Operations Research

Publication details, including instructions for authors and subscription information:
http://pubsonline.informs.org

Stackelberg-Nash-Cournot Equilibria: Characterizations and Computations

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(Received August 1981; accepted April 1982)

The supply side of an oligopolistic market supplying a homogeneous product noncooperatively is modeled. In this market, there is one leader and N followers. The followers operate under the Cournot assumption of zero conjectural variation and are accordingly called Cournot firms. The leader, called a Stackelberg firm, specifically takes into account the reaction of the Cournot firms to its output. For this situation, we study the behavior and implications of the joint Cournot reaction curve as generated by plausible economic market assumptions. In particular, we study the existence and uniqueness of a Stackelberg-Nash-Cournot equilibrium. In addition, we prescribe an efficient algorithm to determine a set of equilibrating output quantities for the firms.

A QUANTITY-ORIENTED mathematical analysis of an oligopolistic market in which a few, say N, major firms supply a homogeneous product in a noncooperative manner is presented. In addition, there is one particular firm, supplying the same product without any collusion with the other firms, which sets production levels in an optimal (profit maximizing) fashion by explicitly considering the reaction of the other firms to its output variations. In contrast to this, the other N firms in the industry attempt to maximize their individual profits by producing quantities under the Cournot assumption that the remaining firms will hold their outputs at existing levels. Accordingly, we will call these firms Cournot firms. On the other hand, the single, less naive firm will be called the Stackelberg firm. Stackelberg [1934] first proposed such a situation in the context of a duopoly, by way of an objection to the uniform Cournot assumption of zero conjectural variation. Here, the conjectural variation of firm i with respect to firm j is designated to be the estimated

Subject classification: 131 Stackelberg leader-follower model, 622 equilibrating fixed-points

Operations Research
Vol. 31, No 2, March-April 1983

0030-364X/83/3102-0253 $01.25

or actual (partial) rate of change of output of the ith firm with respect to a change in the jth firm's output.

The objective of this paper is threefold. First, we characterize and investigate the behavior of the joint reaction curve of the Cournot firms, that is, their joint response to an extraneous market supply, namely, the output of the Stackelberg firm, subject to some plausible market assumptions. Second, we study the implications of this joint reaction curve on the Stackelberg firm. And finally, we use this characterization in the development of an algorithm to determine equilibrating market shares of the oligopolists. At such an equilibrium solution, which we call the Stackelberg-Nash-Cournot equilibrium (after Stackelberg, Nash [1951] and Cournot [1838], each Cournot firm has no incentive to change its output since its current output maximizes its profits given that the other firms do not alter their production levels. Furthermore, the Stackelberg firm has maximized its profits after manipulating its output based upon the knowledge about the production policies of the Cournot firms in response to its own. We will also assert weaker than usual conditions on the existence and uniqueness of such an equilibrium solution.

In terms of Friedman's [1977] classification scheme for oligopolistic market games, the Stackelberg model exhibits a "leader-follower" behavior. The followers are the Cournot firms which react to the production strategies of the leader, namely, the Stackelberg firm. The Stackelberg-Nash-Cournot equilibrium, alluded to above, then becomes a viable solution in a situation in which the leader announces its output first, and then followers simultaneously announce their respective production levels. In this context, all firms are assumed to have a complete knowledge about every other firm.

We reiterate that our modeling is done with quantities or output levels as decision variables as opposed to prices as first advocated for economic models by Bertrand [1883], and later more fully by Edgeworth [1925], and Shubik [1959]. Our choice is directed by Friedman's remark that a quantity-model is more meaningful in situations where a homogeneous product is being supplied with a perfectly competitive market mechanism on the demand side. Such a mechanism causes a common market price to prevail, irrespective of the individual supplier. We assume such a situation in the present paper.

Intriligator [1971] has presented a mathematical analysis of the Stackelberg-Cournot duopoly in the simple case of linear cost and demand functions and derives a closed form equilibrium solution. Other than such simplified situations, few previous attempts have been made at actually computing an equilibrium solution. Instead, the mainstream effort has been to identify sufficient conditions for the existence and uniqueness of an equilibrium solution. In the context of quantity models which deal with nonlinear demand and supply curves, we are aware of one such attempt by Okuguchi [1976] which considers a discrete, dynamic Stackelberg process. (Note that our model is static.) However, as pointed out by Furth [1979], this model is not a true leader-follower situation. In the context of price models (see Furth, Okuguchi [1978]) and some extensions to the Stackelberg quantity model in Okuguchi [1976], a typical assumption is made of linear reaction curves with known or estimated (constant) conjectural variations. In contrast, studying the existence and uniqueness of an equilibrium solution in the presence of nonlinear demand and supply functions, we attempt in this paper to investigate the actual properties of the Cournot-firms' joint reaction curve as dictated by plausible market assumptions, rather than make a priori assumptions on the reaction curve itself. In addition, we describe a mechanism for actually computing such an equilibrium solution.

Our paper is organized as follows. In the next section, we present a mathematical description of the problem along with our assumptions. Thereafter, we investigate the properties of the Cournot-firms' joint reaction curve, and their implications on the model behavior. Next, we devise an algorithm to compute the Stackelberg-Nash-Cournot equilibrium. Finally, we present an illustrative example.

1. MATHEMATICAL DESCRIPTION OF THE PROBLEM

For each Cournot firm $i = 1, \cdots, N$, let $f_i(q_i)$ be the total cost of supplying q_i units of the product. In a similar fashion, for the Stackelberg firm, let $f(x)$ represent the total cost for supplying x units of the product. (We use the symbol x for the output of the Stackelberg firm in order to conspicuously differentiate it from other supply quantities.) We make the normal assumption that each of the functions $f(\cdot)$ and $f_i(\cdot)$ $i = 1, \cdots, N$ are convex and twice differentiable.

Next, let $p(\cdot)$ represent the industry (inverse) demand curve. Hence $p(Q)$ is the price at which consumers will demand and purchase a quantity Q. We make the typical assumption that $p(\cdot)$ is a strictly decreasing function. In addition, we assume that $p(\cdot)$ is twice differentiable and that the following inequality holds:

$$p'(Q) + Qp''(Q) \leq 0 \quad \text{for each} \quad Q \geq 0. \tag{1.1}$$

Equation 1.1 has also been used by Okuguchi [1976] in the context of an oligopolistic market analysis. We will endeavor to offer an economic interpretation for (1.1). But first consider the following important implication of (1.1) regarding the industry revenue curve $Qp(Q + K)$ for a fixed extraneous supply $K \geq 0$.

LEMMA 1. *Let $p(\cdot)$ be a strictly decreasing, twice differentiable function and assume that* (1.1) *holds. Then for each fixed $K \geq 0$, the industry* *revenue function* $R_K(Q) = Qp(Q + K)$ *is a strictly concave function of* Q *over* $Q \geq 0$.

Proof. It is sufficient to show that

$$R_K''(Q) = 2p'(Q + K) + Qp''(Q + K) < 0 \quad \text{for each} \quad Q \geq 0. \tag{1.2}$$

Hence, for a given $Q \geq 0$, if $p''(Q + K) \leq 0$ then (1.2) clearly holds since $p'(Q + K) < 0$. Thus, suppose $p''(Q + K) > 0$. But this means that

$$\begin{aligned} R_K''(Q) &\leq 2p'(Q + K) + (Q + K)p''(Q + K) \\ &< p'(Q + K) + (Q + K)p''(Q + K) \end{aligned}$$

and the proof now follows directly from (1.1).

Note that the condition (1.1) is satisfied, for example (though not interestingly), by demand functions which are concave in addition to being decreasing. In general, condition (1.1) has the following interpretation. Consider an (assumed) monopolist facing the demand curve $p(\cdot)$, and suppose that there exists an extraneous supply of $K \geq 0$ units in the market. At an output of Q units, the revenue of the monopolist is $Qp(Q + K)$ and hence, his marginal revenue is $p(Q + K) + Qp'(Q + K)$. The rate of change of this marginal revenue with an increase in the extraneous supply K is

$$p'(Q + K) + Qp''(Q + K). \tag{1.3}$$

When $K = 0$, (1.1) implies that (1.3) is nonpositive. When $K > 0$ and $p''(Q + K) \leq 0$, then (1.3) is negative since $p'(Q + K) < 0$. When $K > 0$ and $p''(Q + K) > 0$, then (1.3) is strictly less than $p'(Q + K) + (Q + K)p''(Q + K)$ which is nonpositive by (1.1). Consequently, (1.1) implies via (1.3) that for any output level Q, the monopolist's marginal revenue decreases when K is increased. (Hahn [1962] has the same interpretation for an assumption similar to (1.1) used in the context of a Cournot oligopoly.) Now, since an increase in K denotes a leftward shift in the effective demand curve $p(Q + K)$, $Q \geq 0$ faced by the monopolist, it follows from (1.3) that the monopolist's marginal revenue curve uniformly decreases when the demand curve is shifted to the left. Such a uniform decrease has an important implication. Since the monopolist determines output so that marginal cost is equal to marginal revenue, and assuming marginal cost is nondecreasing, it follows that a leftward shift of the demand curve results in a decrease in the monopoly output. Hence, if assumption (1.1) holds, then the (assumed) monopolist would decrease output as additional units are introduced into the market.

