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TWO-LEVEL LINEAR PROGRAMMING*

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Decentralized planning has long been recognized as an important decision making problem. Many approaches based on the concepts of large-scale system decomposition have generally lacked the ability to model the type of truly independent subsystems which often exist in practice. Multilevel programming models partition control over decision variables among ordered levels within a hierarchical planning structure. A planner at one level of the hierarchy may have his objective function and set of feasible decisions determined, in part, by other levels. However, his control instruments may allow him to influence the policies at other levels and thereby improve his own objective function. This paper examines the special case of the two-level linear programming problem. Geometric characterizations and algorithms are presented with some examples. The goal is to demonstrate the tractability of such problems and motivate a wider interest in their study.
(PROGRAMMING—LINEAR, ALGORITHMS)

1. Introduction

Many planning problems require the synthesis of decisions of several, interacting individuals or agencies. Often, these groups are arranged within a hierarchical administrative structure, each with independent, and perhaps conflicting, objectives.

Multilevel decision-making has always been regarded as an important, although sometimes overlooked, aspect of the planning process. Frequently, the impact of directives from superiors and the reactions of subordinates have been viewed as externalities, beyond the control of a planner. However, there have been attempts to model the ability of one planner to indirectly influence the decisions of others to his own benefit.

Because of the pervasive nature of the topic, the term, "multilevel planning," has appeared in a variety of settings, clothed in different mathematical raiments. With few exceptions, multilevel methods for optimizing hierarchical systems have rested heavily on the decomposition method of Dantzig and Wolfe (1960) (see, for example, Balas 1966, Cooper *et al.* 1964, Geoffrion 1970a, b, 1971 and Lasdon 1970). Multilevel decomposition methods easily lend themselves to an economic interpretation of the algorithmic process. The procedure is viewed as an adjustment phase with the superior planner sending tentative information to the lower level subunits, observing their reactions, and then updating the corporate information. This information can take the form of placing prices on the scarce resources (see Baumol and Fabian 1964, Charnes *et al.* 1967), or partitioning the resources among the subunits (see Freeland and Baker 1975, Kornai and Liptak 1965).

One criticism of decomposition methods is that although the solution process can be interpreted to mimic the behavior of a multilevel organization, in fact, one is solving an optimization problem, perhaps very large, with a single objective function over a fixed feasible region. The assumption is that either the single corporate objective

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decomposes into the objectives of the subunits, or that the central planner is willing to accept the aggregated objectives of the subunits as his own (Dirickx and Jennergren 1979).

Multiobjective programming approaches (e.g., Geoffrion and Hogan 1972 and Keeney and Raiffa 1976) seek to find a simultaneous compromise among the various goals of different divisions. However, once again, such techniques assume that all objectives are those of a single planner, or a harmonious bevy of planners. They do not usually account for possible independent actions from each division (Wendell 1980).

This paper will offer an approach to a large class of multilevel planning problems which are applicable to certain hierarchical decision-making systems. These problems are characterized by a hierarchy of planners, each independently controlling a set of decision variables, disjoint from the others. Their final decisions are executed sequentially within the hierarchy, from the highest to lowest levels. The objective function and set of feasible decisions of any one planner will be determined, in part, by decisions made at other levels.

The proposed formulation is intended to supplement the long-standing decomposition approach, not supplant it. Much of its philosophy is underpinned by the work on Stackelberg games (see Simaan and Cruz 1973). To some, the problem will be a dynamic programming problem, although difficult to solve using traditional techniques. This paper will view the problem as a nonconvex programming problem with very special structure.

Similar hierarchical planning methods have been proposed by Goreux and Manne (1973) and Hax and Meal (1975). However, usually such methods consider only downward parametric linkages. A noteworthy exception is a model proposed by Cassidy *et al.* (1971). Also, techniques in the fields of control theory and dynamic games (see Ho *et al.* 1980 and Papavassilopoulos 1980) are close companions of the methods presented here. Recent work in this area has also been conducted by Haimes (1982), Sage (1977) and Singh and Titli (1979).

The next section will provide a definition for a general two-level problem. The discussion will then focus on the linear case, and the solution to the nonconvex programming problem which arises. The goal of the presentation is to demonstrate the tractability of this nonconvex problem, and thus promote a wider interest in the analysis of multilevel decentralized planning problems.

2. A Definition of the Problem

An important feature of so-called "multilevel programming" problems is that a planner at one level of a hierarchy may have his objective function *and* decision space determined, in part, by other levels. In addition, each planner's control instruments may allow him to influence the policies at other levels, and thereby improve his own objective function. These instruments may include the allocation and use of resources at lower levels, and the benefits conferred upon other levels.

The problems we want to consider have the following common characteristics:

1. The system has interacting decision-making units within a hierarchical structure.
2. Each subordinate level executes its policies after, and in view of, the decisions of superior levels.
3. Each unit maximizes net benefits independently of other units, but may be affected by the actions and reactions of those units.
4. The external effect on a decision-maker's problem can be reflected in both his objective function and his set of feasible decisions.

A general definition for such problems involving n-levels has been offered by Bialas and Karwan (1982). The two-level problem presented here is merely a special case.

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Let the control over all real-valued decision variables in the vector $x \equiv (x_1^1, x_2^1, \ldots, x_{N(1)}^1, x_1^2, x_2^2, \ldots, x_{N(2)}^2)$ be partitioned between two planners, hereafter known as level one (the superior or "top" planner) and level two (the inferior or "bottom" planner). That is, level one controls the first $N(1)$ components of the vector x, and level two controls the remaining $N(2)$ components. Let $x^1 \equiv (x_1^1, x_2^1, \ldots, x_{N(1)}^1)$ and $x^2 \equiv (x_1^2, x_2^2, \ldots, x_{N(2)}^2)$. When the policies are finally executed, level one will first specify x^1, and level two will then specify x^2, with full knowledge of level one's decision.

Let $S \subset \mathbf{R}^{N(1)+N(2)}$ denote the feasible choices of (x^1, x^2). For the discussion here, we will assume that S is closed and bounded. For any fixed choice of x^1, level two will choose a value of x^2 to maximize the objective function $f_2(x^1, x^2)$. Hence, for each feasible value of x^1, level two will react with a corresponding value of x^2 (assume for a moment that this value exists and is unique for each x^1). This induces a functional relationship between the decisions of level one, x^1, and the reactions of level two, say $x^2 = \psi(x^1)$. We will assume that the reaction function, $\psi(\cdot)$, is completely known by level one. Hence, level one is really restricted to choosing a point in the set given by the following:

DEFINITION 2.1. The set $\Psi_{f_2}(S)$ given by $\Psi_{f_2}(S) \equiv \{(x_*^1, x_*^2) \in S \mid f_2(x_*^1, x_*^2) = \max_{x^2} f_2(x_*^1, x^2)\}$ is the *set of rational reactions* of f_2 over S.

