Theorem 2: The stable part of U is given by

$$[U(s)]_+ = [I - \tilde{B}^* V_1 (sI + \Lambda_1)^{-1} \Lambda_1 Y_1 \tilde{A}^{-1} \tilde{B}]\Theta. \tag{27}$$

Proof: It follows from (20), (22), (25), and (26) that the stable part is given by

$$[U(s)]_+ = [I - \tilde{B}^* V_1 (sI + \Lambda_1)^{-1} Y_1 \tilde{F}^{-1} \tilde{B}]\Theta. \tag{28}$$

Since Γ is of full rank, $\tilde{A}$ is invertible. We, thus, have that

$$F^{-1} = V\Lambda_0 V^{-1} \tilde{A}^{-1}$$

and (27) readily follows from (28).

III. Conclusions

A simple expression is derived in closed form for the all-pass transfer function matrix U that satisfies directional interpolation requirements. The number of the unstable poles of U is a function of the scalar parameter λ and it varies from zero, for large enough values of this parameter, to ρ, the number of the interpolation conditions that is obtained for λ that is close to zero. In spite of the fact that a Pick matrix F is used to obtain the all-pass matrix and its properties, we have not used here the Nevanlinna–Pick iterative approach. The state-space realization of U is directly expressed in terms of the interpolation requirements and the matrix F, which enables an easy evaluation of the required all-pass matrix and provides a clear insight to the structure of the interpolation solution.

The above result has been derived under the assumption of simple interpolation requirements for U. More involved requirements that correspond to nondistinct zeros in the H_∞ interpolation problem can be readily found using the method of [18].

Except for the case where λ equals λ_i, $i = 1, \cdots, \rho$ of (19), the order of U is ρ and a clear distinction is observed between the stable and the unstable poles of U. As λ comes closer, from above, to one of the above λ_i, one of the poles of U in the left half plane tends to infinity and, as λ crosses this value of λ_i, the corresponding pole enters the right half plane from infinity. At $\lambda = \lambda_i$ the all-pass matrix becomes of order $\rho - 1$.

References

[1] B. A. Francis and G. Zames, "On H^∞-optimal sensitivity theory for SISO feedback systems," *IEEE Trans. Automat. Contr.*, vol. AC-29, pp. 9–16, 1984.

[2] B. C. Chang and J. B. Pearson, "Optimal disturbance reduction in linear multivariable systems," *IEEE Trans. Automat. Contr.*, vol. AC-29, pp. 880–887, 1984.

[3] A. Tannenbaum, "Modified Nevanlinna–Pick interpolation of linear plants with uncertainty in the gain factor," *Int. J. Contr.*, vol. 36, pp. 331–336, 1982.

[4] H. Kimura, "Robust stabilization for a class of transfer functions," *IEEE Trans. Automat. Contr.*, vol. AC-29, pp. 788–793, 1984.

[5] D. C. Youla and M. Saito, "Interpolation with positive-real functions," *J. Franklin Inst.*, vol. 284, pp. 77–108, 1967.

[6] J. A. Ball and J. W. Helton, "Lie group over the field of rational functions, signed spectral factorization, signed interpolation and amplifier design," *J. Operator Theory*, vol. 8, pp. 19–64, 1982.

[7] H. Kimura, "Directional interpolation approach to H^∞-optimization and robust stabilization," *IEEE Trans. Automat. Contr.*, vol. AC-32, pp. 1085–1093, 1987.

[8] R. Nevanlinna, "Uber Beschrankte analytische Funktionen," *Ann. Acad. Sci. Fenn.*, vol. A32, pp. 1–75, 1929.

[9] Ph. Delsarte, Y. Genin, and Y. Kamp, "The Nevanlinna–Pick problem for matrix-valued functions," *SIAM J. Appl. Math.*, vol. 36, pp. 47–61, 1979.

[10] M. G. Safonov and M. Verma, "L_∞-sensitivity optimization and Hankel approximation," *IEEE Trans. Automat. Contr.*, vol. AC-30, pp. 279–280, 1985.

[11] K. Glover, "All optimal Hankel-norm approximations of linear multivariable systems and their L_∞-error bounds," *Int. J. Contr.*, vol. 39, pp. 1115–1193, 1984.

[12] Ph. Delsarte, Y. Genin, and Y. Kamp, "On the role of the Nevanlinna-Pick problem in circuit and system theory," *Circuit Theory Appl.*, vol. 9, pp. 177–187, 1981.

