



# The steepest descent direction for the nonlinear bilevel programming problem

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## Abstract

In this paper, we give necessary optimality conditions for the nonlinear bilevel programming problem. Furthermore, at each feasible point, we show that the steepest descent direction is obtained by solving a quadratic bilevel programming problem. We give indication that this direction can be used to develop a descent algorithm for the nonlinear bilevel problem.

**Key words:** Bilevel programming; Parametric analysis; Global optimization

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## 1. Introduction

We consider the nonlinear bilevel programming problem (BLP):

$$\min_x F(x, y), \quad x \in \mathbb{R}^n, \quad y \in \mathbb{R}^m,$$

where  $y$ , slightly abusing notation, is solution of the lower level program

$$\min_y f_0(x, y)$$

$$\text{LLP}(x): \quad \text{s.t. } f_i(x, y) \leq 0, \quad i \in I$$

$$f_i(x, y) = 0, \quad i \in J$$

and  $I, J$  are finite sets of indices. For a fixed  $x$ , we denote  $y(x)$  an optimal solution of the lower level program and  $\Omega(x) = \{y : f_i(x, y) \leq 0, i \in I \text{ and } f_i(x, y) = 0, i \in J\}$ , the lower level feasible region. A point  $(x, y(x))$  is called a rational point.

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Bilevel programming is an important area of current research. It is characterized by the presence of two optimization problems where some of the first level variables are implicitly defined by the lower level. The nonconvexity and nondifferentiability of BLP, even in the linear case, make the problem a difficult one. Recently, many algorithmic approaches have been proposed for the special case of linear bilevel programming, (see [5, 6, 8, 9, 11, 14, 18]), where the functions are all linear. For the nonlinear bilevel problem, Bard and Moore [6] and Bard [4] have proposed exact algorithms for, respectively, the linear-quadratic and the convex-quadratic problems, while some local algorithms have been proposed for the general nonlinear case (see [1–4]). Bard [3, 5] are good references for an introduction to the subject.

In general, the BLP is a difficult problem and has been shown to be *strongly NP-hard* [14], even for the linear case. Recently, Dempe [10] has proposed necessary optimality conditions for the BLP based on the nonexistence of a descent direction in a combinatorial system. However, no method is suggested for verifying the existence or the nonexistence of the given direction. The purpose of this paper is (i) to give new necessary optimality conditions for BLP based on the directional derivative of the optimal solution of  $y(x)$  that can be verified efficiently, (ii) to characterize the steepest descent direction and (iii) to propose a descent parametric method to determine a local optimum for BLP. In this method, we consider the lower level program as a parametric optimization problem where the optimal solution  $y(x)$  depends on the parameter  $x$ . Hogan [15] and Tanino and Ogawa [20] have proposed the use of the directional derivative for the solution of programs with implicitly defined constraints, in the specific case where the optimal value of a lower level problem,

$$\phi(x) = \min_y f_0(x, y), \quad y \in \Omega(x),$$

is used instead of the optimal solution,

$$y(x) = \operatorname{argmin}_y f_0(x, y), \quad y \in \Omega(x)$$

in the first level objective. These problems are convex under the convexity assumption on the first and lower level problems and the steepest descent direction can be found by solving a linear program. This structure appears frequently in the decomposition of mathematical programs. However, under the same assumptions, the general bilevel problem is not convex and we show in this paper that the steepest descent direction is found by solving a linear-quadratic bilevel program.

The paper is organized as follows. In Section 2 we give new necessary optimality conditions for the general nonlinear bilevel programming problem and propose an efficient method to compute the steepest descent direction at any point. An algorithm using this direction is discussed in Section 3 and a numerical example is given to illustrate the results obtained.

## 2. Necessary optimality conditions for the BLP and the steepest descent direction

**Assumption A<sub>1</sub>:** The problem is well-posed, i.e. that for any  $x$ , the optimal solution  $y(x)$  of the lower level problem is unique.

