problem $P1$ can be represented as a basic feasible solution to problem P5.

This result follows since W_*, defined above, is a feasible basis for problem $P5$. Therefore, as Falk[6] has already noted, an examination of all feasible basic solutions to problem $P5$ constitutes an algorithm that will find the solution to $P1$ in a finite number of steps. We can add that examination of all behavioral optimal bases will also find the solution to $P1$ in a finite number of steps.

Whereas the branch and bound algorithm presented by Falk, for solving a subset of our two-level programming problem $P1$, concentrates on the relationships between $P2$, $P5$ and $P1$, the algorithm presented here concentrates on the relationship between $P4$ and $P1$.

Also, if W_* is an optimal feasible basis for the policy L.P. problem $P4$ corresponding to (BOB): B_*, then:

$$(x_1^{\dagger**}, x_2^{\dagger**}) = W_*^{-1}b \text{ for } (x_1^{\dagger**}, x_2^{\dagger**} | H_{1j}, H_{2q} \in W_*) \tag{9}$$
$$= 0 \text{ otherwise,}$$

is an optimal feasible solution to the two-level programming problem $P1$.

If the optimal setting for the policy variables is unique, then:

$$x_{2q}^{\dagger} = x_{2q}^{\dagger*} \text{ for } H_{2q} \in W_* \tag{10}$$
$$= 0 \text{ otherwise,}$$

and,

$$x_{1j}^{\dagger} = x_{1j}^{\dagger*} \text{ for } H_{1j} \in B_*, W_* \tag{11}$$
$$= 0 \text{ otherwise.}\dagger$$

3. A HIGH POINT AND NEIGHBORING BASES

The algorithm being developed, involves an implicit search of all bases in H_1. The number of bases to be searched explicitly is restricted to those which satisfy (global) necessary conditions for a better basis. Three types of necessary conditions, all involving restrictions on the vectors in H_1 which must enter the basis are developed. These sets T^1, T^2, and T^3 (depending on their derivation), are described intuitively in the following paragraphs. Having given this overview, we then return to the rigorous definition of the sets, and their use.

If B_l is (FBOB), then we refer to the solution of $P4$ as the *high point* in basis B_l. From the updated L.P. tableau corresponding to a high point we can derive a set of non-basic behavioral variables, from H_1, at least one of which must enter the basis if a higher high point is to be found. This set is called set T_l^1, and restricts the range of bases to be examined in the remaining implicit search.

Given a basis B_l which satisfies all existing $T_1^1, T_2^1, \ldots, T_{l-1}^1$, sets, which is (BOB), either it can be made (FBOB), in which case a new set T_l^1, can be generated, or it cannot be made feasible. In the latter case we can define a set of non-basic behavioral variables, from H_1, at least one of which must enter the basis in order that any basis B_l be feasible. This set is called set T_l^2, and restricts the choice of bases to be examined in exactly the same way as the T^1 sets.

§Clearly $f_2^{\dagger*} \geq f_2^{\dagger}$ since $(x_1^{\dagger}, x_2^{\dagger})$ is a feasible solution to $P4$ given (BOB): B_*.

†See [4, Appendix 1].

The T^1 and T^2 sets provided necessary conditions (so that all such T-sets must be satisfied) if a better FBOB, B_l, is to be found. We may, however, find that a set of $p \leq m$ vectors, say C_l, which satisfy the existing T-sets cannot be augmented to yield a behaviorally optimal basis (BOB), yet alone an (FBOB). In this case we conclude that any (BOB) must contain at most p-1 of the vectors in C_l. This leads to the third and final type of T-sets, T_l^3.

Whilst initially finding m vectors from H_1, which satisfy the T-sets is a trivial problem, formally it corresponds to an integer programming problem.

We now turn to the existence of a high-point in a given (FBOB): B_l, and the set T_l^1 associated with it.

Let B_l be a (BOB) and assume that

$$S_l:[x_2|B_l^{-1}H_2x_2 \leq B_l^{-1}b;\, x_2 \geq 0]$$

is not empty, i.e. corresponding to B_l there is at least one set of values for x_2 (the policy variables) such that there exists a feasible solution to the multi-level programming problem, $P1$.

Consider now the policy L.P. problem, $P4$, corresponding to B_l. Since we have assumed that there is a feasible solution to $P4$,† then either $P4$ is unbounded or it has an optimum basic feasible solution. If $P4$ is unbounded, then $P1$ is also unbounded. If $P4$ has an optimal basic feasible solution, we refer to it as the *high-point* in basis B_l, in the sense that it yields the highest value of f_2, (the outer objective function for our two-level programming problem $P1$), that can be achieved without a change in the behavioral basis. Theorem 1 has already shown that the solution to $P1$, will occur at a high point.

Let W_l be an optimal feasible basis for the policy L.P. problem $P4$; i.e. W_l is an optimal feasible basis set from $[B_l, H_2]$ where we assume S_l is not empty. The corresponding (high-point) solution values for $P4$ are:

$$(x_1^{(l)}, x_2^{(l)}) = W_l^{-1}b \text{ for } H_{1j}, H_{2q} \in W_l \tag{12}$$
$$= 0 \text{ otherwise.}$$

Denote the corresponding high-point value for the policy objective function by $f_2^{*(l)}$.

The set of reduced cost values for the policy objective function, (the objective function for $P4$ and $P1$), associated with W_l may be classified as follows:‡

$$f_{2q}^{(l)} = 0 \quad H_{2q} \in W_l \tag{13.1}$$

$$f_{2q}^{(l)} \geq 0 \quad H_{2q} \notin W_l \tag{13.2}$$

$$f_{2j}^{(l)} = 0 \quad H_{1j} \in B_l, W_l \tag{13.3}$$

$$f_{2j}^{(l)} \geq 0 \quad H_{1j} \in B_l, \notin W_l \tag{13.4}$$

$$f_{2j}^{(l)} \gtreqless 0 \quad H_{1j} \notin B_l. \tag{13.5}$$

Conditions (13.1) to (13.4) hold since W_l was assumed to be an optimal (feasible) basis for the policy L.P. problem $P4$ corresponding to (BOB): B_l. Condition (13.5) holds since the behavioral vectors (activities) that did not belong to B_l, were not considered as eligible activities for the policy L.P. corresponding to (BOB): B_l.