[13] Y. V. Genin and S. Y. Kung, "A two-variable approach to the model reduction problem with Hankel norm criterion," *IEEE Trans. Circuits Syst.*, vol. CAS-28, pp. 912–924, 1981.

[14] U. Shaked, "The explicit structure of inner matrices and its application in H^∞ optimization," *IEEE Trans. Automat. Contr.*, vol. AC-34, pp. 734–738, 1989.

[15] A. Ostrowski and H. Schneider, "Some theorems on the inertia of general matrices," *J. Math. Anal. Appl.*, vol. 4, pp. 72–84, 1962.

[16] R. Bellman, *Introduction to Matrix Analysis.* New York: McGraw-Hill, 1960.

[17] D. Carlson and R. Loewy, "On ranges of Lyapunov transformations," *Linear Algebra Appl.*, vol. 8, pp. 237–248, 1974.

[18] U. Shaked, "The structure of inner matrices that satisfy multiple directional interpolation requirements," *IEEE Trans. Automat. Contr.*, vol. 34, pp. 1293–1296, 1989.

[19] H. Kimura, "Directional interpolation in the state-space," *Syst. Contr. Lett.*, vol. 10, pp. 317–324, 1988.

[20] ——, "Conjugation, interpolation, and model-matching in H^∞," *Int. J. Contr.*, vol. 49, pp. 269–307, 1989.

A Solution Method for the Linear Static Stackelberg Problem Using Penalty Functions

G. ANANDALINGAM and D. J. WHITE

***Abstract*—This note presents a new solution technique for the linear constrained static Stackelberg problem. The duality gap of the follower's problem is appended to the leader's objective with a penalty. This structure leads to the decomposition of the composite problem into a series of linear programs leading to an efficient algorithm. We prove that local optimality is reached for an exact penalty function and illustrate the method with some examples.**

I. Introduction

Stackelberg games can be static or dynamic (see [8] for a classification). Dynamic Stackelberg games have been the focus of much research by control theorists in recent years [7], [9]–[13]. There is related research in the area of *static* Stackelberg problems [1], [4], which in mathematical programming focuses on linear systems [2], [5], [6].

In the static Stackelberg problem [11], the "leader" controls decision vector $x \in R_+^N$ and the "follower" controls $y \in R_+^M$, respectively. The leader is assumed to select his decision vector first and the follower to select his decision vector after that.

Using this notation, the *linear* Stackelberg problem is formulated as follows.

P1:

$$\operatorname*{Max}_x F(x, y(x)) = ax + by(x) \tag{1}$$

subject to

$$G(x, y(x)) = Ax + By(x) - p \leq 0 \tag{2}$$

where, for given x, y solves

$$\operatorname*{Max}_y f(x, y) = cx + dy \tag{3}$$

subject to

$$g(x, y) = Cx + Dy - q \leq 0 \tag{4}$$

$$x \geq 0, \qquad y \geq 0. \tag{5}$$

We make the following usual *assumption*.

A1: For any fixed $x \in R_+^N$, an optimal solution $y(x) \in R_+^M$ exists. It need not be unique.

In this note, we provide a new algorithm (cf. [2], [5], [6]) based on a penalty function approach, for solving the *linear* Stackelberg problem which is different from others in the literature. For a given value of x, the leader's decision vector, the follower is at his rational reaction set when the duality gap of the second-level problem becomes zero. We solve the

Manuscript received March 24, 1989; revised August 4, 1989 and February 16, 1990.

G. Anandalingam is with the Department of Systems, University of Pennsylvania, Philadelphia, PA 19104.

D. J. White is with the Department of Decision Theory, University of Manchester, Manchester, M13 9PL England.

IEEE Log Number 9036822.

0018-9286/90/1000-1170$01.00

outer problem by appending to the leader's objective, a function that minimizes the duality gap of the follower's problem.

There are related papers by Aiyoshi and Shimuzu [1] and [11] that use a penalty function approach for solving a problem with *nonlinear* objectives. These papers are significantly different from ours. First, the penalty function in the linear Stackelberg problem is not convex and, thus, solution techniques have to be devised that take into account this particularity. This is the essence of our note. In the Aiyoshi-Shimizu papers, the penalty function is convex, thus, making the problems much easier to solve using, for instance, the conjugate gradient method. Second, our method solves a sequence of linear programs and is more elegant than the methods proposed by Aiyoshi-Shimizu. Third, while Aiyoshi-Shimizu [1] prove that their algorithm will obtain the optimal solution as the penalty (barrier) tends to infinity (zero), in our methodology there is an exact penalty function.