Note that the maximization in the above definition is only taken over x^2 for each fixed x_*^1. With the reaction function $\psi(\cdot)$ in hand, one can also represent the rational reaction set as $\Psi_{f_2}(S) \equiv \{(x_*^1, x_*^2) \in S \mid x_*^2 = \psi(x_*^1)\}$.

EXAMPLE 2.1. Let x^1 and x^2 be single component vectors. Suppose $S = \{x \in \mathbf{R}^2 \mid A^1x^1 + A^2x^2 \leq b\}$ is the polyhedron shown in Figure 2.1. Let $f_2(x) = cx$. Then, for any fixed, feasible choice of x_*^1, level two solves the linear programming problem

$$\max cx = c^1x_*^1 + c^2x^2 \qquad \text{st:} \quad A^2x^2 \leq b - A^1x_*^1.$$

The solution x_*^2, together with x_*^1, results in a point (x_*^1, x_*^2) which is an element of $\Psi_{cx}(S)$, the hatched region in Figure 2.1.

If level one wishes to maximize its objective function, $f_1(x^1, x^2)$, by controlling only the vector x^1, it must solve the mathematical programming problem

$$\max f_1(x^1, x^2) \qquad \text{st:} \quad (x^1, x^2) \in \Psi_{f_2}(S), \tag{2.1}$$

or equivalently

$$\max f_1(x^1, \psi(x^1)) \qquad \text{st:} \quad (x^1, \psi(x^1)) \in S.$$

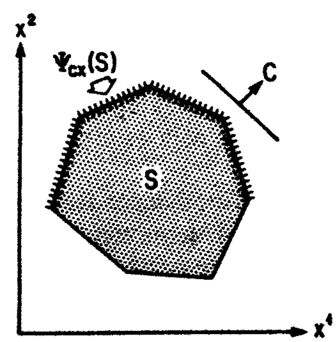

FIGURE 2.1 Example of a Rational Reaction Set for a Two-Level Problem.

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The problem given by (2.1) is called a *two-level programming problem* and may be more explicitly written as

$$\max_{x^1} f_1(x^1, x^2) \qquad \text{where } x^2 \text{ solves} \tag{P^1}$$

$$\max_{x^2} f_2(x^1, x^2) \qquad \text{st:} \ (x^1, x^2) \in S. \tag{P^2}$$

For convenience of notation and terminology we will refer to $S^1 \equiv \Psi_{f_2}(S)$ as the *level-one feasible region*, and $S^2 \equiv S$ as the *level-two feasible region*.

3. Two Examples

The Two-Level Linear Resource Control Problem

The *two-level linear resource control problem* is a two-level problem where

$$f_k(x) = c^k x = c^{k1}x^1 + c^{k2}x^2 \qquad (k = 1,2) \quad \text{and}$$

$$S^2 = \{x \mid A^1x^1 + A^2x^2 \leqq b, (x^1, x^2) \geqq 0\}.$$

In this problem, level one controls x^1 which, in turn, varies the resource space of level two by restricting $A^2x^2 \leqq b - A^1x^1$. Note, however, that the value of the level-one objective function is determined by x^2 as well as x^1. This problem may be written as

$$\max_{x^1} c^{11}x^1 + c^{12}x^2 \qquad \text{where } x^2 \text{ solves} \tag{P^1}$$

$$\max_{x^2} c^{21}x^1 + c^{22}x^2 \tag{P^2}$$

$$\begin{aligned} \text{st:} \quad A^1x^1 + A^2x^2 &\leqq b, \\ x^1 \qquad &\geqq 0, \\ x^2 &\geqq 0. \end{aligned}$$

This model is similar to one proposed by Cassidy *et al.* (1971) to analyze the distribution of a federal budget among several states. Although their problem was formulated as an integer programming problem, the essential aspects are the same. The vector x^1 is the allocation of the budget to individual states. At level two, each state can choose to independently fund projects within its borders. The level of funding is given by the vector, x^2. The federal government's objective function depends on the ultimate project allocation decisions made by the states, and may vary its allocation, x^1, to the states to improve that objective function.

The Two-Level Linear Price Control Problem

The *two-level linear price control problem* is another special case of the general two-level programming problem, with

$$f_1(x) = c^{11}x^1 + c^{12}x^2, \tag{3.1}$$

$$f_2(x) = x^1x^2, \quad \text{and} \tag{3.2}$$

$$S^2 = \{x \mid Ax \leqq b, x \geqq 0\}.$$

Note that in equation (3.2) x^1 is transposed to conform with x^2 in the inner product.

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This problem may also be written as

$$\max_{x^1} c^{11}x^1 + c^{12}x^2 \qquad \text{where } x^2 \text{ solves} \tag{P^1}$$

$$\max_{x^2} x^1x^2 \qquad \text{st:} \quad Ax \leqq b, \quad x \geqq 0. \tag{P^2}$$

In this problem, level one controls the cost coefficients of level two. This model is useful in the analysis of tax and subsidy programs. Upon examining the nature of $f_2(x)$, one might argue that the problem should not be called "linear." However, for fixed x^1, problem P^2 is simply a linear programming problem.

This type of formulation may be used to determine optimal effluent charge schedules for regional wastewater treatment plants. In such a model, level one is a regional authority which seeks to minimize regional treatment costs, and at the same time assure that regional environmental quality standards are maintained. Among the authority's control instruments are the fees, x^1, it charges to various waste producers for treating their effluent at the regional plant. However, the producers may alternatively construct their own waste treatment facilities, avoiding the regional effluent charges and pollution taxes imposed by level one.

4. Characterization of Solutions

Effects of Multiple Optima

For any two-level programming problem, care must be taken when the solution to (P^2) is not unique for fixed x^1 (i.e., $\psi(\cdot)$ does not exist). Although not affecting the value of the level-two objective function, $f_2(x)$, these solutions can have a greatly varying impact on the objective at level one. Therefore, the control over the choice among multiple optima at level two may have to be delegated to level one or an outside referee.

