Self-Enforced Job Matching*

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Abstract

Complementarities and peer effects are common in matching markets, yet incorporating them often leads to the nonexistence of stable matchings. We observe that matching is often an ongoing process rather than a static allocation, where long-lived firms interact over time with short-lived workers. We show that when wages are flexible and firms are sufficiently patient, a dynamically stable solution always exists in many-to-one matching markets—even with complementarities and peer effects. Flexible wages are crucial to our result, as they not only facilitate surplus extraction when firms cooperate in no-poaching agreements but also enhance the threat of punishment through bidding wars.

Keywords: matching, repeated games, no-poaching agreements

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1 Introduction

Two-sided many-to-one matching models with monetary transfers are crucial for understanding a variety of transactional environments, including auctions, labor markets, and housing markets. In their seminal work, Kelso and Crawford (1982) established the existence of stable matchings, provided that firms' production technologies satisfy gross substitutability (no workers serve as complementary inputs in a firm's production) and workers' preferences do not exhibit peer effects (workers ignore their colleagues' identities and characteristics).

However, complementarities and peer effects are present in many important matching markets. In industries such as manufacturing, healthcare, and software development, it is common for tasks to require the collaboration of workers with complementary skills. In labor markets such as academia and startups, colleagues are an important consideration when choosing where to work. Although recent literature has obtained positive existence results by modeling large markets or considering alternative assumptions on preferences, accommodating arbitrary production technologies and peer effects has been challenging.

In this paper, we propose an approach that does not rely on restrictions on market size, technologies, or preferences. We consider matching as an ongoing process in which long-lived players on one side of the market interact over time with short-lived players or objects on the other side. This dynamic is salient in many matching environments: Long-lived sellers can serve a sequence of short-lived buyers, firms can hire a new batch of interns in every recruitment season, and brokers can purchase newly issued securities in every trading cycle. To fix ideas, we focus on labor markets with long-lived *firms* and short-lived *workers*. We find that, in the presence of complementarities and peer effects that destabilize static matchings, stability can be maintained through (tacit) no-poaching agreements among long-lived firms. Such agreements allow firms to eliminate wage competition and extract surplus from workers, and are enforced by the threat of bidding wars in the event of noncompliance. The key idea is that dynamic incentives can act as both carrots and sticks to deter firms' deviations. We provide an illustrative example at the end of this section.

The driving force behind our result is wage flexibility, which not only enables firms to extract surplus from workers when colluding in a no-poaching agreement, but also enhances their ability to punish defectors through bidding wars. In fact, as we illustrate with a counterexample in Section 4.1, our existence result does not extend to a rigid-wage environment with peer effects and complementarities. This observation aligns with the tendency of markets experiencing chaos and "unraveling" to be those with wage rigidity, such as the markets for medical interns prior to the National Resident Matching Program (Roth 1984) and for federal judicial law clerks (Avery, Jolls, Posner and Roth 2001).[1]

No-poaching agreements appear in many labor markets and have attracted intense antitrust scrutiny (Krueger and Ashenfelter 2022). While specific practices and implementations vary, no-poaching agreements are, in essence, market-sharing arrangements in which firms avoid competing for each other's workers to suppress wages.[2] For example, in *U.S. v. Adobe Systems Inc., et al.*, tech companies are alleged to have established such agreements informally via emails among CEOs and HR officers. A similar phenomenon occurs in college admissions, where universities reduce competition by avoiding "poaching" each other's students with better financial aid offers. Despite antitrust exemptions under the Improving America's Schools Act of 1994, there has been a surge in antitrust cases that scrutinize universities' collusive practices in setting financial aid. Notably, in *Henry, et al. v. Brown University, et al.*, universities are alleged to have colluded to use a common aid formula to "eliminate financial aid as a locus of competition." Likewise, *Hansen et al. v. Northwestern University, et al.* scrutinizes universities for taking "concerted action" to reach an "agreed pricing strategy" that reduces aid to students with divorced parents.[3] Our findings add a new perspective to this debate: While such agreements are inherently anti-competitive, they may play a vital role in sustaining market stability, especially in environments lacking static stable matchings due to significant complementarities and peer effects.

From a technical standpoint, our setting differs from standard repeated noncooperative games since our stage game is a cooperative game. In standard repeated games, the non-cooperative stage game is guaranteed to have a Nash equilibrium: Playing the same Nash

[1]Medical residents are compensated primarily through educational opportunities and prospects for advancement, with monetary compensation close to the binding minimum wage. Law clerks receive compensation mainly in the form of prestige, with wages determined by government pay scales (United States Courts 2024).

[2]Importantly, no-poaching agreements differ from noncompete clauses, in which firms directly contract with workers to restrict their employment options.

[3]Similarly, in *Seaman v. Duke University* and *Binotti v. Duke University*, disputes centered on coordinated baseline wages by university and hospital administrators.

equilibrium at every history delivers a subgame perfect Nash equilibrium. In our setting, by contrast, since the cooperative stage game may not have a stable outcome, the existence of a dynamically stable matching process cannot be guaranteed simply by forming a static stable matching at every history. We establish existence by explicitly constructing a dynamic matching process. We do this in three steps. First, we define and characterize the analog of "minmax payoffs" in the cooperative stage game. We then use a folk-theorem-like construction to show that payoff profiles above these minmax payoffs can be sustained in a dynamically stable matching process when firms are sufficiently patient. Finally, we show that in a no-poaching agreement implemented through a random serial dictatorship, each firm obtains random payoffs that first-order stochastically dominate their respective minmax payoffs.[4]

An Example. Two long-lived firms f_1 and f_2 each offer two internship positions every period. Each firm treats workers as complements: It generates a revenue of \$6 only when both of its vacancies are filled and is unproductive otherwise. On the other side of the market, every period, three new identical workers w_1, w_2, and w_3 look for positions. For simplicity, assume that workers' payoffs are equal to the wages they receive. If the matching market in every period is treated as an isolated one-shot interaction, no static matching is stable: In any static matching, at most one firm is productive, and for both firms to break even, there must be one worker earning zero and another worker earning no more than \$3. The unproductive firm can then form a blocking coalition with these two workers.

Instead of static matchings, consider the following history-dependent matching process (illustrated in Figure 1). In each period, the market is in one of four possible states: two wage-suppressing collusion states C1 and C2 and two punishment states P1 and P2. Each collusion state has a dominant firm, f_1 in C1 and f_2 in C2, and each punishment state has a punished firm, f_1 in P1 and f_2 in P2. The market starts at and remains in state C1 until a firm deviates. Whenever a firm deviates, the market transitions to the punishment state for that firm and remains there for four periods; if a firm deviates during a punishment phase,

[4] In a random serial dictatorship, an ordering of firms is randomly drawn, and firms take turns to hire their favorite set of workers from the remaining pool. This arrangement entails a no-poaching agreement, as firms agree not to poach any workers hired by firms with higher priorities according to the ordering. An example of serial dictatorship in labor markets is the new player drafts in major North American professional sports (e.g., NBA, NFL, NHL, and MLB).

then the process transitions to the punishment phase for that firm. After this punishment phase, the market enters the collusion state in which the non-deviating firm assumes the role of the dominant firm and stays there until a firm deviates.

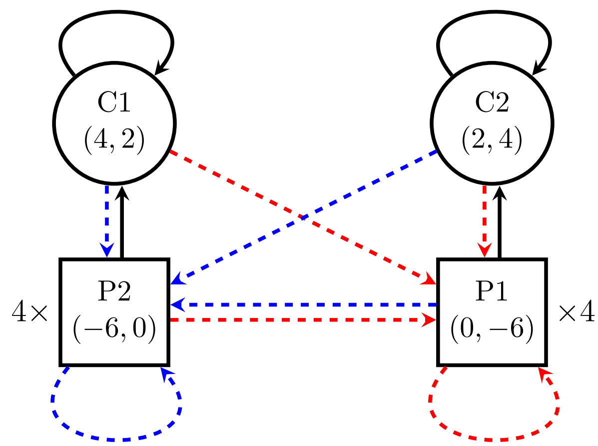

FIGURE 1. A stable matching process. The first number in parentheses is the stage-game payoff of firm f_1, and the second number is that of firm f_2. Solid arrows represent transitions when no firm deviates, dashed arrows that point to P1 (in red) represent transitions to P1 after firm f_1's deviations, and dashed arrows that point to P2 (in blue) represent transitions to P2 after firm f_2's deviations.

At the beginning of every period in collusion states C1 and C2, a biased coin toss determines which firm secures the hiring rights for the current period. The dominant firm wins with a probability of 2/3, and the nondominant firm wins with a probability of 1/3. The firm that wins the coin toss hires two workers at zero wage, while the losing firm temporarily shuts down. Therefore, the dominant firm obtains \$4 on average, while the other obtains \$2. Notice that the collusion states feature a no-poaching agreement: In each period, the non-hiring firm refrains from soliciting workers from the other firm, thus forgoing an immediate gain in the current period. In punishment states P1 and P2, the firm being punished shuts down, while the punishing firm hires two workers each at a wage of \$6. Figure 1 depicts firms' payoffs in each state.

To see that this matching process is dynamically stable, we use a one-shot deviation principle (Lemma 3), which establishes dynamic stability by checking two requirements at every history: (1) no worker wishes to unilaterally leave her matched firm to be unemployed and (2) no firm has a profitable one-shot deviation with a group of workers who also find this deviation profitable. Note that the first requirement is satisfied in every state of the matching process since all workers are weakly better off than being unemployed. We next verify that the second requirement is also met in every state.

In punishment states, any deviation by the punished firm improves neither its current stage-game payoff nor its continuation value. The punishing firm can gain a profit in the current period by deviating, but by doing so it would forgo the advantage of being the dominant firm in future collusive periods. As a result, for high values of δ, the punishing firm does not find one-shot deviations profitable.