II. A Penalty Function Approach

For a given x, cx in (3) is just a constant and, thus, the follower's *primal* problem becomes the following.

P2:

$$\operatorname*{Max}_{y} dy \tag{6}$$

subject to

$$Dy \leq q - Cx \tag{7}$$

$$y \geq 0. \tag{8}$$

The dual of P2 is the following.

P3:

$$\operatorname*{Min}_{w} w'(q - Cx) \tag{9}$$

subject to

$$w'D \geq d \tag{10}$$

$$w \geq 0. \tag{11}$$

Note that, given x and some values of w and y that satisfy the dual and primal constraints of the follower's problem, the optimal solution lies in the interval $[dy, w'(q - Cx)]$. When the duality gap given by $\pi(x, y, w) = [w'(q - Cx) - dy]$ is equal to zero, then the follower's optimal solution (for the given $x \in R_+^N$) would be reached. We can employ a penalty function approach and formulate the overall problem P1 as follows.

P4:

$$P(K) = \operatorname*{Max}_{x,y,w} \hat{F}(x, y, w, K) = (ax + by) - K[w'(q - Cx) - dy] \tag{12}$$

subject to

$$Ax + By \leq p \tag{13}$$

$$Cx + Dy \leq q \tag{14}$$

$$w'D \geq d \tag{15}$$

$$x, y, w \geq 0 \tag{16}$$

where K is a large positive constant. Clearly, P4 will reach optimality when $[w'(q - Cx) - dy] \to 0$.

III. Theoretical Results

We will now derive the theory necessary for developing an algorithm for solving the linear Stackelberg problem using the penalty function approach.

Definition: The *feasible region of* w is given by

$$W = \{w\colon\ w'D \geq d,\ w \geq 0\} \tag{17}$$

and the *feasible region of* $z = (x, y)$ is given by

$$Z = \{(x, y)\colon\ Ax + By \leq p,\ Cx + Dy \leq q,\ x \geq 0,\ y \geq 0\}. \tag{18}$$

We assume the following.

A2: W and Z are nonempty bounded polyhedra, i.e., polytopes.

We will denote the extreme points of W and Z by W_v and Z_v, respectively. Since W and Z are polyhedra, W_v and Z_v characterize the set of vertices of these polyhedra.

Theorem 1: For a given value of w and fixed K, suppose we define

$$\Theta(w, K) = \operatorname*{Max}_{x,y} \hat{F}(x, y, w, K) \qquad \text{subject to } (x, y) \in Z. \tag{19}$$

Then $\Theta(w, K)$ is convex, and a solution to the problem

$$\operatorname*{Max}_{w} \Theta(w, K) \qquad \text{subject to } w \in W$$

will occur at some $w^* \in W_v$.

Proof: Let w_1 and w_2 be feasible, $l(x, y) = ax + (b + Kd)y$, and $\lambda \in (0, 1)$. Then

$$\begin{aligned}
&\Theta[\lambda w_1 + (1-\lambda)w_2, K] \\
&\quad = \operatorname*{Max}_{(x,y)\in Z} \{l(x, y) - K[\lambda w_1 + (1-\lambda)w_2]'(q - Cx)\} && (20)\\
&\quad = \operatorname*{Max}_{(x,y)\in Z} \{\lambda[l(x, y) - Kw_1'(q - Cx)] \\
&\qquad + (1-\lambda)[l(x, y) - Kw_2'(q - Cx)] && (21)\\
&\quad \leq \lambda \operatorname*{Max}_{(x,y)\in Z} [l(x, y) - Kw_1'(q - Cx)] \\
&\qquad + (1+\lambda) \operatorname*{Max}_{(x,y)\in Z} [l(x, y) - Kw_2(q - Cx)] \\
&\quad \leq \lambda\Theta(w_1, K) + (1-\lambda)\Theta(w_2, K).
\end{aligned}$$

Thus, $\Theta(w, K)$ is convex. By assumption A2, maximizing a convex function $\Theta(w, K)$ over the nonempty compact polyhedron W will yield an optimal solution at a vertex of W, i.e., at a point of W_v (see [3, Theorem 3.4.6.]). ⊗

Theorem 1 yields the following theorem.