Bialas and Karwan (1978) proposed an incentive scheme to overcome the problem of multiple optima. This method perturbs the level-two objective function, replacing the original by $f_2^*(x) = f_2(x) + \epsilon f_1(x)$ where the value of $\epsilon > 0$ is suitably small. This would require that level one "kick-back" a small portion of its earnings to encourage level two to choose a desirable solution.

Note that, in general, such a perturbation method may still not determine a unique solution since level one may have the same objective function value for a number of level two multiple optima. However, any of these solutions would be satisfactory for level one.

Nonconvexity

The geometric properties of the two-level mathematical programming problem are more complex than familiar mathematical programming problems. For example, consider the simplest, nontrivial case, i.e., the two-level linear resource control problem. Although its overall feasible region, S^2, is a convex polyhedron and all objectives are linear, the actual problem is a nonconvex programming problem since the set of rational reactions, S^1, is nonconvex.

Consider the example of a two-level linear resource control problem shown in Figure 4.1. In this simple example, x^1 and x^2 are single component vectors. For any fixed choice of x^1, level two will choose the value of x^2 which maximizes c^2x. This results in the rational reaction set, S^1, which is the hatched region in Figure 4.1. The obvious choice of x^1 for level one is that which yields the maximum value of c^1x with respect to the hatched region. This requires the maximization of a linear objective function over a nonconvex set.

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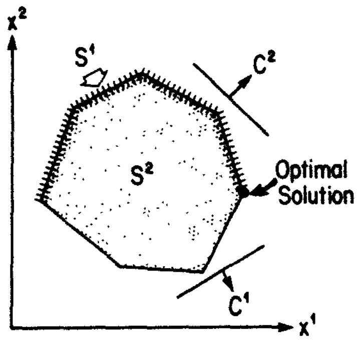

FIGURE 4.1
A Two-Level Linear Resource Control Problem.

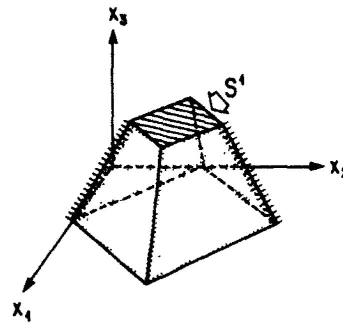

FIGURE 4.2
A Rational Reaction Set in Three Dimensions.

In problems of higher dimensions, the rational reaction set is composed of edges and faces of the convex polyhedron, S^2. In the following three-dimensional example:

$$\max_{(x_2)} x_2 + x_3 \qquad \text{where } (x_1, x_3) \text{ solves:} \qquad (P^1)$$

$$\max_{(x_1,x_3)} x_3 \qquad (P^2)$$

$$\begin{aligned} \text{st:} \quad x_1 + x_2 + x_3 &\leqq 3, \\ x_1 + x_2 - x_3 &\geqq 1, \\ x_1 - x_2 + x_3 &\leqq 1, \\ -x_1 + x_2 + x_3 &\leqq 1, \\ x_3 &\leqq 1/2, \\ x_i \geqq 0, \quad i &= 1,2,3, \end{aligned}$$

the rational reaction set is the hatched nonconvex set shown in Figure 4.2.

5. The Geometry of the Two-Level Linear Resource Control Problem

Although the set of rational reactions for level two in the linear resource control problem may be nonconvex, it does possess some of the important properties of convex sets. This section will highlight some of the currently known results regarding the geometric properties of this problem. These features will be exploited by some of the algorithms in the next section.

The following theorem and its corollaries help to characterize both S^1 and the optimal solution for P^1 in the two-level linear resource control problem. Their proofs appear in Bialas and Karwan (1982).

THEOREM 5.1. *Suppose $S^2 = \{x \mid Ax \leqq b, x \geqq 0\}$ is bounded. Let $S^1 = \Psi_{c^2x}(S^2)$. Let $y_1, \ldots, y_r$ be any r points of S^2 and $\lambda_1, \ldots, \lambda_r > 0$ be scalars with $\sum_{i=1}^r \lambda_i = 1$, such that $\sum_{i=1}^r \lambda_i y_i \in S^1$. Then $y_i \in S^1$ for all $i = 1, \ldots, r$.*

The set, S^1, possesses a weak convex-like property with respect to the set S^2. Any points of S^2 which strictly contribute in a convex combination to form a point in S^1 must also be elements of S^1. Since this is true of any point of S^2, it must be true for the extreme points of S^2 which results in the following corollary:

COROLLARY 5.2. *If x is an extreme point of S^1, then x is an extreme point of S^2.*

Using the above results, one can conclude that S^1 is a very special portion of the boundary of S^2. Using the above corollary and noting that the two-level linear resource control problem may be reformulated as $\max c^1x$ st: $x \in S^1$ the following result is obtained:

COROLLARY 5.3. *An optimal solution to the two-level linear resource control problem (if one exists) occurs at an extreme point of the constraint set of all variables,* S^2.

This is an important result since it justifies extreme point search procedures as a basis for algorithms to solve the two-level linear resource control problem. An alternate method of proof to Corollary 5.3 is offered by Candler and Townsley (1982).

6. Algorithms for Two-Level Linear Programming

This section will present several methods to solve two-level linear programming problems. Most of the algorithmic development, to date, has been devoted to the two-level linear resource control problem which can be written as

$$\max_{x^1} c^{11}x^1 + c^{12}x^2 \qquad \text{where } x^2 \text{ solves:} \qquad (P^1)$$

$$\max_{x^2} c^{21}x^1 + c^{22}x^2 \qquad (P^2)$$

$$\begin{aligned} \text{st:}\quad A^1x^1 + A^2x^2 &= b,\\ x^1 \qquad &\geqq 0,\\ x^2 &\geqq 0. \end{aligned} \qquad (6.1)$$

Note that if the constraints $Ax = b$ originally arose from inequalities, we can without loss of generality assign all slack variables to the control of level two. Furthermore, since x^1 is fixed when P^2 is solved by level two, we can assume $c^{21} = 0$.

Four algorithmic approaches will be presented. Although the discussion will center on the problem given in (6.1), some of the procedures can also be used to solve the two-level linear price control problem and hybrids of the two formulations. Two of the methods can only guarantee local optimal solutions to the nonconvex programming problems. The remaining two provide global optimal solutions. The local optimal procedures are important because of their computational efficiency and their employment within algorithms to find global optima.