In collusion states, regardless of which firm has the hiring rights for the period, the dominant firm finds no profitable one-shot deviations when δ is high: Any deviation would lead to the loss of its position as the dominant firm and thus a lower long-run payoff. The nondominant firm also has no incentive to deviate: Even though it could potentially obtain a current-period gain of \$6 by poaching workers from the winning firm, this would be followed by a net loss of \$2 for each of the next four periods (it gets \$0 from shutting down, compared with \$2 in the collusive state). When firms are patient, the loss outweighs the potential gain, so the nondominant firm also has no profitable one-shot deviations.

Let us briefly discuss two features of our example. The coin tosses in the collusion states are public randomization devices that convexify players' payoffs. Public randomization simplifies our exposition but is not essential to our results, since we can also convexify payoffs using sequences of play. As noted earlier, our results rely on firms' ability to flexibly adjust workers' wages, which facilitates both extracting surplus in the collusion states and rendering workers unavailable to deviators in the punishment states.

Related Literature. The literature on static job matching has guaranteed the existence of stable matching by (i) imposing restrictions on preferences (Hatfield and Milgrom 2005; Sun and Yang 2006; Echenique and Yenmez 2007; Hatfield and Kojima 2008; Pycia 2012; Hatfield, Kominers, Nichifor, Ostrovsky and Westkamp 2013; Rostek and Yoder 2020; Kojima, Sun and Yu 2020, 2024; Pycia and Yenmez 2023); (ii) studying large markets (Kojima, Pathak and Roth 2013; Ashlagi, Braverman and Hassidim 2014; Azevedo and Hatfield 2018; Che, Kim and Kojima 2019); (iii) making minimal adjustments to quotas (Nguyen and Vohra

2018); or (iv) considering strategically consistent beliefs (Rostek and Yoder 2024).[5] In this paper, we propose a different approach based on firms' dynamic incentives.

Our paper also contributes to the literature on dynamic matching. Prior studies by Corbae, Temzelides and Wright (2003), Kurino (2020), and Doval (2022) have explored dynamic matching as a complete contingent plan with a focus on one-to-one matching. Both Corbae et al. (2003) and Doval (2022) impose a perfection requirement, which we also adopt in our analysis. However, Corbae et al. (2003) examine a setting in which all players are long-lived and matchings can be revised over time, focusing on implications for monetary economics. Doval (2022) considers a setting with arrivals, where matchings are one-time and irrevocable.[6] In contrast, we study a setting in which each long-lived firm repeatedly matches with multiple short-lived workers in every period. Another approach is to study dynamic matching not as a contingent plan but as a path of realized matchings. Damiano and Lam (2005) consider blocking plans by coalitions subject to a coalition-proofness requirement (Bernheim, Peleg and Whinston 1987). Kadam and Kotowski (2018a,b) examine pairwise stability in a framework that allows for complex preferences that are not necessarily time-separable; in the same environment, Kotowski (2024) considers a notion of stability that combines a perfection requirement with conservative conjectures about the future.

The works most closely related to this paper are Ali and Liu (2020) and Liu (2023), both of which consider repeated games with moves by coalitions. Ali and Liu (2020) examine repeated games where all players are long-lived, and the stage game can be either a cooperative game or a normal-form game. They study dynamic contingent plans that are immune to blocking coalitions in this general setting. Liu (2023) considers dynamic matching with long-lived firms and short-lived workers, which is similar to our setting. However, Liu (2023) focuses on matching *without* transfers, which significantly constrains the scope of dynamic punishments and rewards. In Section 4.1 we present an example with peer effects in which dynamic incentives fail to ensure stability in the absence of transfers. Liu (2023) relies on assumptions on payoffs (i.e., gross substitutability and no peer effects) to guarantee the existence of

[5] Relatedly, Echenique and Yenmez (2007) consider matching with peer effects and propose an algorithm that finds all static stable matchings whenever they exist.

[6] Du and Livne (2016) study a similar problem without transfers, while Altınok (2020) extends this approach to many-to-one matching.

static stable matchings in the stage game, and to characterize when the limited scope of dynamic incentives can or cannot overcome the constraints of the Rural Hospital Theorem. By contrast, our paper studies matching markets *with* transfers, which substantially expand the scope of dynamic enforcement. One main contribution of the current paper is therefore to shed light on the stabilizing role of dynamic incentives in matching markets with transfers, especially when static stability breaks down due to nonstandard preferences.

Our approach is also related to several important papers on classic repeated games. Specifically, our model involves long-run firms playing against short-run workers, making it closely related to the folk theorems for repeated games with long-run and short-run players (Fudenberg, Kreps and Maskin 1990; Fudenberg and Levine 1994). However, we consider deviations by coalitions consisting of both long-run and short-run players. The coalition deviations in our paper are also related to Rubinstein (1980), who considers strong perfect equilibria, which require that players' continuation strategies form a strong Nash equilibrium (Aumann 1959) in the continuation game following every history. The key difference is that, at every history, a strong perfect equilibrium considers coalitions that are arbitrary subsets of the finite set of long-lived players. In our model, after any given history, we consider coalitions consisting of one long-lived player (a firm) and infinite generations of future workers.

The rest of the paper is organized as follows. Section 2 introduces the model. Section 3 describes the main results and their proofs. Section 4 discusses crucial assumptions of the model. Section 5 concludes. Appendix A collects omitted proofs.

2 Model

Players. At the beginning of every period $t = 0, 1, 2, \ldots$, a new generation of workers enters the market to match with a fixed set of long-lived firms. Let $\mathcal{F}$ denote the finite set of firms. Matching is many-to-one: In every period, each firm $f \in \mathcal{F}$ has $q_f \in \mathbb{Z}_{++}$ hiring slots to fill. Workers are short-lived: They remain in the market for only one period. For expositional convenience, we use the same notation $\mathcal{W}$ to denote the finite set of workers in every period.

A finite set Θ describes the set of *states of the world*. In each period, a state $\theta \in \Theta$ is randomly drawn from a distribution $\pi \in \Delta(\Theta)$ and then publicly observed. The state space Θ

is quite general and can encompass both common and individual shocks faced by the players.[7] It is important to note that while we introduce random θ to mirror real-world markets, our results do not depend on this randomness. In fact, all results hold in the special case when π is a degenerate distribution.

Each firm f has a stage-game revenue function $\widetilde{u}_f : 2^{\mathcal{W}} \times \Theta \to \mathbb{R}$ defined over subsets of workers and states. We normalize the revenue of staying unmatched, $\widetilde{u}_f(\emptyset, \theta)$, to 0 for every $\theta \in \Theta$. Note that for each state, we do not require firms' revenue functions to satisfy the gross substitutes condition. Firms share a common discount factor δ and evaluate a sequence of flow utilities through exponential discounting.

Each worker cares about both her employer and her colleagues, which we will refer to collectively as her *work environment*. Let $\Phi_w = (\mathcal{F} \times 2^{\mathcal{W}\setminus\{w\}}) \cup \{(\emptyset, \emptyset)\}$ denote the set of possible work environments of worker w, where $(\emptyset, \emptyset)$ represents staying unmatched. Each worker w has a payoff function $\widetilde{v}_w : \Phi_w \times \Theta \to \mathbb{R}$ over work environments and states. For every worker $w \in \mathcal{W}$ and state $\theta \in \Theta$, $\widetilde{v}_w(\emptyset, \emptyset, \theta)$ is normalized to 0.

Stage-Game Matching. The outcome in every period is a static many-to-one matching among the firms and workers. Formally, a stage-game matching $m = (\phi, p)$ is described by an *assignment* ϕ and a *wage vector* p. In particular, ϕ is a mapping defined on the set $\mathcal{F} \cup \mathcal{W}$ such that (i) for every $w \in \mathcal{W}$, $\phi(w) \in \Phi_w$; (ii) for every $f \in \mathcal{F}$, $\phi(f) \subseteq \mathcal{W}$ and $|\phi(f)| \leq q_f$; and (iii) for every $w \in \mathcal{W}$ and every $f \in \mathcal{F}$, $w \in \phi(f)$ if and only if $\phi(w) = (f, \phi(f)\setminus\{w\})$. The wage vector $p = (p_{fw}) \in \mathbb{R}^{|\mathcal{F}|\times|\mathcal{W}|}$ describes the transfers from firms to workers. We assume that any nonzero transfer occurs only between a firm and its own employees: $p_{fw} = 0$ for every $w \notin \phi(f)$.[8]

[7]For example, one can let Θ_0 denote the set of economy-wide shocks, Θ_f the set of productivity levels of firm f, and Θ_w the set of qualities of worker w; the state space can then be constructed as a product set $\Theta \equiv \Theta_0 \times \prod_{f\in\mathcal{F}} \Theta_f \times \prod_{w\in\mathcal{W}} \Theta_w$. In addition, our model can also accommodate fluctuations in the number of workers across periods by introducing states with unavailable workers, so that matching with any unavailable worker renders all matched players strictly worse off (before transfers) than staying unmatched.

[8]Allowing transfers among firms expands the set of feasible outcomes without changing firms' minmax payoffs, and as our results indicate, this would expand the set of self-enforced outcomes as $\delta \to 1$. Hence, by only allowing transfers between firms and their employees, we show that this more restrictive form of transfers suffices to guarantee the existence of self-enforced outcomes. Another reason we choose to model only transfers between firms and employees is that this also aligns with the model of Kelso and Crawford (1982), which our paper generalizes.