Theorem 2: For a fixed K, an optimal solution to problem P4 is achievable in $Z_v \times W_v$ and $Z_v \times W_v = (Z \times W)_v$.

Proof: Let $z = (x, y)$. For $w = w^* \in \arg\max [\Theta(w, K)]$, $\hat{F}(z, w^*, K)$ is an *affine* function of z. Because Z is a polytope, maximizing $\hat{F}(z, w^*, K)$ subject to $z \in Z$ will have a solution $z^* \in Z_v$. This proves the first part.

In order to prove the second part, we have to prove the following. i) $Z_v \times W_v \subset (Z \times W)_v$ and ii) $(Z \times W)_v \subset Z_v \times W_v$.

i) Let $(z, w) \in (Z_v \times W_v)\backslash(Z \times W)_v$ then $(z, w) \in (Z \times W)\backslash (Z \times W)_v$ and $\exists\alpha \in (0, 1)$ and $(z^1, w^1), (z^2, w^2) \in Z \times W$ with $(z^1, w^1) \neq (z^2, w^2)$ such that

$$(z, w) = \alpha(z^1, w^1) + (1-\alpha)(z^2, w^2). \tag{22}$$

Then

$$z = \alpha z^1 + (1-\alpha)z^2$$

$$w = \alpha w^1 + (1-\alpha)w^2$$

and either $z^1 \neq z^2$ and/or $w^1 \neq w^2$. Thus, either $z \notin Z_v$ and/or $w \notin W_v$, which is a contradiction. Thus, $(z, w) \in Z_v \times W_v \Rightarrow (z, w) \in (Z \times W)_v$. This proves i).

ii) Now, let

$$(z, w) \in (Z \times W)_v \backslash (Z_v \times W_v). \tag{23}$$

Then *either* $\exists z^1, z^2 \in Z$, $z^1 \neq z^2$ such that

$$z = \alpha z^1 + (1-\alpha)z^2, \qquad \alpha \in (0, 1) \tag{24}$$

or $\exists w^1, w^2 \in W$, $w^1 \neq w^2$ such that

$$w = \beta w^1 + (1-\beta)w^2, \qquad \beta \in (0, 1). \tag{25}$$

Suppose (24) is true; then

$$(z, w) = \alpha(z^1, w) + (1-\alpha)(z^2, w).$$

However, $(z^1, w), (z^2, w) \in Z \times W$, but $(z^1, w) \neq (z^2, w)$. This contradicts (23). Now suppose (25) is true; then

$$(z, w) = \beta(z, w^1) + (1-\beta)(z, w^2).$$

However, $(z, w^1), (z, w^2) \in Z \times W$, but $(z, w^1) \neq (z, w^2)$. This contradicts (23). Hence, $(z, w) \in (Z \times W)_v \subset Z_v \times W_v$. ⊗

Theorems 1 and 2 were based on a fixed value of K. We now show that a finite value of K would yield an exact solution to the overall problem P4 where the duality gap $[w'(q - Cx) - dy]$ becomes zero.

Theorem 3: There exists a finite value of K, K^* say, for which the solution to the penalty function problem P4 yields a solution to the overall linear Stackelberg problem P1.

Proof: We can write $\hat{F}(x, y, w, K)$ in (12) as

$$\hat{F}(x, y, w, K) = (ax + by) - K\pi(x, y, w) \tag{26}$$

where $\pi(x, y, w)$ is the duality gap.

Suppose (x^*, y^*, w^*) is the solution to problem P1, the linear Stackelberg problem, then the duality gap is zero, i.e., $\pi(x^*, y^*, w^*) = 0$.

For $(x, y, w) \in \operatorname{argMax} \hat{F}(x, y, w, K)$

$$ax + by - K\pi(x, y, w) \geq ax^* + by^*.$$

Thus,

$$\pi(x, y, w) \leq [ax + by - ax^* - by^*]/K$$

$$\pi(x, y, w) \leq \max[(ax + by) - (ax^* + by^*)]/K \leq k/K \tag{27}$$

where k is some constant. Note that $\pi(x, y, w) \geq 0$ for all $(x, y, w) \in ZxW$. Thus, as $K \to \infty$, $\pi(x, y, w) \to 0$. However, since $Z_v \times W_v$ is finite, $\pi(x, y, w) = 0$ for some large finite value of K, say K^*. ⊗

We now show that, by increasing K monotonically, we can achieve the local optimal solution of the linear Stackelberg game. For that, we need the following theorem that is the essence of penalty function methods.