Algorithm L1

Assume $S^2 = \{x \mid Ax = b,\ x \geqq 0\}$ is bounded with no degeneracy. Furthermore, assume P^2 has a unique solution for each feasible x^1. Algorithm $L1$ will obtain a local optimal solution to the two-level linear resource control problem, P^1. On rare occasions, it can identify a global optimal solution.

This algorithm uses most of the standard tools of the simplex method for bounded variables. Let the jth column of A be denoted by a_j, and let $B \subseteq A$ be a basis. Let c_B^1 represent those elements of c^1 which correspond to B. Since we will be imposing simple bounds on x^1, let $x_{N_1}^1, x_{N_2}^1, \ldots, x_{N_k}^1$ represent those nonbasic level-one variables which are at nonzero values. Let $r_j = c_B^1 B^{-1}a_{N_j} - c_{N_j}^1$ $(j = 1, \ldots, k)$ denote the reduced cost of nonbasic variable $x_{N_j}^1$ with respect to c^1. Also, in the spirit of the simplex method, define

$$y_j \equiv B^{-1}a_j \qquad (j = 1, \ldots, n), \qquad \bar{b} \equiv B^{-1}b - \sum_{j=1}^{k} B^{-1}a_{N_j}x_{N_j}^1.$$

Under the conditions stated above, the following algorithm will obtain a local optimal solution:

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Step 1. Solve the following problem via the simplex method:

$$\max c^1x \qquad \text{st:}\quad Ax = b, \quad x \geqq 0,$$

with optimal solution $\hat{x} = (\hat{x}^1, \hat{x}^2)$.

Step 2. Set $x^1 = \hat{x}^1$, and solve the following problem via bounded simplex ($l = u = \hat{x}^1$), beginning with basic feasible solution $(\hat{x}^1, \hat{x}^2)$:

$$\max c^2x \qquad \text{st:}\quad Ax = b, \quad x^1 = \hat{x}^1, \quad x^2 \geqq 0.$$

Let the optimal solution be $\bar{x}$. If $\bar{x} = \hat{x}$, stop; $\hat{x}$ is a *global* optimal solution. Otherwise, go to Step 3a with current basis $\bar{B}$, and relax the constraints $x^1 = \hat{x}^1$.

Step 3a. If all nonbasic variables are equal to zero, go to Step 4 with current basis $\bar{B}$. Otherwise, go to Step 3b.

Step 3b. If $\bar{b}_i > 0$ for all i, then go to Step 3c. Otherwise consider $\bar{b}_s = 0$. Choose y_{s,N_j} such that $1 \leqq j \leqq k$ and $y_{s,N_j} \neq 0$. Bring $x^1_{N_j}$ into the basis via a degenerate pivot. Go to Step 3a.

Step 3c. Consider any nonbasic level one variable which is at a strictly positive value, say $x^1_{N_j}$. If $r_j \leqq 0$, then increase the value of $x^1_{N_j}$ bringing it into the basis until another variable is forced out of the basis. If $r_j > 0$, then decrease the value of $x^1_{N_j}$ until it either reaches zero, or forces another variable out of the basis. Go to Step 3a.

Step 4. Beginning with the current basis $\bar{B}$, solve the following problem via a modified simplex procedure:

$$\max c^1x \qquad \text{st:}\quad Ax = b, \quad x \geqq 0.$$

The modification is as follows: Given a candidate to enter the basis (one for which c^1x will increase), only allow it to enter if the resulting basic solution, $\tilde{x}$, will be contained in S^1. This is determined by obtaining the solution, x^*, to the following linear programming problem:

$$\max c^2x \qquad \text{st:}\quad A^2x^2 = (b - A^1\tilde{x}^1), \quad x^1 = \tilde{x}^1, \quad x^2 \geqq 0.$$

If $x^* = \tilde{x}$, then enter the candidate into the basis. Repeat Step 4 until no more candidates exist which satisfy the above conditions, then stop. The resulting solution is a local optimal solution to P^1.

A complete proof of convergence of the above procedure is provided by Bialas and Karwan (1978). The algorithm begins by finding the maximum of c^1x over the entire feasible region, S^2. A check in Step 2 determines if the resulting solution is also an element of S^1. If it is, the algorithm terminates with a *global* (not simply local) optimal solution. Otherwise, the algorithm finds a point, $\bar{x}$, in S^1 whose x^1 coordinates are the same as the solution in Step 1.

The purpose of Step 3 is to relax the constraint $x^1 = \bar{x}^1$, and move to an extreme point, x^*, of S^1 such that $c^1x^* \geqq c^1\bar{x}$. The vector $\bar{x}$ is guaranteed to be an element of S^1 simply by its definition. The extreme point x^* must also be in S^1 by Theorem 5.1 since x^* positively contributes in a convex combination which forms $\bar{x}$.

Once the algorithm has found an extreme point in S^1, it then moves among extreme points of S^1, never allowing c^1x to decrease. The algorithm terminates with an extreme point solution in S^1 which has the property that all adjacent extreme points either lead to a decrease in c^1x or do not belong to S^1. Thus a local optimal solution is obtained.

Algorithm L2

Algorithm *L2* takes a different approach to find local, and potentially global, optimal solutions. Again assume S^2 is bounded, and a unique solution exists for P^2 for

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any feasible x^1. For any fixed x^1, level two will choose that value of x^2 which solves

$$\max c^{21}x^1 + c^{22}x^2 \qquad (P^2)$$

$$\text{st:} \quad A^2x^2 = (b - A^1x^1), \quad x^2 \geqq 0,$$

where $c^{21}x^1$ is a constant.

Under the usual linear programming conditions, a basis $B \subset A^2$ can be an optimal basis for P^2 only if

$$c_B^2 B^{-1}a_j - c_j^2 \geqq 0 \qquad \text{for all} \quad j = 1, \ldots, N(2) \tag{6.2}$$

where c_B^2 is a vector of elements of c^2 corresponding to the basis B, the vector a_j is the jth column of A^2 ($j = 1, \ldots, N(2)$), and $N(2)$ is the number of level-two variables, including slacks.

Let $\mathscr{B} \equiv \{B_1, B_2, \ldots, B_L\}$ be the collection of all bases which satisfy condition (6.2). An element of $\mathscr{B}$ is called a *level-two optimal basis*.