If in (13.5) $f_{2j}^{(l)} \geq 0$ for all $H_{ij} \notin B_l$, then clearly W_l is an optimal feasible basis for the L.P. problem $P5$. But the solution to the L.P. problem $P5$ is an upper bound on f_2 in $P2$, and hence, under this condition, W_l is a global optimum feasible basis for the multi-level programming problem $P2$.

Unfortunately, $f_{2j} \geq 0$ for (13.5) is a sufficient, but *not* a necessary condition for a solution to the multi-level programming problem, $P2$.

†We have assumed $S_l \neq \emptyset$ for (BOB): B_l.

‡Remembering that the subscripts j and q denote behavioral and policy variables, respectively.

This sufficient condition does yield a necessary condition for a better solution. The basis for any higher value of f_2 must contain at least one vector such that $f_{2j}^{(l)} < 0$ in (13.5). We now define a set T_l^1 corresponding to the high-point solution in (FBOB): B_l.

$$T_l^1 = [H_{1j} | f_{2j}^{(l)} < 0]. \tag{14}$$

Thus, any *behavioral* optimum basis (BOB) leading to a better high-point value for $f_2 > f_2^{*(l)}$, must contain at least one vector from T_l^1. Unfortunately, this is a necessary and *not* a sufficient condition for obtaining a better high-point value of f_2. This is, however, a necessary condition which applies globally:

Theorem 2

If W_k and W_l are optimal feasible bases for problem $P4$ corresponding to (BOBs): B_k and B_l respectively, and $f_2^{*(k)} > f_2^{*(l)}$, then B_k contains at least one $H_{1j} \in T_l^1$.

Proof.

$$f_2 = f_2^{*(l)} - \sum_{j \notin W_l} f_{2j}^{(l)} x_{1j} - \sum_{q \notin W_l} f_{2q}^{(l)} x_{2q}. \tag{15}$$

Now, from equations (13) we have

$$f_{2q}^{(l)} \geq 0 \quad \text{all } q \tag{16.1}$$

$$f_{2j}^{(l)} \geq 0 \quad H_{1j} \in B_l. \tag{16.2}$$

Thus, if $f_2 > f_2^{*(l)}$, then $x_{1j} > 0$, $f_{2j}^{(l)} < 0$ for at least one $H_{1j} \notin B_l$. But if $f_2^{*(k)} > f_2^{*(l)}$, this is exactly the requirement that W_k, and hence B_k, contain at least one $H_{1j} \in T_l^1$.

As a corollary to Theorem 2, we have:

Corollary

Given a set of (FBOBs): $B_1, \ldots, B_l$, and their corresponding T^1 sets: $T_1^1, \ldots, T_l^1$; then if (FBOB): B_{l+1} is to have a higher high-point value for f_2 *than any previous (FBOB)*, then it is necessary that B_{l+1} contain an element of T_1^1, *and* an element of T_2^1, *and*..., *and* an element of T_l^1.†

It should be emphasized that, due to the non-convex problem structure of $P1$, T_l^1 only provides a necessary condition for improvement in the value of f_2. For a given linear two-level programming problem there may be many (FBOBs) which contain at least one vector from T_l^1 and have high-point values with $f_2 \leq f_2^{*(l)}$. There will not, however, be any (FBOBs) with high-point values with $f_2 > f_2^{*(l)}$ which do not include at least one vector from T_l^1.

If there is no (FBOB): B_{l+1} satisfying the necessary T^1 set constraints for a better (FBOB) than any previous (FBOB), then the optimal solution to the linear rwo-level programming problem $P1$ is given by the previous best high-point solution.

We write $[T^1]_k$ for the set of k constraints corresponding to $T_1^1, \ldots, T_k^1$.

Given a high-point we next examine the opportunity for making some monotonic improvement in the value of the objective function for the linear two-level programming problem, using the information contained in only the most recent T^1 set. Subsequent sections examine the problem of constructing a (FBOB) subset to the constraint of satisfying all previous T^1 sets, i.e. satisfying $[T^1]_k$.

A high point in basis, B_l, is defined as the optimal solution to problem $P4$. It is possible that this occurs "at the origin in policy space", $x_2^{(l)} = 0$. In this case, provided (13.2) and (13.4) hold as strong inequalities (i.e. in the absence of degeneracy in the policy objective function), the high-point will be a local optimum.‡ In the presence of degeneracy, at least one policy variable can be introduced into the basis, with the implications discussed below.

†Of course, if a particular behavioral activity, say H_{1r}, was a member of each of the T^1 sets: $T_1^1, \ldots, T_l^1$; then inclusion of this one behavioral activity in B_{l+1} would satisfy the necessary T^1 set conditions.

‡Since any small increase in a policy variable (and therefore of a linear combination of them) will reduce f_2.

In general $x_{2_q} > 0$ for at least one policy variable at a high-point, and correspondingly, an equal number of $x_{1j} = 0$, $H_{1j} \in B_l$, [4, Appendix I]. Thus at a high-point $x_2^{(l)}$, there will in general be several zero elements on the RHS of the behavioral problem $P2$, and there will be the opportunity for alternative bases to be (FBOB) at the high-point. Any basis to the behavioral problem $P2$ which can be obtained by swapping activities at zero level at $x_2^{(l)}$, will be called a (primal) *feasible basis* at $x_2^{(l)}$. Any feasible basis at $x_2^{(l)}$, which is also optimal will be called an *adjacent basis* at $x_2^{(l)}$. All adjacent bases are thus FBOB. Any adjacent basis at $x_2^{(l)}$ which can be obtained by a *single* pivot operation will be called a *neighboring basis* for B_l.