Theorem 4: The leader's objective $F(x, y)$ and the duality gap of the follower's rational reaction problem $\pi(x, y, w)$ are both monotonically nonincreasing in the penalty value K.

Proof: The proof is fairly straightforward (see [3], for instance).

Theorem 5: The penalty function approach yields the optimal solution of P1.

Proof: We will provide a logical proof. Optimality is reached when one can get the largest possible value of $F(x, y)$ and also satisfy the optimality conditions of the follower which is achieved when $\pi(x, y, w)$ goes to zero. The latter is achieved monotonically (by Theorem 4) and at a finite K (by Theorem 3). ⊗

Let $\{x(w), y(w)\}$ be the solution to $\Theta(w)$ (19). In the latter case, we suppress the K for notational simplicity.

Theorem 6: For $u, w \in W$

$$\Theta(u) \geq \Theta(w) - K(u - w)(q - Cx(w)). \tag{28}$$

Proof: Since $\Theta(u)$ is convex (by Theorem 1) for all $u \in W$

$$\Theta(w) = ax(w) + by(w) - K(w'(q - Cx(w)) - dy(w)) \tag{29}$$

and

$$\Theta(u) \geq ax(w) + by(w) - K(u'(q - Cx(w)) - dy(w)). \tag{30}$$

Subtract (29) from (30) and the result follows. ⊗

From Theorem 6, the following is clear.

$$\text{If } \chi(u, w) = \operatorname*{Min}_u (u - w)(q - Cx(w)) < 0, \qquad \text{some } u \in W,$$

$$\text{then } w \neq w^* \in \operatorname{argMax}\{\Theta(w)\colon w \in W\}. \tag{31}$$

By (31), if $\chi(w^*) < 0$, and is reached at $u = u^*$, then we can select u^* as the next vertex to go to. Conversely, if $\chi(w^*) \geq 0$, w^* remains unchanged, and subsequently, the solution to P4 remains unchanged at $[x(w^*), y(w^*)]$. Thus, $[x(w^*), y(w^*)]$ is the best solution for the current value of K. At this value of K, if the duality gap $\pi(x(w^*, y(w^*), w^*))$ is zero, then by Theorem 4, we are at a *local* optimal solution for the problem. If not, we increase K, and go through another iteration.

IV. The Algorithm

Step 0 (Initialization):
$i = 0$.
Choose $K = 0$, $w^0 \in W$, and λ.

Step 1
Solve $\text{Max}_{(x,y)\in Z}\{(a + Kw^iC)x + (b + Kd)y\}$
Obtain $\{x(w^i, K), y(w^i, K)\}$.

Step 2
Solve $\chi(w^i, K) = \text{Min}_{w\in W}(w - w^i)[q - Cx(w^i, K)]$
Obtain $w^*(w^i, K)$.

Step 3 (Optimality Test):
i) If $\chi(w^i, K) < 0$
Then set $w^{i+1} = w^*(w^i, K)$
$i = i + 1$, Go to Step 1.
ii) If $\chi(w^i, K) \geq 0$
and $\pi\{x(w^i, K), y(w^i, K), w^i\} > 0$
$K = K + \lambda$
$i = 0$, Go to Step 1.
iii) If $\chi(w^i, K) \geq 0$
and $\pi\{x(w^i, K), y(w^i, K), w^i\} = 0$
Then Optimality Reached
Optimal Solution $=$
$\{x(w^i, K), y(w^i, K), w^i\}$.

In this version of the algorithm, we propose to increase the penalty K by discrete small steps λ. However, K can be increased by using some intelligent heuristic, many of which have been suggested in the nonlinear programming literature [3]. Note that if assumption A2 is violated, and the polyhedron Z (W) is empty or unbounded, the algorithm will terminate at Step 1 (Step 2) with an infeasible or unbounded solution, respectively.

V. Illustrative Examples

We illustrate the method with an example and some computational runs. In all these problems, the constraint sets of the leader $G(\cdot)$ and the follower $g(\cdot)$ *coincide*. This does not detract from the essential method outlined in this note.