Players have quasilinear utilities: For every stage-game matching $m = (\phi, p)$ and state θ,

$$u_f(m,\theta) \equiv \widetilde{u}_f(\phi(f),\theta) - \sum_{w' \in \mathcal{W}} p_{fw'} \quad \text{and} \quad v_w(m,\theta) \equiv \widetilde{v}_w(\phi(w),\theta) + \sum_{f' \in \mathcal{F}} p_{f'w}$$

are the stage-game payoffs received by firm f and worker w, respectively. Note that although the expression for v_w above sums over wage payments from all firms, by assumption, at most one of these payments can be positive. While we adopt quasilinear utility for tractability, the key driving force is wage flexibility, and our results extend as long as players' payoffs are monotone and unbounded in wages (see the discussion in Section 4.1).

A static stable matching is immune to three kinds of deviation: (i) a unilateral deviation by a firm f to fire all its employees; (ii) a unilateral deviation by a worker w, who leaves her employer and remains unmatched; and (iii) a coalitional deviation $(f, W, p_f) \in \mathcal{F} \times 2^{\mathcal{W}} \times \mathbb{R}^{|\mathcal{W}|}$ with $|W| \leq q_f$ and $p_{fw} = 0$ for every $w \notin W$, where firm f and workers W match at wages specified in p_f and abandon any other pre-existing match partners. A deviation is profitable if all participants strictly prefer the deviating matching to the original matching.[9] A stage-game matching is *individually rational* if no unilateral deviations are profitable, and is *stable* if none of the three kinds of deviations above are profitable.

When studying the stability of static matchings, there is no need to specify how other players will be matched after a deviation by a coalition. However, in our dynamic setting, players' future behavior is influenced by past histories, so to study the stability of matching processes, we need to specify the realized stage-game outcome after a deviation. To this end, we adopt the following assumption.

Assumption 1. *Let $[m, (\widehat{f}, \widehat{W}, \widehat{p}_{\widehat{f}})] \in M$ denote the stage-game matching that is realized after coalitional deviation $(\widehat{f}, \widehat{W}, \widehat{p}_{\widehat{f}})$ from stage-game matching $m = (\phi, p)$, and let (ϕ', p') denote the assignment and wages in $[m, (\widehat{f}, \widehat{W}, \widehat{p}_{\widehat{f}})]$. We assume that the assignment ϕ' satisfies $\phi'(\widehat{f}) = \widehat{W}$ and $\phi'(f) = \phi(f) \backslash \widehat{W}$ for all $f \neq \widehat{f}$; furthermore, we assume that the wages satisfy $p'_{\widehat{f}} = \widehat{p}_{\widehat{f}}$, while $p'_{fw} = p_{fw}$ for all $f \neq \widehat{f}$ and $w \in \phi'(f)$.*

Assumption 1 says that in the stage-game matching that results from a coalitional

[9] In our model with transfers, this formulation of profitable deviations is equivalent to the one that requires all participants be weakly better off, with at least one member in the coalition becoming strictly better off.

deviation, the deviators are matched together, players abandoned by the deviators remain unmatched, and those untouched by the deviation remain matched with their original partners. The sole purpose of this assumption is to ensure "perfect monitoring": When a matching m is blocked by a coalition (f, W', p'_f), the firm in the deviating coalition is identifiable. Lemma 4 formally proves this identifiability property from Assumption 1. Any alternative assumption that delivers the same identifiability property will not change our results.[10]

Repeated Matching. To convexify players' stage-game payoffs, we employ a public randomization device on the unit interval $\Gamma = [0, 1]$ endowed with the Lebesgue measure. The use of public randomization streamlines our proofs, but our results do not depend on it.[11]

The timing in each period is as follows. First, a new cohort of workers arrives. A state of the world $\theta \in \Theta$ is drawn according to the distribution π, and a public randomization $\gamma \in \Gamma$ is realized. All players observe the realized (θ, γ), and a stage-game matching is recommended for them based on (θ, γ). Players then decide whether to deviate from this recommendation, which determines the outcome of the stage game.

A t-period ex ante history $\overline{h} = (\theta_\tau, \gamma_\tau, m_\tau)_{\tau=0}^{t-1}$ specifies a sequence of past realizations of the state, the public randomization device, and the stage-game matching up to period $t-1$.[12] We write $\overline{\mathcal{H}}_t$ for the set of all t-period ex ante histories, with $\overline{\mathcal{H}}_0 = \{\emptyset\}$ being the singleton set comprising the null history. Let $\overline{\mathcal{H}} \equiv \bigcup_{t=0}^{\infty} \overline{\mathcal{H}}_t$ be the set of all ex ante histories. Moreover, let $\mathcal{H}_t \equiv \overline{\mathcal{H}}_t \times \Theta \times \Gamma$ denote the set of t-period ex post histories, and $\mathcal{H} \equiv \overline{\mathcal{H}} \times \Theta \times \Gamma$ the set of all ex post histories.

A *matching process* $\mu : \mathcal{H} \to M$ specifies a stage-game matching for every ex post history. It represents a shared understanding among players regarding how past histories impact future employment. For every ex post history $h \in \mathcal{H}$, we let $\mu(f|h) \in 2^{\mathcal{W}} \times \mathbb{R}^{|\mathcal{W}|}$ denote the

[10]Similar assumptions appear in Mauleon, Vannetelbosch and Vergote (2011) and Liu (2023). As an example, an alternative to Assumption 1 is to suppose that if m is blocked by (f, W', p'_f), any firm that loses workers to f fires all its remaining workers, whereas all firms untouched by this coalition retain their original hiring decisions. One can easily verify that identifiability still holds under this specification.

[11]For example, we could follow arguments in Sorin (1986) and Fudenberg and Maskin (1991) to convexify players' payoffs using sequences of play instead.

[12]Note that this assumes observable wages. However, since workers are short-lived and none of the self-enforcing matching processes constructed in this paper make use of information about wages, the existence result continues to hold even if wages are unobservable.

matching partners of firm f and the wages it pays in the stage-game matching $\mu(h)$.

Let $\overline{\mathcal{H}}_\infty = (\Theta \times \Gamma \times M)^\infty$ be the set of outcomes h_∞ of the repeated matching game. Let $m_t(h_\infty)$ denote the stage-game matching in the t-th period of h_∞. Following every t-period (ex ante or ex post) history $\widehat{h} \in \overline{\mathcal{H}} \cup \mathcal{H}$, let

$$U_f(\widehat{h} \mid \mu) \equiv (1-\delta)\mathbb{E}_\mu\Big[\sum_{\tau=t}^{\infty} \delta^{\tau-t} u_f(m_\tau(h_\infty), \theta_\tau) \,\Big|\, \widehat{h}\Big]$$

denote the continuation payoff firm f obtains from the matching process μ following $\widehat{h}$, where the expectation is taken with respect to the measure over $\overline{\mathcal{H}}_\infty$ induced by μ conditional on $\widehat{h}$.

Deviation Plan. We make two observations that allow us to tractably analyze firms' deviations in the repeated game. Recall that there are three kinds of deviations within a period. We first observe that a unilateral deviation by firm f to fire all its employees is equivalent to a deviation by the coalition consisting of f and an empty set of workers. It is therefore without loss to focus on two kinds of deviation: (i) unilateral deviations by workers and (ii) deviations by coalitions consisting of a firm and a (possibly empty) set of workers in the stage game.

Our second observation is that as a long-lived player, a firm can participate in a sequence of deviations by forming coalitions with workers across periods. Each of these coalitions must be immediately profitable for the participating short-lived workers but not necessarily for the firm, since the firm cares about the profit it collects from the entire sequence.

Motivated by this second observation, we define a *deviation plan* for firm f as a complete contingent plan that specifies, at every ex post history, a set of workers to recruit and their wage offers. Formally, a deviation plan for firm f is a pair $(d : \mathcal{H} \to 2^{\mathcal{W}}, \eta : \mathcal{H} \to \mathbb{R}^{|\mathcal{W}|})$ such that $|d(h)| \leq q_f$ for all h, and $\eta_w(h) \neq 0$ only if $w \in d(h)$ for all h. Together with the original matching process, a deviation plan generates a distribution over the outcomes of the game $\overline{\mathcal{H}}_\infty$. Given a matching process μ and f's deviation plan (d, η), the *manipulated matching process*, denoted by $[\mu, (f, d, \eta)] : \mathcal{H} \to M$, is a matching process defined by

$$\Big[\mu, (f, d, \eta)\Big](h) \equiv \Big[\mu(h),\ \Big(f, d(h), \eta(h)\Big)\Big] \quad \forall h \in \mathcal{H}.$$

Firm f's deviation plan (d, η) from μ is *feasible* if at every ex post history $h = (\overline{h}, \theta, \gamma)$ such that $(d(h), \eta(h)) \neq \mu(f|h)$,

$$v_w\Big(\Big[\mu, (f, d, \eta)\Big](h),\, \theta\Big) > v_w\Big(\mu(h), \theta\Big) \quad \forall w \in d(h).$$

That is, workers participate in the deviation only if they find the new stage-game matching strictly better than the recommendation from μ. Lastly, the deviation plan (d, η) is *profitable* if there exists an ex post history h such that $U_f\big(h \,\big|\, [\mu, (f, d, \eta)]\big) > U_f(h|\mu)$.

Self-Enforcing Matching Process. The two observations above motivate our notion of dynamic stability.

Definition 1. Matching process μ is *self-enforcing* if (i) $v_w(\mu(h), \theta) \geq 0$ for every $w \in \mathcal{W}$ at every ex post history $h \in \mathcal{H}$ and (ii) no firm has a feasible and profitable deviation plan.

The first requirement guards against deviations by a single worker in any generation, and the second guards against deviations by firms. These requirements are imposed on the matching process at *all* ex post histories, including those that are off path. This restriction on all ex post histories embeds a form of sequential rationality in the same way as subgame perfection does in a repeated noncooperative game. It is also worth noting that Definition 1 focuses on stability against coalitions consisting of a single firm and multiple workers: This criterion is stronger than pairwise stability but weaker than group stability (Roth and Sotomayor 1990, p. 130), which requires immunity to blocking by arbitrary subsets of players. In dynamic environments, these stability notions are not equivalent.[13] Nevertheless, in one-period settings, Definition 1 coincides with the definition of core allocation in Kelso and Crawford (1982), so our definition can be viewed as its dynamic generalization.