**Assumption A<sub>2</sub>:** If  $I(x) = \{i \in I \mid f_i(x, y(x)) = 0\}$  denotes that the set of indices for the binding inequality constraints at  $(x, y(x))$ , then we have that the vectors

$$\nabla_y f_i(x, y(x)), \quad i \in I(x) \cup J$$

are linearly independent. This implies the existence of a unique Kuhn–Tucker vector  $\lambda(x)$  corresponding to the optimal solution  $y(x)$ . We denote by

$$E(x) = \{v \in \mathbb{R}^m \mid \nabla_y f_i(x, y(x))v = 0, \quad i \in I(x) \cup J\},$$

the tangent subspace at  $y(x)$ .

**Assumption A<sub>3</sub>:** We assume a second-order sufficient optimality condition:

$$v^T \nabla_y^2 L(x, y(x); \lambda(x))v > 0, \quad \forall v \in E(x), v \neq 0,$$

where  $L(x, y; \lambda) = f_0(x, y) + \sum_{i \in I(x) \cup J} \lambda_i f_i(x, y)$  is the Lagrangian corresponding to the lower level program LLP(x).

The first assumption ensures that the bilevel problem has a solution. Otherwise, the first level cannot impose his own choice within the set  $y(x)$  which can lead to an ill-posed problem [5]. The second assumption assumes a first-order necessary condition for the lower level problem. The second and third assumptions guarantee at least Lipschitzian behavior of the function  $y(x)$ . The second assumption can be replaced by the Mangasarian–Fromowitz constraint qualification, in which case the Kuhn–Tucker vectors are not necessarily unique, at the cost of a lightly more complicated formulation of the quadratic program defined below. The third assumption cannot be significantly relaxed.

Under the previous assumptions, we have the following first-order necessary optimality conditions.

**Theorem.** Let  $(x^*, y(x^*))$  be an optimal solution for BLP. Then for any first level direction  $z \in \mathbb{R}^n$  at  $x^*$ , the directional derivative of the objective function of the first level problem satisfies:

$$F'(x^*, y(x^*); z) = \nabla_x F(x^*, y(x^*))z + \nabla_y F(x^*, y(x^*))w(x^*; z) \geq 0,$$

where  $w(x^*; z)$  is the optimal solution for  $x = x^*$  of the quadratic program:

$$\min_w (z^T, w^T) \nabla_{(x,y)}^2 L(x, y(x); \lambda(x))(z, w), \quad w \in \mathbb{R}^m$$

s.t.

$$(QP(x; z)): \quad \nabla_y f_i(x, y(x))w \leq -\nabla_x f_i(x, y(x))z, \quad i \in I(x)$$

$$\nabla_y f_i(x, y(x))w = -\nabla_x f_i(x, y(x))z, \quad i \in J$$

$$\nabla_y f_0(x, y(x))w = -\nabla_x f_0(x, y(x))z + \nabla_x L(x, y(x); \lambda(x))z.$$

**Proof.** At a locally optimal solution  $(x^*, y(x^*))$  of BLP, we have for any direction  $z \in \mathbb{R}^n$  and for some  $t_0 > 0$ ,

$$F(x^*, y(x^*)) \leq F(x^* + tz, y(x^* + tz)), \quad t \in (0, t_0),$$

where  $y(x^* + tz)$  is the optimal solution of the lower level program, for  $x^* + tz$  fixed. It follows that

$$F'(x^*, y(x^*); z) = \lim_{t \downarrow 0} [F(x^* + tz, y(x^* + tz)) - F(x^*, y(x^*))]/t \geq 0.$$

