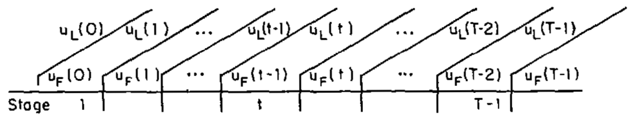

Fig. 4. Decomposition of a multistage problem.

A11):

$$\eta_L(t): x(0),\ldots,x(t), u_L(0),\ldots,u_L(t-1)$$
$$\eta_F(t): x(0),\ldots,x(t), u_F(0),\ldots,u_F(t-1),$$

i.e., each player has perfect memory of the state (x) and his own control history. The point here is that given $x(t)$, $u_L(t-1)$, $x(t-1)$, we can in general calculate $u_F(t-1)$ or vice versa [4]. As shown in Fig. 4, the leader at time t can choose his decision $u_L(t)$ based on $u_F(t-1)$ (or the entire past decisions of the follower), thus L essentially imposes a kind of reversed information structure on F. Note that whoever gets to declare his strategy first becomes the leader. This approach requires separate treatments at $t=T-1$ and 0 by solving $u_F(T-1)$ first which has a permanently optimal solution, and considering $u_F(-1)$ to be fixed at zero as is evident in the works of Basar and Tolwinski. Also, bear in mind that the distinction between closed-loop Stackelberg controls and Stackelberg feedback strategies [1] still exists. With this understanding, closed-loop Stackelberg strategy for linear quadratic deterministic problem can be solved using the basic idea discussed in Section III. It is clear that many strategies γ_L^s are possible due to the enormous flexibility here. For example, L's strategy may punish a nonrational behavior of F for one stage only (as in [4]), two stages, etc., or for the rest of the game (as in [2]). It is thus possible that different γ_L^s may enjoy various advantages.

VIII. Conclusion

In this paper we have identified two reasons why L may be interested in the decision of F. First, knowing F's strategy and his decision may enable L to infer the states of nature [R1)]. Second, F's decision may directly affect L's payoff [R2)]. We then discussed mechanisms by which L can induce F to behave cooperatively. In case R1) the mechanism is to transform F's payoff function so that it looks like L's own. In case R2) it is to make directly any choice of F's strategy other than the cooperative one unpalatable. In either case the crucial requirement is that we have the reversed information structure as defined in Section III. It serves as a unifying ingredient in diverse applications.

Acknowledgment

The authors would like to thank Dr. G. J. Olsder for many insightful comments on this subject.

References

[1] M. Simaan and J. B. Cruz, Jr., "Additional aspects of the Stackelberg strategy in nonzero-sum games," *J. Optimiz. Theory Appl.*, vol. 11, pp. 613–626, June 1973.
[2] T. Basar and H. Selbuz, "Closed-loop Stackelberg strategies with applications in the optimal control of multilevel systems," *IEEE Trans. Automat. Contr.*, vol. AC-24, pp. 166–179, Apr. 1979.
[3] G. P. Papavassilopoulos and J. B. Cruz, Jr., "Nonclassical control problems and Stackelberg games," *IEEE Trans. Automat. Contr.*, vol. AC-24, pp. 155–166, Apr. 1979.
[4] B. Tolwinski, "Closed-loop Stackelberg solution to multi-stage linear-quadratic game," *J. Optimiz. Theory Appl.*, to be published.
[5] M. Spence, "Nonlinear pricing and welfare," *J. Public Econ.*, vol. 8, pp. 1–18, Aug. 1977.
[6] T. Groves and M. Loeb, "Incentives in a divisionalized firm," *Management Sci.*, vol. 25, pp. 221–230, Mar. 1979.
[7] H. von Stackelberg, *The Theory of the Market Economy*. Oxford, England: Oxford Univ. Press, 1952.
[8] C.-I Chen and J. B. Cruz, Jr., "Stackelberg solution for two-person games with biased information patterns," *IEEE Trans. Automat. Contr.*, vol. AC-17, pp. 791–798, Dec. 1972.
[9] M. Simaan and J. B. Cruz, Jr., "On the Stackelberg strategy in nonzero-sum games," *J. Optimiz. Theory Appl.*, vol. 11, pp. 533–555, May 1973.
[10] T. Basar, "Information structures and equilibria in dynamic games," in *New Trends in Dynamic System Theory and Economics*, M. Aoki and A. Marzollo, Eds. New York: Academic, 1978.
[11] T. Basar and G. J. Olsder, "Team-optimal closed-loop Stackelberg strategies in hierarchial control problems," Memorandum NR. 242, Dep. Appl. Math., Twente Univ. of Technol., The Netherlands, Memo. NR 242, Feb. 1979, pp. 1–25.
[12] Y-C. Ho and K-C. Chu, "Team decision theory and information structures in optimal control problems, Part I," *IEEE Trans. Automat. Contr.*, vol. AC-17, pp. 15–22, Feb. 1972.
[13] Y-C. Ho, D. M. Chiu, and R. Muralidharan, "A model for optimal peak load pricing by electric utilities," in *Proc. Lawrence Symp. on Syst. and Decision Sci.*, Berkeley, CA, Oct. 1977, pp. 16–23.
[14] I. Pressman, "A mathematical formulation of the peak-load pricing problem," *Bell J. Econ. Management Sci.*, vol. 1, pp. 304–326, Autumn 1970.
[15] P. Dasgupta, P. Hammond, and E. Maskin, "The implementation of social choice rules: Some general results on incentive compatibility," *Rev. Econ. Studies*, vol. 46, pp. 185–216, Apr. 1979.
[16] T. Basar, "Hierarchical decisionmaking under uncertainty," in *Dynamic Optimization and Mathematical Economics*, P. T. Liu, Ed. New York: Plenum, 1980.
[17] G. P. Papavassilopoulos and J. B. Cruz, Jr., "Sufficient conditions for Stackelberg and Nash strategies with memory," *J. Optimiz. Theory Appl.*, Sept. 1980.
[18] Y-C. Ho, P. B. Luh, and G. J. Olsder, "A control theoretic view on incentives," in *Proc. 4th Int. Conf. on Analysis and Optimiz. Sys.*, INRIA, Versailles, France Dec. 1980; also Springer-Verlag Lecture Notes Series, to be published.

A New Computational Method for Stackelberg and Min–Max Problems by Use of a Penalty Method

KIYOTAKA SHIMIZU and EITARO AIYOSHI

***Abstract*—This paper is concerned with the Stackelberg problem and the min–max problem in competitive systems. The Stackelberg approach is applied to the optimization of two-level systems where the higher level determines the optimal value of its decision variables (parameters for the lower level) so as to minimize its objective, while the lower level minimizes its own objective with respect to the lower level decision variables under the given parameters. Meanwhile, the min–max problem is to determine a min–max solution such that a function maximized with respect to the maximizer's variables is minimized with respect to the minimizer's variables. This problem is also characterized by a parametric approach in a two-level scheme. New computational methods are proposed here; that is, a series of nonlinear programming problems approximating the original two-level problem by application of a penalty method to a constrained parametric problem in the lower level are solved iteratively. It is proved that a sequence of approximated solutions converges to the correct Stackelberg solution, or the min–max solution. Some numerical examples are presented to illustrate the algorithms.**

I. Introduction

This paper is concerned with the Stackelberg problem and the min–max problem in competitive systems.