3 Results

Minmax Payoffs. We begin by defining firms' minmax payoffs. For every realized state

[13]In Section 4.3, we discuss the conceptual and practical challenges posed by multi-firm deviation plans and assess the extent to which our results extend to settings that allow for such deviations.

$\theta \in \Theta$, let

$$M^{\circ}(\theta) \equiv \{m \in M : v_w(m,\theta) \geq 0 \text{ for all } w \in \mathcal{W}\}$$

denote the set of stage-game matchings that are individually rational for workers. From Definition 1, a self-enforcing matching process can only recommend stage-game matchings in $M^{\circ}(\theta)$. Moreover, for every firm $f \in \mathcal{F}$ and recommended stage-game matching $m = (\phi, p)$, let

$$D_f(m,\theta) \equiv \left\{ (W', p'_f) : \begin{array}{c} |W'| \leq q_f,\ p'_{fw} = 0 \text{ if } w \notin W', \text{ and} \\ \widetilde{v}_w(f, W', \theta) + p'_{fw} > v_w(m,\theta) \text{ for every } w \in W' \end{array} \right\}$$

denote the set of feasible stage-game deviations for f at state θ. Similar to the definition of feasible deviation plans, a deviation in the stage-game matching is feasible if (i) the deviating firm does not exceed its quota in hiring, and (ii) any worker involved should be strictly incentivized compared to the recommended stage-game matching. Firm f's minmax payoff is its payoff from "best responding" to the worst recommendation.

Definition 2. Firm f's minmax payoff at state θ is

$$\underline{u}_f(\theta) \equiv \inf_{m \in M^{\circ}(\theta)} \sup_{(W', p'_f) \in D_f(m,\theta)} u_f([m, (f, W', p'_f)], \theta).$$

To characterize this minmax payoff, let

$$s(f, W, \theta) \equiv \widetilde{u}_f(W, \theta) + \sum_{w \in W} \widetilde{v}_w(f, W \backslash \{w\}, \theta)$$

denote the total surplus of coalition (f, W) at state θ. The following lemma characterizes each firm's minmax payoff.

Lemma 1. *Let $Q \equiv \sum_{f' \in \mathcal{F}} q_{f'}$ represent the sum of all firms' hiring quotas. For every firm f and state θ, f's minimax payoff satisfies*

$$\underline{u}_f(\theta) = \min_{W' \subseteq \mathcal{W}, |W'| \leq Q} \ \max_{W \subseteq \mathcal{W} \backslash W', |W| \leq q_f} s(f, W, \theta). \tag{1}$$

Lemma 1 states that a firm's minmax payoff equals the maximum surplus it can generate after $Q = \sum_{f' \in \mathcal{F}} q_{f'}$ workers have been removed from the market in an adversarial manner.

The intuition is as follows. To punish f, other firms can offer high wages to $\sum_{f' \neq f} q_{f'}$ workers, which renders them unattractive as potential partners for f. In addition, the employees of f demand high punitive wages from f. Since current employees require consent to join a deviation, if f blocks, it would be better off abandoning all current employees and looking for cheaper workers elsewhere. Altogether, this excludes $\sum_{f'} q_{f'}$ workers from f's potential match pool in case of a deviation. Instead, f will select its partners from the remaining workers and extract all the matching surplus by offering those workers zero payoff.

The following example illustrates Lemma 1 while also highlighting an important feature of the minmax stage: Because of the way punishment works in our matching environment, the minmaxed player might have to temporarily receive a stage-game payoff that is even lower than their minmax payoff. Nonetheless, dynamic incentives ensure that the player is still willing to accept this lower payoff, as shown later in Proposition 1.

Example 1. *Consider an economy with a singleton state space $\Theta = \{\theta_0\}$. Since preferences are identical across periods, we omit the dependence on θ_0 from our notation for brevity. There is a single firm f, and three workers $\mathcal{W} = \{w_1, w_2, w_3\}$ enter the market in each period. The firm can hire at most two workers, and each hired worker generates a revenue of \$2. Workers care only about wages. According to Lemma 1, f's minmax payoff is $\underline{u}_f = 2$: The total hiring capacity is $Q = 2$, and the maximum surplus f can generate with the remaining worker is* 2.

This minmax payoff is secured by the minmax matching $\underline{m}_f$: Two workers (say, w_1 and w_2) match with f while each demanding a punitive wage of 2 *from f. This makes them "too expensive" to be included in f's blocking coalition. Consequently, if f were to deviate from $\underline{m}_f$, its best response would be to abandon $\{w_1, w_2\}$ and instead hire w_3 at a wage of* 0*, which yields the minmax payoff $\underline{u}_f = 2$ for f.*

Note that the minmax payoff describes what a firm can obtain from its most profitable feasible deviation, rather than from implementing the minmax matching. In this example, f's payoff from the minmax matching, $u_f(\underline{m}_f) = 0$, is lower than its actual minmax payoff $\underline{u}_f = 2$. This contrasts with standard normal-form games, where the minmaxed player receives exactly the minmax payoff from the minmaxing action profile. The reason for this key difference is that, in our cooperative stage game, the minmax matching is implemented collectively by all *firms in the game, including the firm being minmaxed. This can be seen from Equation (1), where all firms' hiring quotas are used in the minmax matchings.*

Characterization. We first introduce some notation. For every matching $m \in M^{\circ}(\theta)$ and state $\theta \in \Theta$, let $u(m,\theta) \equiv (u_f(m,\theta))_{f\in\mathcal{F}}$ denote firms' payoff profile under matching m at state θ. For every $\theta \in \Theta$, let

$$\mathcal{U}(\theta) \equiv \mathrm{co}\left(\left\{ u \in \mathbb{R}^{\mathcal{F}} : u = u(m,\theta) \text{ for some } m \in M^{\circ}(\theta) \right\} \right)$$

denote the convex hull of these payoff profiles. In addition, define

$$\mathcal{U}^* \equiv \left\{ \sum_{\theta\in\Theta} \pi(\theta) u(\theta) : u(\theta) \in \mathcal{U}(\theta) \text{ for every } \theta \in \Theta \right\}.$$

Finally, for every firm f, let

$$\underline{u}_f^* \equiv \mathbb{E}_{\pi}[\underline{u}_f(\theta)]$$

denote its expected minmax payoff over states of the world. The next result characterizes the continuation payoffs from self-enforcing matching processes.

Proposition 1. *(i) If $u \in \mathcal{U}^*$ satisfies $u_f > \underline{u}_f^*$ for all $f \in \mathcal{F}$, then there is a $\underline{\delta} \in (0,1)$ such that for every $\delta \in (\underline{\delta}, 1)$, there exists a self-enforcing matching process with firms' continuation payoffs u at the beginning of period 0. (ii) Suppose μ is a self-enforcing matching process for a given $\delta \in (0,1)$. For every ex ante history $\overline{h} \in \overline{\mathcal{H}}$, firms' continuation payoff profile satisfies $\left(U_f(\overline{h} \,|\, \mu)\right)_{f\in\mathcal{F}} \in \mathcal{U}^*$ and $U_f(\overline{h} \,|\, \mu) \geq \underline{u}_f^*$ for every $f \in \mathcal{F}$.*

To prove statement (ii), note that given realized state θ, by the definition of minmax payoffs, every firm f can secure the minmax payoff $\underline{u}_f(\theta)$ by deviating with workers. Taking expectation over the distribution of states delivers the result.

To prove statement (i), we first show that $\mathcal{U}^*$ satisfies the NEU condition (Abreu, Dutta and Smith 1994), which requires that firms' payoffs are not completely aligned with each other. For every payoff profile $u^* \in \mathcal{U}^*$ that is strictly above every firm's average minmax payoff, this misalignment in payoffs allows us to perturb u^* in different directions, constructing firm-specific payoff profiles that serve as punishments for deviating firms. Building on a construction similar to the one used in the folk theorem for repeated games with perfect monitoring (Fudenberg and Maskin 1986), any firm that deviates is minmaxed for a finite number of periods to offset the gains from deviation, before permanently transitioning to its firm-specific punishment.

The cooperative stage game gives rise to a technical difficulty in the minmax phase that is not present in standard folk theorems. As illustrated in Example 1, when a firm f is being minmaxed, it may be necessary for f to accept a stage-game payoff from the minmax matching $\underline{m}_f(\theta)$ that is even lower than its minmax payoff $\underline{u}_f(\theta)$. If the gap between $u_f(\underline{m}_f(\theta), \theta)$ and $\underline{u}_f(\theta)$ is constant across all realizations of θ, firm f can be compensated through continuation value to refrain from deviating from $\underline{m}_f(\theta)$ and obtaining $\underline{u}_f(\theta)$. However, if $\pi \in \Delta(\Theta)$ is non-degenerate, this can create incentives for f to deviate from $\underline{m}_f(\theta)$ when the realized gap is larger than its ex ante expectation. We tackle this by calibrating wages in the minmax matchings $\underline{m}_f(\theta)$ so that this gap is equalized across all realizations of θ. For further details, see Lemma 5(iii); we also provide an illustrative example of this construction in Appendix A.3.

Proposition 1, however, does not guarantee the existence of self-enforcing matching processes. In fact, statement (i) would be vacuously true if there is no feasible payoff profile $u \in \mathcal{U}^*$ that satisfies $u_f > \underline{u}_f^*$ for all $f \in \mathcal{F}$. Note that this is not a concern for noncooperative games, but in two-sided matching environments, such scenarios can indeed arise, for example, when the market has no wages or there are matching externalities. We provide detailed illustrations of these possibilities in Section 4.