Finally, we assume that there exists a quantity $q^u \geq 0$ such that

$$f'(q) \geq p(q) \quad \text{and} \quad f_i'(q) \geq p(q), \qquad i = 1, \cdots, N, \quad \text{for each} \quad q \geq q^u. \tag{1.4}$$

That is to say, none of the firms would ever wish to produce more than some (large) quantity q^u. Henceforth, we will say that $p(\cdot)$, $f(\cdot)$ and $f_i(\cdot)$ $i = 1, \cdots, N$ satisfy *Assumptions A* if they satisfy the above set of assumptions.

Now, let us define what we mean by an equilibrium solution.

DEFINITION. *For a given $x \geq 0$, let $[q_1(x), \cdots, q_N(x)]$ be a set of quantities (if they exist) such that for each individual $i = 1, \cdots, N$, assuming that $q_j(x)$, $j \neq i$ are fixed, it turns out that $q_i = q_i(x)$ solves the following (univariate) problem:*

$$CP_i(x)\text{: maximize}_{q_i \geq 0}\{q_i p[q_i + x + \textstyle\sum_{j=1, j\neq i}^{N} q_j(x)] - f_i(q_i)\}. \quad (1.5)$$

Accordingly, denote

$$Q(x) = \textstyle\sum_{i=1}^{N} q_i(x). \quad (1.6)$$

Then, a set of quantities $(x^*, q_1^*, \cdots, q_N^*)$ is said to be a Stackelberg-Nash-Cournot (SNC) *equilibrium solution if*

$$x^* \textit{ solves Problem SP}\text{: maximize}_{x\geq 0}\{xp[x + Q(x)] - f(x)\} \quad (1.7)$$

and moreover,

$$(q_1^*, \cdots, q_N^*) \equiv [q_1(x^*), \cdots, q_N(x^*)]. \quad (1.8)$$

Some clarification is necessary in regard to the above definition. First of all, note that the objective functions in (1.5) and (1.7) are profit expressions. Moreover, from Lemma 1 and assumption (1.4), observe that each of the problems $CP_i(x)$ defined in (1.5) involve the maximization of a strictly concave objective function, essentially over the interval $[0, q^u]$, a closed, convex, compact set. Hence a unique optimum exists. Thus, if one can assert the existence of a unique set of quantities $[q_1(x), \cdots, q_N(x)]$ which simultaneously solve (1.5) for each $i = 1, \cdots, N$, then it follows that the functions $q_i(\cdot) i = 1, \cdots, N$, and hence the function $Q(\cdot)$ of Equation (1.6), are well defined. Indeed, as Theorem 1 below indicates, this is the case since $[q_1(x), \cdots, q_N(x)]$ is simply a Nash-Cournot equilibrium (see Nash) with an extraneous market supply of x units, and our assumptions guarantee the existence of a unique equilibrium. We will refer to the functions $q_1(x), \cdots, q_N(x)$ as the joint reaction functions of the firms $1, \cdots, N$ and we will call $Q(x)$ the aggregate reaction function. That is to say, $[q_1(x), \cdots, q_N(x)]$ is the joint equilibrating reaction or response of the Cournot firms to a Stackelberg output of x units, with $Q(x)$ given by (1.6) being the aggregate reaction or response. Hence, the Stackelberg (leader) firm specifically considers this aggregate reaction in its problem SP defined in (1.7). Thus, if x^* solves SP, then the equilibrating response $[q_1(x^*), \cdots, q_N(x^*)] \equiv (q_1^*, \cdots, q_N^*)$, say, of the Cournot firms yields a SNC equilibrium solution $(x^*, q_1^*, \cdots, q_N^*)$.

Now let us turn our attention to Problems $CP_i(x)$ of (1.5) and discuss a recent study by Murphy et al. [1982] which shows how to determine the quantities $[q_1(x), \cdots, q_N(x)]$ in addition to establishing the existence of a unique solution of this type. This method concocts the following family of mathematical programs (EP for Equilibrating Program) in order to determine $[q_1(x), \cdots, q_N(x)]$:

$$\begin{aligned} \mathrm{EP}(x, Q)\text{: } & \textit{maximize } p(Q+x) \textstyle\sum_{i=1}^{N} q_i + (1/2)p'(Q+x) \sum_{i=1}^{N} q_i^2 \\ & \qquad - \textstyle\sum_{i=1}^{N} f_i(q_i) \\ & \textit{subject to } \textstyle\sum_{i=1}^{N} q_i = Q \qquad\qquad (1.9) \\ & \qquad q_i \geq 0 \quad \text{for} \quad i = 1, \cdots, N. \end{aligned}$$

For a fixed $x \geq 0$, $Q \geq 0$, each problem EP(x, Q) in this family involves the maximization of a strictly concave objective function over a nonempty, compact, convex feasible region, and hence attains a unique local and global optimum. However, given $x \geq 0$, the idea is to determine that $Q = Q(x)$ for which the optimal Lagrange multiplier $\lambda[Q(x)]$, say, associated with constraint (1.9) is zero, since then, one may easily verify that the Kuhn-Tucker conditions for Problem EP$[x, Q(x)]$ essentially replicate those of problems (1.5) with $q_i \equiv q_i(x)$ for $i = 1, \cdots, N$. Indeed, this fact guides the derivation of Problems EP(x, Q). This idea is embodied in the following result.

THEOREM 1. *For a fixed $x \geq 0$, consider Problem EP(x, Q) and suppose that $p(\cdot)$, $f(\cdot)$ and $f_i(\cdot)$ $i = 1, \cdots, N$ satisfy Assumptions A. Denote by $q_i^*(x, Q)$ $i = 1, \cdots, N$ the unique optimal solution to EP(x, Q) and let $\lambda^*(x, Q)$ be the corresponding optimal Lagrange multiplier associated with constraint* (1.9). *(In case $Q = 0$, since alternative optimal multipliers associated with* (1.9) *exist, let $\lambda^*(x, Q)$ be the minimum nonnegative optimal Langrange multiplier associated with* (1.9).*) Then,*

(i) *$q_i^*(x, Q)$ is a continuous function of Q, $Q \geq 0$, for $i = 1, \cdots, N$.*
(ii) *$\lambda^*(x, Q)$ is a continuous, strictly decreasing function of Q, for $Q > 0$. Moreover, there exist quantities Q_l and Q_u such that $\lambda^*(x, Q_l) \geq 0$ and $\lambda^*(x, Q_u) \leq 0$.*
(iii) *A set of output quantities $[q_1(x), \cdots, q_N(x)]$ optimal to Problem EP$[x, Q(x)]$, where $Q(x) = \sum_{i=1}^{N} q_i(x)$, satisfy the Nash-Cournot conditions* (1.5) *if and only if $\lambda^*[x, Q(x)] = 0$, whence, $q_i(x) \equiv q_i^*[x, Q(x)]$ $i = 1, \cdots, N$.*

Proof. The proof follows from the development of Murphy et al. by noting two facts. First, that for a fixed extraneous supply x, the residual demand function $p_x(Q) \equiv p(Q + x)$ for the Cournot firms is strictly decreasing, twice differentiable and from Lemma 1, results in $Qp_x(Q)$ being a strictly concave function of Q for $Q \geq 0$. Second, that the necessary and sufficient Kuhn-Tucker conditions for EP(x, $\cdot$) imply (using (1.4)) that $Q_l = 0$, $Q_u = Nq^u$ satisfy $\lambda^*(x, Q_l) \geq 0$ and $\lambda^*(x, Q_u) \leq 0$.

COROLLARY. *There exists a unique set of quantities* $[q_1(x), \cdots, q_N(x)]$ *satisfying the conditions* (1.5) *for each fixed* $x \geq 0$. (The proof follows from parts (ii) and (iii) of Theorem 1.)

Hence, Theorem 1 embodies an efficient method of finding $Q(x)$ for each fixed $x \geq 0$. Namely, one simply needs to conduct a univariate bisection search, for example, in order to find the unique root $Q(x)$ of $\lambda^*(x, \cdot)$. In fact, part (iii) of Theorem 1 provides an alternative definition of $[q_1(x), \cdots, q_N(x)]$ and $Q(x)$.

In concluding this section let us make a remark about the SNC equilibrium in the light of Fellner's [1949] and Friedman's discussions. Consider a dynamic process in which each firm continually readjusts output and suppose it converges to a SNC equilibrium. Then at every stage, the Cournot firms are making a wrong assumption about the other firms, namely, one of zero conjectural variation. Hence, at an equilibrium, the Cournot firms turn out to be "right for the wrong reason." However, the Stackelberg firm is making a correct judgment on the other firms and turns out to be "right for the right reason" at an equilibrium. In other words, at an equilibrium solution, each Cournot firm is producing at an output level which maximizes its profits, given the outputs of the other firms. Hence, no Cournot firm wishes to unilaterally change its output. On the other hand, the Stackelberg firm has maximized its profits while explicitly considering the reaction of the Cournot firms to variations in its own outputs. Behaviorally, the Cournot firms are following the conservative viewpoint that they will not be able to manipulate the market by adjusting their output levels, whereas the Stackelberg firm is recognizing its ability to do so.