In addition to satisfying (6.2), for any fixed x^1, level two will choose a feasible basis $\hat{B} \in \mathscr{B}$, i.e., $\hat{B}^{-1}(b - A^1x^1) \geqq 0$. Knowing these rational reactions of level two, level one can solve L linear programming problems, P_l^1 ($l = 1, \ldots, L$), of the form

$$\max c^{11}x^1 + c^{12}B_l^{-1}(b - A^1x^1) \qquad (P_l^1)$$

$$\text{st:} \quad B_l^{-1}(b - A^1x^1) \geqq 0, \quad x^1 \geqq 0.$$

Solving P^1 is then equivalent to finding the maximum value of $c^{11}x^1 + c^{12}B_l^{-1}(b - A^1x^1)$ for all P_l^1 ($l = 1, \ldots, L$). Note that the linear constraint set implies that the region in which a given choice of x^1 yields the same level-two optimal basis is a convex set. The following algorithm searches adjacent regions of this type to find a local optimal solution:

Step 1. Set $k = l = 1$, $\bar{z} = -\infty$, and choose an initial basis, B_1, for P^2 which satisfies $c_B^2 B_1^{-1}a_j - c_j^2 \geqq 0, j = 1, \ldots, N(2)$.

Step 2. Solve the problem P_l^1 and denote its maximum objective function value by z_l. If $z_l > \bar{z}$, then replace $\bar{z}$ by z_l, set $k = l$ and go to Step 3. Otherwise set $q = q + 1$ and go to Step 4. (Note that this second branch is impossible at the first iteration when q is undefined.)

Step 3. Compute $y_j = B_k^{-1}a_j$ and $c_{B_k}^2 y_j - c_j^2$ for $j = 1, 2, \ldots, N(2)$, i.e., compute the corresponding level-two tableau. Generate a list of pivot elements for $q = 1, 2, \ldots, q_k$ which lead to adjacent optimal level-two bases via degenerate dual simplex pivots (see the following discussion). Set $q = 1$ and go to Step 4.

Step 4. If $q = q_k + 1$ (all adjacent level-one bases have been considered with no improvement in $\bar{z}$), stop. Otherwise, let $l = l + 1$ and find B_l by pivoting on element q of the kth level-two tableau. Go to Step 2.

A brute-force implementation of this algorithm is not recommended. For example, in Step 2, the matrix $B_l^{-1}A^1$ can be obtained by updates from previous iterations.

Step 3 also offers a number of opportunities for computational refinements. Given a basis, B_l, employed in the problem P_l^1, one can construct a list of pivot elements for the corresponding problem P^2 to obtain all adjacent level-two optimal bases. Note that if the values of x^1 resulting from P_l^1 were used in problem P^2, an optimal basis for P^2 would be B_l. Degeneracy in P^2 would correspond to binding constraints in P_l^1. Let T_l be the index set of binding constraints in P^1, i.e., $T_l \equiv \{i \mid (B_l^{-1}(b - A^1x^1))_i = 0\}$. Compute $y_j = B_l^{-1}a_j$ and $c_{B_l}^2 y_j - c_j^2$ for all $j = 1, 2, \ldots, N(2)$, which do not correspond to columns contained in B_l. For each $i \in T_l$, compute $\min_{1 \leqslant j \leqslant N(2)} \{(c_{B_l}^2 y_j - c_j^2)/|y_{ij}| \mid y_{ij} < 0\}$ with the minimum occurring for index $j(i)$, not necessarily unique. Form a list of each combination of i and $j(i)$ for all $i \in T_l$. This list identifies all pivot

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elements in the current representation of P^2 (with basis B_l), which produce adjacent level-two optimal bases corresponding to problem P_l^1.

Of course, if one wished, a complete enumeration could be conducted to find all level-two optimal bases, and thus solve the problem P^1 with a global optimal solution. Candler and Townsley (1982) developed necessary conditions for a new level-two basis to allow for an improvement in the level-one objective function. They are then able to limit such a search to an implicit enumeration of all level-two optimal bases.

The "Kth-Best" Algorithm

Assume S^2 is bounded, and a unique solution exists for P^2 for any feasible x^1. From Corollary 5.2, a solution to P^1 must occur at an extreme point of S^2. Let $\hat{x}_{[1]}$, $\hat{x}_{[2]}, \ldots, \hat{x}_{[N]}$ denote the N ordered basic feasible solutions to the linear programming problem

$$\max c^1x \qquad \text{st:} \quad Ax = b, \quad x \geqq 0 \tag{6.3}$$

such that $c^1\hat{x}_{[i]} \geqq c^1\hat{x}_{[i+1]}$ $(i = 1, \ldots, N-1)$. Then solving P^1 is equivalent to finding the index $K^* \equiv \min\{i \in \{1, \ldots, N\} \mid \hat{x}_{[i]} \in S^1\}$ yielding the global optimal solution $\hat{x}_{[K^*]}$. This requires finding the (K^*)th best extreme point solution to the problem given in (6.3).

The "Kth-Best" Algorithm performs this search and thus obtains a global optimal solution.

Step 1. Set $i = 1$. Solve problem (6.3) with optimal solution $\hat{x}_{[i]}$ via the simplex method. Let $W = \{\hat{x}_{[i]}\}$ and $T = \emptyset$. Go to Step 2.

Step 2. Solve the following linear programming problem via the bounded simplex method:

$$\max c^2x \qquad \text{st:} \quad Ax = b, \quad x^1 = \hat{x}^1_{[i]}, \quad x^2 \geqq 0. \tag{6.4}$$

Let $\tilde{x}$ denote the optimal solution to (6.4). If $\tilde{x} = \hat{x}_{[i]}$, stop; $\hat{x}_{[i]}$ is a global optimal solution with $K^* = i$. Otherwise, go to Step 3.

Step 3. Let $W_{[i]}$ denote the set of extreme points x which are adjacent to $\hat{x}_{[i]}$ and such that $c^1x \leqq c^1\hat{x}_{[i]}$. Let $T = T \cup \{\hat{x}_{[i]}\}$ and $W = (W \cup W_{[i]}) \cap T^c$. Go to Step 4.

Step 4. Set $i = i + 1$ and choose $\hat{x}_{[i]}$ so that $c^1\hat{x}_{[i]} = \max_{x \in W}\{c^1x\}$. Go to Step 2.

Step 2 repeatedly tests extreme points to determine if they are elements of S^1. Since each successive pair of points is adjacent, this process can be efficiently carried out by dual simplex. Also for the same reason, a simple implementation of this algorithm can consist of storing the basis column indices for the first $(K-1)$ best points, and then determining which basis adjacent to these yields the maximum level-one objective function value. Another approach would actually store column indices and reduced costs from the $(K-1)$ best bases. The computational performance of this latter method is presented in §7 along with a sample problem.