If $x_2^{(l)}$ is the high-point for all adjacent bases at $x_2^{(l)}$, then $x_2^{(l)}$ corresponds to a local optimum of $P2$, since any small change in x_2 away from $x_2^{(l)}$ will lead to lower values of f_2.

In order to reach a feasible basis to B_l at $x_2^{(l)}$, it is only necessary to replace a vector $H_{1j} \in B_l$, by another vector $H_{1r} \notin B_l$, where H_{1j} is the basis at zero level. Provided H_{1r} is not a linear combination of the m-1 vectors remaining in the basis, this substitution will give an alternative basis B_{lr}. Since $x_2^{(l)}$ is a high-point, or point of "induced degeneracy" from the viewpoint of the behavioral problem, and given that H_{1j} was in the basis at zero level, H_{1r} will equally be in the basis at zero level, and the alternative basis will be feasible. As defined above, a neighboring basis is, however, not only feasible, but also (FBOB).

The set of vectors $H_{1j} \in B_l$, such that $x_{1j}^{(l)} = 0$, (defined in (17)) are thus candidates for removal from B_l to give another feasible basis. A fairly simple calculation of reduced cost (18) indicates which vector $H_{1j} \notin B_l$ must enter the basis for each vector removed, in order to give an (FBOB) basis. If this entering vector is a member of T_l^1, then the new basis will have a higher high-point, and monotonic local improvement is possible.

Consider now the solution to the behavioral L.P. problem, $P2$, corresponding to B_l, and given $x_2 = x_2^{(l)}$. Define:

$$R_l = [i | x_{1i}^{(l)} = 0, H_{1i} \in B_l] \tag{17}$$

where, $x_1^{(l)} = B_l^{-1}b - B^{-1}H_2x_2^{(l)}$. Thus R_l identifies the degenerate rows (vectors) of the updated tableau of the behavioral L.P. problem $P2$, corresponding to (FBOB): B_l, at the high-point setting for the policy variables $x_2^{(l)}$. As indicated above, a sufficient condition for $R_l \neq \emptyset$ is that W_l contain at least one vector from H_2.

If H_{1i}, $i \in R_l$, can be replaced by $H_{1r} \notin B_l$ to give a neighboring basis B_{lr}, and if H_{1r} is a member of T_l^1 (defined in (14)), then B_{lr} will have a high-point such that $f_2^{*(lr)} \geq f_2^{*(l)}$. For convenience we refer to B_{lr} satisfying these conditions as a *better* neighboring basis to B_l.

Let:

$$d_{ij} = \left(\frac{-f_{1j}}{t_{ij}} \middle| t_{ij} < 0,\ i \in R_l, H_{1j} \notin B_l \right) \tag{18}$$

where t_{ij} is the ith element of $B_l^{-1}H_{1j}$, and $f_{1j} > 0$† is the corresponding reduced cost value for the behavioral objective function. Then H_{1r} can replace H_{1i} in B_l to yield a neighboring basis B_{lr}, where H_{1r} is identified by:

$$d_{ir} = \min_j d_{ij}. \tag{19}$$

If $H_{1r} \in T_l^1$, then B_{lr} is a better neighboring basis to B_l. Also, for given i and the absence of dual degeneracy in the behavioral L.P. problem, H_{1r} identified by (19) will be unique. Clearly, if $t_{ij} \geq 0$ for all j, H_{1i} cannot be replaced in B_l to yield a neighboring basis.

The vectors in B_l corresponding to $i \in R_l$, can be examined in turn to derive the set of neighboring bases to B_l. If none of the neighboring bases to B_l contain a member of T_l^1, then B_l yields the high-point for all neighboring bases. However, this high-point is not necessarily a local optimum since one or more *adjacent* bases (requiring the substitution of two or more vectors in B_l) may have a higher high-point.

Assuring that a high point is local optimum, involves the solution of a dual problem, for the dual variables with $i \in R_l$ [1]. Since we are concerned with global optimality, we restrict ourselves to the limited search involved in the examination of neighboring bases. This allows

†As noted earlier, if there is dual degeneracy, i.e. $f_{1j} = 0$ for at least one $H_{1j} \notin B_l$, then the response of the inner decision maker to given policy setting $\bar{x}_2$ is indeterminate (since the inner decision maker is indifferent as between degenerate choices), and $P1$ is not well defined for these values of x_2.

the possibility of making at least some monotonic improvement in the value of the objective function for $P1$.

4. GLOBAL OPTIMALITY

We have just seen that T_l^1, associated with (FBOB): B_l, enables us to identify a better neighboring (FBOB) to B_l, should one exist. In this section we discuss the problem of finding (FBOB): B_l satisfying all previous T^1 set constraints. Our procedure is to find a set of vectors from H_1 satisfying all previous T-sets, and to then find a (BOB) that includes this set. We then check this (BOB) for feasibility.

Initially, we finesse the problem of finding a (BOB) which satisfies $[T^1]_k$. Suppose we have a (BOB): B_l, satisfying $[T^1]_k$. *If* B_l is not (FBOB), we can use the objective function minimized in Phase I of the simplex method (i.e. sum of infeasibilities) for $P4$, to identify the sub-set of those behavioral vectors not belonging to B_l, (and hence excluded from $P4$), which could restore feasibility. One or more of these vectors must be a member of subsequent (BOB's) as a nesessary condition for feasibility.