The Stackelberg solution [14], [12], [13], [8] is the most rational one to answer a question: what will be the best strategy for Player 1 who knows Player 2's objective function and has to choose his strategy first, while Player 2 chooses his strategy after announcement of Player 1's strategy. A problem in the field of competitive economics is one such problem.

The min–max problem [3], [2], [4], [9], [6] is formulated so that a function, maximized with respect to the maximizer's variables, is minimized with respect to the minimizer's variables. The min–max solution is optimal for the minimizer against the worst possible case that might be taken by the opponent (the maximizer). Thus, the min–max concept plays an important role in game theory.

Many articles on the equilibrium solutions, such as the Nash solution and the saddle-point solution, have been published. However, the Stackel-

Manuscript received September 4, 1979; revised February 18, 1980 and June 30, 1980. Paper recommended by A. J. Laub, Chairman of the Computational Methods and Discrete Systems Committee.

The authors are with the School of Engineering, Keio University, 3-14-1 Hiyoshi Kohoku-ku Yokohama, Japan.

0018-9286/81/0400-0460$00.75 ©1981 IEEE

berg and the min–max solutions differ from them in that Player 2 makes his decision after Player 1 does, i.e., a precedence in decision-making order exists. Therefore, both the Stackelberg and the min–max problems can be represented by hierarchical optimization problems, where a part of constraints in the upper level problem consists of a parameterized optimization problem in the lower level. The problems of this type can be solved by a parametric approach in principle. However, that approach causes difficulties in developing an available computational method also. Some computational methods for the min–max problem were proposed in [4], [6], [9], [11], where a gradient type algorithm was discussed [4], [6] and a relaxation method [9], [11]. In [15], a penalty method was applied to calculate a saddle-point solution. However, papers on the Stackelberg and the min–max solutions are comparatively few, most of them limited to separate constraints cases when the constraints for each player do not depend on another player's strategies.

In this paper, we consider the most general problems with unseparate constraints, and propose new computational methods based on the theory of a barrier method (an interior penalty method [1], [5]). Our method is based on that a series of nonlinear programming problems approximating the original hierarchical one are solved iteratively by applying the barrier method to the lower level problem. It proves that a sequence of approximate solutions converges to the true solution for the original problem.

II. Formulation of the Stackelberg Problem

In a two-player game, Player 1 has all information about Player 2's objective function and constraints, while Player 2 knows nothing about player 1 but the strategy announced by Player 1. Then, until Player 1 announces his own optimal strategy, Player 2 cannot solve his optimizing problem responding to the Player 1's strategy. In such a situation, Player 1 has the leadership in playing the game and can decide the best strategy in anticipation of the Player 2's act. The optimal solution for Player 1 with precedence in decision is called a Stackelberg Solution with Player 1 as leader. The Stackelberg solution is the optimal strategy for the leader when the follower reacts by playing optimally.

Let $x \in R^{n_1}$, $f_1 \in R$ and $g_1 \in R^{m_1}$ be the decision variable vector, the objective function and the vector constraint function for Player 1, respectively, while $y \in R^{n_2}$, $f_2 \in R$ and $g_2 \in R^{m_2}$ for Player 2. Two functions, $f_1(x, y)$ and $f_2(x, y)$, map $R^{n_1} \times R^{n_2}$ into the real lines such that Player 1 wishes to minimize f_1 and Player 2 wishes to minimize f_2. Then, Player 2's minimal solution $\check{y}(x)$ responding to a Player 1's strategy x satisfies the following relation:

$$f_2(x, \check{y}(x)) \leqslant f_2(x, y) \quad \text{for } \forall y \in Y \text{ satisfying } g_2(x, y) \leqslant 0. \tag{1}$$

For such $\check{y}(x)$, if there exists x^{s_1} such that

$$f_1(x^{s_1}, \check{y}(x^{s_1})) \leqslant f_1(x, \check{y}(x)) \quad \text{for } \forall x \in X \text{ satisfying } g_1(x, \check{y}(x)) \leqslant 0, \tag{2}$$

x^{s_1} is called the Stackelberg solution for Player 1 as a leader and $y^{s_1} \equiv \check{y}(x^{s_1})$ is an optimal solution for Player 2 in response to x^{s_1}. In other words, the Stackelberg solution x^{s_1} can be defined to be a solution to the following two-level decision problem:

$$\min_x f_1(x, \check{y}(x)) \tag{3a}$$

$$\text{subject to } x \in X \tag{3b}$$

$$g_1(x, \check{y}(x)) \leqslant 0 \tag{3c}$$

$$f_2(x, \check{y}(x)) = \min_y f_2(x, y) \tag{3d}$$

$$\text{subject to } y \in Y \tag{3e}$$

$$g_2(x, y) \leqslant 0 \tag{3f}$$

where $\check{y}(x)$ denotes a parametric minimal solution to the lower level problem (3d)–(3f). Here, we assume existence of $(x, \check{y}(x))$ satisfying (3b)–(3f). For simplicity of notation, we shall denote x^{s_1} and y^{s_1} with x^s and y^s, respectively, in the following section.

III. Approximate Technique Using a Barrier Method for the Stackelberg Solution

Our approach to the problem (3) begins with replacing the lower level problem by an unconstrained problem based on the theory of a barrier method. Then, the parametric minimization problem (3d)–(3f) is transformed into an unconstrained parametric minimization problem with the augmented objective function P^r combined with the constraint functions. The parametric solution $\check{y}(x)$, which is regarded as a function of x, can be approximated by an implicit function satisfying a stationary condition for the resulting unconstrained lower level problem. A sequence of approximated problems will be solved by use of appropriate nonlinear programming techniques.

Let the feasible region Y be given with the vector function $h_2 \in R^{q_2}$ as $Y = \{y \mid h_2(y) \leqslant 0\}$, and let

$$S(x) = \{y \mid h_2(y) \leqslant 0, g_2(x, y) \leqslant 0\}.$$

We begin with imposing the following assumption:

a) int $S(x)$, the interior of $S(x)$, which is given by

$$\operatorname{int} S(x) = \{y \mid h_2(y) < 0, g_2(x, y) < 0\},$$

is not empty for any fixed x and its closure becomes $S(x)$.

Let us define the augmented objective function on $x \times \operatorname{int} S(x)$, where x is any point in R^{n_1}, as

$$P^r(x, y) = f_2(x, y) + r\phi(g_2(x, y), h_2(y)) \tag{4}$$

where $r > 0$ and ϕ is such a continuous function defined on the negative domain that

$$\phi(g_2(x, y), h_2(y)) \geqslant 0, \qquad \text{as } y \in \operatorname{int} S(x)$$

$$\phi(g_2(x, y), h_2(y)) \to +\infty, \qquad \text{as } y \to \partial S(x).$$

Here, $\partial S(x)$ denotes the boundary of $S(x)$.

We consider an unconstrained minimization problem with the augmented objective function:

$$\min_y P^r(x, y) \tag{5}$$

in place of the constrained minimization problem:

$$\min_y f_2(x, y) \quad \text{subject to } y \in S(x). \tag{6}$$

In order to apply the theory of a barrier method, we impose the following assumptions.

b) The functions $f_2(x, y)$ and $g_2(x, y)$ are continuous at any $(x, y) \in R^{n_1} \times R^{n_2}$ and $h_2(y)$ is continuous at any $y \in R^{n_2}$.

c) The set Y is compact. This implies that $S(x)$ is also compact.