The Kuhn–Tucker Approach

Another solution technique for the two-level linear resource (and price) control problem replaces the level-two problem, P^2, by its Kuhn–Tucker conditions (Bard and Falk 1982, Bialas, *et al.* 1980, Fortuny and McCarl 1981). The two-level problem

$$\max_{x^1} c^{11}x^1 + c^{12}x^2 \qquad \text{where } x^2 \text{ solves:} \tag{P^1}$$

$$\max_{x^2} c^{21}x^1 + c^{22}x^2 \tag{P^2}$$

$$\begin{aligned} \text{st:} \quad A^1x^1 + A^2x^2 &\leqq b, \\ x^1 &\geqq 0, \\ x^2 &\geqq 0, \end{aligned}$$

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can be equivalently written as

$$\max c^{11}x^1 + c^{12}x^2 \qquad (KT)$$

$$\text{st:} \quad A^1x^1 + A^2x^2 + u = b, \quad A^2w - v = c^{22},$$

$$wu = 0, \quad x^2v = 0, \quad x, u, v, w \geqq 0, \qquad (6.5)$$

where vectors and matrices are transposed as necessary to form inner products. Note that the third and fourth sets of constraints are complementary slackness conditions which make (KT) a nonconvex programming problem.

Bard and Falk (1982) solve problem (6.5) using a nonconvex programming algorithm based on branch and bound techniques. The feasible region is enclosed within a linear polyhedron which is then partitioned. A global solution is obtained to a piecewise linear approximation to problem (KT). However, limited computational experience is reported, and it is anticipated that only small problems could be solved with reasonable computational resources.

Fortuny and McCarl (1981), who also include the price control aspects in the level two problem, enforce the complementarity conditions by transforming the formulation (6.5) into a much larger mixed integer programming problem:

$$\begin{aligned} &\max c^{11}x^1 + c^{12}x^2 && (FM)\\ &\text{st:} \quad A^1x^1 + A^2x^2 + u = b, \\ &\qquad A^2w - v = c^{22}, \\ &\qquad w \leqq M\eta, \quad u \leqq M(1-\eta), \\ &\qquad x^2 \leqq M\xi, \quad v \leqq M(1-\xi), \\ &\qquad x, u, v, w \geqq 0, \quad \eta, \xi \text{ binary.} \end{aligned}$$

In this method, a zero-one variable, η_j or ξ_k, is added for each constraint $w_ju_j = 0$ or $x_k^2v_k = 0$, respectively. In addition, each of these constraints is replaced by two linear inequalities involving η or ξ, and M, a large positive constant.

Fortuny and McCarl do not offer empirical evidence to demonstrate the efficiency of their procedure. However, the large size of the augmented problem, (FM), required for this approach may limit its application. Fourer (1981) has also pointed out that if M is chosen large enough, some integer programming solution methods just reduce to an enumeration (implicit or explicit) of all the combinations of $w_j = 0$ or $u_j = 0$ and $x_k^2 = 0$ or $v_k = 0$.

The Kuhn-Tucker method has been used by Bialas *et al.* (1980) in their Parametric Complementary Pivot (PCP) Algorithm. The algorithm considers the following system of equalities and inequalities which is similar to problem (6.5):

$$A^1x^1 + A^2x^2 + u = b, \qquad -\epsilon Hx^1 + A^2w - v = c^{22}, \qquad (KT')$$

$$c^{11}x^1 + c^{12}x^2 \geqq \alpha, \qquad wu = x^2v = 0, \qquad x, u, v, w \geqq 0. \qquad (6.6)$$

The matrix, H, may be any negative definite matrix, and ϵ is a suitably small positive scalar so that the term ϵH leads only to a small perturbation of the problem (6.5). The implementation of the algorithm which produced the results in §7 used $H = -I$ and it was found that values of ϵ ranging from 10^{-2} to 10^{-4} times the absolute average parameter value rarely changed the final solution.

If $c^{12} \geqq 0$, then a solution to system (6.6) can be obtained (if one exists) by using a restricted basis entry simplex procedure which implicitly enforces $wu = x^2v = 0$. A solution to (6.6) is a point in S^1 with the level one objective function greater than or equal to α. If $c^1 \geqq 0$, the initial choice of α may be zero. Once a solution to (KT') is obtained, the value of α can be parametrically increased until the current complementary basis for (KT') is no longer feasible. Then the *PCP* Algorithm finds another complementary basis to satisfy (KT').

Wolfe (1959) used a restricted basis entry simplex procedure to solve quadratic programming problems after they were placed in the form of their Kuhn-Tucker conditions. These conditions include complementarity conditions which are implicitly enforced by a two-phase scheme. In the *PCP* Algorithm, formulation (KT) has an objective function for level one, as well as the set of equalities representing the Kuhn-Tucker conditions for the second level problem. However, the Parametric Complementary Pivot Algorithm takes advantage of Wolfe's approach by adding the cut $c^{11}x^1 + c^{12}x^2 \geqq \alpha$ and increasing α parametrically to improve the level one objective, c^1x.

The following steps describe a simplified version of the Parametric Complementary Pivot Algorithm which, in the absence of degeneracy and multiple level-two optima, will obtain a global optimal solution to (6.1):

Step 0. Let $\alpha = \alpha_{\min}$, where $\alpha_{\min}$ is a lower bound for c^1x.

Step 1. Obtain a feasible solution to

$$\begin{aligned} A^1x^1 + A^2x^2 \qquad + u \qquad &= b, \\ -c^{11}x^1 - c^{12}x^2 \qquad\qquad + s \quad &= -\alpha, \\ -\epsilon Hx^2 + A^2w \qquad\quad - v &= c^{22}, \end{aligned} \tag{6.7}$$

using the first phase of a two-phase simplex procedure under the side conditions:

(a) if x_k^2 is in the basis and is not the leaving variable for the current pivot, do not admit v_k into the basis, and vice versa, and

(b) if u_i is in the basis and is not the leaving variable for the current pivot, do not admit w_i into the basis, and vice versa.

If a feasible solution exists, go to Step 2. Otherwise go to Step 3.