Let W_l be the basis for problem $P4$, corresponding to B_l, for which the sum of infeasibilities is a minimum; and let

$$F_l = [i|\, W_l^{-1}b < 0]. \tag{20}$$

Then we define:

$$T_l^2 = [H_{1j}|\sum_{i\in F_l} t_{ij} < 0] \tag{21}$$

where

$$t_{ij} = W_l^{-1}H_{1j}. \tag{22}$$

Clearly $\sum_{i\in F_l} t_{ij} \geqslant 0$ for all $H_{ij} \in B_l$, since W_l already minimizes the sum of infeasibilities. Given a (BOB) satisfying $[T^1]_k$ and $[T^2]_k$, the above allows us to either find a high point, and generate a new T_l^1, or prove infeasibility and generate a new T_l^2.

.Next we finesse the problem of finding a set of at most m vectors, C_l, from H_1 which satisfy $[T^1]_k$ and $[T^2]_k$ and consider the problem of finding a (BOB) which includes C_l. If C_l contains m vectors, then it is a simple matter to check whether C_l provides an (FBOB) by solution of $P4$. In general C_l will contain less than m vectors, in which case we write $H_1 = [C_l, D_l]$ and $c_1 = (c_{1c}, c_{1d})$ and write the dual of $P2$ as:

$P6$. For any given (lth) set of vectors from H_1, C_l, find values of u such that:

$$uC_l = c_{1c}; \tag{6.1}$$

$$uD_l \geqslant c_{1d} \tag{6.2}$$

$$u \geqslant 0 \tag{6.3}$$

If this problem has a feasible solution, u_l, and in the absence of degeneracy then $H_{1j} \in B_l$; $u_lH_{1j} = c_{1j}$; and $H_{1j} \notin B_l$; $u_lH_{1j} > c_{1j}$ provides a (BOB) satisfying $[T^1]_k$ and $[T^2]_k$.

If $P6$ is infeasible, then if u_l is the solution which minimizes the sum of the infeasibilities, we know that at least one of the vectors from C_l must be *excluded* from the basis. This allows us to define a new T-set:

$$T_l^3 = [C_l]. \tag{23}$$

Given a set, C_l, of at most m vectors from H_{1j}, satisfying all existing $[T^1]_k$, $[T^2]_k$ and $[T^3]_k$, the above allows us to either find a (BOB): B_l or to generate a new T-set T_l^3. The additional T-sets ensure that cycling will not occur. Denote the sets of constraints given by $[T^1]_k$, $[T^2]_k$, and $[T^3]_k$ by $[T]_k$.

Initially, the selection of vectors from H_1 to satisfy the T-sets $[T]_k$, is trivial, but as the search proceeds and the T-sets become more restrictive, we note that solution of the following integer programming problem is sufficient to select a set C_l, satisfying all existing T-sets:

$P7$. Find a vector y such that:

$$\sum_j y_j\delta_{ij} \geq 1;\ \delta_{ij} = 1 \text{ if } H_{1j} \in T_i^1, = 0 \text{ otherwise} \tag{7.1}$$

$$\sum_j y_j\delta_{ij} \geq 1;\ \delta_{ij} = 1 \text{ if } H_{1j} \in T_i^2, = 0 \text{ otherwise} \tag{7.2}$$

$$\sum_j y_j\delta_{ij} \leq \sum_j \delta_{ij} - 1;\ \delta_{ij} = 1 \text{ if } H_{1j} \in T_i^3, = 0 \text{ otherwise} \tag{7.3}$$

$$\sum_j y_j \leq m \tag{7.4}$$

$$y_j = 0 \text{ or } 1. \tag{7.5}$$

If $P7$ is feasible the solution $H_{1j} \in C_l$ if $y_j = 1$, $H_{1j} \notin C_l$ if $y_j = 0$, provides a set of vectors from H_1 satisfying all existing T-sets.

If $P7$ is infeasible, then the implicit search is complete, and the highest known high-point is the global solution.

5. ALGORITHMIC PRINCIPLES

Since problem $P1$ can have local optima (and hence there is little hope of continuous monotonic improvement in the objective function), some form of implicit enumeration has to be relied upon, using the global necessary condition information generated in the search for locally better solutions.

The T^1 set: T_k^1 generated at the high-point in any (FBOB): B_k, constitutes a requirement that some subset of the $H_{1j} \notin B_k$ be in any (FBOB) leading to a better high-point value of f_2. Hence B_k does not satisfy T_k^1. By paying attention to all T^1 sets generated to date in the solution process, we can avoid returning to any previously explored (FBOB).

Similarly, by satisfying the constraint sets $[T^3]_k$ and $[T^2]_k$, we can avoid returning to any previously explored set of vectors, e.g. C_k, that cannot be part of a (BOB): $B_k = [C_k, D_k]$, or any (BOB) that cannot be made feasible.

These observations lead us to propose the following three broad steps in a solution algorithm for the linear two-level programming problem.

Step 1. Attempt to find a set of ($\leq m$) vectors, C_l, from H_1, that satisfy the existing T-set constraints: $[T]_k$.

Step 2. Given C_l, attempt to find a behaviorally optimum basis; (BOB): $B_l = [C_l, D_l]$.

Step 3. Given (BOB): B_l attempt to solve the corresponding policy L.P. problem ($P4$), to find the high-point in B_l. This sequence is illustrated in Figure 2.

If Step 1 cannot be completed, then the best high-point solution obtained to date is the global optimum solution to $P1$. If Step 2 cannot be completed, we generate T_l^3, add the corresponding constraint to $[T]_k$, and return to Step 1. If we cannot find a feasible solution to $P4$ in Step 3, we generate T_l^2, add the corresponding constraint to $[T]_k$ and return to Step 1. If B_l is (FBOB), then from the high-point solution information we generate T_l^1, add the corresponding constraint to $[T]_k$, to obtain $[T]_l$, and return to Step 1.

Since there exists a finite number of behaviorally optimal bases, and since the constraint set generated at any stage prevents us from returning to the set of vectors used to generate the T-set, the proposed algorithm will find the global optimum solution to the linear two-level programming problem, $P1$, in a finite number of iterations. The proposed algorithm also examines neighboring bases at the high-point in any (FBOB): B_l for the opportunity of making at least some monotonic improvement in the value of f_2, while using only the T_l^1 set necessary conditions.