We will now proceed to study some important properties of the aggregate Cournot reaction curve $Q(x)$, to be used later in our analysis.

2. SOME PROPERTIES OF THE AGGREGATE COURNOT REACTION CURVE

Specifically, in this section, we will establish that $Q(x)$ is a continuous, strictly decreasing function of x over its positive range, for $x \geq 0$, and moreover, $Q^+(x) \geq -1$, $x \geq 0$, where $Q^+(x)$ denotes the right hand derivative of $Q(x)$ with respect to x, that is, the rate of increase of $Q(x)$ with an increase in x. In other words, we show that $x + Q(x)$ is a nondecreasing function of x. The main result of this section is formally summarized below. A reader not interested in the details of the proof may either skip the rest of this section or simply read the annunciations of the lemmas which lead to the following result.

THEOREM 2. *Let the functions $p(\cdot)$, $f(\cdot)$ and $f_i(\cdot)$ $i = 1, \cdots, N$ satisfy Assumptions A and let $Q(x)$, $x \geq 0$ be a function of x as defined through* (1.5), (1.6). *Then $Q(x)$ is a continuous function of x, for $x \geq 0$ and moreover,*

$$\begin{aligned} -1 < Q^{+}(x) < 0 \quad &\text{if} \quad Q(x) > 0 \\ Q^{+}(x) = 0 \quad &\text{if} \quad Q(x) = 0, \text{ for each } x > 0. \end{aligned} \tag{2.1}$$

Proof. This result is established through the use of the following four lemmas.

LEMMA 2. *Suppose that the functions $p(\cdot)$, $f(\cdot)$ and $f_i(\cdot)$ $i = 1, \cdots, N$ satisfy Assumption A. Consider Problem EP(x, Q) and let $q_i^*(x, Q)$ $i = 1, \cdots, N$ and $\lambda^*(x, Q)$ be as defined in Theorem 1. Then, $q_i^*(x, Q)$ is continuous over $x \geq 0$, $Q \geq 0$, for $i = 1, \cdots, N$, and $\lambda^*(x, Q)$ is continuous over $x \geq 0$, $Q \geq 0$.*

Proof. Theorem 1 asserts that for a fixed $x \geq 0$, $q_i^*(x, Q)$ $i = 1, \cdots, N$ are continuous over $Q \geq 0$ and $\lambda^*(x, Q)$ is continuous over $Q > 0$. Hence, let us show that for a fixed $Q \geq 0$, $q_i^*(x, Q)$ $i = 1, \cdots, N$ are continuous over $x \geq 0$ and for a fixed $Q > 0$, $\lambda^*(x, Q)$ is continuous over $x \geq 0$.

Denoting $q = (q_1, \cdots, q_N)$, let $G(\bar{x}, q)$ be the objective function of Problem EP$(\bar{x}, Q)$ for some fixed $\bar{x} \geq 0$, $Q \geq 0$. Furthermore, let $q^*(\bar{x} + \epsilon, Q)$ be the unique optimal solution vector to EP$(\bar{x} + \epsilon, Q)$ where $\epsilon > 0$ if $\bar{x} = 0$ and $|\epsilon| > 0$ is small enough so that $\bar{x} + \epsilon > 0$ otherwise. We will first demonstrate that $q_i^*(\cdot, Q)$ $i = 1, \cdots, N$ are continuous at $x = \bar{x}$. Toward this end, suppose by contradiction that

$$\lim_{\theta \to 0} q^*(\bar{x} + \theta\epsilon, Q) \neq q^*(\bar{x}, Q). \tag{2.2}$$

Thus, given a sequence $\{\theta_n\} \to 0$, since $\{q^*(\bar{x} + \theta_n\epsilon, Q)\}$ is a bounded sequence, it has a convergent subsequence. Assuming for notational simplicity that $\{\theta_n\}$ itself gives this convergent subsequence, (2.2) asserts that

$$\{q^*(\bar{x} + \theta_n\epsilon, Q)\} \to \bar{q} \neq q^*(\bar{x}, Q). \tag{2.3}$$

Note that $q^*(\bar{x} + \theta_n\epsilon, Q) \geq 0$, and $\sum_{i=1}^{N} q_i^*(\bar{x} + \theta_n\epsilon, Q) = Q$ for each n further implies that

$$\bar{q} \geq 0 \quad \text{and} \quad \textstyle\sum_{i=1}^{N} \bar{q}_i = Q. \tag{2.4}$$

Now, by the optimality of $q^*(\bar{x} + \theta_n\epsilon, Q)$, we get

$$G[\bar{x} + \theta_n\epsilon, q^*(\bar{x} + \theta_n\epsilon, Q)] \geq G[\bar{x} + \theta_n\epsilon, q^*(\bar{x}, Q)]. \tag{2.5}$$

Therefore, by continuity of $G[\cdot, \cdot]$ and by (2.3), as $\{\theta_n\} \to 0$, inequality (2.5) becomes

$$G[\bar{x}, \bar{q}] \geq G[\bar{x}, q^*(\bar{x}, Q)]. \tag{2.6}$$

But (2.4) and (2.6) imply that $\bar{q}$ and $q^*(\bar{x}, Q)$ are distinct alternative optimal solutions to $\mathrm{EP}(\bar{x}, Q)$, a contradiction. Therefore, given $Q \geq 0$, $q_i^*(x, Q)$ is a continuous function of x, $x \geq 0$ for each $i = 1, \cdots, N$.

Now consider the continuity of $\lambda^*(x, Q)$ at $x = \bar{x} \geq 0$ for a fixed $Q > 0$. From the Kuhn-Tucker conditions for Problem $\mathrm{EP}(\bar{x}, Q)$, we have $\lambda^*(\bar{x}, Q) = p(Q + \bar{x}) + q_i p'(Q + \bar{x}) - f_i'(q_i)$ at optimality for each $i = 1, \cdots, N$ such that $q_i > 0$. Since each $p(\cdot)$, $p'(\cdot)$ and $f_i'(\cdot)$ $i = 1, \cdots, N$ is continuous and $q_i^*(\cdot, Q)$ $i = 1, \cdots, N$ are continuous, and furthermore, $Q > 0$ means that at least one $q_i > 0$, $i \in \{1, \cdots, N\}$, it then follows that $\lambda^*(x, Q)$ is continuous at $x = \bar{x}$. This completes the proof.

LEMMA 3. *Suppose that the functions $p(\cdot)$, $f(\cdot)$ and $f_i(\cdot)$ $i = 1, \cdots, N$ satisfy Assumption A. Then $Q(x)$ defined by* (1.5) *and* (1.6) *is a continuous function of x, $x \geq 0$.*

Proof. From Theorem 1 (and its corollary), note that $Q(x)$ is a well defined function of x, and that $\lambda^*[x, Q(x)] \equiv 0$ for each $x \geq 0$.

Now, consider a sequence $\{x_n\} \to \bar{x}$, where $x_n, \bar{x} \geq 0$. Define $Q_n = Q(x_n)$. Since $\{Q_n\}$ is bounded, there exists a convergent subsequence. Assuming that the subscript n refers to this convergent subsequence and that $\{Q_n\} \to \bar{Q}$, it is sufficient to show that $Q(\bar{x}) = \bar{Q}$. But we know that $\lambda^*(x_n, Q_n) = 0$ for each n and $(x_n, Q_n) \to (\bar{x}, \bar{Q})$. Hence, if $\bar{Q} > 0$, then by the continuity of $\lambda^*(\cdot, \cdot)$ at $(\bar{x}, \bar{Q})$ from Lemma 2, it follows that $\lambda^*(\bar{x}, \bar{Q}) = 0$, which in turn from Theorem 1 part (iii) implies that $Q(\bar{x}) \equiv \bar{Q}$. On the other hand, suppose that $\bar{Q} = 0$. The Kuhn-Tucker conditions for $\mathrm{EP}(x_n, Q_n)$ imply that

$$0 \equiv \lambda^*(x_n, Q_n) \geq p(Q_n + x_n) + q_{ni} p'(Q_n + x_n) - f_i'(q_{ni}), \; i = 1, \cdots, N. \tag{2.7}$$

By the continuity of $p(\cdot)$, $p'(\cdot)$, $f_i'(\cdot)$, $i = 1, \cdots, N$ noting that as $n \to \infty$, $x_n \to \bar{x}$, $Q_n \to 0$ and so, $q_{ni} \to 0$ for each $i = 1, \cdots, N$, (2.7) yields

$$0 \geq p(\bar{x}) - f_i'(0) \quad \text{for each} \quad i = 1, \cdots, N. \tag{2.8}$$

But the Kuhn-Tucker conditions for $\mathrm{EP}(\bar{x}, 0)$ assert that an optimal multiplier associated with (1.9) must satisfy $\lambda^*(\bar{x}, 0) \geq p(\bar{X}) - f_i'(0)$, $i = 1, \cdots, N$, which from our convention stated in Theorem 1 and from (2.8) implies that $\lambda^*(\bar{x}, 0) \equiv 0$. Hence again, $Q(\bar{x}) = \bar{Q} \equiv 0$. This completes the proof.