By the chain rule, we obtain the first-order necessary optimality condition

$$\nabla_x F(x^*, y(x^*))z + \nabla_y F(x^*, y(x^*))y'(x^*; z) \geq 0,$$

where

$$y'(x; z) = \lim_{t \downarrow 0} [y(x + tz) - y(x)]/t$$

is the directional derivative of the optimal solution  $y(x + tz)$  at  $t = 0^+$  in the direction  $z$ . According to a result about the directional derivative of this optimal solution, originally stated in Jittorntrum [17], and given in Gauvin and Janin [12, 13], it happens that the quadratic program QP(x; z) has, under the given assumptions, a unique optimal solution  $w(x; z)$  which turns out to be the directional derivative  $y'(x; z)$ .  $\square$

The computation of the steepest descent direction for BLP is now straightforward. For a nonoptimal point  $(x, y(x))$ , a direction of descent at  $x$  is a  $z \in \mathbb{R}^n$  such that

$$\nabla_x F(x, y(x))z + \nabla_y F(x, y(x))w(x; z) < 0.$$

Therefore, the steepest descent direction at  $x$  for the BLP is given by any optimal solution of the following linear-quadratic bilevel program:

$$\min_z \nabla_x F(x, y(x))z + \nabla_y F(x, y(x))w(x; z),$$

QBLP( $x$ ): with  $-1 \leq z_i \leq 1$ ,  $i = 1, \dots, n$ ,

where  $w(x; z)$  solves QP( $x; z$ ).

This quadratic bilevel program can be solved by some efficient algorithms such as those recently developed by Bard and Moore [6] or Jaumard, Savard and Xiong [16]. These methods use an efficient implicit enumeration scheme over the complementarity condition derived from the sufficient Kuhn–Tucker conditions of the lower level quadratic problem.

### 3. An overview of a descent method

In this section we show in some general manner how the previous result can be exploited to develop a descent method for BLP. Let  $(x_k, y(x_k))$  denote the solution at iteration  $k$ .

*Step 1:* Steepest descent direction. Solve the quadratic bilevel programming problem QBLP( $x_k$ ) to obtain the steepest descent direction  $z_k \in \mathbb{R}^n$  with the corresponding direction  $w(x_k; z_k)$  for the lower level program. If it happens that the optimal value of the quadratic QBLP( $x_k$ ) is nonnegative then stop;  $(x_k, y(x_k))$  satisfies the necessary optimality condition. Otherwise go to step 2.

*Step 2:* Step length computation. Compute a step length  $t_k$  such that

$$F(x_k + t_k z_k, y(x_k + t_k z_k)) < F(x_k, y(x_k)).$$

Return to Step 1 with the feasible point  $(x_{k+1}, y(x_{k+1})) = (x_k + t_k z_k, y(x_k + t_k z_k))$  and with  $k = k + 1$ .

If convergence is obtained, the steepest descent algorithm will usually yield a locally optimal solution for BLP. Some global optimization scheme is then needed to explore for a global solution.

**Remark 1.** At Step 2 of the algorithm, it should be noticed that whenever  $F(x_k + t z_k, y(x_k + t z_k))$  is required, the lower level program LLP( $x_k + t z_k$ ) must be solved. The assumptions made on the problem assure that there exists  $t^* > 0$  such that for all  $t$  with  $0 \leq t < t^*$ , the feasible set for the lower level  $\Omega(x_k + t z_k)$  is nonempty and the lower level's optimal solution  $y_k = y(x_k + t z_k)$  exists (Gauvin and Janin [12]). Hence, the use of an approximate line search technique, adapted for nondifferential function, is justified (Lemaréchal [19]).

**Remark 2.** In the implementation of the algorithm, it will not be necessary to solve QBLP( $x_k$ ) exactly, as any rational solution with nonpositive objective yields a descent direction; this direction has less chance to be extremal. In any way, an efficient implementation of the algorithm must exploit the structure of the feasible set.

As an illustration, we present the following example (see Fig. 1). In this example, as the feasible set is a polyhedron, the maximum step length allowed is the one that keeps  $(x_k + t z_k, y_k + t w_k)$  feasible.