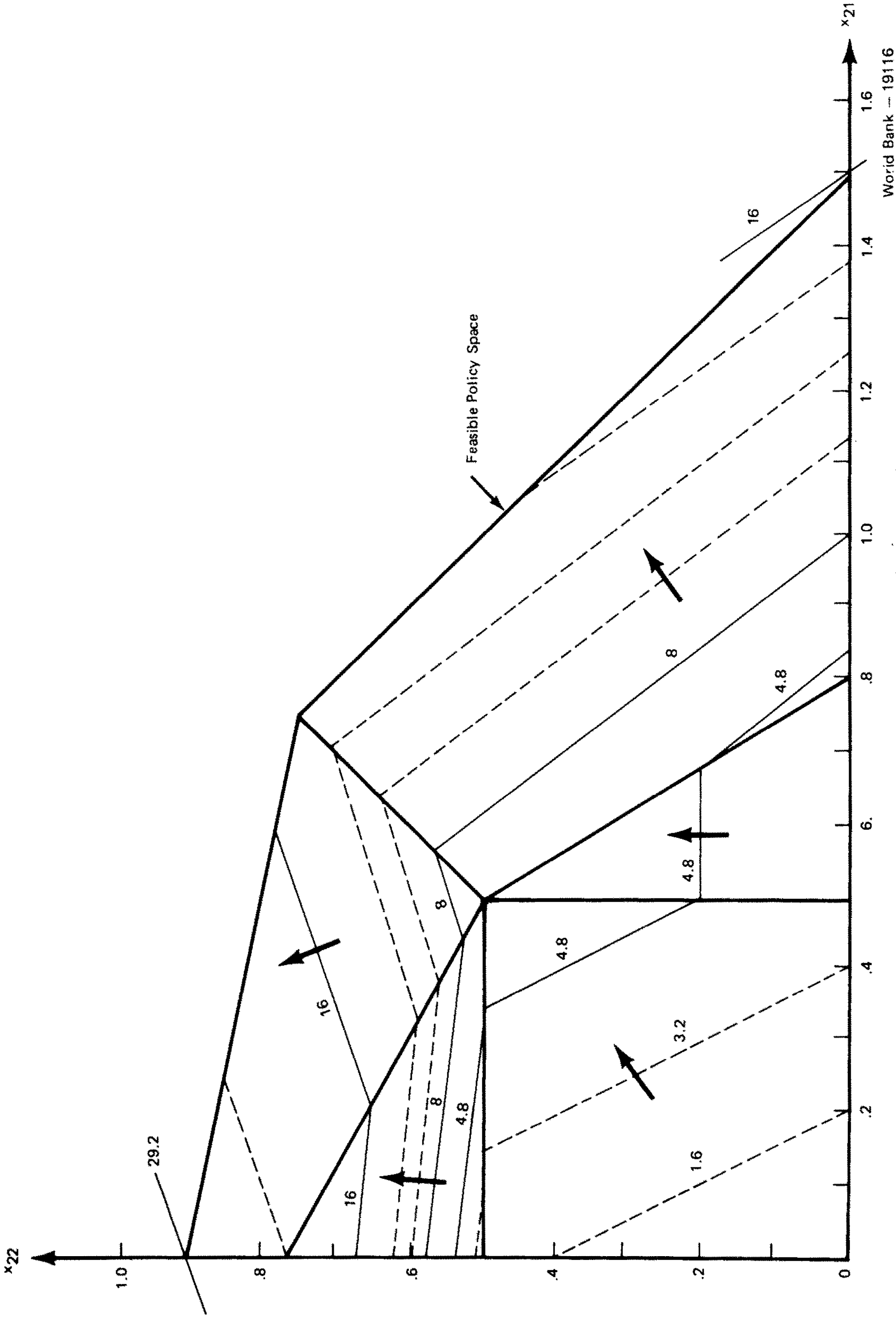

Fig. 1. Policy objective function contours given optimal feasible bases for numerical example.

6. ALGORITHM

Steps 1 to 4 below serve to initialize the problem, and Steps 11 to 14 involve a search of neighboring bases for the possibility of monotonic improvement in f_2. This leaves Steps 6 to 10, as the major implicit search sequence, illustrated schematically in Fig. 2.

(1) Set $k = 1$, and $f_2^* = -\infty$, then attempt to solve $P5$. If $P5$ is infeasible, go to Step 2. If $P5$ is unbounded, go to Step 3. Otherwise go to Step 4.

(2) Since $P5$ is infeasible, so is problem $P1$. Stop.

(3) Impose an arbitrary upper limit on f_2, and go to Step 1.

(4) The solution to $P5$ provides an upper limit on f_2. Given the solution values for the policy variables from Step 1, $x_2 = \bar{x}_2^{(k)}$, solve $P2$. Denote the optimal feasible basis: B_k. If $P2$ is unbounded, place an arbitrary upper limit on f_1 and go to Step 4. Otherwise go to Step 9.

(5) Set $k = k + 1$ and solve $P7$ to select a set of vectors C_k which satisfy $[T]_k$. If this is impossible, go to Step 6; otherwise go to Step 7.

(6) The implicit search is complete, and f_2^* is the optimal value of the policy objective function for $P1$; B_* and W_* are the corresponding optimal bases. Stop.

(7) Solve $P6$ to find (BOB): B_k, and go to Step 9. If this is not possible, go to Step 8.

(8) Generate T_k^3 as in (23) and add the corresponding constraint to $[T]_k$. Go to Step 5.

(9) Attempt to find a feasible solution to $P4$, given (BOB): B_k, by minimizing the sum of infeasibilities, (phase I, simplex method). If $P4$ is feasible, go to Step 10. If $P4$ is infeasible, define T_k^2 as in (21), add this to the set of constraints $[T]_k$, and go to Step 5.