Then, it is assured that the problems (5) and (6) have their optimal solutions $\check{y}^r(x) \in \operatorname{int} S(x)$ and $\check{y}(x) \in S(x)$, respectively, for any fixed x. Let us consider a sequence of the optimal solutions $\{\check{y}^{r^k}(x)\}$ for the problem (5) in response to a positive parameter sequence $\{r^k\}$ strictly decreasing to zero. When the parameter x is fixed, it follows directly from a theory of a barrier method (an interior penalty function method [1], [5]) that any accumulation point of the sequence $\{\check{y}^{r^k}(x)\}$ is optimal for the problem (6). On the other hand, when the parameter x varies as a sequence $\{x^k\}$, the convergence about $\{\check{y}^{r^k}(x^k)\}$ is left unsettled. So, we prepare the following lemma.

Lemma 1: Let $\{\check{y}^{r^k}(x^k)\}$ be a sequence of the optimal solutions to the problem (5) in response to a sequence $\{x^k\} \subset X$ converging to $\bar{x}$ and a positive sequence $\{r^k\}$ strictly decreasing to zero. If assumptions a)–c) are satisfied, then the sequence $\{\hat{y}^{r^k}(x^k)\}$ has an accumulation point, any of which is optimal for the problem (6) with $\bar{x}$.

Proof (Existence of an Accumulation Point): A barrier method implies that $\hat{y}^{r^k}(x^k)\in \text{int}\, S(x^k)\subset Y$ for any $x^k\in\{x^k\}$. This fact and compactness of Y assure the existence of an accumulation point of $\{\hat{y}^{r^k}(x^k)\}$.

(Feasibility of the Accumulation Point): Denote anyone of the accumulation points by $\bar{y}$ and a convergent subsequence of $\{\hat{y}^{r^k}(x^k)\}$ to $\bar{y}$ by $\{\hat{y}^{r^k}(x^k)\}$ renewedly. When the function g_2 is continuous at any (x,y), the point-to-set map $S(x)$ becomes closed[1] at any x, as shown by Hogan [7]. So, by definition of closedness, $\{x^k\}\subset X$, $x^k\to\bar{x}$, $\hat{y}^{r^k}(x^k)\in S(x^k)$ and $\hat{y}^{r^k}(x^k)\to\bar{y}$ imply that $\bar{y}\in S(\bar{x})$. That is, the accumulation point $\bar{y}$ becomes feasible to the problem (6) with $\bar{x}$.

(Optimality of the Accumulation Point): Suppose that $\bar{y}$ does not solve the problem (6) with $\bar{x}$, then there exists $y'\in S(\bar{x})$ such that

$$f_2(\bar{x},y')<f_2(\bar{x},\bar{y}).$$

Here, consider an open ball $B(y';\delta)\subset R^{n_2}$ around y' with radius δ, then there exists a number $\delta>0$ such that

$$f_2(\bar{x},y)<f_2(\bar{x},\bar{y}) \qquad \text{for } \forall y\in B(y';\delta).$$

Furthermore, seeing that $S(\bar{x})=cl(\text{int}\, S(\bar{x}))$ in the assumption a), we find that $\text{int}\, S(\bar{x})\cap B(y';\delta)\neq\phi$. This fact enables us to choose another point y'' such that

$$y''\in \text{int}\, S(\bar{x})\cap B(y';\delta)\subset S(\bar{x}).$$

For such y'', it holds that

$$f_2(\bar{x},y'')<f_2(\bar{x},\bar{y}).$$

Let

$$f_2(\bar{x},\bar{y})-f_2(\bar{x},y'')=2\epsilon, \qquad \text{where } \epsilon>0. \tag{7}$$

On the other hand, the assumption a) and continuity of g_2 at (x,y) in the assumption b) assure openness[1] of $S(x)$, which is also indebted to Hogan [7]. Therefore, $\{x^k\}\subset X$, $x^k\to\bar{x}$ and $y''\in S(\bar{x})$ imply that there exists a sequence $\{y^k\}\subset Y$ and a positive integer K_1 such that

$$y^k\in S(x^k) \qquad \text{for } \forall k>K_1$$

and that $y^k\to y''$.

Note that since $y''\in\text{int}\, S(\bar{x})$, $\phi(g_2(\bar{x},y''),h_2(y''))$ is finite, that is, the point $(\bar{x},y'')$ is located on the domain of definition of the composite function $\phi(g_2(x,y),h_2(y))$. From the continuity of g_2, h_2 and ϕ, $\phi(g_2(x,y),h_2(y))$ also becomes continuous at any point on its domain of definition. From these facts, it follows that

$$\phi\big(g_2(x^k,y^k),h_2(y^k)\big)\to\phi\big(g_2(\bar{x},y''),h_2(y'')\big),$$

as $x^k\to\bar{x}$ and $y^k\to y''$. Therefore, as $k\to\infty(r^k\to 0)$,

$$r^k\phi\big(g_2(x^k,y^k),h_2(y^k)\big)\to 0,$$

that is,

$$P^{r^k}(x^k,y^k)\to f_2(\bar{x},y''),$$

which implies the existence of a positive integer K_2 such that

$$|P^{r^k}(x^k,y^k)-f_2(\bar{x},y'')|<\epsilon \qquad \text{for } \forall k>K_2 \tag{8}$$

for ϵ in (7). Besides, from the continuity of f_2 at any (x,y), we have the existence of a positive integer K_3 such that

$$|f_2(\bar{x},\bar{y})-f_2\big(x^k,\hat{y}^{r^k}(x^k)\big)|<\epsilon \qquad \text{for } \forall k>K_3. \tag{9}$$

Set $K=\max(K_1,K_2,K_3)$. Then, using (8), (7), and (9) in turn, we have the following relations for all $k>K$:

$$P^{r^k}(x^k,y^k)<f_2(\bar{x},y'')+\epsilon$$
$$=f_2(\bar{x},\bar{y})-\epsilon<f_2\big(x^k,\hat{y}^{r^k}(x^k)\big). \tag{10}$$

Since $\phi\geq 0$,

$$f_2\big(x^k,\hat{y}^{r^k}(x^k)\big)\leq P^{r^k}\big(x^k,\hat{y}^{r^k}(x^k)\big). \tag{11}$$

Equations (10) and (11) yield that

$$P^{r^k}(x^k,y^k)<P^{r^k}\big(x^k,\hat{y}^{r^k}(x^k)\big) \qquad \text{for } \forall k>K.$$

This relation contradicts that $\hat{y}^{r^k}(x^k)$ is optimal for the problem (5) in response to x^k and r^k. Thus, we can conclude that any accumulation point is optimal for the problem (6) with $\bar{x}$. ∎

In addition to assumptions a) and c), the following assumption is imposed.

d) The function $f_2(x,y)$ is strictly convex in y, $g_2(x,y)$ and $h_2(y)$ are convex in y. Furthermore, $\phi(g_2,h_2)$ is a increasing convex function in (g_2,h_2). (These imply that the function P^r becomes strictly convex in y.) This convexity holds for any fixed x.