Step 2. Record $x_* = (x_*^1, x_*^2)$ as the incumbent solution. Let B denote the current basis and let $\beta = (b, -\alpha, c^{22})$ represent the right-hand side vector in (6.7). Let $\bar{\beta} = B^{-1}\beta$ and let the column corresponding to s in the simplex tableau be the kth column denoted by y_k. (Note y_k is simply $B^{-1}e_{m+1}$ where e_{m+1} is a unit vector.) Choose

$$\Delta\alpha = \min_i \left\{ \bar{\beta}_i / y_{ki} \,|\, y_{ki} > 0 \right\}.$$

Let $\alpha = (\alpha + \Delta\alpha)(1 + \gamma)$ where γ is a suitably small positive scalar. Go to Step 1.

Step 3. Stop; x_* is a solution which is feasible to S^1 and c^2x_* is within $\gamma \times 100\%$ of the optimal objective function value.

As already mentioned, this is a simplified version of the algorithm. In the actual implementation of the procedure described by Bialas *et al.* (1980), the system (6.7) need not be completely re-solved at each iteration. Instead, a complementary pivoting rule is employed which permits the use of the incumbent complementary basis. The theorem which guarantees global optimality is similar to a theorem by Wolfe [33].

The Parametric Complementary Pivot Algorithm can be viewed as an implicit enumeration of the level-two optimal bases. However, it does not require the use of a branch and bound technique employed in other procedures. All of the computations for the *PCP* Algorithm may be performed within the framework of a simplex-like tableau whose size is roughly that of system (6.7). In addition, this method can be easily extended to solve hybrid two-level linear resource and price control problems

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(Bialas *et al.* 1980). An example of its use to solve resource control problems is presented in the next section.

7. A Sample Problem and Computational Experience

Consider the following two-level linear resource control problem which will be solved using the "*K*th-Best" and *PCP* Algorithms (see Figure 7.1):

$$\begin{aligned}
&\max_{(x_1)} x_2\\
&\text{where } x_2 \text{ solves:}\\
&\max_{(x_2)} -x_2\\
&\text{st:}\quad -x_1-2x_2 \leqq -10,\\
&\qquad x_1-2x_2 \leqq 6,\\
&\qquad 2x_1-x_2 \leqq 21,\\
&\qquad x_1+2x_2 \leqq 38,\\
&\qquad -x_1+2x_2 \leqq 18,\\
&\qquad x_1, x_2 \geqq 0.
\end{aligned}\tag{7.1}$$

In this problem, level one controls the vector $x^1 = (x_1)$ and level two controls $x^2 = (x_2)$.

Using the "Kth-Best" Algorithm

The first step of the "*K*th-Best" Algorithm is to solve the linear programming problem

$$\begin{aligned}
&\max x_2\\
&\text{st:}\quad -x_1-2x_2 \leqq -10,\\
&\qquad x_1-2x_2 \leqq 6,\\
&\qquad 2x_1-x_2 \leqq 21,\\
&\qquad x_1+2x_2 \leqq 38,\\
&\qquad -x_1+2x_2 \leqq 18,\\
&\qquad x_1, x_2 \geqq 0.
\end{aligned}$$

with solution $\hat{X}_{[1]} = (10, 14)$, the "first best" solution. Set $W = \{(10, 14)\}$ and $T = \emptyset$.

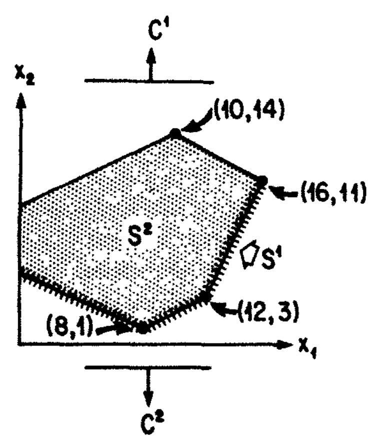

FIGURE 7.1 A Graphical Representation of Problem (7.1).

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The algorithm then tests to determine if $\hat{x}_{[1]}$ is an element of S^1 by solving

$$\begin{aligned} &\max -x_2 \\ &\text{st:}\quad -x_1 - 2x_2 \leqq -10, \\ &\qquad x_1 - 2x_2 \leqq 6, \\ &\qquad 2x_1 - x_2 \leqq 21, \\ &\qquad x_1 + 2x_2 \leqq 38, \\ &\qquad -x_1 + 2x_2 \leqq 18, \\ &\qquad x_1 = 10, \\ &\qquad x_1, x_2 \geqq 0. \end{aligned}$$

In this case, the resulting solution is $\tilde{x} = (10, 2) \neq \hat{x}_{[1]}$. Hence, $\hat{x}_{[1]} \notin S^1$.

In Step 3, $W_{[1]} = \{(16, 11), (0, 9)\}$, $T = \{(10, 14)\}$, and $W = \{(16, 11), (0, 9)\}$. In Step 4, $\hat{x}_{[2]}$ is chosen to equal (16, 11) as the "second best" solution, and the algorithm returns to Step 2.

Now solving the linear programming problem

$$\begin{aligned} &\max -x_2 \\ &\text{st:}\quad -x_1 - 2x_2 \leqq -10, \\ &\qquad x_1 - 2x_2 \leqq 6, \\ &\qquad 2x_1 - x_2 \leqq 21, \\ &\qquad x_1 + 2x_2 \leqq 38, \\ &\qquad -x_1 + 2x_2 \leqq 18, \\ &\qquad x_1 = 16, \\ &\qquad x_1, x_2 \geqq 0, \end{aligned}$$

yields a solution $\tilde{x} = (16, 11) = \hat{x}_{[2]}$. Hence, $x_* = (16, 11)$ is a global optimal solution to problem (7.1).

Using the PCP Algorithm

The Parametric Complementary Pivot Algorithm examines the system of equations

$$\begin{aligned} -x_1 - 2x_2 + u_1 &= -10, \\ x_1 - 2x_2 + u_2 &= 6, \\ 2x_1 - x_2 + u_3 &= 21, \\ x_1 + 2x_2 + u_4 &= 38, \\ -x_1 + 2x_2 + u_5 &= 18, \\ -x_2 + s &= -\alpha, \\ -\epsilon x_2 - 2w_1 - 2w_2 - w_3 + 2w_4 + 2w_5 - v_1 &= -1, \\ u_1 w_1 = u_2 w_2 = u_3 w_3 = u_4 w_4 = u_5 w_5 = x_2 v_1 &= 0, \\ x, u, s, w, v &\geqq 0, \end{aligned} \tag{7.2}$$

with $\epsilon = 10^{-8}$. Since $c^1 \geqq 0$, we may choose $\alpha_{\min} = 0$. In Step 1, a solution to (7.2) with $\alpha = 0$ is

$$x = (8, 1), \qquad w = (0.5, 0, 0, 0, 0),$$
$$u = (0, 0, 6, 28, 24), \qquad v = (0),$$
$$s = (1),$$

where the value of w is being expressed only with an accuracy of three decimal places. In Step 2, $\Delta\alpha = 1$ and α is set to $(\alpha + \Delta\alpha)(1 + 0.01) = 1.01$. Here, the value of γ is 0.01.