Proving the existence of a self-enforcing matching process therefore amounts to showing the existence of $u \in \mathcal{U}^*$ that satisfies $u_f > \underline{u}_f^*$ for all $f \in \mathcal{F}$. We will show that such a payoff profile can arise from a random serial dictatorship among the firms.

Existence. Let a bijection $o : \mathcal{F} \to \{1, \ldots, |\mathcal{F}|\}$ denote an ordering of all firms where $o(f)$ represents the ranking of firm f. Write $\mathcal{O}$ for the set of all such orderings. In the serial dictatorship corresponding to an ordering $o \in \mathcal{O}$, firms choose workers sequentially based on the ordering o while setting all of their matched workers' payoffs to zero, thereby capturing the entire matching surplus.

Definition 3 (Serial Dictatorship)**.** Given state $\theta \in \Theta$ and ordering $o \in \mathcal{O}$, the stage-game matching induced by serial dictatorship, $\widehat{m}(\theta, o)$, is the output of the following procedure. Initialize $W_0^{\#} \equiv \emptyset$. For every step $i = 1, \ldots, |\mathcal{F}|$, denote $\widehat{f}_i \equiv o^{-1}(i)$, and let

$$\widehat{W}_i \in \underset{W \subseteq \mathcal{W} \setminus W_{i-1}^{\#}, |W| \leq q_{\widehat{f}_i}}{\arg\max} s(\widehat{f}_i, W, \theta).$$

Set $\phi(\widehat{f}_i) = \widehat{W}_i$, $p_{\widehat{f}_i w} = -\widetilde{v}_w(\widehat{f}_i, \widehat{W}_i \backslash \{w\}, \theta)$ for every $w \in \widehat{W}_i$, and $p_{\widehat{f}_i w} = 0$ for every $w \notin \widehat{W}_i$; Update $W_i^{\#} \equiv W_{i-1}^{\#} \cup \widehat{W}_i$.

In a *random serial dictatorship (RSD)*, firms randomize over $\mathcal{O}$ and match according to $\widehat{m}(\theta, o)$ based on the realized order o. Formally speaking, at an ex ante history $\overline{h} \in \overline{\mathcal{H}}$, we say players match *according to the outcomes of an RSD* if for each state θ, (i) there is a partition over the realizations of the public randomization device $\Gamma = \bigcup_{o \in \mathcal{O}} \Gamma(\theta, o)$ consisting of disjoint measurable sets, and (ii) players follow the matching $\widehat{m}(\theta, o)$ at the ex post history $(\overline{h}, \theta, \gamma)$ when the realization $\gamma \in \Gamma$ lies in $\Gamma(\theta, o)$. Hence, the probability of $\widehat{m}(\theta, o)$ being recommended at $\overline{h}$ corresponds to the Lebesgue measure of $\Gamma(\theta, o)$.

The example below illustrates how firms match according to the outcomes of an RSD; in addition, it also shows that the firms' expected payoff profile from an RSD needs not be Pareto efficient.

Example 2. *Consider two firms $\mathcal{F} = \{f_1, f_2\}$, each with quota 2, and four workers $\mathcal{W} = \{w_1, w_2, w_3, w_4\}$. Suppose the state space is a singleton $\Theta = \{\theta_0\}$. Given θ_0, firms' revenue functions are additive, and for each firm $f \in \mathcal{F}$, the revenue generated from a single employee is given by*

	w_1	w_2	w_3	w_4
f_1	6	2	1	1
f_2	2	6	1	1

Workers only care about wages. Then, each firm's expected payoff in the uniform RSD is 5 *(where each firm with equal probabilities obtains either (i) workers $\{w_1, w_2\}$ yielding payoff* 8 *or (ii) workers $\{w_3, w_4\}$ yielding payoff* 2*). However, consider the deterministic matching $\widehat{m}$ where firm f_1 matches with workers $\{w_1, w_3\}$ and firm f_2 matches with workers $\{w_2, w_4\}$,* *so each firm ends up with a payoff of 7. The payoff profile generated by the uniform RSD is Pareto dominated by that from $\widehat{m}$, so RSD does not always yield a Pareto efficient payoff profile.*

We make an assumption to simplify our existence proof. Let

$$\overline{u}_f(\theta) \equiv \max_{W \subseteq \mathcal{W}, |W| \leq q_f} s(f, W, \theta) \tag{2}$$

denote firm f's maximum feasible payoff at state θ. It is easy to see that $\overline{u}_f(\theta) \geq \underline{u}_f(\theta)$ for every $\theta \in \Theta$ and $f \in \mathcal{F}$, since removing Q workers in an adversarial manner reduces f's maximum surplus. For now, we assume that for every firm, this inequality is strict with a positive probability.

Assumption 2. *For every firm f, there exists $\theta \in \Theta$ with $\pi(\theta) > 0$ such that $\overline{u}_f(\theta) > \underline{u}_f(\theta)$.*

Assumption 2 holds generically, for example, when players' payoffs in every state θ are randomly drawn from a continuous distribution. This assumption greatly simplifies the exposition of Proposition 2 but is not crucial to our results. In Appendix B.1, we show that our results hold even without this assumption.

The following lemma shows that when firms randomize uniformly over serial dictatorships at every state (i.e., when the measure of $\Gamma(\theta, o)$ equals $\frac{1}{|\mathcal{O}|}$ for every $\theta \in \Theta$ and $o \in \mathcal{O}$), they can simultaneously obtain payoffs that strictly exceed their expected minmax payoffs. Figure 2 provides an illustration of this result when the market has only two firms.

Lemma 2. *Under Assumption 2, for every firm f,*

$$\frac{1}{|\mathcal{O}|} \sum_{o \in \mathcal{O}} \mathbb{E}_\pi \Big[u_f\big(\widehat{m}(\theta, o), \theta\big) \Big] > \underline{u}_f^*.$$

Let us explain the intuition for Lemma 2. If firms randomize uniformly over $\mathcal{O}$ for every realized state θ, then each firm f receives the payoff on the left side. Fix f and θ. In this RSD, the worst-case scenario for f arises when f is ranked last in o (i.e., $o(f) = |\mathcal{F}|$): By the time f selects its employees, $Q_{-f} \equiv \sum_{f' \neq f} q_{f'}$ workers are already off the market. However, for f, this worst case under RSD is still weakly more desirable than being minmaxed, in

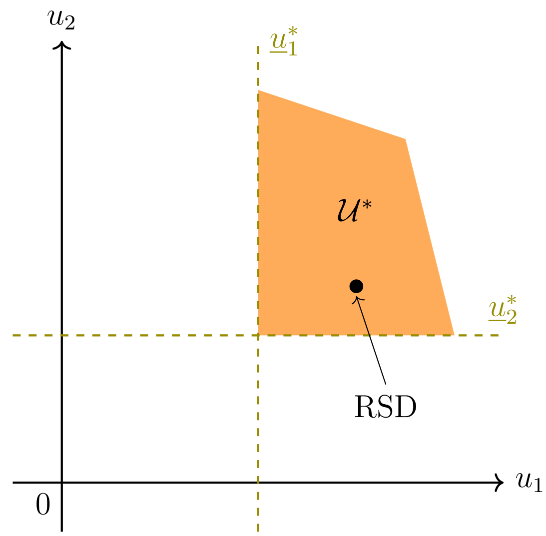

FIGURE 2. Firms' payoff space. Payoff profiles in the orange region $\mathcal{U}^*$ (feasible and strictly above firms' expected minmax payoffs $\underline{u}_1^*$ and $\underline{u}_2^*$) can be sustained by Proposition 1. Lemma 2 ensures that region $\mathcal{U}^*$ is nonempty, which includes a payoff profile generated by an RSD. Note that the RSD outcome may or may not be Pareto optimal; see Example 2.

which case f must choose its employees after $Q = \sum_{f'} q_{f'}$ workers have been eliminated in an *adversarial* manner. Therefore, the distribution of payoffs f obtained under the uniform RSD first-order stochastically dominates the distribution of its minmax payoffs, so in expectation, every firm prefers this RSD over being minmaxed.

Combined with Proposition 1, Lemma 2 delivers our existence result below.

Proposition 2. *Under Assumption 2, when firms are sufficiently patient, there exists a self-enforcing matching process in which players match according to the outcome of an RSD in every period on path.*

To understand the intuition of this result, consider the uniform RSD. For every state θ and every ordering o, matching $\widehat{m}(\theta, o)$ is in $M^\circ(\theta)$, meaning that randomization over $\mathcal{O}$ places firms' payoff profile in $\mathcal{U}(\theta)$. Taking the expectation over θ, we know that firms' payoff profile from the uniform RSD is in $\mathcal{U}^*$. Furthermore, according to Lemma 2, each firm's payoff is strictly higher than its average minmax payoff. Proposition 1(i) then delivers the result: A self-enforcing matching process exists where firms' continuation payoffs on path are determined by the uniform RSD.

We highlight that matching according to the outcomes of an RSD can be viewed as a specific form of no-poaching agreement, where firms collude to suppress wages and extract surplus from workers. In an RSD, once an ordering has been established, each firm refrains from soliciting workers already employed by firms with higher priority, even if it is feasible and profitable to do so. This arrangement eliminates wage competition and facilitates surplus extraction from workers. Of course, RSDs are not the only kind of no-poaching agreements; firms can achieve similar effects by, for example, committing to offer predetermined wages without engaging in competitive bidding—as seen in cases involving financial aid in college admissions that we discussed earlier. Our result demonstrates that the uniform RSD, as a specific form of no-poaching agreement, always gives rise to a payoff profile that can be sustained dynamically, even in markets that exhibit complementarities and peer effects.