Then the following corollary can be immediately obtained from Lemma 1.

Corollary 1: Let $\{\hat{y}^{r^k}(x^k)\}$ be a sequence of the optimal solution to the problem (5) in response to a sequence $\{x^k\}\subset X$ converging to $\bar{x}$ and a positive sequence $\{r^k\}$ strictly decreasing to zero. If the assumptions a)–d) are satisfied, then it holds that

$$\lim_{k\to\infty}\hat{y}^{r^k}(x^k)=\hat{y}(\bar{x}).$$

Proof: Since the optimal solution $\hat{y}(\bar{x})$ to the problem (6) with $\bar{x}$ is unique under the assumption d), the accumulation point of $\{\hat{y}^{r^k}(x^k)\}$ is also unique. Therefore, the accumulation point becomes a limit point of $\{\hat{y}^{r^k}(x^k)\}$. ∎

Next, let us consider the following two level problem by replacing the parametric problem (3d)–(3f) with an unconstrained problem (5):

$$\min_x f_1(x,\hat{y}^r(x)) \tag{12a}$$

$$\text{subject to } x\in X \tag{12b}$$

$$g_1(x,\hat{y}^r(x))\leq \mathbf{0} \tag{12c}$$

$$P^r(x,\hat{y}^r(x))=\min_y P^r(x,y). \tag{12d}$$

Here, let us presuppose that the problem (12) is feasible and attains its minimum for all r below some positive number R. Let $\{x^{r^k}\}$ be the sequence of optimal solutions generated from the problem (12) in response to a positive sequence $\{r^k\}$ strictly decreasing to zero. In order to prove the relation between the sequence $\{x^{r^k}\}$ and the optimal solution to the original two-level problem (3), we impose the following assumptions.

e) The functions $f_1(x,y)$ and $g_1(x,y)$ are continuous at any (x,y).

f) The set X is compact and connected.

g) The closure of the set $\{x\in X\,|\,g_1(x,\hat{y}(x))<\mathbf{0}\}$ becomes the set $\{x\in X\,|\,g_1(x,\hat{y}(x))\leq\mathbf{0}\}$. In other words, it holds that $\{x\in X\,|\,g_1(x,\hat{y}(x))<\mathbf{0}\}\cap B(x;\delta)\neq\phi$ for any $x\in\{x\in X\,|\,g_1(x,\hat{y}(x))\leq\mathbf{0}\}$ and for all open ball $B(x;\delta)\subset R^{n_1}$ around the point x.

Assumption g) excludes peculiar cases such that the set

[1]The closedness and the openness of a point-to-set map $S(x)$, which associates a subset of Y with each point of X, are defined as follows.

1) $S(x)$ is closed at a point $\bar{x}\in X$ if $\{x^k\}\subset X$, $x^k\to\bar{x}$, $y^k\in S(x^k)$ and $y^k\to\bar{y}$ imply that $\bar{y}\in S(\bar{x})$.

2) $S(x)$ is open at a point $\bar{x}\in X$ if $\{x^k\}\subset X$, $x^k\to\bar{x}$, $\bar{y}\in S(\bar{x})$ imply the existence of a sequence $\{y^k\}\subset Y$ and an integer K such that $y^k\in S(x^k)$ for $k\geq K$ and $y^k\to\bar{y}$.

$$\{(x, \check{y}(x)) \mid \check{y}(x) \text{ minimizes } f_2(x, y) \text{ subject to } y \in Y \text{ and } g_2(x, y) \leq 0\}$$

intersects the set $\{(x, y) \mid g_1(x, y) \leq 0\}$ only on its boundary $\{(x, y) \mid g_1(x, y) = 0\}$, for instance.

Then, we have the following theorem.

Theorem 1: Let $\{x^{r^k}\}$ be the sequence of optimal solutions generated from the problem (12) in response to a positive sequence $\{r^k\}$ strictly decreasing to zero. If the assumptions a)–g) are satisfied, then the sequence $\{x^{r^k}\}$ has an accumulation point, any of which is optimal for the Stackelberg problem (3).

Proof (Feasibility of the Accumulation Point): Since $\{x^{r^k}\}$ belongs to the compact set X, it has an accumulation point. Denote anyone of the accumulation points by $\bar{x}$ and a convergent subsequence of $\{x^{r^k}\}$ to $\bar{x}$ by $\{x^{r^k}\}$ renewedly. Since $\check{y}^{r^k}(x^{r^k}) \to \check{y}(\bar{x})$ as $x^{r^k} \to \bar{x}$, which is proved in Corollary 1, $\bar{x}$ satisfies (3d)–(3f). Furthermore, since $g_1(x^{r^k}, \check{y}^{r^k}(x^{r^k})) \leq 0$, the above convergence and continuity of g_1 imply that

$$g_1(\bar{x}, \check{y}(\bar{x})) \leq 0.$$

Therefore, these facts together with $\bar{x} \in X$ assure feasibility of $\bar{x}$ for the problem (3).

(Optimality of the Accumulation Point): Suppose that $\bar{x}$ does not solve the problem (3), then there exists $x' \in X$ such that

$$g_1(x', \check{y}(x')) \leq 0 \tag{13}$$

and

$$f_1(x', \check{y}(x')) < f_1(\bar{x}, \check{y}(\bar{x})). \tag{14}$$

Since $\check{y}(x)$ is continuous at any x under the assumptions a)–d), as proved by Hogan [7], $f_1(x, \check{y}(x))$ becomes also continuous at any x under the continuity of $f_1(x, y)$ at any (x, y). Therefore, consider an open ball $B(x'; \delta) \subset R^{n_1}$ around x' with radius δ, then there exists a number $\delta > 0$ such that

$$f_1(x, \check{y}(x)) < f_1(\bar{x}, \check{y}(\bar{x})) \qquad \text{for } \forall x \in B(x'; \delta).$$

Furthermore, (13) and assumption g) imply that there exists another point $x'' \in \{x \in X \mid g_1(x, \check{y}(x)) < 0\} \cap B(x'; \delta)$. That is, there exists $x'' \in X$ such that

$$g_1(x'', \check{y}(x'')) < 0 \tag{15}$$

and

$$f_1(x'', \check{y}(x'')) < f_1(\bar{x}, \check{y}(\bar{x})).$$

Here, let

$$f_1(\bar{x}, \check{y}(\bar{x})) - f_1(x'', \check{y}(x'')) = 2\epsilon, \qquad \text{where } \epsilon > 0. \tag{16}$$

By the theory of a barrier method, we have that $\check{y}^{r^k}(x'') \to \check{y}(x'')$ as $k \to \infty$, and Corollary 1 says that $(x^{r^k}, \check{y}^{r^k}(x^{r^k})) \to (\bar{x}, \check{y}(\bar{x}))$. These two convergences and the continuity of f_1 imply the existence of positive integers K'' and $\tilde{K}$ such that

$$|f_1(x'', \check{y}^{r^k}(x'')) - f_1(x'', \check{y}(x''))| < \epsilon \qquad \text{for } \forall k > K'' \tag{17}$$

and

$$|f_1(x^{r^k}, \check{y}^{r^k}(x^{r^k})) - f_1(\bar{x}, \check{y}(\bar{x}))| < \epsilon \qquad \text{for } \forall k > \tilde{K}. \tag{18}$$