Example-E1 [5]:

$$\operatorname*{Max}_{x_1} F = x_1 + 3x_2$$

$$\operatorname*{Max}_{x_2} f = x_1 - 3x_2$$

$$\text{s.t.} \quad -x_1 - 2x_2 \leq -10$$

$$x_1 - 2x_2 \leq 6$$

$$2x_1 - x_2 \leq 21$$

$$x_1 + 2x_2 \leq 38$$

$$-x_1 + 2x_2 \leq 18$$

$$x_1, x_2 \geq 0.$$

We will now present the steps in solving the example. The dual space of the follower's problem for E1 is given by

$$W = \{(w_1, w_2, w_3, w_4, w_5)'\colon\ -2w_1 - 2w_2 - w_3 + w_4 + 2w_5 \geq -3,\quad w_i \geq 0,\ i = 1, \cdots 5\}.$$

The solution proceeds as follows.

Iteration 1

Step 0 $K = 0$, $\lambda = 0.1$, $w_0^t = (1.5, 0, 0, 0, 0) \in W$.

TABLE I
ALGORITHMIC PERFORMANCE (cpu s)

Size	kth Best	Grid Search	Penalty Method
(3, 7, 4)	59.23	12.17	8.76
(5, 10, 6)	127.55	33.41	21.85
(6, 14, 8)	111.74	57.29	44.19
(8, 17, 10)	186.23	74.87	74.01
(15, 30, 20)	1200.87	129.33	119.52

Step 1 $a + Kw_0A = a = 1$
$b + Kd = b = 3$
Thus $\text{argMax}\,\{x + 3y\colon\ (x, y) \in Z\}$
$= x(w_0, K = 0) = 10,\ y(w_0, 0) = 14.$

Step 2 $[p - Ax(w_0, 0)]^t = [0, -4, 1, 28, 28]$
Thus $\text{argMin}\,\{-4w_2 + w_3 + 28w_4 + 28w_5\}$
$= w^*(w_0, 0)^t = (0, 1.5, 0, 0, 0).$

Step 3 $\chi(w_0) = -6$
$w_1^t = (0, 1.5, 0, 0, 0)$
$i = 1$, Go to Step 1.

Iteration 2

Step 1 $a + Kw_1^tA = a = 1$
$b + Kd = b = 3$
Thus $x(w_1, 0) = 10,\ y(w_1, 0) = 14.$

Step 2 remains unchanged

Step 3 $\chi(w_1) = 0$
$\pi\{x(w_1), y(w_1), w_1\} = 36 > 0$
So Optimal solution NOT found.
$K = 0.1$, Go to Step 1.

Iteration 3

Step 1 $a + Kw_1^tA = 1.15$
$b + Kd = 2.7$
$x(w_1, 0.1) = 10,\ y(w_1, 0.1) = 14.$

Step 2 remains unchanged

Step 3 remains unchanged
$K = 0.2$, Go to Step 1.

Iteration 4

Step 1 $a + Kw_1^tA = 1.30$
$b + Kd = 2.40$
$x(w_1, 0.2) = 16,\ y(w_1, 0.2) = 11.$

Step 2 $[p - Ax(w_1, 0.2)]^t = [6, -10, -11, 22, 34]$
Thus $w^*(w_1, 0.2)^t = (0, 0, 3, 0, 0)$

Step 3 $\chi(w_1, 0.2) = -18$
Thus $w_2^t = w^*(w_1, 0.2)^t = (0, 0, 3, 0, 0)$
Go to Step 1.

Iteration 5

Step 1 $a + Kw_2^tA = 2.2$
$b + Kd = 2.4$
$x(w_2, 0.2) = 16,\ y(w_2, 0.2) = 11.$

Step 2 remains unchanged

Step 3 $\chi(w_2, 0.2) = 0.$
$\pi(w_2, 0.2) = 0$
Thus optimality is reached
Optimal solution is $x = 16,\ y = 11.$

The optimal solution from our method is the same for this problem as that obtained by Bialas and Karwan [5]. The total number of iterations, including those required to solve the linear programs was 28.

Table I provides the results of a computational exercise of running 50 randomly selected problems of five different sizes for each method. The size of the problem is denoted (n_1, n_2, m) where n_1 and n_2 stand for the dimension of the leader's and follower's decision vectors, respectively, and m is the number of constraints. The problems were run on an ATT PC6300+ microcomputer with Intel 80286 microprocessor, and 80287 math coprocessor. Each linear program was solved using the LINDO package, and Pascal programs linked the different components of the iteration together. The penalty function method proposed in this note marginally outperformed Bard's [2] grid search method and easily outperformed the kth best method [5]. In a companion paper [14], we provide an algorithm to find global optimal solutions for these problems.