(10) Find the optimum solution to $P4$. Define T_k^1 according to (14). If $T_k^1 = \phi$, go to Step 6, otherwise add T_k^1 to the set of constraints $[T]_k$, and go to Step 11.

(11) Construct R_k as in (17) and calculate d_{ij} as in (18). For each $i \in R_k$, identify the neighboring vector H_{1r} (should one exist), by (19), and check whether $H_{1r} \in T_k^1$. As soon as a neighboring vector belonging to T_k^1 is found, go to Step 14. If there are no neighboring vectors belonging to T_k^1, go to Step 12.

(12) If $f_2^{*(k)} < f_2^*$, go to Step 5. Otherwise go to Step 13.

(13) Set $f_2^* = f_2^{*(k)}$, $B_* = B_k$, $W_* = W_k$; go to Step 5.

(14) Set $k = k + 1$. Obtain a better neighboring basis by substituting H_{1r} for H_{1i}. Call the new basis B_k, and go to Step 10.

7. NUMERICAL EXAMPLE

As a specific numerical example of the application of the algorithm consider the problem:

Find $\quad x_{1j} (j = 1, \ldots, 6)$ and $x_{2q} (q = 1, 2)$ such that:

$$f_2 = \max_{x_{2q}} (-4x_{11} + 40x_{12} + 4x_{13} + 8x_{21} + 4x_{22}) \tag{24}$$

subject to

$$f_1 = \max_{x_{1j}} (-x_{11} - x_{12} - 2x_{13} - x_{21} - 2x_{22}) \tag{25}$$

and

$$H_1 x_1 + H_2 x_2 = b \tag{26}$$

$$x_1, x_2 \geq 0 \tag{27}$$

where

$$H_1 = \begin{bmatrix} -1 & 1 & 1 & 1 & 0 & 0 \\ -1 & 2 & -0.5 & 0 & 1 & 0 \\ 2 & -1 & -0.5 & 0 & 0 & 1 \end{bmatrix}$$

$$H_2 = \begin{bmatrix} 0 & 0 \\ 2 & 0 \\ 0 & 2 \end{bmatrix} \quad b = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

A contour of f_2 for this problem in the "policy space" (x_{21}, x_{22}) is given in Fig. 1.

Applying the algorithm:

Step 1. We solve problem $P5$, i.e. we maximize f_2 with respect to all x's subject to (26) and (27). This yields the first simplex tableau in Table 1, and provides an upper bound of 58.0 on f_2. In Table 1, b is the right hand side and H_{14}, H_{15} and H_{16} are the "disposal" activities. Since the solution to problem $P5$ is feasible and bounded, we go to Step 4.

Step 4. In tableau $n = 1$ of Table 1, $x_{21} = x_{22} = 0$. Fixing these values of the policy variables, x_{21} and x_{22}, we solve $P2$ to get a BOB. This solution value is shown in tableau $n = 2$ of Table 1. (BOB): $B_1 = [H_{14}, H_{15}, H_{16}]$. Since a BOB has been found, go to Step 9.

Step 9. The solution obtained in Step 4 already FBOB, go to Step 10.

Step 10. We find the optimum solution to $P4$ for B_1. This optimal solution is given in tableau $n = 3$ of Table 1. The only $f_{2j} < 0$ in tableau $n = 3$ is for H_{12}, hence $T_1{}^1 = [H_{12}]$, and we know that any better basis must contain H_{12}. Go to Step 11.

Step 11. $R_1 = [H_{15}, H_{16}]$ since the levels of these vectors in B_1 have been driven to zero. For row 2 of tableau $n = 2$ (i.e. for the BOB tableau) we have:

$$d_{21} = -\frac{1}{-1} = 1$$

$$d_{23} = -\frac{2}{-0.5} = 4$$

$$d_{21} = \min_j d_{2j}, H_{11} \notin T_1{}^1$$

and for row 3, we have:

$$d_{32} = -\frac{1}{-1} = 1$$

$$d_{33} = -\frac{2}{-0.5} = 4$$

$$d_{32} = \min_j d_{3j}, H_{12} \in T_1{}^1.$$

Table 1. Simplex tableaus used in solution of numerical example

$n = 1$	b	H_{14}	H_{15}	H_{16}	H_{11}	H_{12}	H_{13}	H_{21}	H_{22}	Remarks
H_{11}	1.5	.5	.17	.83	1			.33	1.67	Optimum solution to P5
H_{12}	1.5	.5	.5	.5		1		1	1	$(f_{2j} \geq 0,\ f_{2q} \geq 0)$.
H_{13}	1	1	-.33	.33			1	-.67	.67	Upper bound on $f_2 = 58$.
$z-c_1$	-5	-3	0	-2				1	- 2	
$z-c_2$	58	22	18	18				28	32	
$n = 2$										
H_{14}	1	1			-1	1	1			BOB $(f_{1j} \geq 0)$
H_{15}	1		1		-1	2	-.5	2		Bases which by inspection, are
H_{16}	1			1	2	-1	-.5		2	also FBOB $[H_{14}\ H_{11}\ H_{16}]$ $[H_{14}\ H_{15}\ H_{12}]$
$z-c_1$	0				1	1	2	1	2	
$z-c_2$	0				4	-40	-4	-8	-4	