Set $K = \max(\tilde{K}, K'')$. Then, using (17), (16), and (18) in turn, we have the following relations for all $k > K$:

$$\begin{aligned} f_1(x'', \check{y}^{r^k}(x'')) &< f_1(x'', \check{y}(x'')) + \epsilon \\ &= f_1(\bar{x}, \check{y}(\bar{x})) - \epsilon < f_1(x^{r^k}, \check{y}^{r^k}(x^{r^k})) \end{aligned} \tag{19}$$

To have a contradiction, it is enough to show that $(x'', \check{y}^{r^k}(x''))$ satisfies (12c). From (15), the continuity of g_1 at any (x, y) and the fact that $\check{y}^{r^k}(x'') \to \check{y}(x'')$ as $k \to \infty$, there exists a positive integer K' such that

$$g_1(x'', \check{y}^{r^k}(x'')) \leq 0 \qquad \text{for } \forall k > K'. \tag{20}$$

Consider any integer $k > \max(K, K')$; then (19) and (20) contradict the fact that x^{r^k} is optimal for the problem (12). This completes the proof. ■

Theorem 1 says that a limit point of $\{x^{r^k}\}$ which are generated by a series of the problem (12) corresponding to a sequence $\{r^k\}$ such that $r^k > r^{k+1} > 0$ and $r^k \to 0$, if it exists, is a Stackelberg solution. Consequently, if we provide a positive number for r so that the solution to the problem (12) may sufficiently approximate to the Stackelberg solution of the problem (3), all we have to do will be to develop a solution method for the problem (12). So let the parameter r be fixed in the following, and let us impose the following assumption.

h) For any fixed x, the function $P^r(x, y)$ is differentiable at $y \in \operatorname{int} S(x)$.

In order that $\check{y}^r(x)$ solves the problem (5) under the strict convexity of P^r in y, it is necessary and sufficient that $\check{y}^r(x)$ satisfies a stationary condition with respect to y:

$$\nabla_y P^r(x, y) = \nabla_y f_2(x, y) + r \nabla_y \phi(g_2(x, y), h_2(y)) = 0. \tag{21}$$

Remark: We shall use the following notations in terms of a p-dimensional vector function $f(x, y)$ defined on $R^n \times R^m$ which is differentiable at $(\bar{x}, \bar{y})$. The i-jth element of the matrix is given as

$$[\nabla_x f(\bar{x}, \bar{y})]_{ij} = \frac{\partial f_i(\bar{x}, \bar{y})}{\partial x_j}, \quad [\nabla_y f(\bar{x}, \bar{y})]_{ij} = \frac{\partial f_i(\bar{x}, \bar{y})}{\partial y_j}$$

and

$$\nabla f(\bar{x}, \bar{y}) = (\nabla_x f(\bar{x}, \bar{y}), \nabla_y f(\bar{x}, \bar{y})).$$

We also define for a scalar function $f(x, y)$

$$[\nabla^2_{xx} f(\bar{x}, \bar{y})]_{ij} = \frac{\partial^2 f(\bar{x}, \bar{y})}{\partial x_i \partial x_j}, \quad [\nabla^2_{xy} f(\bar{x}, \bar{y})]_{ij} = \frac{\partial^2 f(\bar{x}, \bar{y})}{\partial x_i \partial y_j}.$$

Under assumption d), since P^r is convex on $S(x)$, we need only $y \in S(x)$ for the sufficiency. Therefore, the solution to the problem (12) solves following problem under assumption d):

$$\min_{x, y} f_1(x, y) \tag{22a}$$

$$\text{subject to } x \in X \tag{22b}$$

$$g_1(x, y) \leq 0 \tag{22c}$$

$$\nabla_y P^r(x, y) = 0 \tag{22d}$$

where let the constraint set X be given with the vector function $h_1 \in R^{q_1}$ as $X = \{x \mid h_1(x) \leq 0\}$. Conversely, when the solution to the problem (22) satisfies $y \in S(x)$, it solves the problem (12). Here, we assume the following.

i) There exist $\nabla^2_{yy} P^r(x, y)$ and $\nabla^2_{xy} P^r(x, y)$, and the former is nonsingular for any (x, y).

Then, regarding x as an independent variable vector and y as a dependent one, by the implicit function theorem there exists an implicit vector function $y = \eta(x)$, having continuous partial derivatives, such that

$$\nabla_y P^r(x, \eta(x)) = 0 \tag{23}$$

and its Jacobian is given as

$$\nabla \eta(x) = [\nabla^2_{yy} P^r(x, y)]^{-1} \nabla^2_{xy} P^r(x, y) \tag{24}$$

where $y = \eta(x)$. Consequently, we have the following assumption.

j) If the functions f_1 and g_1 are differentiable with respect to (x, y). Then the function $\tilde{f}_1(x) = f_1(x, \eta(x))$ and $\tilde{g}_1(x) = g_1(x, \eta(x))$ will also become differentiable, whose gradient and Jacobian at (x, y) can be given as follows, respectively,

$$\nabla \tilde{f}_1(x) = \nabla_x f_1(x, y) + \nabla_y f_1(x, y) \nabla \eta(x)$$
$$= \nabla_x f_1(x, y) - \nabla_y f_1(x, y) \left[\nabla_{yy}^2 P^r(x, y) \right]^{-1} \nabla_{xy}^2 P^r(x, y) \tag{25}$$

$$\nabla \tilde{g}_1(x) = \nabla_x g_1(x, y) - \nabla_y g_1(x, y) \left[\nabla_{yy}^2 P^r(x, y) \right]^{-1} \nabla_{xy}^2 P^r(x, y). \tag{26}$$

Then, the problem (22) is equivalent to the following problem under the relation (23):

$$\min_x f_1(x, \eta(x)) \tag{27a}$$

$$\text{subject to } h_1(x) \leq 0 \tag{27b}$$

$$g_1(x, \eta(x)) \leq 0. \tag{27c}$$

From now on, let us try to solve the problem (27) instead of the problem (12). Note, however, it is impossible to represent the function $\eta(x)$ in an explicit form. This matter causes difficulties in directly applying nonlinear programming to the problem (27). In the most of iterative methods of NLP, however, the data needed for its computation are values of gradients of the objective and the constraint functions as well as their function values at the current point. Let x^t be the current point. Then, in our case, the value $y^t \equiv \eta(x^t)$ coincides with the solution $\hat{y}^r(x^t)$ to the problem (5) with $x = x^t$, which can be solved easily. By use of this value y^t, we can evaluate $\nabla \tilde{f}_1(x^t)$ and $\nabla \tilde{g}_1(x^t)$ by (25) and (26) along with

$$\tilde{f}_1(x^t) = f_1(x^t, y^t), \quad \tilde{g}_1(x^t) = g_1(x^t, y^t).$$

Only after the preparation of those data, we can apply the existing nonlinear programming, such as Zoutendijk's feasible direction method [16], to the problem (27).