Thus far, we have established that the aggregate reaction curve $Q(x)$, $x \geq 0$ and the joint reaction curves $q_i(x) \equiv q_i^*[x, Q(x)]$, $x \geq 0$, $i = 1, \cdots, N$ are continuous functions. In essence, the following two lemmas show that wherever the right hand derivatives $q_i^+(x)$, $i = 1, \cdots, N$ and $Q^+(x)$ exist for $x > 0$, they are bounded above and below by 0 and -1 respectively. We will assume henceforth that the right hand derivatives of $q_i(x)$, $i = 1, \cdots, N$ and $Q(x)$ exist for $x \geq 0$.

LEMMA 4. *Suppose that the functions* $p(\cdot)$, $f(\cdot)$ *and* $f_i(\cdot)$ $i = 1, \cdots, N$ *satisfy Assumptions A, and let* $Q(x)$, $x \geq 0$ *be defined as in* (1.5), (1.6). *Then*

$$Q^+(x) < 0 \text{ if } Q(x) > 0 \text{ and } x > 0$$

and (2.9)

$$Q^+(x) = 0 \text{ if } Q(x) = 0, \text{ and } x \geq 0.$$

Proof. First of all note that since $Q(x)$ is continuous and $Q(x) \geq 0$ for $x \geq 0$, it is sufficient to show that $Q^+(x) < 0$ whenever $x > 0$ and $Q(x) > 0$. This follows for if $Q(x) = 0$ and $x \geq 0$, $Q^+(x)$ must be nonnegative; and, if $Q^+(x) > 0$, then for some $\epsilon > 0$, one would obtain $Q^+(x + \epsilon) > 0$ with $Q(x + \epsilon) > 0$, a contradiction.

Let us write the Kuhn-Tucker conditions for the family of programs EP$[x, Q(x)]$, $x \geq 0$:

$$p[x + Q(x)] + q_i(x)p'[x + Q(x)] - f_i'[q_i(x)] + u_i(x) = 0 \quad \text{for each} \quad i = 1, \cdots, N \tag{2.10}$$

$$\textstyle\sum_{i=1}^N q_i(x) = Q(x) \tag{2.11}$$

$$u_i(x)q_i(x) = 0 \qquad i = 1, \cdots, N \tag{2.12}$$

$$u(x), q(x) \geq 0. \tag{2.13}$$

Hence, for any $x \geq 0$, there exists a unique $Q(x)$ and correspondingly, uniquely defined $q_i(x)$, $u_i(x)$ $i = 1, \cdots, N$ for which the dual variable associated with (1.9) in EP$[x, Q(x)]$ is zero. These functions satisfy (2.10) through (2.13) for each $x \geq 0$. Moreover, from Lemmas 2 and 3, $Q(x)$, $q_i(x)$, $i = 1, \cdots, N$ are all continuous functions of x. (Recall that $q_i(x) \equiv q_i^*[x, Q(x)]$, $i = 1, \cdots, N$.) Hence from (2.10) and the continuity of $p(\cdot)$, $p'(\cdot)$ and $f_i'(\cdot)$, it follows that $u_i(x)$, $i = 1, \cdots, N$ are also continuous functions. Further from (2.10), since $p(\cdot)$, $p'(\cdot)$ and $f_i'(\cdot)$ are differentiable, and since for $x > 0$, $q_i^+(x)$ and $Q^+(x)$ are assumed to exist as argued above, $u_i^+(x)$ also exists for $x > 0$.

Now consider a fixed $\bar{x} > 0$ such that $Q(\bar{x}) > 0$. Define the set

$$I(\bar{x}) = \{i: q_i^+(\bar{x}) > 0\} \tag{2.14}$$

where as before, $q_i^+(\cdot)$ denotes the right-hand derivative of $q_i(\cdot)$, $i = 1$,

$\cdots, N$. Let us first show that

$$u_i^+(\bar{x}) = 0 \quad \text{for each} \quad i \in I(\bar{x}) \cup \{i: q_i(\bar{x}) > 0\}. \tag{2.15}$$

Note from Equation 2.12 and the continuity of $q_i(\cdot)$ that (2.15) holds for each i satisfying $q_i(\bar{x}) > 0$. Hence, suppose that $i \in I(\bar{x})$ and $q_i(\bar{x}) = 0$. Again, we may assert from (2.12) that $u_i(x) = 0$ for each $\bar{x} < x \leq \bar{x} + \epsilon$ for some $\epsilon > 0$ for otherwise (2.12) would be violated for $x \in (\bar{x}, \bar{x} + \epsilon]$. Therefore, by the continuity of $u_i(\cdot)$ at $x = \bar{x}$, we must have $u_i(\bar{x}) = 0$. Thus, $u_i(x) = 0$ for $x \in [\bar{x}, \bar{x} + \epsilon]$ implies that (2.15) holds.

Now, let us differentiate Equations 2.10 and 2.11 with respect to (an increase in) x and evaluate it at $x = \bar{x}$ to obtain

$$\begin{aligned} &p'[\bar{x} + Q(\bar{x})][1 + Q^+(\bar{x})] \\ &\quad + q_i(\bar{x})p''[\bar{x} + Q(\bar{x})][1 + Q^+(\bar{x})] + q_i^+(\bar{x})p'[\bar{x} + Q(\bar{x})] \\ &\quad - f_i''[q_i(\bar{x})][q_i^+(\bar{x})] + u_i^+(\bar{x}) = 0 \quad \text{for each} \quad i = 1, \cdots, N \end{aligned} \tag{2.16}$$

and

$$\textstyle\sum_{i=1}^{N} q_i^+(\bar{x}) = Q^+(\bar{x}). \tag{2.17}$$

Denote the cardinality $|I(\bar{x})|$ of the set $I(\bar{x})$ defined in (2.14) by r. If $r = 0$, then (2.14) and (2.17) imply that $Q^+(\bar{x}) < 0$ or $Q^+(\bar{x}) = 0$. Hence, suppose that $Q^+(\bar{x}) = 0$ which means that $q_i^+(\bar{x}) = 0$ for each $i = 1, \cdots, N$ (since $r = 0$). But by hypothesis, $Q(\bar{x}) > 0$ and so there exists a firm i such that $q_i(\bar{x}) > 0$. For such a firm, (2.15) and (2.16) yield

$$p'[\bar{x} + Q(\bar{x})] + q_i(\bar{x})p''[\bar{x} + Q(\bar{x})] = 0$$

or multiplying by $[\bar{x} + Q(\bar{x})]/q_i(\bar{x})$,

$$\begin{aligned} &[(\bar{x} + Q(\bar{x}))/q_i(\bar{x})]p'[\bar{x} + Q(\bar{x})] \\ &\qquad + [\bar{x} + Q(\bar{x})]p''[\bar{x} + Q(\bar{x})] = 0. \end{aligned} \tag{2.18}$$

But since $[\bar{x} + Q(\bar{x})]/q_i(\bar{x}) > 1$, and $p(\cdot)$ is strictly decreasing, it follows that Equations 1.1 and 2.18 are contradictory. Hence, we have shown that if $r = 0$, then $Q^+(\bar{x}) < 0$.

Thus, suppose that $r \geq 1$. Then, by adding (2.16) over $i \in I(\bar{x})$ and using Equation 2.15 and the facts that for $i \in I(\bar{x})$, $p'[\bar{x} + Q(\bar{x})]q_i^+(\bar{x}) < 0$, $f_i''[q_i(\bar{x})]q_i^+(\bar{x}) \geq 0$ (since $f_i(\cdot)$ is convex), we obtain

$$\begin{aligned} &rp'[\bar{x} + Q(\bar{x})][1 + Q^+(\bar{x})] \\ &\qquad + [\textstyle\sum_{i\in I(\bar{x})} q_i(\bar{x})]p''[\bar{x} + Q(\bar{x})][1 + Q^+(\bar{x})] > 0. \end{aligned} \tag{2.19}$$

Now, if $\sum_{i\in I(\bar{x})} q_i(\bar{x}) = 0$, then we must have $1 + Q^+(\bar{x}) < 0$ or $Q^+(\bar{x}) < -1$. Hence, suppose that $\sum_{i\in I(\bar{x})} q_i(\bar{x}) > 0$. By multiplying (2.19) throughout by $[\bar{x} + Q(\bar{x})]/\sum_{i\in I(\bar{x})} q_i(\bar{x}) > 1$, we obtain,

$$[1 + Q^+(\bar{x})]\{(r[\bar{x} + Q(\bar{x})]/\textstyle\sum_{i\in I(\bar{x})} q_i(\bar{x}))p'[\bar{x} + Q(\bar{x})] + [\bar{x} + Q(\bar{x})]p''[\bar{x} + Q(\bar{x})]\} > 0.$$

Again, since $r[\bar{x} + Q(\bar{x})]/\sum_{i\in I(\bar{x})} q_i(\bar{x}) > 1$, Equation 1.1 implies that the term $\{\cdot\}$ above is negative, and so $Q^+(\bar{x}) < -1$. This completes the proof.