4 Discussions

In this section, we examine three aspects of the model that are important to our analysis. First, we show that flexible wages and the absence of externalities are necessary for our existence result, and we provide counterexamples illustrating the failure of existence when these conditions are violated. Finally, we discuss our focus on deviation plans by single firms, and explore how our results can be extended to accommodate multi-firm deviation plans.

4.1 Role of Transfers

Our results hinge on wage flexibility in two key respects: (1) it allows firms to extract surplus from workers in an RSD, and (2) it ensures that, when a firm f faces a minmax punishment, sufficiently high wages price out certain workers regardless of mutual preferences, yielding the minmax payoffs in Lemma 1. Note that quasilinearity is not required to fulfill these roles—only that monetary preferences are monotone and unbounded. By contrast, without wage flexibility, firms' ability to extract surplus and enforce punishments is severely limited. In particular, when minmaxing a firm f, rival firms can no longer simply price specific workers out of f's choice set; instead, the set of workers available to f is determined by workers' preferences, which generally raises firms' minmax payoffs and may preclude the existence of a self-enforcing matching process.

To illustrate potential nonexistence without wage flexibility, consider wages fixed at 0. Let $\mathcal{W} = \{w_1, w_2, w_3\}$ and $\mathcal{F} = \{f_1, f_2\}$, each with quota 2. A firm's stage-game payoff is 2 if it hires two workers and 1 if it hires one, regardless of identities. On the other hand, workers' preferences are determined by their colleagues: w_1 prefers to work with w_2, w_2 prefers w_3, and w_3 prefers w_1. Moreover, having any coworker is strictly better than working alone or being unemployed. Workers may possess strict preferences for firms, but this aspect is dominated by their preferences for coworkers.

In this example, the minmax payoff of each firm is exactly 2. To see this, note that each firm can secure a deviating payoff of 2 in the stage game by hiring an unemployed worker w together with the worker w' who prefers w. This is true regardless of the other firm's hiring decision, so the minmax payoff must be at least 2. On the other hand, since the maximum payoff in a stage game is 2, the minmax payoff cannot exceed this value.

As a result, any self-enforcing matching process must deliver to each firm a continuation payoff weakly higher than 2. For if not, the firm would have a feasible and profitable deviation plan such that at every ex post history, it deviates from the stage-game matching by securing a stage-game payoff of 2. However, there is no feasible way to generate a payoff weakly higher than 2 for every firm in the stage-game matching even with a randomizing device. Hence, no self-enforcing matching process exists for any value of δ in this no-transfers setting. Figure 3 illustrates this non-existence issue.

In a no-transfers setting with neither complementarities nor peer effects, Liu (2023) provides a formal characterization of minmax payoffs and an analog of Proposition 1. Even with complementarities and peer effects, as long as there is no random θ, it is straightforward to extend Liu (2023)'s characterization of minmax, and the analog of Proposition 1 will hold—that is, any feasible payoff profile that strictly dominates these minmax payoffs will be sustainable when $\delta \to 1$, and any self-enforcing matching process must generate payoff profiles that weakly dominate these minmaxes. In this sense, the nonexistence issue illustrated in the example above arises because without transfers, the minmaxes are too high, so the feasible and weakly individually rational payoff set is empty.

However, extending Proposition 1 to the general environment with *random* θ, comple-

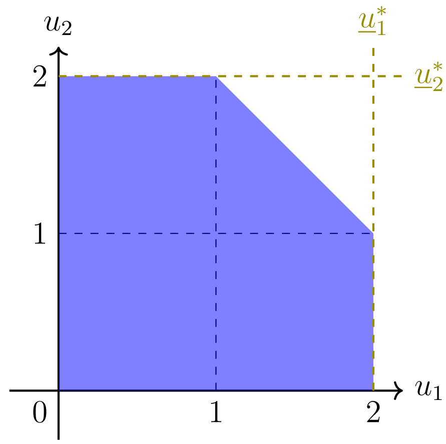

FIGURE 3. Matching without transfers. The blue region includes the set of firm payoffs that can be achieved via public randomization. The green lines represent the firms' minmax payoffs. There is no feasible payoff vector such that both firms are weakly better off than receiving the minmax payoffs.

mentarities, and peer effects remains challenging in the rigid-wage setting. The difficulty arises because we do not currently have a method to address the complications arising from minmaxing players, as highlighted after Proposition 1, without the flexibility provided by adjustable wages. We leave it as an open question for future research.

4.2 Externalities

In our setting, we assume that firms are exclusively concerned with the identities and types of their own employees, while workers only care about their own work environments. However, in a more general setting, players' payoffs may be contingent on the entire matching assignment, including the hiring decisions of other firms. If players are affected by externalities only through the assignment but not wages set by other firms, the analogs of Lemma 1 (characterization of minmax payoffs) and Proposition 1 (characterization of equilibrium payoffs) still hold.[14] However, it may not be feasible to provide all firms with payoffs weakly higher than their corresponding minmaxes, and therefore a self-enforcing matching process no longer exists by the necessary conditions in Proposition 1(ii). We provide an example below to illustrate this nonexistence issue.

[14]In the analog of Lemma 1, the minimization would be taken over all assignments ϕ instead of W', and the surplus function s would depend on the entire assignment ϕ.

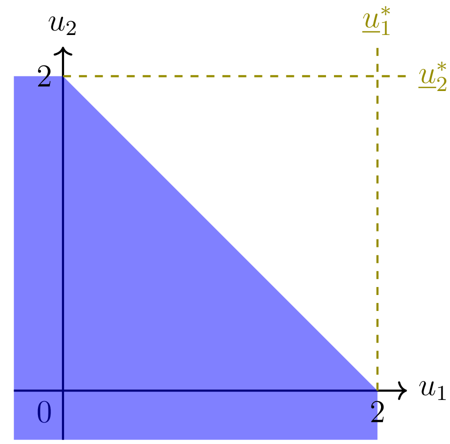

Figure 4. Matching with externalities. The blue region includes the set of firm payoffs that can be achieved via public randomization. The green lines represent the firms' minmax payoffs. There is no feasible payoff vector such that both firms are weakly better off than receiving the minmax payoffs.

Consider a matching market with flexible wages in which each firm's stage-game payoff depends on the assignments of other firms. Suppose there are three workers $\mathcal{W} = \{w_1, w_2, w_3\}$ and two firms $\mathcal{F} = \{f_1, f_2\}$. Each firm can only hire one worker. The workers are indifferent between being unemployed and working for any firm at a payment of 0. Firm f_1 obtains a stage-game payoff of 2 if it hires the same number of workers as firm f_2 does and a stage-game payoff of 0 otherwise. On the other hand, firm f_2 gets 2 if the two firms hire different numbers of workers and 0 otherwise.

In this example, each firm can ensure a deviating payoff arbitrarily close to 2 regardless of the other firm's hiring decisions: For example, firm f_1 can always turn to the unemployed worker and offer an infinitesimal wage if f_2 has an employee, and it can choose not to hire and shut down if f_2 does the same. Therefore, the minmax payoffs of both firms should be equal to 2. However, there is no feasible way to generate a payoff of 2 for both firms in the stage-game matching, because either they hire the same number of workers or they do not. In other words, the frontier of the feasible payoff set has an intercept of 2 and a slope of -1; see Figure 4 for an illustration. Hence, no self-enforcing matching process exists for any value of δ with the presence of matching externalities.

To understand the intuition behind this nonexistence issue, consider Lemma 2, which is a crucial piece that leads to our existence result. In a serial dictatorship without externalities, a firm's payoff is monotonic with respect to its priority ranking. When the firm holds the highest priority, it can secure its maximum payoff in the stage game; when the firm is last to hire, its payoff becomes weakly lower but cannot be worse than its minmax. However, in the presence of externalities, this monotonicity in a serial dictatorship no longer holds. For instance, in our example, each firm can achieve a strictly higher payoff if it makes its choice after the other firm. As a result, a random serial dictatorship may not produce a payoff profile that strictly dominates the minmax payoffs for all firms, and we cannot apply the sufficient conditions in Proposition 1(i) to establish existence.

4.3 Deviations by Multiple Firms

Our notion of dynamic stability requires immunity to deviation plans conceived by individual firms. As discussed after Definition 1, this mirrors classic stability concepts in static settings, which likewise focus on immunity to single-firm coalitions. In static settings, the rationale is often that larger coalitions are harder to organize due to the coordination required among multiple firms. In dynamic settings, an additional concern is that multi-firm deviation plans may themselves be vulnerable to further deviations and thus may not constitute credible deviations in the first place.

Apart from these philosophical considerations, there are two practical challenges when assessing multi-firm deviations. One difficulty in evaluating multi-firm deviation plans is the lack of a one-shot deviation principle: for coalitions consisting of multiple firms, ruling out profitable one-shot deviations alone does not rule out the possibility of a profitable deviation plan, which complicates verifying stability. The second challenge lies in specifying an appropriate observability assumption for intertemporal coordination. For instance, consider firms $\{f_1, f_2\}$ executing a joint deviation in which f_1 blocks in period one and f_2 blocks in period two. Other players may be unable to determine whether f_2's action is part of a coordinated deviation by $\{f_1, f_2\}$ or an independent deviation following f_1's move.

Although it remains open whether our existence result extends to environments where intertemporal coordination may be unobservable, we are able to show that it continues to hold if firms' membership in any blocking plan is common knowledge. Specifically, the assumption requires that whenever the realized stage-game matching differs from the recommendation by the matching process, all firm in the deviation plan are publicly identified and recorded in the stage-game outcome, regardless of whether they directly took part in the current-period deviation. Despite the absence of a one-shot deviation principle, we show that under this observability assumption, the matching process constructed in the proof of Proposition 2 can be adapted to withstand multi-firm deviation plans.