IV. An Application to the Min–Max Problem

We consider the min–max problem as a special case of the Stackelberg problem (3). Setting $f_1 = -f_2 = f$ in the problem (3), we have the following min–max problem:

$$\min_x f(x, \hat{y}(x)) \tag{28a}$$

$$\text{subject to } x \in X \tag{28b}$$

$$g_1(x, \hat{y}(x)) \leq 0 \tag{28c}$$

$$f(x, \hat{y}(x)) = \max_y f(x, y) \tag{28d}$$

$$\text{subject to } y \in Y \tag{28e}$$

$$g_2(x, y) \leq 0 \tag{28f}$$

where $Y = \{ y | h_2(y) \leq 0 \}$ and $\hat{y}(x)$ is a parametric maximum solution to the lower level problem (28d)–(28f). The solution to the problem (28), which we call a min–max solution, is the best for Player 1 against the worst possible case that might be taken by Player 2, when Player 1 knows nothing about Player 2.

In a special case, when the constraints on x and y are independent (separate constraints), the min–max problem is formulated simply as

$$\min_{x \in X} \max_{y \in Y} f(x, y)$$

where $X = \{ x | h_1(x) \leq 0 \}$ and $Y = \{ y | h_2(y) \leq 0 \}$

Most studies on min–max problems have been limited to the separate type not including unseparate constraints (28c) and (28f). Danskin [3] presented the directional derivative of the maximal valued function $\phi(x) = \max_{y \in Y} f(x, y) = f(x, \hat{y}(x))$. Using this, Bram [2] and Schmitendorf [10] presented optimality conditions for the min–max problem of the separate type in a form like the optimality conditions for usual nonlinear programming problems. As to computational methods, however, only few methods were proposed, based on a gradient method [4], [6], and a relaxation method [9], [11], that are not applicable to min–max problems subject to the unseparate constraints. So, we apply the solution method presented for the Stackelberg problem to the min–max problem (28).

Let us define the augmented objective function for the lower level problem as

$$P^r(x, y) = f(x, y) - r\phi(g_2(x, y), h_2(y)).$$

and replace the constrained maximization problem in (28d)–(28f) with unconstrained maximization problem

$$\max_y P^r(x, y). \tag{29}$$

Furthermore, since the objective function is common in both the upper and lower levels, this peculiarity makes it possible for us to transform the min–max problem (28) into the following problem:

$$\min_x P^r(x, \hat{y}^r(x)) \tag{30a}$$

$$\text{subject to } x \in X \tag{30b}$$

$$g_1(x, \hat{y}^r(x)) \leq 0 \tag{30c}$$

$$P^r(x, \hat{y}^r(x)) = \max_y P^r(x, y). \tag{30d}$$

Notice that $f(x, \hat{y}(x))$ in (28a) is replaced with the maximal valued function $P^r(x, \hat{y}^r(x))$ in (30a). This point distinguishes between the Stackelberg problem and the min–max problem.

Here, we impose the following assumptions corresponding to b) and d) in Section III.

b') The functions $f(x, y)$ and $g_2(x, y)$ are continuous in $(x, y) \in R^{n_1} \times R^{n_2}$ and $h_2(y)$ are continuous in $y \in R^{n_2}$.

d') The function $f(x, y)$ is strictly concave in y, $g_2(x, y)$ and $h_2(y)$ are convex in y. Furthermore, $\phi(g_2, h_2)$ is an increasing convex function in (g_2, h_2).

Then, the following lemma can be obtained.

Lemma 2: Let $\{ \hat{y}^{r^k}(x^k) \}$ be a sequence of the optimal solution to the problem (29) in response to a sequence $\{ x^k \} \subset X$ converging to $\bar{x}$ and a positive sequence $\{ r^k \}$ strictly decreasing to zero. If assumptions a), b'), c), and d') are satisfied, then it holds that

i) $\lim_{k \to \infty} \hat{y}^{r^k}(x^k) = \hat{y}(\bar{x})$

ii) $\lim_{k \to \infty} P^{r^k}(x^k, \hat{y}^{r^k}(x^k)) = f(\bar{x}, \hat{y}(\bar{x}))$.

Proof:

i) Setting $f_2 = -f$ in Corollary 1 in Section III, we have this result immediately.

ii) Suppose that $P^{r^k}(x^k, \hat{y}^{r^k}(x^k))$ does not converge to $f(\bar{x}, \hat{y}(\bar{x}))$, then there exist a positive number ϵ and a positive integer K_1 such that

$$\left| P^{r^k}\left(x^k, \hat{y}^{r^k}(x^k)\right) - f(\bar{x}, \hat{y}(\bar{x})) \right| \geq 2\epsilon \qquad \text{for } \forall k > K_1. \tag{31}$$

Here, consider an open ball $B(\hat{y}(\bar{x}); \delta) \subset R^{n_2}$ around $\hat{y}(\bar{x})$, then the continuity of f at y implies the existence of a positive δ such that

$$|f(\bar{x}, y) - f(\bar{x}, \hat{y}(\bar{x}))| < \epsilon \qquad \text{for } \forall y \in B(\hat{y}(\bar{x}); \delta).$$

Assumption a) enables us to choose another point y'' such that $y'' \in \operatorname{int} S(\bar{x}) \cap B(\hat{y}(\bar{x}); \delta)$. For such y'', it holds that

$$|f(\bar{x}, y'') - f(\bar{x}, \hat{y}(\bar{x}))| < \epsilon. \tag{32}$$

By the same reason, as in the proof of Lemma 1, we have the existence of a sequence $\{ y^k \} \subset Y$ and a positive integer K_2 such that

$$y^k \in S(x^k) \qquad \text{for } \forall k > K_2$$

and that $y^k \to y''$. Since $y'' \in \operatorname{int} S(\bar{x})$, we have the existence of a positive integer K_3 such that

$$|P^{r^k}(x^k, y^k) - f(\bar{x}, y'')| < \epsilon \qquad \text{for } \forall k > K_3 \tag{33}$$

in the similar manner to the proof of Lemma 1. Set $K = \max(K_1, K_2, K_3)$. Then, using (33), (32), and (31) in turn, we have the following relations for all $k > K$.

$$P^{r^k}(x^k, y^k) > f(\bar{x}, y'') - \epsilon$$
$$> f(\bar{x}, \hat{y}(\bar{x})) - 2\epsilon \geq P^{r^k}(x^k, \hat{y}^{r^k}(x^k)).$$

This contradicts the fact that $\hat{y}^{r^k}(x^k)$ is a maximum solution to the problem (29) in response to x^k and r^k. Thus, we can conclude that $P^{r^k}(x^k, \hat{y}^{r^k}(x^k))$ converges to $f(\bar{x}, \hat{y}(\bar{x}))$. ■

Let us impose the following assumption.

e′) The functions $f(x, y)$ and $g_1(x, y)$ are continuous in (x, y). Then, the following theorem can be obtained.

Theorem 2: Let $\{x^{r^k}\}$ be the sequence of optimal solutions generated from the problem (30) in response to a positive sequence $\{r^k\}$ strictly decreasing to zero. If the assumptions a), b′), c), d′), e′), f), and g) are satisfied, then the sequence $\{x^{r^k}\}$ has an accumulation point, any of which is optimal for the min-max problem (28).

Proof (Feasibility of the Accumulation Point): By a similar way in Theorem 1, we can prove the feasibility.