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Obtaining a solution to (7.2) with the new value of α results in

$$x = (12, 3), \qquad w = (0, 0.5, 0, 0, 0),$$
$$u = (8, 0, 0, 20, 24), \qquad v = (0).$$
$$s = (1.99),$$

Computing $\Delta\alpha = 1.99$ and resetting α to $(1.01 + 1.99)(1.01) = 3.03$, we re-solve system (7.2) once again to obtain

$$x = (16, 11), \qquad w = (0, 0, 1, 0, 0),$$
$$u = (28, 12, 0, 0, 12), \qquad v = (0),$$
$$s = (7.97),$$

with $\Delta\alpha = 7.97$ and $\alpha = 11.11$.

Returning to Step 1 one more time, we find that no solution to (7.2) exists for $\alpha = 11.11$. Hence, the incumbent solution, $x_* = (16, 11)$ solves the two-level problem.

Computational Experience

A sample of the computational experience with the "*K*th-Best" and *PCP* Algorithms is shown in Table 7.1. The execution times for 40, randomly generated, two-level problems are shown for both algorithms written in FORTRAN IV on a CDC Cyber 174 under the NOS 1.4 operating system (Bialas *et al.* 1980).

The performance for both algorithms is apparently affected by the fraction of variables controlled by level two. This is a reasonable conjecture, since one would surmise that two-level problems involving fewer level-two variables should be "easier" to solve. In such cases, the preemptive control exercised by level one can produce an optimal solution which is closer to $\hat{x}_{[1]}$, the "first best" solution.

TABLE 7.1

Comparison of Computation Times for the "Kth-Best" and PCP Algorithms

Number of Variables[a]	% Variables Controlled by Level 2	Algorithm	Problem Number[b] 1	2	3	4	5
20	30	*PCP*	0.17	1.38	0.20	0.14	0.40
		*K*th-Best	0.12	0.92	0.15	0.09	0.09
30	30	*PCP*	4.28	5.40	0.29	5.96	0.23
		*K*th-Best	66.79	(5.0%)	0.21	(9.8%)	0.15
40	30	*PCP*	10.08	0.50	0.77	0.57	10.39
		*K*th-Best	2.07	0.41	0.64	0.45	5.44
50	30	*PCP*	0.93	16.01	27.28	18.00	20.16
		*K*th-Best	0.18	(10.8%)	31.14	(4.6%)	(17.2%)
20	40	*PCP*	1.48	0.17	0.16	4.28	1.65
		*K*th-Best	0.28	0.11	0.09	12.65	0.22
30	40	*PCP*	3.49	0.24	3.78	0.30	5.55
		*K*th-Best	0.59	0.17	0.30	0.24	73.86
40	40	*PCP*	0.72	18.48	11.44	11.30	14.21
		*K*th-Best	0.61	(12.1%)	0.79	7.35	1.62
50	40	*PCP*	0.79	35.58	31.15	13.84	1.27
		*K*th-Best	0.62	(2.7%)	(9.3%)	(2.6%)	1.04

Notes: (a) Number of constraints = 0.4 × (Number of Variables).

(b) The parentheses, ( ), denote problems for which the "*K*th-Best" Algorithm exceeded a 80 CPU-second time limit. In all such cases, upper and lower bounds were obtained for the global optimal solution, and a feasible, near-optimal solution provided. The number in the parentheses is the maximum percentage deviation between the near-optimal and optimal objective function values.

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Experience with the "*K*th-Best" Algorithm has demonstrated that it can be erratic. Quite often it will perform as well as or better than the *PCP* Algorithm, while, for some problems, it fails to find a solution given reasonable computer time. The reasons for this are largely speculative, but are likely due once again to the relative position of the solution to $\hat{x}_{[1]}$ and the degree of "control" exercised by level one over the entire problem. In this light, the *PCP* Algorithm might be seen as a more reliable approach for solving two-level programming problems.

8. Conclusions and Beginnings

This paper has presented an approach to model and solve certain multilevel decision-making problems. The proposed formulation partitions control over decision variables among ordered levels within a hierarchical planning structure. The planner at each level attempts to maximize his individual objective function which may depend, in part, on variables controlled at other levels. His control instruments may allow him to influence rather than dictate the policies at other levels, and thereby improve his own objective function. The ultimate execution of the decisions is sequential, from highest to lowest level. The philosophy behind this approach is not new, and we have attempted to place it in perspective with its predecessors.

The discussion here has focused on the two-level linear resource control problem. Although the actual problem is a nonconvex programming problem, the feasible region possesses properties which make the solution to the problem tenable. Four algorithms have been described to produce such solutions; two of them can provide local optimal solutions, while the remaining two yield global optima.

The most obvious challenge is to develop methods to solve more general multilevel planning problems. Even for the n-level linear case, the geometry is more complex than that suggested by the results of §5 for two levels. In addition, there is a need to relax the linearity assumptions demanded for the algorithms presented here. Another natural extension would restrict the variables to integers to model problems similar to the one posed by Cassidy *et al.* (1971).

Another interesting feature of multilevel programming problems also requires further study: their solutions may not be Pareto-optimal. That is, there may exist feasible decisions where some levels can increase their objective function values without decreasing the objective function of any level. This is not a shortcoming of the formulation. Instead, this implies that for those systems which can be represented by a multilevel programming problem, the resulting behavior may be economically inadmissible. There is, of course, widespread anecdotal evidence that the decisions arising from hierarchical bureaucracies are often inefficient. Mechanisms for obtaining Pareto-optimal solutions through the formation of coalitions among levels have been investigated by Bialas and Chew (1982).

There is no claim that this approach will be appropriate for all multilevel decision-making problems. As of yet, only modest applications of these techniques have been developed. The greatest barrier to the effective use of these concepts is the lack of efficient algorithm procedures to solve the associated mathematical programming problems. It is hoped that this paper helps demonstrate that such problems are tractable.[1]

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