LEMMA 5. *Suppose that* $p(\cdot)$, $f(\cdot)$ *and* $f_i(\cdot)$, $i = 1, \cdots, N$ *satisfy Assumption A. Let* $Q(x)$ *be defined as in* (1.5) *and* (1.6). *Then,*

$$Q^+(x) > -1 \text{ for each } x > 0. \tag{2.20}$$

Proof. By Lemma 4, if $Q(x) = 0$, then $Q^+(x) = 0$. Hence, for a given $\bar{x} > 0$ assume that $Q(\bar{x}) > 0$. Again, by Lemma 4 $Q^+(\bar{x}) < 0$. Then, from (2.11) and (2.17) it follows that for some $k \in \{1, \cdots, N\}$

$$q_k^+(\bar{x}) < 0, \quad \text{and hence,} \quad q_k(\bar{x}) > 0. \tag{2.21}$$

Now, combining (2.15) and (2.21) imply that (2.16) for firm k is

$$p'[\bar{x} + Q(\bar{x})][1 + Q^+(\bar{x})] + q_k(\bar{x})p''[\bar{x} + Q(\bar{x})][1 + Q^+(\bar{x})] + q_k^+(\bar{x})p'[(\bar{x} + Q(\bar{x})] - f_k''[q_k(\bar{x})][q_k^+(\bar{x})] = 0. \tag{2.22}$$

But from (2.21), $q_k^+(\bar{x}) < 0$ and Assumptions A imply that $p'[\bar{x} + Q(\bar{x})] < 0$ and $f_k''[q_k(\bar{x})] \geq 0$. Thus, (2.22) gives

$$[1 + Q^+(\bar{x})]\{p'[\bar{x} + Q(\bar{x})] + q_k(\bar{x})p''[\bar{x} + Q(\bar{x})]\} < 0. \tag{2.23}$$

Now, if $p''[\bar{x} + Q(\bar{x})] \leq 0$, then the term $\{\cdot\}$ in (2.23) is negative and so $Q^+(\bar{x}) > -1$. Hence, suppose $p''[\bar{x} + Q(\bar{x})] > 0$. In this case, from (1.1), the term $\{\cdot\}$ in (2.23) satisfies the following since $\bar{x} + Q(\bar{x}) > q_k(\bar{x})$:

$$\{\cdot\} < p'[\bar{x} + Q(\bar{x})] + [\bar{x} + Q(\bar{x})]p''[\bar{x} + Q(\bar{x})] \leq 0. \tag{2.24}$$

Thus, again from (2.23) and (2.24), we have $Q^+(\bar{x}) > -1$. This completes the proof.

COROLLARY. *For* $x > 0$, $q_i^+(x) \leq 0$ *for each* $i = 1, \cdots, N$.

Proof. If for any $x = \bar{x} > 0$, $q_i^+(\bar{x}) > 0$ for some $i \in \{1, \cdots, N\}$, then $|I(\bar{x})| = r \geq 1$, where $I(\bar{x})$ is defined in (2.14). But the proof of Lemma 4 for $r \geq 1$ leads to the conclusion that $Q^+(\bar{x}) < -1$, a contradiction to Lemma 5. This completes the proof.

Thus, the above corollary implies that as the extraneous supply increases, each Cournot firm's equilibrating output uniformly decreases.

Lemmas 2 through 5 have established Theorem 2. We will now proceed to discuss the implications of the properties of $Q(\cdot)$ as stated in Theorem 2.

3. IMPLICATIONS OF THE AGGREGATE COURNOT REACTION CURVE ON THE STACKELBERG PROBLEM

In this section, we will establish the existence of a SNC equilibrium under Assumptions A and we will also state a sufficient condition for the uniqueness of such an equilibrium solution. In accomplishing this task, we will also derive lower and upper bounds on the Stackelberg equilibrating output. These bounds have an economic interpretation. Basically, our development in this section will lay the foundation for an algorithm to estimate a SNC equilibrium solution.

To begin with, let us first of all establish the rather straightforward result that the Stackelberg firm makes more profit than it would have as an $(N + 1)$st Cournot firm. This, of course, gives it the incentive to obtain and exploit the aggregate Cournot reaction curve $Q(x)$.

LEMMA 6. *Consider the Stackelberg problem SP defined in* (1.7). *Then the profit at optimality is greater than or equal to the profit it would have made were it also a Cournot firm.*

Proof. Let x^* solve Problem SP and suppose that $(\hat{x}, \hat{q}_1, \cdots, \hat{q}_N)$ is a Nash-Cournot equilibrium for the $(N + 1)$ firm oligopoly. Then, in the spirit of problem (1.5), $\sum_{i=1}^{N} \hat{q}_i \equiv Q(\hat{x})$ and moreover,

$$\hat{x} \text{ solves: maximize}_{x \geq 0} xp[x + Q(\hat{x})] - f(x). \tag{3.1}$$

But since x^* solves SP, we must have

$$x^*p[x^* + Q(x^*)] - f(x^*) \geq \hat{x}p[\hat{x} + Q(\hat{x})] - f(\hat{x}).$$

This completes the proof.

We will now proceed to establish lower and upper bounds on the optimal Stackelberg output.

LEMMA 7. *Let the functions $p(\cdot)$, $f(\cdot)$ and $f_i(\cdot)$ $i = 1, \cdots, N$ satisfy Assumptions A. Furthermore, let $(\hat{x}, \hat{q}_1, \cdots, \hat{q}_N)$ be the unique (see Murphy et al.) Nash-Cournot equilibrium for an $N + 1$ (Cournot) oligopoly, and suppose that x^* solves Problem SP. Then*

$$x^* \geq \hat{x}. \tag{3.2}$$

Proof. If $\hat{x} = 0$, then (3.2) is trivially true. Hence suppose that $\hat{x} > 0$. Then the necessary first order optimality condition for (3.1) yields

$$p[\hat{x} + Q(\hat{x})] + \hat{x}p'[\hat{x} + Q(\hat{x})] - f'(\hat{x}) = 0. \tag{3.3}$$

Now, let us denote the Stackelberg objective function in (1.7) by

$$g(x) = xp[x + Q(x)] - f(x). \tag{3.4}$$

Then, the right-hand derivative of $g(x)$ is

$$g^+(x) = xp'[x + Q(x)][1 + Q^+(x)] + p[x + Q(x)] - f'(x). \quad (3.5)$$

From (3.3), we obtain

$$g^+(\hat{x}) = \hat{x}p'[\hat{x} + Q(\hat{x})]Q^+(\hat{x}). \quad (3.6)$$

From (2.1), it follows that $g^+(\hat{x}) \geq 0$. In order to prove this lemma, it is sufficient to show that

$$g^+(x) \geq 0 \qquad \text{for each} \qquad 0 < x \leq \hat{x}. \quad (3.7)$$

Toward this end, define the continuous function

$$h(x) = xp'[x + Q(x)] + p[x + Q(x)] - f'(x) \quad (3.8)$$

so that from (3.5), (2.1), we have,

$$g^+(x) = h(x) + xp'[x + Q(x)]Q^+(x) \geq h(x). \quad (3.9)$$

But from (3.3), note that $h(\hat{x}) = 0$. Thus, in order to prove (3.7), it is sufficient to show that

$$h^+(x) < 0 \qquad \text{for each} \qquad x > 0. \quad (3.10)$$

Hence, from (3.8), we obtain

$$h^+(x) = p'[x + Q(x)] + [1 + Q^+(x)]\{p'[x + Q(x)] + xp''[x + Q(x)]\} - f''(x).$$

Since $p(\cdot)$ is decreasing, $f(\cdot)$ is convex, we obtain

$$h^+(x) < [1 + Q^+(x)]\{p'[x + Q(x)] + xp''[x + Q(x)]\}. \quad (3.11)$$

Now, for a given $x > 0$, if $p''[x + Q(x)] \leq 0$, then from (2.1) and the fact that $p(\cdot)$ is decreasing, we obtain $h^+(x) < 0$. On the other hand, if $p''[x + Q(x)] > 0$ for a given $x > 0$, then again from (2.1), (3.11),

$$h^+(x) < [1 + Q^+(x)]\{p'[x + Q(x)] + [x + Q(x)]p''[x + Q(x)]\}. \quad (3.12)$$

Using (1.1), it follows that $h^+(x) < 0$ and the proof is complete.

We will now derive an upper bound for x^*, possibly lesser than the upper bound q^u of (1.4). First of all, let us establish that

$$x^* \leq q^u. \quad (3.13)$$

Using Equation 1.4 and the fact that $p(\cdot)$ is decreasing, we have,

$$p[x + Q(x)] - f'(x) \leq p(x) - f'(x) \leq 0 \qquad \text{for each} \qquad x \geq q^u. \quad (3.14)$$

Hence, using (3.14) in Equation (3.5) along with (2.1), we obtain

$$g^+(x) \leq 0 \qquad \text{for each} \qquad x \geq q^u \tag{3.15}$$

which in turn implies (3.13).