(Optimality of the Accumulation Point): Suppose that $\bar{x}$ does not solve the problem (28); then in the same manner as in Theorem 1, we have the existence of $x'' \in X$ such that

$$g_1(x'', \hat{y}(x'')) < 0 \tag{34}$$

and

$$f(x'', \hat{y}(x'')) < f(\bar{x}, \hat{y}(\bar{x})).$$

Here, let

$$f(\bar{x}, \hat{y}(\bar{x})) - f(x'', \hat{y}(x'')) = 2\epsilon, \qquad \text{where } \epsilon > 0. \tag{35}$$

As x is fixed at x'', the theory of a barrier method yields that as $k \to \infty$,

$$P^{r^k}(x'', \hat{y}^{r^k}(x'')) \to f(x'', \hat{y}(x'')).$$

That is, there exists a positive integer K'' such that

$$|P^{r^k}(x'', \hat{y}^{r^k}(x'')) - f(x'', \hat{y}(x''))| < \epsilon \qquad \text{for } \forall k > K''. \tag{36}$$

On the other hand, ii) in Lemma 2 implies the existence of a positive integer $\tilde{K}$ such that

$$|P^{r^k}(x^{r^k}, \hat{y}^{r^k}(x^{r^k})) - f(\bar{x}, \hat{y}(\bar{x}))| < \epsilon \qquad \text{for } \forall k > \tilde{K}. \tag{37}$$

Set $K = \max(K'', \tilde{K})$. Then, using (36), (35), and (37) in turn, we have the following relations for all $k > K$:

$$P^{r^k}(x'', \hat{y}^{r^k}(x'')) < f(x'', \hat{y}(x'')) + \epsilon$$
$$= f(\bar{x}, \hat{y}(\bar{x})) - \epsilon < P^{r^k}(x^{r^k}, \hat{y}^{r^k}(x^{r^k}))$$

Since we find that $(x'', \hat{y}^{r}(x''))$ satisfies (30c) in the similar manner to the proof in Theorem 1, this relation contradicts the fact that $\{x^{r^k}\}$ solves the problem (30). ■

TABLE I

r^k	x^{r^k}	$\check{y}^{r^k}(x^{r^k})$	$f_1(x^{r^k}, \check{y}^{r^k}(x^{r^k}))$	$P^{r^k}(x^{r^k}, \check{y}^{r^k}(x^{r^k}))$
200	8.430	8.425	73.55	126.9
50	8.992	8.988	81.88	44.05
10	9.414	9.413	88.96	13.63
1	9.730	9.729	94.74	2.707
0.1	9.873	9.872	97.49	0.5589
0.01	9.944	9.943	98.88	0.1194
True	x^s	$\check{y}(x^s)$	$f_1(x^s, \check{y}(x^s))$	$f_2(x^s, \check{y}(x^s))$
Values	10.00	10.00	100.00	0.0000

The computational aspect for the problem (30) is similar to that for the problem (12) in Section III. So, we omit the computational consideration for the min-max problem.

V. Some Numerical Examples

In order to illustrate the convergence properties, we shall give some examples for the Stackelberg problems.

Example 1: As the Stackelberg problem of the unseparate type, we present the following simple problem:

$$\min_x x^2 + (\check{y}(x) - 10)^2$$
$$\text{subject to } -x + \check{y}(x) \leq 0$$
$$0 \leq x \leq 15$$
$$(x + 2\check{y}(x) - 30)^2 = \min_y (x + 2y - 30)^2$$
$$\text{subject to } x + y - 20 \leq 0$$
$$0 \leq y \leq 20.$$

The Stackelberg solution of this problem is $x^s = 10$ and $\check{y}(x^s) = 10$, which can be found by geometrical consideration. Applying the presented algorithm in Section III by use of the penalty function ϕ of SUMT [5], we obtain the computational result shown in Table I. It has been certified that $(x^{r^k}, \check{y}^{r^k}(x^{r^k}))$ approaches gradually to the Stackelberg solution of the original problem in proportion as r^k decreases.

Example 2: As the Stackelberg problem of the separate type, we present the following problem with two-dimensional variables:

$$\min_x (x_1 - 30)^2 + (x_2 - 20)^2 - 20y_1 + 20y_2$$
$$\text{subject to } x_1 + 2x_2 \geq 30$$
$$x_1 + x_2 \leq 25$$
$$x_2 \leq 15$$
$$(x_1 - \check{y}_1(x))^2 + (x_2 - \check{y}_2(x))^2 = \min_y (x_1 - y_1)^2 + (x_2 - y_2)^2$$
$$\text{subject to } 0 \leq y_1 \leq 10$$
$$0 \leq y_2 \leq 10.$$

Our method exhibits the power, as there exists no other appropriate manner to calculate the Stackelberg solution even for the separate type. Table II illustrates the convergence property in this case, too.

TABLE II

r^k	$x_1^{r^k}$	$x_2^{r^k}$	$\check{y}_1^{r^k}(x^{r^k})$	$\check{y}_2^{r^k}(x^{r^k})$	$f_1(x^{r^k}, \check{y}^{r^k}(x^{r^k}))$	$P^{r^k}(x^{r^k}, \check{y}^{r^k}(x^{r^k}))$
500	18.44	6.563	6.339	5.173	291.0	564.0
200	19.03	5.968	7.288	5.228	276.0	319.8
50	20.00	5.000	8.553	4.976	253.4	191.4
10	20.00	5.000	9.326	4.821	234.9	133.9
1	20.00	5.000	9.775	4.959	228.7	109.5
True	x_1^s	x_2^s	$\check{y}_1(x^s)$	$\check{y}_2(x^s)$	$f_1(x^s, \check{y}(x^s))$	$f_2(x^s, \check{y}(x^s))$
Values	20.00	5.000	10.000	5.000	225.0	100.0

Some computational experiments for the min–max problems confirmed the convergence property and availability of our method also.

VI. Conclusion

In this paper, we have presented two solution methods for the most general types of the Stackelberg and the min–max problems, both of which are formulated as two-level optimization problems to be solved in principle by a parametric approach. Our methods apply the barrier method to transform the two-level problems into the ordinary nonlinear programming problems. Some numerical examples were given to illustrate the proposed algorithms. In view of the fact that there are scarcely any available solution methods so far, we are convinced of the significance of our methods. Nevertheless, there are also troubles in practical computations. For instance, when r is set sufficiently small, the rapid increase or decrease of the function $P^r(x, y)$ in the vicinity of the boundary of the feasible region troubles us to obtain an accurate solution to the problem (5) or (29), and the increase of sensitivity of $P^r(x, y)$ introduces roundoff errors in the calculation of the gradient of the implicit function. Those troubles give serious influence on the computational results. It is hoped, however, that we will overcome the above stated troubles, which are originated in disadvantages peculiar to the barrier method, with various existing techniques for improvement of the barrier method (SUMT).

Acknowledgment

The authors wish to thank the reviewers for their many constructive comments.