Table 1 (cont.d) Simplex Tableaus Used in Solution of Numerical Example

n = 3	b	H_{14}	H_{15}	H_{16}	H_{11}	H_{12}	H_{13}	H_{21}	H_{22}	Remarks
H_{14}	1	1			-1	1	1			High Point ($f_{2j} \geq 0$ $H_{1j} \in B_1$
H_{21}	.5		.5		-.5	1	-.25	1		
H_{22}	.5			.5	1	-.5	-.25		1	$f_{2q} \geq 0$).
Z-C_1	-1.5		-.5	-1	-.5	1	2.75			T_1^1 = [H_{12}]
Z-C_2	6		4	2	4	-34	7			
n = 4										
H_{14}	2	1		1	1		.5		2	BOB ($f_{1j} \geq 0$ all H_{1j})
H_{15}	3		1	2	3		-1.5	2	4	Not feasible, but know, by construction that is
H_{12}	-1			-1	-2	1	.5		-2	feasible for $x_{21} = x_{22} = .5$
Z - C_1	1			1	3		1.5	1	4	
Z-C_2	-40			-40	-76		16	-8	-84	
n = 5										
H_{14}	.5	1	-.5		-1		1			High Point ($f_{2j} \geq 0$,
H_{12}	.5		.5		-.5	1	-.25	.5		$H_{1j} \in B_2$, $f_{2q} \geq 0$).
H_{22}	.75		.25	.5	.75		3.75	.25	1	T_2^1 = [H_{11}, H_{13}]
Z - C_1	-2		-1	-1	0		3	-1		
Z - C_2	23		21	2	-13		-15.5	34		
n = 6										
H_{14}	3	1	.33	1.67	2			.67	3.33	BOB. Not feasible, but
H_{12}	0		.33	-.33	-1	1		.67	-.67	know by construction, feasibl
H_{13}	-2		-.67	-1.33	-2		1	-1.33	-2.67	for $x_{21} = 0$, $x_{22} = .75$
Z-C_1	4		1	3	6			3	8	
Z-C_2	-8		10.66	-18.67	-44			13.33	-33.33	
n = 7										
H_{12}	.6	.2	.4		-.6	1		.8		High Point.
H_{13}	.4	.8	-.4		-.4		1	-.8		T_3^1 = [H_{11}]
H_{22}	.9	.3	.1	.5	.6			.2	1	
Z-C_1	-3.2	-2.4	.2	-1	1.2			1.4		
Z-C_2	29.2	12.4	14.8	2	-19.2			21.6		
n = 8										
H_{11}	-1	-2	1		1		2.5	1.33		Unbounded, unless T-sets violated. Since no positive
H_{12}	0	-1	1			1	-1.5	2	1	pivot for H_{15}.
H_{16}	3	3	-1	1			3	-2	2	
Z-C_1	1	3	-2				6	-3	2	
Z-C_2	4	-32	-36				-54	64	-4	

Since $H_{12} \in T_1^1$, we know that there is a better neighboring (FBOB) at $x_{21} = x_{22} = 0.5$. We go to Step 14.

Step 14. Set $k = k+1 = 2$. Define a new basis $B_2 = B_1 + H_{12} - H_{16} = [H_{14}, H_{15}, H_{12}]$. The corresponding (FBOB) is given in tableau $n = 4$ of Table 1. This basis is not feasible in tableau $n = 4$, but by construction we know it would be feasible for $x_{21} = x_{22} = 0.5$. Go to Step 10.

Step 10. Solution of problem $P4$ for B_2 leads to tableau $n = 5$ in Table 1. The new T-set is $T_2^1 = [H_{11}, H_{13}]$. (At this stage we know that any better basis has H_{12} (from T_1^1) and "H_{11} or H_{13} or both" (from T_2^1)). Go to Step 11.

Step 11. $R_2 = [H_{15}]$ since only this member of B_2 has been driven to zero. For row 2 of tableau $n = 4$ (i.e. for the BOB tableau) we have

$$d_{23} = -\frac{1.5}{-1.5} = 1 = \min_j d_{2j}$$

and

$$H_{13} \in T_2^1.$$

Since $H_{13} \in T_2^1$, we know that there is a "better" neighboring (FBOB) at $x_{21} = 0$, $x_{22} = 0.75$. We go to Step 14.

Step 14. Set $k = k+1 = 3$. Define a new basis $B_3 = B_2 + H_{13} - H_{15} = [H_{14}, H_{12}, H_{13}]$. The corresponding (FBOB) is given in tableau $n = 6$ of Table 1. This tableau is not feasible, but since it is a neighboring basis to an (FBOB), it must itself be feasible for appropriate values of the policy variables. Go to Step 10.

Step 10. Solution of problem $P4$ for B_3 leads to tableau $n = 7$ in Table 1. The new T-set is $T_3^1 = [H_{11}]$ and we see that T_3^1 dominates T_2^1. We now know that any better basis contains H_{11} (from T_3^1) and H_{12} (from T_1^1). Go to Step 11.

Step 11. $R_3 = [H_{14}]$. Since the H_{14} row in tableau $n = 6$, (the BOB tableau) has no negative elements, ratio d_{ij}, and hence $\min_j d_{ij}$, are not defined, and we know that there is no better neighboring basis. (Indeed no better adjacent basis.) Go to Step 12.

Step 12. $f_2^{*(3)} = 29.2 > f_2^* = -\infty$, hence go to Step 13.

Step 13. $f_2^* = f_2^{*(3)} = 29.2$, $B_* = [H_{14}, H_{12}, H_{13}]$ and $W_* = [H_{12}, H_{13}, H_{22}]$. Go to Step 5.

Step 5. Set $k = k+1 = 4$. We revert to basis $B_4 = [H_{11}, H_{12}, H_{13}]$ which satisfies T_1^1 and T_3^1 (and hence T_2^1) and is recorded in tableau $n = 1$. Go to Step 7.

Step 7. From the first tableau in Table 1, we carry out a dual pivot step (where H_{11} and H_{12} are not considered candidates for removal, since this would violate the T-sets), to replace H_{13} by H_{16} in the basis. This results in tableau $n = 8$, where $f_{15} < 0$, but only positive pivots would remove H_{11} or H_{12} from the basis (thus violating the T-sets). Thus a basis containing both H_{11} and H_{12} cannot be made BOB. Thus $T_4^3 = [H_{11}, H_{12}]$. Go to Step 8.