Now, it turns out that, if $f(\cdot)$ is strictly convex, we can do better than q^u, or at least as good. Recall that $\hat{x}$, the lower bound, arose from a follower-follower model in which each firm behaved as a Cournot firm. There is another follower-follower model discussed by Sherali et al. [1980] which yields an upper bound on x^*. In this model, the N-firms behave as Cournot firms, but the "Stackelberg" firm behaves as though it belonged to the competitive fringe (see Fellner or Sherali et al.), that is, it is content at an equilibrium to have adjusted its output to the level for which marginal cost equals price.

To formalize this statement, consider a set of output quantities ($\bar{x}$, $\bar{q}_1, \cdots, \bar{q}_N$) such that for each $i = 1, \cdots, N$, assuming $\bar{x}$ and $\bar{q}_j$, $j \neq i$ as fixed,

$$\bar{q}_i \text{ solves: } \text{maximize}_{q_i \geq 0}\{q_i p(q_i + \bar{x} + \textstyle\sum_{j=1, j\neq i}^{N} \bar{q}_j) - f_i(q_i)\}. \tag{3.16}$$

In other words, from (1.5), (1.6), $Q(\bar{x}) \equiv \sum_{i=1}^{N} \bar{q}_i$. In addition, for the "Stackelberg" firm, let $\bar{x}$ satisfy

$$\begin{aligned} &p[\bar{x} + Q(\bar{x})] = f'(\bar{x}) \qquad \text{if} \qquad \bar{x} > 0 \\ \text{and} \qquad &p[\bar{x} + Q(\bar{x})] \leq f'(\bar{x}) \qquad \text{if} \qquad \bar{x} = 0. \end{aligned} \tag{3.17}$$

Sherali et al. have shown that if in addition to Assumption A, $f(\cdot)$ is strictly convex, then a unique solution ($\bar{x}$, $\bar{q}_1, \cdots, \bar{q}_N$) satisfying (3.16) and (3.17) exists. It is obtained through the following family of mathematical programs, each of which involves the maximization of a strictly concave objective function over a nonempty convex, compact feasible region (where EPF stands for Equilibrating Program with a Fringe).

$$\begin{aligned} \text{EPF}(\bar{Q})\text{: maximize}&\{xp(\bar{Q}) - f(x)\} + \textstyle\sum_{i=1}^{N}\{q_i p(\bar{Q}) \\ &+ (1/2)q_i^2 p'(\bar{Q}) - f_i(q_i)\} \\ &\text{subject to } \textstyle\sum_{i=1}^{N} q_i + x = \bar{Q} \\ &\qquad x \geq 0, \quad q_i \geq 0 \qquad i = 1, \cdots, N. \end{aligned} \tag{3.18}$$

Basically, problems EPF$(\cdot)$ are used in the same spirit as problems EP$(\cdot, \cdot)$ were used in Theorem 1. Namely, one manipulates $\bar{Q}$ so that the optimal Lagrange multiplier $\bar{\lambda}(\bar{Q})$ associated with (3.18) is driven to zero. This is again an easy task since $\bar{\lambda}(\cdot)$ is shown to be strictly decreasing and continuous, with $\bar{\lambda}(0) \geq 0$ and $\bar{\lambda}(q^u) \leq 0$. This in fact asserts that $\bar{x} \leq q^u$. However, for the sake of completeness, we give a more direct proof of this as a part of the following result.

LEMMA 8. *Suppose that* $p(\cdot)$, $f(\cdot)$ *and* $f_i(\cdot)$ $i = 1, \cdots, N$ *satisfy*

Assumptions A and let $f(\cdot)$ *in addition be strictly convex. Define* $(\bar{x}, \bar{q}_1, \cdots, \bar{q}_N)$ *to be the (unique) solution to* (3.16), (3.17), *and let* q^u *be as given by* (1.4). *Then,*

$$x^* \leq \bar{x} \leq q^u. \tag{3.19}$$

Proof. In order to prove that $x^* \leq \bar{x}$, it is sufficient to show that

$$g^+(x) \leq 0 \qquad \text{for each} \qquad x > \bar{x} \tag{3.20}$$

where $g(\cdot)$ and $g^+(\cdot)$ are defined in (3.4) and (3.5), respectively. Toward this end, define the continuous function

$$v(x) = p[x + Q(x)] - f'(x), \qquad x \geq 0 \tag{3.21}$$

so that from (3.5), (2.1) and the fact that $p(\cdot)$ is decreasing, we have,

$$\begin{aligned} g^+(x) = xp'[x + Q(x)][1 + Q^+(x)] \\ + v(x) \leq v(x) \qquad \text{for all} \qquad x > 0. \end{aligned} \tag{3.22}$$

But from (3.17), $v(\bar{x}) \leq 0$. Thus, in order to show (3.20), it is sufficient to prove that

$$v^+(x) < 0 \qquad \text{for each} \qquad x > 0 \tag{3.23}$$

since then (3.22) and (3.23), along with the fact that $v(\bar{x}) \leq 0$, imply (3.20). Now, from (3.21),

$$\begin{aligned} v^+(x) = p'[x + Q(x)][1 + Q^+(x)] \\ - f''(x) < 0 \qquad \text{for each} \qquad x > 0 \end{aligned} \tag{3.24}$$

since $p(\cdot)$ is decreasing, $[1 + Q^+(x)] > 0$ from (2.1) and since $f''(x) > 0$ for each $x > 0$. This proves $x^* \leq \bar{x}$.

Now, in order to prove that $\bar{x} \leq q^u$ note that if $\bar{x} = 0$ then this is trivially true and if $\bar{x} > 0$, then from (3.17),

$$f'(\bar{x}) = p[\bar{x} + Q(\bar{x})] \leq p(\bar{x}). \tag{3.25}$$

But (1.4) asserts that

$$f'(q^u) \geq p(q^u). \tag{3.26}$$

Since $f'(\cdot)$ is increasing and $p(\cdot)$ is decreasing, it follows that $\bar{x} \leq q^u$ and the proof is complete.

We are now ready to discuss the existence and uniqueness of a SNC equilibrium and to present an algorithm to estimate such a solution.

THEOREM 3. *Let* $p(\cdot)$, $f(\cdot)$ *and* $f_i(\cdot)$, $i = 1, \cdots, N$ *satisfy Assumptions A. Then a Stackelberg-Nash-Cournot equilibrium exists. Furthermore, if* $Q(x)$ *turns out to be convex, then this equilibrium is unique.* (For example, as will be shown in Section 5, $Q(\cdot)$ is convex for the case of a linear demand curve and quadratic cost functions.)

Proof. Lemma 7 and Equation 3.13 assert that Problem SP defined in (1.7) essentially involves the maximization of a continuous objective function ($p[x + Q(x)]$ is continuous since $Q(x)$ is continuous) over a compact set $[\hat{x}, q^u]$ and, hence, an optimal solution x^* exists. (If $f(\cdot)$ is strictly convex, then Lemma 8 asserts that $x^* \in [\hat{x}, \bar{x}] \subseteq [\hat{x}, q^u]$.) Furthermore, for such an x^*, Theorem 1 shows that one can find (unique) quantities $[q_1(x^*), \cdots, q_N(x^*)]$ which simultaneously solve problems $CP_i(x^*)$ $i = 1, \cdots, N$ defined in (1.5). By definition then, $[x^*, q_1(x^*), \cdots, q_N(x^*)]$ would be a SNC equilibrium solution.

In order to prove the uniqueness assertion, it is sufficient to show that $Q(x)$ being convex implies that the (continuous) objective function $g(x)$ (Equation 3.4) of Problem SP is strictly concave on $(0, \infty)$. To prove this in turn, we will show that $g^+(\cdot)$ defined in (3.5) is strictly decreasing on $(0, \infty)$. Let us rewrite $g^+(x)$ in terms of $v(x)$ as in Equations 3.21, 3.22 and note that as shown in (3.23), $v(x)$ is strictly decreasing for $x > 0$. Hence, all we need to prove is that

$$w(x) = xp'[x + Q(x)][1 + Q^+(x)] \tag{3.27}$$

is a nonincreasing function of x on $(0, \infty)$. Toward this end, let $x, \epsilon > 0$. We will show that

$$\Delta = w(x + \epsilon) - w(x) \le 0. \tag{3.28}$$

From Equation 3.27, we obtain

$$\Delta = (x + \epsilon)p'[x + \epsilon + Q(x + \epsilon)][1 + Q^+(x + \epsilon)] - xp'[x + Q(x)][1 + Q^+(x)].$$

Rearranging this expression, we obtain

$$\Delta = xp'[x + Q(x)][Q^+(x + \epsilon) - Q^+(x)] + [1 + Q^+(x + \epsilon)] \cdot \{(x + \epsilon)p'[x + \epsilon + Q(x + \epsilon)] - xp'[x + Q(x)]\}. \tag{3.29}$$

But since $Q(\cdot)$ is convex then $Q^+(\cdot)$ is nondecreasing, and since $p(\cdot)$ is decreasing, we know that

$$xp'[x + Q(x)][Q^+(x + \epsilon) - Q^+(x)] \le 0. \tag{3.30}$$

Furthermore, from (2.1) and our assumption (1.1), it is readily verified that $qp'[q + Q(q)]$ is a nonincreasing function of q, and since from (2.1), $[1 + Q^+(x + \epsilon)] > 0$, we obtain

$$[1 + Q^+(x + \epsilon)]\{(x + \epsilon)p'[x + \epsilon + Q(x + \epsilon)] - xp'[x + Q(x)]\} \le 0. \tag{3.31}$$

Using (3.30) and (3.31) in Equation 3.29 shows that (3.28) is true, and this completes the proof.