ACKNOWLEDGMENT

The authors thank, without implicating, K. Jornsten, J. Judice, M. Karwan, H. Sherali, and two anonymous referees for comments and suggestions. They also thank R. Nigam for computer programming assistance.

REFERENCES

[1] E. Aiyoshi and K. Shimizu, "A solution method for the static constrained Stackelberg problem via penalty method," *IEEE Trans. Automat. Contr.*, vol. AC-29, pp. 1111-1114, Dec. 1984.
[2] J. F. Bard, "An efficient point algorithm for a linear two-stage optimization problem," *Operations Res.*, pp. 670-684, July-Aug. 1983.
[3] M. Bazaraa and C. M. Shetty, *Nonlinear Programming: Theory and Algorithms*. New York: Wiley, 1979.
[4] W. F. Bialas and M. H. Karwan, "On two-level optimization," *IEEE Trans. Automat. Contr.*, vol. AC-27, pp. 211-214, 1982.
[5] —, "Two-level linear programming," *Management Sci.*, vol. 30, no. 8, pp. 1004-1020, Aug. 1984.
[6] W. Candler and R. Townsley, "A linear two-level programming problem," *Comput. Operations Res.*, vol. 9, no. 1, pp. 59-76, 1982.
[7] T.-S. Chang and Y. C. Ho, "Stochastic Stackelberg games: Nonnested multistage multiagent incentive problems," *IEEE Trans. Automat. Contr.*, vol. AC-28, pp. 477-488, Apr. 1983.
[8] Y. C. Ho, P. Luh, and R. Muralidharan, "Information structures, Stackelberg games, and incentive controllability," *IEEE Trans. Automat. Contr.*, vol. AC-26, pp. 454-460, Feb. 1981.
[9] P. B. Luh, S.-C. Chang, and T.-S. Chang, "Solutions and properties of multistage Stackelberg games," *Automatica*, vol. 20, no. 2, pp. 251-256, 1984.
[10] G. P. Papavassilopoulos and J. B. Cruz, "Nonclassical control problems and Stackelberg games," *IEEE Trans. Automat. Contr.*, vol. AC-24, no. 2, pp. 155-166, Feb. 1979.
[11] K. Shimizu and E. Aiyoshi, "A new computational method for Stackelberg and mini-max problems by the use of a penalty method," *IEEE Trans. Automat. Contr.*, vol. AC-26, no. 2, pp. 460-466, Feb. 1981.
[12] M. Simaan and J. B. Cruz, "On the Stackelberg strategy in nonzero-sum games," *J. Optimiz. Theory Appl.*, vol. 11, no. 5, pp. 533-555, 1973.
[13] B. Tolwinski, "Closed-loop Stackelberg solution to multistage linear quadratic games," *J. Optimiz. Theory Appl.*, vol. 34, no. 4, pp. 485-501, 1981.
[14] D. J. White and G. Anandalingam, "A penalty function approach for solving bi-level linear programs," Dep. Syst., Univ. Pennsylvania, Philadelphia, working paper, Nov. 1989.

Application of Lyapunov Functionals to Studying Stability of Linear Hyperbolic Systems

MARIUSZ ZIÓŁKO

***Abstract*—The Lyapunov functional method is used to prove the stability conditions for Cauchy problems and initial-boundary value problems if the system is described by a set of linear first-order partial differential equations of the hyperbolic type. Although the considered system is linear, it is possible to obtain necessary and sufficient conditions for stability only when matrices in differential equations and boundary conditions have some special properties.**

I. INTRODUCTION

The stability of a system described by a set of first-order partial differential equations of hyperbolic type is an interesting problem in control theory and arises in stability theory of numerical methods for hyperbolic systems. Gunzburger [2] considered the stability of initial-boundary value problems for a system consisting of two equations. The application of the Lyapunov functional method to stability of linear hyperbolic systems with more than two equations leads to the searching for functionals with diagonal matrices. The questions of whether or not there exists a positive

Manuscript received March 24, 1989; revised September 29, 1989.
The author is with the Institute of Automatic Control, Technical University of Mining and Metallurgy, 30-058 Kraków, Poland.
IEEE Log Number 9036823.