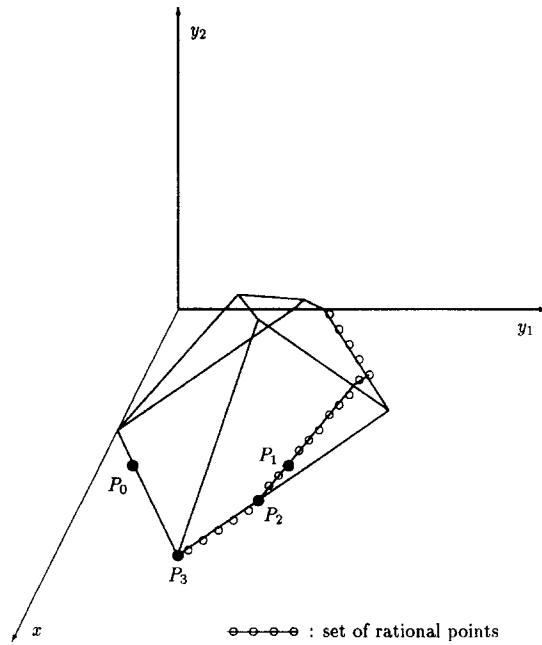


Fig. 1. Example.

**Example.**

$$\begin{aligned}
 & \min_x (x - 1)^2 + 2y_1^2 - 2x, \\
 & \min_y (2y_1 - 4)^2 + (2y_2 - 1)^2 + xy_1, \\
 & \text{s.t. } 4x + 5y_1 + 4y_2 \leq 12 \\
 & \quad -4x - 5y_1 + 4y_2 \leq -4 \\
 & \quad 4x - 4y_1 + 5y_2 \leq 4 \\
 & \quad -4x + 4y_1 + 5y_2 \leq 4 \\
 & \quad x \geq 0 \\
 & \quad y_i \geq 0, \quad i = 1, 2.
 \end{aligned}$$

We have, in this example:

$$\begin{aligned}
 L(x, y; \lambda) &= (2y_1 - 4)^2 + (2y_2 - 1)^2 + xy_1 + \lambda_1(4x + 5y_1 + 4y_2 - 12) \\
 &\quad + \lambda_2(-4x - 5y_1 + 4y_2 + 4) + \lambda_3(4x - 4y_1 + 5y_2 - 4) \\
 &\quad + \lambda_4(-4x + 4y_1 + 5y_2 - 4) - \lambda_5x - \lambda_6y_1 - \lambda_7y_2, \\
 \nabla_{(x,y)}L(x, y; \lambda) &= (y_1 + 4\lambda_1 - 4\lambda_2 + 4\lambda_3 - 4\lambda_4 - \lambda_5, \\
 &\quad 4(2y_1 - 4) + x + 5\lambda_1 - 5\lambda_2 - 4\lambda_3 + 4\lambda_4 - \lambda_6, \\
 &\quad 4(2y_2 - 1) + 4\lambda_1 + 4\lambda_2 + 5\lambda_3 + 5\lambda_4 - \lambda_7),
 \end{aligned}$$

$$\nabla_{(x,y)}^2 L(x, y; \lambda) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 8 & 0 \\ 0 & 0 & 8 \end{pmatrix},$$

$$\nabla_{(x,y)} F(x, y) = (2(x - 1) - 2, 4y_1, 0),$$

$$\nabla_{(x,y)} f_0(x, y) = (y_1, 4(2y_1 - 4) + x, 4(2y_2 - 1)).$$

We get the following sequence.