Online Appendix B.2 contains the formal definitions and proofs, but let us illustrate with the example in the introduction. Specifically, we modify the matching process in Figure 1 so that if no firm or a single firm blocks, the original transition applies, but if firms $\{f_1, f_2\}$ jointly deviate, then the transition follows the red arrows to P1 so f_1 is punished. To overcome the lack of a one-shot deviation principle for multi-firm deviation plans, we will show that any profitable multi-firm deviation plan implies a profitable *single-firm* deviation plan, which in turn yields a profitable one-shot deviation by that firm. Since the introduction shows that the process admits no such deviations, this leads to a contradiction.

How can one reduce a profitable multi-firm deviation plan to a profitable single-firm plan? Suppose $\{(d_1, \eta_1), (d_2, \eta_2)\}$ is a profitable multi-firm deviation plan, where each (d_i, η_i) is a complete contingent plan specifying the workers and wages that firm i intends to employ. A key observation is that in the stage game, joint blocking with f_2 never enlarges firm f_1's set of feasible blockings. Hence, we construct an alternative deviation plan (d_1', η_1') for firm 1 alone that mimics (d_1, η_1) but ignores f_2's blocks. Under both plans, f_1 is punished whenever the outcome differs from the one recommended by the process; however, since (d_1', η_1') omits f_2's blocking, f_1 is punished less often, yielding a higher payoff for f_1. Therefore, (d_1', η_1') is a profitable single-firm deviation for f_1.

5 Conclusion

In this paper, we propose a new approach of studying stability in matching markets with complementarities and peer effects, which have traditionally challenged the existence of static stable outcomes. Our approach treats stability as the result of a dynamic process that is self-sustained by expectations; importantly, these expectations must also be themselves consistent with stability.

We leverage the repeated interactions of long-lived firms to discipline these market participants and sustain stability over time. As a result, we can define notions such as minmax payoffs and feasible payoff sets that are analogous to those from standard repeated games. However, in our model firms and workers play in a cooperative game every period. This difference makes the nonissue of equilibrium existence in noncooperative setting more challenging in our cooperative setting. We prove the existence of a stable matching process by explicitly constructing a feasible payoff profile through random serial dictatorship where firms take turns to receive their more favorable outcome. This can be interpreted as a form of no-poaching agreement, which is prevalent and subject to intense legal debates in real-world contexts.

From a theoretical standpoint, the existence of self-enforcing matching processes reconciles the lack of static stable matching and absence of complete chaos in many matching markets. From a practical point of view, the sustainability of outcomes through no-poaching agreements adds a new dimension to the ongoing debate regarding their anti-trust implications. While no-poaching agreements are generally regarded as anti-competitive, our analysis suggests they may play a crucial role in preserving market stability.

An interesting open question for future research is the comparative statics in the minimum amount of patience needed to sustain stable a self-enforcing matching process. In addition, it would be interesting to see if such a dynamic approach would help solve the nonexistence of stable outcomes in roommate problems or networks with externalities.

A Appendix

A.1 Intermediate Results

Proof of Lemma 1. Fix $f \in \mathcal{F}$, $\theta \in \Theta$, and stage-game matching $m = (\phi, p) \in M^\circ(\theta)$. Denote the set of employed workers by $W_m \equiv \{w \in \mathcal{W} : \phi(w) \neq (\emptyset, \emptyset)\}$. Hence, $|W_m| \leq Q$. Hiring from $\mathcal{W} \backslash W_m$ at infinitesimal wages is always a feasible deviation for f, which means

$$\begin{aligned}
\sup_{(W', p'_f) \in D_f(m,\theta)} u_f([m, (f, W', p'_f)], \theta) &\geq \max_{W' \subseteq \mathcal{W} \backslash W_m, |W'| \leq q_f} s(f, W', \theta) \\
&\geq \min_{W \subseteq \mathcal{W}, |W| = |W_m|} \max_{W' \subseteq \mathcal{W} \backslash W, |W'| \leq q_f} s(f, W', \theta) \\
&\geq \min_{W \subseteq \mathcal{W}, |W| \leq Q} \max_{W' \subseteq \mathcal{W} \backslash W, |W'| \leq q_f} s(f, W', \theta).
\end{aligned}$$

Taking infimum over $m \in M^\circ(\theta)$ on the LHS yields

$$\underline{u}_f(\theta) \geq \min_{W \subseteq \mathcal{W}, |W| \leq Q} \max_{W' \subseteq \mathcal{W} \backslash W, |W'| \leq q_f} s(f, W', \theta).$$

For the other direction, take any $W \subseteq \mathcal{W}$ with $|W| \leq Q$. We can always construct a stage-game matching $\widehat{m} = (\widehat{\phi}, \widehat{p}) \in M^\circ(\theta)$ such that (i) $\widehat{\phi}$ assigns all workers in W to firms, and (ii) all workers in W receive sufficiently high wages so that f never finds it profitable to deviate with any workers in W. Therefore,

$$\begin{aligned}
\max_{W' \subseteq \mathcal{W} \backslash W, |W'| \leq q_f} s(f, W', \theta) &= \sup_{(W', p'_f) \in D_f(\widehat{m},\theta)} u_f([\widehat{m}, (f, W', p'_f)], \theta) \\
&\geq \inf_{m \in M^\circ(\theta)} \sup_{(W', p'_f) \in D_f(m,\theta)} u_f([m, (f, W', p'_f)], \theta) = \underline{u}_f(\theta).
\end{aligned}$$

Minimizing over W on the LHS yields the other direction. □

Lemma 3. *Deviation plan (d, η) is a* one-shot deviation *from matching process μ if there is a unique ex post history $\widehat{h}$ where $[\mu, (f, d, \eta)](\widehat{h}) \neq \mu(\widehat{h})$. Matching process μ is self-enforcing if and only if (i) $v_w(\mu(h), \theta) \geq 0$ for every $w \in \mathcal{W}$ at every $h \in \mathcal{H}$; and (ii) no firm has a* *feasible and profitable one-shot deviation.*

Proof. Standard arguments (Blackwell 1965) suffice. We include them for completeness. Suppose one-shot deviation plan (d,η) from matching process μ for firm f is feasible and profitable. Since stage-game payoffs are bounded for firm f and there is discounting, the standard one-shot deviation principle for individual decision-making (Blackwell 1965) implies that there exists an ex post history $\widehat{h} = (\widehat{\overline{h}}, \widehat{\theta}, \widehat{\gamma})$ such that

$$(1-\delta)\left[\widetilde{u}_f(d_f(\widehat{h}),\widehat{\theta}) - \sum_{w\in d(\widehat{h})} \eta_w(\widehat{h})\right] + \delta U_f\big(\big[\mu(\widehat{h}),(f,d(\widehat{h}),\eta(\widehat{h}))\big] \,\big|\, \mu\big) > U_f(\widehat{h}\,|\,\mu).$$

Consider deviation plan (d'_f, η'_f) that satisfies

$$[\mu,(f,d'_f,\eta'_f)](h) = \begin{cases} [\mu,(f,d,\eta)](h) & \text{if } h=\widehat{h}, \\ \mu(h) & \text{otherwise.} \end{cases}$$

Then (d'_f,η'_f) is a profitable one-shot deviation plan for firm f. $\square$

Lemma 4. *For any stage-game matching m, $[m,(f_1,W_{f_1},p_{f_1})] = [m,(f_2,W_{f_2},p_{f_2})] \neq m$ implies $f_1 = f_2$.*

Proof. Let $m=(\phi,p)$, $[m,(f_1,W'_{f_1},p'_{f_1})] = (\overline{\phi},\overline{p})$, and $[m,(f_2,W'_{f_2},p'_{f_2})] = (\widehat{\phi},\widehat{p})$. Toward contradiction, suppose $f_1\neq f_2$, but $[m,(f_1,W'_{f_1},p'_{f_1})] = [m,(f_2,W'_{f_2},p'_{f_2})] \neq m$. Then $\overline{\phi}=\widehat{\phi}$ and $\overline{p}_f = \widehat{p}_f$ for every $f\in\mathcal{F}$. Each of the three cases to consider yields a contradiction.

1. Suppose $W'_{f_1}=\phi(f_1)$ and $W'_{f_2}=\phi(f_2)$. Since $[m,(f_1,W'_{f_1},p'_{f_1})]\neq m$, we have $p'_{f_1}\neq p_{f_1}$. Then $\overline{p}_{f_1} = p'_{f_1} \neq p_{f_1} = \widehat{p}_{f_1}$, a contradiction.

2. Suppose $W'_{f_1}\subseteq\phi(f_1)$ and $W'_{f_2}\subseteq\phi(f_2)$ but, without loss of generality, $W'_{f_1}\neq\phi(f_1)$. We have $\overline{\phi}(f_1) = W'_{f_1}\neq\phi(f_1)=\widehat{\phi}(f_1)$, so $\overline{\phi}\neq\widehat{\phi}$, a contradiction.

3. Suppose, without loss of generality, $W'_{f_1}\nsubseteq\phi(f_1)$. Let $w'\in W'_{f_1}\backslash\phi(f_1)$. We have $\overline{\phi}(w')=f_1$, whereas $\widehat{\phi}(w')\in\{\phi(w'),f_2\}$. Since $w'\notin\phi(f_1)$ and $f_1\neq f_2$, $f_1\notin\{\phi(w'),f_2\}$. Hence, $\overline{\phi}\neq\widehat{\phi}$, a contradiction.