References

[1] M. Avriel, *Nonlinear Programming*. Englewood Cliffs, NJ: Prentice-Hall, 1976.
[2] J. Bram, "The Lagrange multiplier theorem for max–min with several constraints," *SIAM J. Appl. Math.*, vol. 14, no. 4, pp. 665–667, 1966.
[3] J. M. Danskin, *The Theory of Max-Min and Its Application to Weapons Allocation Problems*. Berlin: Springer-Verlag, 1967.
[4] V. F. Demjanov, "Algorithms for some minimax problems," *J. Comput. Syst. Sci.*, vol. 2, no. 4, pp. 342–380, 1968.
[5] A. V. Fiacco and G. P. McCormick, "The sequential unconstrained minimization technique for nonlinear programming," *Management Sci.*, vol. 10, no. 2, pp. 601–617, 1964.
[6] J. E. Heller and J. B. Cruz, Jr., "An algorithm for minmax parameter optimization," *Automatica*, vol. 8, no. 3, pp. 325–335, 1972.
[7] W. W. Hogan, "Point-to-set maps in mathematical programming," *SIAM Rev.*, vol. 15, no. 3, pp. 591–603, 1973.
[8] G. P. Papavassilopoulos and J. B. Cruz, Jr., "Nonclassical control problems and Stackelberg games," *IEEE Trans. Automat. Contr.* vol. AC-24, no. 2, pp. 155–166, 1979.
[9] D. M. Salmon, "Minimax controller design," *IEEE Trans. Automat. Contr.*, vol. AC-13, no. 4, pp. 369–376, 1968.
[10] W. E. Schmitendorf, "Necessary conditions and sufficient conditions for static min–max problems," *J. Math. Anal. Appl.*, vol. 57, no. 3, pp. 683–693, 1977.
[11] K. Shimizu and E. Aiyoshi, "Necessary conditions for min–max problems and algorithms by a relaxation procedure," *IEEE Trans. Automat. Contr.*, vol. AC-25, no. 1, 1980.
[12] M. Simaan and J. B. Cruz, Jr., "On the Stackelberg strategy in nonzero-sum games," *J. Optimiz. Theory Appl.*, vol. 11, no. 5, pp. 533–555, 1973.
[13] M. Simaan, "Stackelberg optimization of two-level systems," *IEEE Trans. Syst., Man, Cybern.*, vol. SMC-7, no. 4, pp. 554–556, 1977.
[14] H. Von Stackelberg, *The Theory of the Market Economy*. Oxford, England: Oxford Univ. Press, 1952.
[15] J. J. Strodiot and V. H. Ngyen, "An exponential penalty method for nondifferentiable minmax problem with general constraints," *J. Optimiz. Theory Appl.*, vol. 27, no. 2, pp. 205–219, 1979.
[16] G. Zoutendijk, *Methods of Feasible Directions*. Amsterdam, The Netherlands: Elsevier, 1960.

Finite Chain Approximation for a Continuous Stochastic Control Problem

VIVEK BORKAR AND PRAVIN VARAIYA, FELLOW, IEEE

***Abstract*—A class of feedback policies called finitely switched (FS) policies is introduced. When a one-dimensional nonlinear stochastic system is controlled by FS policies there is a finite state Markov system which is equivalent to the original problem. The finite state problem can be solved by known Markov programming methods.**

Manuscript received February 11, 1980; revised August 8, 1980. Paper recommended by S. I. Marcus, Chairman of the Stochastic Control Committee. This work was supported by the National Science Foundation under Grant 79-03879 and by the Joint Services Electronics Program under Contract F44620-76-C-0100.

The authors are with the Department of Electrical Engineering and Computer Sciences and the Electronics Research Laboratory, University of California, Berkeley, CA 94720.

I. Introduction

We consider the control of the one-dimensional system described by the Itô equation

$$dx_t = m(x_t, u_t)\,dt + \sigma(x_t, u_t)\,dw_t, \qquad t \geq 0 \tag{1.1}$$

in which x_t is the state, u_t is the control, and (w_t) is standard Brownian motion. The control actions are to be selected in feedback form $u_t = \phi(x_t)$ where ϕ belongs to a specified family of feedback policies Φ.

A standard approach to this situation is to treat it as an optimal control problem. One assumes first that Φ is the family of all feedback policies ϕ taking values in a specified control constraint set U. One then seeks ϕ^* in Φ which is optimal in the sense that it minimizes the expected value of a given cost function. ϕ^* can be obtained by solving the Bellman–Hamilton–Jacobi equation. This partial differential equation is usually difficult to solve and one is forced to resort to some approximation.

A different kind of approximation is proposed here. A class Φ of feedback policies called *finitely switched* (FS) policies is introduced. It has the property that the behavior of the state process (x_t) when it is controlled by ϕ in Φ can be summarized by an "equivalent" finite Markov chain. The optimal ϕ^* in an FS family Φ can then be obtained by Markov programming techniques.

The basic idea of this approximation is readily traced to the work of Forestier and Varaiya [1]. The only difference is that there (x_t) itself is a discrete-time finite Markov chain, whereas here it is an Itô process. This difference, of course, leads to changes in the technical argument. Nevertheless, following the discussion in [1], it is reasonable to view the "equivalent" chain obtained here as an "aggregation" of (x_t), and to regard an FS policy as being implemented in a two-layer hierarchy. However, this approach does not generalize easily to vector-valued processes. For some related work, see [6].

FS policies are described in Section II and the equivalent Markov chain is exhibited in Section III. The calculation of the parameters of this chain occupies Section IV. An example is treated in Section V.

II. Finitely Switched Policies

An FS family Φ is defined through a pair (S, Ψ) where $S = \{x_1 < x_2 < \cdots < x_n\}$ is a finite set of states called the *switching set* ("boundary" set in [1]), and Ψ is any specified family of functions $\psi: x \mapsto \psi(x)$. A particularly policy ϕ in $\Phi = \Phi(S, \Psi)$ is obtained by assigning to each $x_i \in S$ a function $\psi_{x_i} \in \Psi$. Then $\Phi = \Psi^S$ is simply the family of all assignments $\phi = (\psi_{x_1}, \cdots, \psi_{x_n})$. To simplify notation such ϕ is usually denoted by $\phi = (\psi_1, \cdots, \psi_n)$.

In the following we suppose that a family $\Phi = \Phi(S, \Psi)$ is given in advance.

Fix a $\phi = (\psi_{x_1}, \cdots, \psi_{x_n})$ in Φ. The control action $(u_t, t \geq 0)$ indicated by ϕ and the resulting state $(x_t, t \geq 0)$ can now be described. Note that (u_t) and (x_t) are stochastic processes which depend on ϕ.

For simplicity suppose that the initial state is a switching state, say $x_0 = x_i$ a.s. Then the control action is given by the feedback policy $u_t = \psi_i(x_t)$ for $0 \leq t \leq T_1$, where T_1 is the first time that x_t reaches another switching state, say $x_j \neq x_i$. The control action is now given by $u_t = \psi_j(x_t)$ for $T_1 \leq t < T_2$, where T_2 is the first time after T_1 that x_t reaches a new switching state $x_k \neq x_j$ when the feedback policy is switched to $u_t = \psi_k(x_t)$, and so on. In brief, as soon as x_t encounters a switching state the control is switched to the corresponding feedback policy. More precisely, define the random variables

$$T_0 = 0 \quad \text{a.s.}$$

$$T_n = \inf\{t > T_{n-1} \mid x_T \in S,\ x_T \neq x_{T_{n-1}}\}, \qquad n \geq 1. \tag{2.1}$$