COROLLARY. *In addition to Assumptions A, if* $f(\cdot)$ *is strictly convex and if* $\bar{x}$ *given through* (3.16), (3.17) *turns out to be zero, then* $[0, q_1(0), \cdots, q_N(0)]$ *is a unique SNC equilibrium.* (The proof follows trivially from (3.19) and Theorem 1.)

Now, Problem SP defined in (1.7) is (to our knowledge) a nonconvex program. The (continuous) objective function $g(x)$ is nondecreasing on $(0, \hat{x}]$ (from the proof of Lemma 7) and is nonincreasing beyond q^u (from Equation 3.15) or beyond $\bar{x}$ if $f(\cdot)$ is strictly convex (from Lemma 8). But in the interval $(\hat{x}, q^u)$ (or $(\hat{x}, \bar{x})$) we are uncertain of its behavior, other than the fact that it is continuous. However, Theorem 3 above provides a strategy for obtaining a SNC equilibrium solution with as close an accuracy as desired. Namely, suppose we approximate $Q(x)$ by a piecewise linear curve which coincides with $Q(x)$ at each breakpoint. Since the approximating $Q(\cdot)$ is convex over each individual segment, a unique optimum to SP may be found for each segment. Observe that this is tantamount to approximating $g(x)$ by a piecewise strictly concave objective function, each such piece coinciding with $g(x)$ at its endpoints. This is the subject matter of the following section.

4. AN ALGORITHM TO SOLVE THE STACKELBERG PROBLEM

Let us begin this section by formalizing the strategy presented above. Toward this end, consider t grid points $x_1 < x_2 < \cdots < x_t$, where $x_1 \equiv \hat{x}$ (Lemma 7) and $x_t = q^u$ (Equation 3.13) or $x_t = \bar{x}$ if $f(\cdot)$ is strictly convex (as in Lemma 8). Furthermore, let us use the following piecewise linear approximation of $Q(x)$ which agrees with $Q(x)$ at each of the grid points $x_1, \cdots, x_t$ and is linear between any two consecutive grid points

$$Q_k(x) = Q(x_k) + \gamma_k(x - x_k) \quad \text{for} \quad x_k \le x \le x_{k+1}, \quad k = 1, \cdots, t-1 \tag{4.1}$$

where

$$\gamma_k = (Q(x_{k+1}) - Q(x_k))/(x_{k+1} - x_k). \tag{4.2}$$

Note from Equation 2.1 that

$$\begin{aligned} \gamma_k = 0 \quad & \text{if} \quad Q(x_k) = 0, \quad \text{whence,} \\ & Q(x) = 0 \quad \text{for each} \quad x \ge x_k \\ -1 < \gamma_k < 0 \quad & \text{if} \quad Q(x_k) > 0. \end{aligned} \tag{4.3}$$

Hence, over the interval $[x_k, x_{k+1}]$, with $Q(x)$ approximated by $Q_k(x)$ of

Equation 4.1, the Stackelberg problem defined in (1.7) becomes

$$\underline{\text{SP}k}\text{: maximize}_{x_k \le x \le x_{k+1}} g_k(x) \equiv \{xp(a_k x + b_k) - f(x)\} \tag{4.4}$$

where,

$$a_k = 1 + \gamma_k > 0 \qquad \text{and} \qquad b_k = Q(x_k) - \gamma_k x_k \ge 0. \tag{4.5}$$

Using Lemma 1, it is easily verified that Problem SPk defined in (4.4) has a strictly concave objective function and is hence readily solvable. Let $x_k{}^*$ be the unique optimum. If $x_k{}^*$ is an endpoint of the interval $[x_k, x_{k+1}]$, then $g(x_k{}^*) \equiv g_k(x_k{}^*)$. However, if $x_k < x_k{}^* < x_{k+1}$, one needs to evaluate $Q(x_k{}^*)$ in order to determine $g(x_k{}^*)$, the estimated optimal objective function value in the interval $[x_k, x_{k+1}]$. This is accomplished as before through the use of Theorem 1.

In this manner, let us suppose that we have evaluated $Q(x)$, and hence $g(x)$, exactly at the grid points $x_1, \cdots, x_k, x_{k+1}, \cdots, x_m$, where $m \ge t$, and where each grid point is either one of the original t grid points or is an interior optimum for some SPk problem. The question we raise is, if one terminates at this stage with the best of these m points as an (estimated) optimal Stackelberg solution, then what is the maximum error or deviation from the actual optimum objective value? The following result addresses this issue.

LEMMA 9. *Consider the Stackelberg Problem SP defined in* (1.7) *and let* $g(x)$ *be its objective function* (*as in Equation* 3.4). *Assume that* $p(\cdot)$, $f(\cdot)$ *and* $f_i(\cdot)$, $i = 1, \cdots, N$ *satisfy Assumptions A. Let* x_k, x_{k+1} *be two consecutive grid points at which* $g(x)$ *has been evaluated* (*exactly*) *and compute*

$$h_k = p[x_{k+1} + Q(x_{k+1})] + x_{k+1} p'[x_{k+1} + Q(x_{k+1})] - f'(x_{k+1}) \le 0 \tag{4.6}$$

and
$$v_k = p[x_k + Q(x_k)] - f'(x_k). \tag{4.7}$$

Let $g_k{}^*$ *be the actual maximum objective function value of Problem SP in the interval* $[x_k, x_{k+1}]$ *and note that our estimate of this value is simply* $\max\{g(x_k), g(x_{k+1})\}$. *Then, the error in our estimate*

$$\Delta_k = g_k{}^* - \max\{g(x_k), g(x_{k+1})\} \tag{4.8}$$

satisfies the following:

$$\Delta_k = 0 \quad \text{if} \quad v_k \le 0 \tag{4.9}$$

$$0 \le \Delta_k \le v_k\{([g(x_{k+1}) - g(x_k)] - h_k(x_{k+1} - x_k))/(v_k - h_k)\} \quad \text{if} \quad v_k > 0 \quad \textit{and} \quad g(x_{k+1}) \le g(x_k) \tag{4.10}$$

and

$$0 \le \Delta_k \le h_k\{([g(x_{k+1}) - g(x_k)] - v_k(x_{k+1} - x_k))/(v_k - h_k)\} \quad \text{if} \quad v_k > 0 \quad \text{and} \quad g(x_{k+1}) \ge g(x_k). \tag{4.11}$$

Proof. The proof is fairly straightforward. Suppose we can show that $g^+(x)$ defined in (3.5) satisfies

$$h_k \le g^+(x) \le v_k \quad \text{for} \quad x_k < x < x_{k+1} \tag{4.12}$$

where $h_k \le 0$. If $v_k \le 0$, then clearly (4.9) holds. On the other hand, if $v_k > 0$, then the maximum possible value of the error Δ_k of (4.8) would occur if $g(\cdot)$ were to increase from $g(x_k)$ at the rate v_k and then decrease

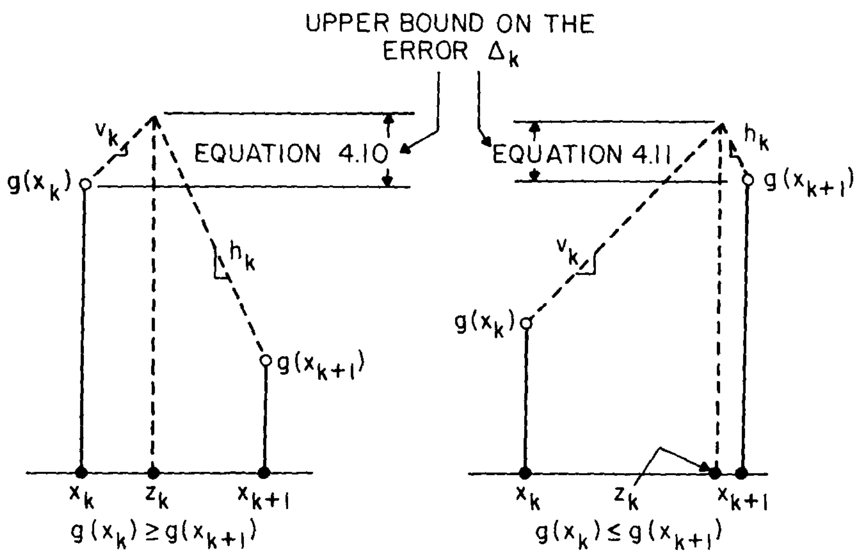

Figure 1. Bounds on the error Δ_k.

at the rate $|h_k|$ to the value $g(x_{k+1})$. This is illustrated in Figure 1. Note that z_k in Figure 1 is defined by

$$g(x_k) + v_k(z_k - x_k) = g(x_{k+1}) - h_k(x_{k+1} - z_k)$$

so that

$$z_k = (g(x_{k+1}) - g(x_k) + v_k x_k - h_k x_{k+1})/(v_k - h_k).$$

It then follows that

$$z_k - x_k = (g(x_{k+1}) - g(x_k) - h_k(x_{k+1} - x_k))/(v_k - h_k)$$

and

$$x_{k+1} - z_k = (-g(x_{k+1}) + g(x_k) + v_k(x_{k+1} - x_k))/(v_k - h_k).$$

The bounds in (4.10) and (4.11) are then simply $v_k(z_k - x_k)$ and $-h_k(x_{k+1} - z_k)$, respectively.