*Step 0:* The initial rational point  $(x, y_1, y_2)_1 = (4/3, 158/123, 5/82)$  ( $P_1$  in Fig. 1) is computed by first considering the following relaxed problem where the lower level's objective is omitted:

$$\begin{aligned} \min_x \quad & (x - 1)^2 + 2y_1^2 - 2x \\ \text{s.t.} \quad & 4x + 5y_1 + 4y_2 \leq 12 \\ & -4x - 5y_1 + 4y_2 \leq -4 \\ & 4x - 4y_1 + 5y_2 \leq 4 \\ & -4x + 4y_1 + 5y_2 \leq 4 \\ & x \geq 0 \\ & y_i \geq 0, \quad i = 1, 2. \end{aligned}$$

The optimal solution is  $(4/3, 1/3, 0)$  ( $P_0$  in Fig. 1). By checking the optimality for the lower level problem for  $x$  fixed at  $4/3$ , LLP( $4/3$ ), we obtain our first rational point  $(x, y_1, y_2)_1 = (4/3, 158/123, 5/82)$ , with  $I(x) = \{1\}$  and  $\lambda_1 = 72/82$ .

*Step 1:* For the current point, problem QBLP( $x$ ) is

$$\begin{aligned} \min_z \quad & -4/3 z + 632/123 w_1 \\ \min_w \quad & 2zw_1 + 8w_1^2 + 8w_2^2 \\ \text{s.t.} \quad & 5w_1 + 4w_2 \leq -4z \\ & -5w_1 - 4w_2 = 4z \\ & -1 \leq z \leq 1. \end{aligned}$$

The optimal solution is  $(z, w_1, w_2)_1 = (1, -22/41, -27/82)$  with a negative value.

*Step 2:* With this direction, we compute the step length

$$\min_{t>0} \{F((x, y_1, y_2)_1 + t(z, w_1, w_2)_1) | (x, y_1, y_2)_1 + t(z, w_1, w_2)_1 \text{ feasible}\}$$

and find  $t^* = 5/27$  to obtain the second point  $(x, y_1, y_2)_2 = (41/27, 32/27, 0)$  ( $P_2$  in Fig. 1) with  $I(x) = \{1, 7\}$ ,  $\lambda_1 = 4$  and  $\lambda_7 = 0$ . We return to step 1.

*Step 1:* For the current point, problem QBLP( $x$ ) is

$$\begin{aligned} \min_z \quad & -26/27 z + 128/27 w_1 \\ \min_w \quad & 2zw_1 + 8w_1^2 + 8w_2^2 \end{aligned}$$

$$\text{s.t. } 5w_1 + 4w_2 \leq -4z$$

$$w_2 \geq 0$$

$$-5w_1 - 4w_2 = 4z$$

$$-1 \leq z \leq 1.$$

The optimal solution is  $(z, w_1, w_2)_2 = (1, -4/5, 0)$  with value  $-642/135 \leq 0$ .

*Step 2:* With this direction, we compute the step length

$$\min_{t>0} \{F((x, y_1, y_2)_2 + t(z, w_1, w_2)_2), (x, y_1, y_2)_2 + t(z, w_1, w_2)_2 \text{ feasible}\}$$

and find  $t^* = 10/27$  to obtain the third point  $(x, y_1, y_2)_3 = (17/9, 8/9, 0)$  ( $P_3$  in Fig. 1), with  $I(x) = \{1, 3, 7\}$ ,  $\lambda_1 = 1.4$ ,  $\lambda_3 = 0$  and  $\lambda_7 = 1.6$ . We return to step 1.

*Step 1:* For the current point, problem QBLP( $x$ ) is

$$\min_z -2/9 z + 32/9 w_1$$

$$\min_w 2zw_1 + 8w_1^2 + 8w_2^2$$

$$\text{s.t. } 5w_1 + 4w_2 \leq -4z$$

$$-4w_1 + 5w_2 \leq -4z$$

$$w_2 \geq 0$$

$$-7w_1 - 4w_2 = -8/9 z$$

$$-1 \leq z \leq 1.$$

The optimal solution is  $(z, w_1, w_2)_3 = (0, 0, 0)$  with value  $\geq 0$ . So the point  $(x, y_1, y_2)_3 = (17/9, 8/9, 0)$  satisfies the necessary optimality conditions. Indeed, we have obtained the global solution of the given bilevel problem.

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