Step 8. We have that H_{11} and H_{12} cannot together be in a (BOB). The full set of T constraints at this stage is given below, where y_j's are (0, 1) variables.

	y_1	y_2	y_3	y_4	y_5	y_6	
T_1^1:		1					≥ 1
T_2^1:	1		1				≥ 1
T_3^1:	1						≥ 1
T_4^3:	1	1					≤ 1
Basis:	1	1	1	1	1	1	≤ 3

It is easily seen in this small example that there is no feasible solution to this set of constraints, i.e. there is no set of vectors C_4 from H_1 that satisfy the existing T-set necessary conditions for a better high-point solution. Go to Step 6.

Step 6. $f_2^* = 29.2$, $B_* = [H_{14}, H_{12}, H_{13}]$, $W_* = [H_{12}, H_{13}, H_{23}]$ Stop.

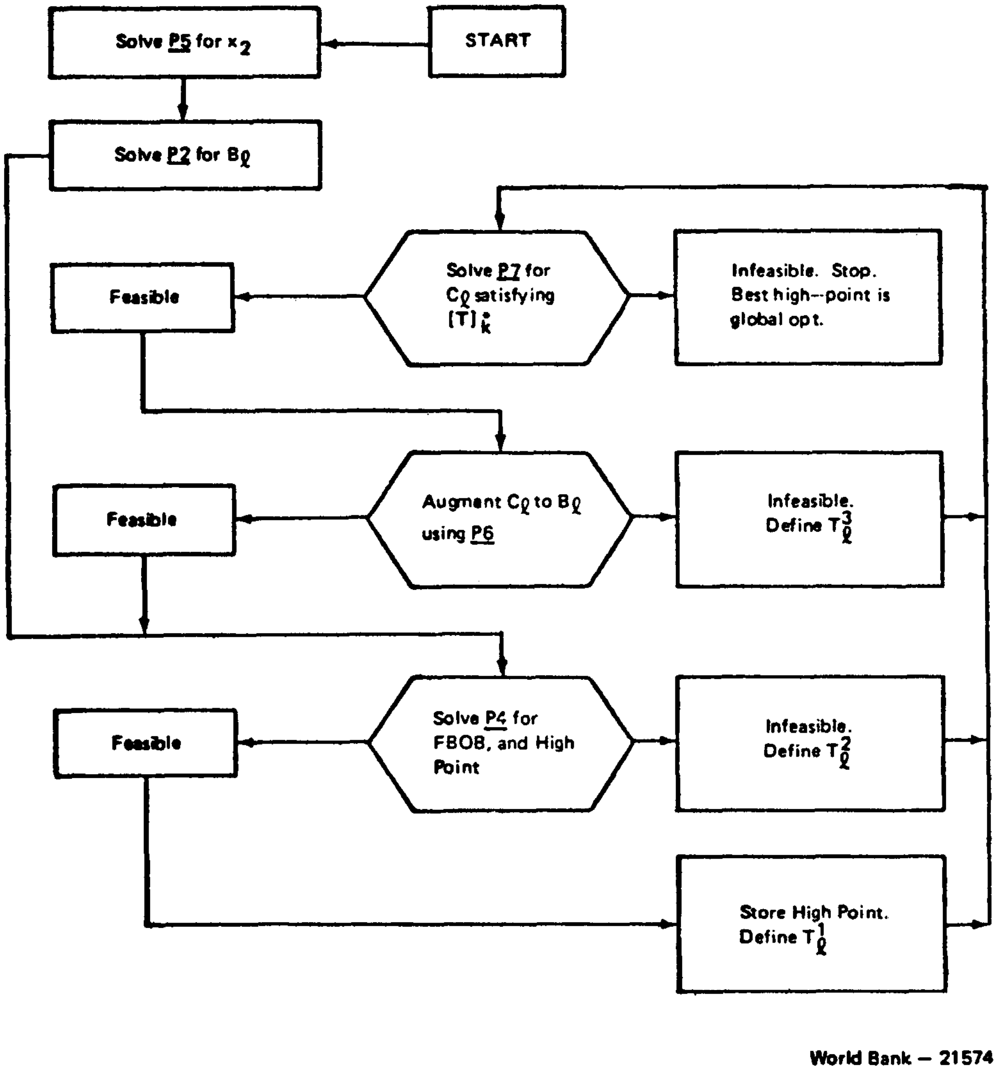

Fig. 2. Schematic of T-set algorithm.

As can be seen from Fig. 1, there is another optimum at $x_{21} = 1.5$, $x_{22} = 0$, but the information contained in the T-sets is sufficient to rule out any need to visit this lower local optima.

7. CONCLUSION

The above algorithm allows an implicit search of the policies in problem $P1$, which affect the right hand side, or resource availabilities for the behavioral problem. It does not apply to the many government policies which operate by affecting prices, and hence the objective function of the inner, behavioral, problem. We have not yet done any work on this policy problem, other than to note that it exists and is, in some sense, the "dual" of the problem considered in this paper. An algorithm due to Fortuny-Amat and McCarl [7] has been presented which allows both the objective function and restraints of the behavioral problem to be influenced by policy-makers. This algorithm [7] allows the inner behavioral problem to be quadratic. Unfortunately, it still requires the outer objective function to be linear, whereas in most situations the quadratic term of program cost (per unit subsidy by quantity subsidized) would enter the outer objective.

Given the crucial role played by the T-sets in our algorithm, it may be conjectured than an algorithm which focused primarily on the tightness of the T-sets which could be generated, would hold promise of a substantial improvement over the algorithm offered here.

A small Fortran computer code has been written to implement the above algorithm for programs where the updated tableau can be carried explicity. No progress has yet been made in incorporation of the algorithm into a large code. Considerable success has, however, been experienced with a heuristic search of a linear two-level programming problem consisting of 53 constraints rows, 314 behavioral activities (excluding slacks) and 8 policy activities. This heuristic search procedure, based on the above ideas, is discussed in Appendix II of [4].

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