Hence, to prove Lemma 9, let us establish (4.12). Toward this end, define the continuous functions (as in Section 3)

$$h(x) = p[x + Q(x)] + xp'[x + Q(x)] - f'(x), \qquad x \geq 0 \quad (4.13)$$

and

$$v(x) = p[x + Q(x)] - f'(x), \qquad x \geq 0. \quad (4.14)$$

Observe from Equations 3.5 and 3.9 that $g^+(x) \geq h(x)$ for all $x > 0$. Moreover, Equation 3.10 asserts that $h^+(x) < 0$, $x > 0$, and Equation 3.3 states that $h(\hat{x}) = 0$. Thus, since $x_k \geq \hat{x}$, $h(x) \leq 0$ for all $x \in (x_k, x_{k+1})$ and moreover,

$$g^+(x) \geq h(x) \geq h(x_{k+1}) \equiv h_k \qquad \text{for all} \quad x \in (x_k, x_{k+1}) \quad (4.15)$$

as defined in (4.6).

In a similar manner, Equation 3.22 asserts that $g^+(x) \leq v(x)$ for all $x > 0$, where $v^+(x) < 0$, $x > 0$ (Equation 3.23). Hence, we obtain (using continuity of v in case $x_k \equiv 0$),

$$g^+(x) \leq v(x) \leq v(x_k) \equiv v_k \qquad \text{for all} \quad x \in (x_k, x_{k+1}) \quad (4.16)$$

as defined in (4.7). Equations 4.15 and 4.16 establish (4.12) and the proof is complete.

In concluding this section, let us point out an expedient, due to Theorem 1, in the evaluation of the quantities $Q(x)$. Observe from (2.1) that since $Q(\cdot)$ is a continuous, monotone decreasing function, if one has already evaluated $Q(x_p)$ and $Q(x_r)$, and if $x_p < x_q < x_r$, then $Q(x_r) \leq Q(x_q) \leq Q(x_p)$. Hence, in evaluating $Q(x_q)$, one may use the bounds $Q_l \equiv Q(x_r)$ and $Q_u \equiv Q(x_p)$ for finding the root of $\lambda(x_q, \cdot)$ as in Theorem 1. One way of using this information is as follows.

First determine $Q(x_1)$ using the bounds $[Q_l, Q_u] \equiv [0, q^u]$. Next, determine $Q(x_t)$ using the bounds $[Q_l, Q_u] \equiv [0, Q(x_1)]$. Thereafter, determine $Q(x_2)$ using the bounds $[Q_l, Q_u] \equiv [Q(x_t), Q(x_1)]$. Next, determine $Q(x_{t-1})$ using the bounds $[Q_l, Q_u] \equiv [Q(x_t), Q(x_2)]$ and so on. Similarly, while solving the interval problem SPk, if $x_k{}^*$ turns out to be an interior point in $[x_k, x_{k+1}]$, then for evaluating $Q(x_k{}^*)$, use the bounds $[Q_l, Q_u] \equiv [Q(x_{k+1}), Q(x_k)]$.

To summarize, one first delineates a set of grid points $x_1, \cdots, x_t$, say, equally spaced in the interval $[\hat{x}, q^u]$ or $[\hat{x}, \bar{x}]$ as the case might be. Then, one evaluates $Q(x_i)$, $i = 1, \cdots, t$ using the above expedient. Next, one solves the interval problems SPk, $k = 1, \cdots, t - 1$ defined in (4.4) and evaluates $Q(x_k{}^*)$ and $g(x_k{}^*)$ for optimal solutions which turn out to be interior points. Again, the above mentioned expedient is used in this evaluation. Thus far, one has evaluated $g(x_i)$ for some grid points $x_1 <$

$x_2 < \cdots < x_m$. One may now either terminate with the best of these grid points as an (estimate of the) optimal solution or choose to further refine the grid in an appropriate region. In any case, on termination, Lemma 9 gives an estimate of the maximum error in the optimal objective function value. Of course, the decision to further refine a grid or terminate may be used on this maximum error estimate and some prespecified error tolerance.

5. ILLUSTRATIVE EXAMPLE AND COMPUTATIONAL CONSIDERATIONS

In order to simplify the presentation, let us consider the case of a linear demand curve and quadratic cost functions. Specifically, let

$$p(q) = a - bq, \qquad \text{where} \quad a > 0, \qquad b > 0 \tag{5.1}$$

and define

$$f_i(q) = (\tfrac{1}{2})cq^2 \qquad \text{for} \quad i = 1, \cdots, N, \qquad \text{with} \quad c > 0 \tag{5.2}$$

and

$$f(q) = (\tfrac{1}{2})dq^2, \qquad d > 0. \tag{5.3}$$

Again, in order to obtain simple closed form expressions, we have assumed that all the Cournot firms are identical. Thus, by symmetry, Equation 1.5 for each Cournot firm i yields $q_i(x)$ as an optimal solution to

$$\text{maximize}_{q_i \geq 0}\{q_i[a - b\{q_i + x + (N-1)q_i(x)\}] - (\tfrac{1}{2})cq_i^2\}. \tag{5.4}$$

It is easily verified through the first order conditions for the convex program (5.4) that

$$q_i(x) = \begin{cases} (a - bx)/(c + b(N+1)) & \text{if} \quad 0 \leq x \leq a/b \\ 0 & \text{otherwise} \end{cases} \tag{5.5}$$

$$\text{for} \quad i = 1, \cdots, N$$

whence,

$$Q(x) = \begin{cases} N(a - bx)/(c + b(N+1)) & \text{if} \quad 0 \leq x \leq a/b \\ 0 & \text{otherwise.} \end{cases} \tag{5.6}$$

Observe that the aggregate reaction curve $Q(\cdot)$ defined in (5.6) is continuous, piecewise linear and convex. Incidentally, $Q(\cdot)$ has a form similar to that in (5.6) even with the Cournot cost functions defined nonidentically as $f_i(q_i) = (\tfrac{1}{2})c_i q_i^2$, $i = 1, \cdots, N$. Observe also that

$$Q^+(x) = \begin{cases} -Nb/(c + b(N+1)) & \text{if} \quad Q(x) > 0 \\ 0 & \text{if} \quad Q(x) = 0 \end{cases} \tag{5.7}$$

which supports (2.1). Furthermore, the Stackelberg problem (1.7) in this case becomes

$$\text{maximize}_{0 \le x \le a/b}\{x[a - b\{x + N(a - bx)/ \\ (c + b(N + 1))\}] - (\tfrac{1}{2})dx^2\}, \tag{5.8}$$

which is a convex program. The (unique) optimal solution to (5.8) is readily obtained as

$$x^* = a(b + c)/(2b(b + c) + d(b + c) + bdN). \tag{5.9}$$

Observe that $0 < x^* < a/b$ so that $q_i(x^*) > 0$ for $i = 1, \cdots, N$, where, $[x^*, q_1(x^*), \cdots, q_N(x^*)]$ is the unique equilibrium solution.

As far as the algorithm of the foregoing section is concerned, any choice of grid points in the interval $[0, a/b]$ would yield the optimal solution since the approximation of $Q(\cdot)$ would be exact in this case.

In general, note that the algorithm of Section 4 involves two types of computations, namely, the evaluation of the function $Q(\cdot)$ and the solution of problems SPk defined in (4.4). The evaluation of $Q(x)$ for a given $x \geq 0$ is done via Problems EP(x, Q), as indicated in Theorem 1, by a univariate line search which finds that value $Q(x)$ of Q which makes the Lagrange multiplier associated with (1.9) equal to zero. The experience reported by Murphy, et al. indicates that this can be done efficiently. Further, problem SPk involves the maximization of a univariate concave function over a compact interval, and a host of well known efficient methods exist for this problem. Thus, by taking advantage of the expedients outlined in Section 4, the algorithm may be implemented in an effective manner. At termination, Lemma 9 gives the maximum error in the optimal objective value, which can be made arbitrarily small at the expense of additional effort by way of further grid refinements.

ACKNOWLEDGMENT

The authors would like to thank Professor William Hogan and two anonymous referees for their painstaking reading and insightful comments on this paper.

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