Therefore, $[m,(f_1,W_{f_1},p_{f_1})] = [m,(f_2,W_{f_2},p_{f_2})]\neq m$ implies $f_1=f_2$. $\square$

A.2 Omitted Proofs for Section 3

A.2.1 Proof of Proposition 1

Lemma 5. *For each $\theta \in \Theta$, there exist stage-game matchings $\{\underline{m}_f(\theta)\}_{f\in\mathcal{F}} \subseteq M^\circ(\theta)$ such that $\forall f \in \mathcal{F}$,*

(i) $\sup_{(W',p'_f)\in D_f(\underline{m}_f,\theta)} u_f([\underline{m}_f(\theta),(f,W',p'_f)],\theta) = \underline{u}_f(\theta)$*;*

(ii) $u_f(\underline{m}_f(\theta),\theta) \leq \underline{u}_f(\theta)$*;*

(iii) $\underline{u}_f(\theta) - u_f(\underline{m}_f(\theta),\theta) = \underline{u}^*_f - \mathbb{E}_\pi[u_f(\underline{m}_f(\theta),\theta)]$.

In this lemma, each $\underline{m}_f(\theta)$ is a minmax matching for firm f. Part (i) says that firm f's best deviation from $\underline{m}_f(\theta)$ yields a payoff at most $\underline{u}_f(\theta)$, i.e., its minmax payoff at state θ. Part (ii) requires that firm f receives a payoff no more than $\underline{u}_f(\theta)$ from following $\underline{m}_f(\theta)$. Part (iii) ensures that gains from the best deviations are equalized across states.

Proof of Lemma 5. Fix $\theta \in \Theta$. For every $f \in \mathcal{F}$, let

$$\underline{W}_f(\theta) \in \mathop{\arg\min}_{W\subseteq\mathcal{W},|W|\leq Q} \max_{W'\subseteq\mathcal{W}\setminus W,|W'|\leq q_f} s(f,W',\theta).$$

Firm f can obtain its minmax payoff by extracting surplus from its favorite workers *not* in $\underline{W}_f(\theta)$, so each $\underline{W}_f(\theta)$ is a set of workers to eliminate to minmax firm f.

For every $f \in \mathcal{F}$, let $\{\underline{W}_f^{f'}(\theta)\}_{f'\in\mathcal{F}}$ be a partition of $\underline{W}_f(\theta)$ such that $\left|\underline{W}_f^f(\theta)\right| \geq 1$ and $\left|\underline{W}_f^{f'}(\theta)\right| \leq q_{f'}$ for every $f' \in \mathcal{F}$. Intuitively, each firm f' is matched with workers in $\underline{W}_f^{f'}(\theta)$, so all firms collectively take away $\underline{W}_f(\theta) = \bigcup_{f'\in\mathcal{F}} \underline{W}_f^{f'}(\theta)$ to minmax firm f. Let

$$B_f \equiv \max_{\theta\in\Theta} \max_{w\in\underline{W}_f^f(\theta)} \left|\underline{W}_f^f(\theta)\right| \left[\overline{u}_f(\theta) - \underline{u}_f(\theta) - \widetilde{v}_w\big(f,\underline{W}_f^f(\theta)\setminus\{w\},\theta\big)\right] + \underline{u}_f(\theta) - \widetilde{u}_f(\underline{W}_f^f(\theta),\theta).$$

Each firm f's payments must satisfy the following conditions: (i) no other firm benefits from deviating with the workers hired by f, (ii) f receives a payoff lower than its minmax value, and (iii) the gains from deviations are equalized across states. The scalar B_f is introduced to help construct these payments for firm f.

For every $\theta \in \Theta$, define stage-game matching $\underline{m}_f(\theta) = (\underline{\phi}^f(\theta), \underline{p}^f(\theta))$, where $\underline{\phi}^f(\theta)(f') = \underline{W}_f^{f'}(\theta)$, and

$$\underline{p}^f_{f'w}(\theta) = \begin{cases} \frac{B_f - \underline{u}_f(\theta) + \widetilde{u}_f(\underline{W}_f^f(\theta), \theta)}{\left|\underline{W}_f^f(\theta)\right|} & \text{if } f' = f \text{ and } w \in \underline{W}_f^f(\theta), \\ \overline{u}_f(\theta) - \underline{u}_f(\theta) - \widetilde{v}_w\big(f', \underline{W}_f^{f'}(\theta) \backslash \{w\}, \theta\big) & \text{if } f' \neq f \text{ and } w \in \underline{W}_f^{f'}(\theta) \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

For any $\theta \in \Theta$, if $w \in \underline{W}_f^f(\theta)$,

$$v_w(\underline{m}_f(\theta), \theta) = \widetilde{v}_w\big(f, \underline{W}_f^f(\theta) \backslash \{w\}, \theta\big) + \frac{B_f - \underline{u}_f(\theta) + \widetilde{u}_f(\underline{W}_f^f(\theta), \theta)}{\left|\underline{W}_f^f(\theta)\right|} \geq \overline{u}_f(\theta) - \underline{u}_f(\theta)$$

by the definition of B_f. If $w \in \underline{W}_f^{f'}(\theta)$ and $f' \neq f$, $v_w(\underline{m}_f(\theta), \theta) = \overline{u}_f(\theta) - \underline{u}_f(\theta)$. Since $\overline{u}_f(\theta) - \underline{u}_f(\theta) \geq 0$, we have $\underline{m}_f(\theta) \in M^\circ(\theta)$.

(i) Consider any feasible deviation $(W', p'_f) \in D_f(\underline{m}_f, \theta)$. Suppose $W' \subseteq \mathcal{W} \backslash \underline{W}_f(\theta)$. By feasibility, every $w \in W'$ finds the deviation individually rational, so $\widetilde{v}_w(f, W' \backslash \{w\}, \theta) + p'_{fw} \geq 0$. This implies

$$\begin{aligned} \widetilde{u}_f(W', \theta) - \sum_{w \in W'} p'_{fw} &\leq \widetilde{u}_f(W', \theta) + \sum_{w \in W'} \widetilde{v}_w(f, W' \backslash \{w\}, \theta) && (3) \\ &= s(f, W', \theta) \\ &\leq \max_{W \subseteq \mathcal{W} \backslash \underline{W}_f(\theta), |W| \leq q_f} s(f, W, \theta) && (4) \\ &= \underline{u}_f(\theta), \end{aligned}$$

where (3) follows from $W' \subseteq \mathcal{W} \backslash \underline{W}_f(\theta)$, and (4) follows from the definition of $\underline{W}_f(\theta)$. Suppose instead $W' \nsubseteq \mathcal{W} \backslash \underline{W}_f(\theta)$. Fix $w \in W' \cap \underline{W}_f(\theta)$. By the construction of $\underline{m}_f(\theta)$, $v_w(\underline{m}_f(\theta), \theta) \geq \overline{u}_f(\theta) - \underline{u}_f(\theta)$. For the deviation to be feasible, w must obtain a payoff weakly higher than in $\underline{m}_f(\theta)$: $\widetilde{v}_w(f, W' \backslash \{w\}, \theta) + p'_{fw} \geq \overline{u}_f(\theta) - \underline{u}_f(\theta)$; meanwhile, every $w' \in W' \backslash \{w\}$ needs to find the deviation individually rational: $\widetilde{v}_{w'}(f, W' \backslash \{w'\}, \theta) +$

$p'_{fw'} \geq 0$. Then

$$\begin{aligned}
\widetilde{u}_f(W',\theta) - \sum_{w'\in W'} p'_{fw'} &= \widetilde{u}_f(W',\theta) - \sum_{w'\in W', w'\neq w} p'_{fw'} - p'_{fw} \\
&\leq \widetilde{u}_f(W',\theta) + \sum_{w\in W'} \widetilde{v}_w(f, W'\backslash\{w\},\theta) - (\overline{u}_f(\theta) - \underline{u}_f(\theta)) \\
&= s(f, W', \theta) - \overline{u}_f(\theta) + \underline{u}_f(\theta) \leq \underline{u}_f(\theta).
\end{aligned}$$

We have shown that for any W', $u_f([\underline{m}_f, (f, W', p'_f)], \theta) \leq \underline{u}_f(\theta)$. Hence,

$$\sup_{(W', p'_f)\in D_f(\underline{m}_f, \theta)} u_f([\underline{m}_f, (f, W', p'_f)], \theta) \leq \underline{u}_f(\theta).$$

By the definition of $\underline{u}_f(\theta)$, the above holds with equality.

(ii) For every $f \in \mathcal{F}$ and $\theta \in \Theta$,

$$\begin{aligned}
u_f(\underline{m}_f(\theta), \theta) &= \widetilde{u}_f(\underline{W}^f_f(\theta), \theta) - \sum_{w\in \underline{W}^f_f(\theta)} \underline{p}^f_{fw}(\theta) \\
&= \widetilde{u}_f(\underline{W}^f_f(\theta), \theta) - \left[B_f - \underline{u}_f(\theta) + \widetilde{u}_f(\underline{W}^f_f(\theta), \theta)\right] \\
&= \underline{u}_f(\theta) - B_f & (5) \\
&\leq \widetilde{u}_f(\underline{W}^f_f(\theta), \theta) + \sum_{w\in \underline{W}^f_f(\theta)} \widetilde{v}_i(f, \underline{W}^f_f(\theta)\backslash\{w\}, \theta) - (\overline{u}_f(\theta) - \underline{u}_f(\theta)) & (6) \\
&= s(f, \underline{W}^f_f(\theta), \theta) - \overline{u}_f(\theta) + \underline{u}_f(\theta) \quad \leq \quad \underline{u}_f(\theta),
\end{aligned}$$

where (6) follows from the definition of B_f and the fact that $\left|\underline{W}^f_f(\theta)\right| \geq 1$.

(iii) By (5), $u_f(\underline{m}_f(\theta), \theta) = \underline{u}_f(\theta) - B_f$ for all $f \in \mathcal{F}$ and $\theta \in \Theta$. Therefore, $\mathbb{E}_\pi[u_f(\underline{m}_f(\theta), \theta)] = \underline{u}^*_f - B_f$, which means $\underline{u}_f(\theta) - u_f(\underline{m}_f(\theta), \theta) = \underline{u}^*_f - \mathbb{E}_\pi[u_f(\underline{m}_f(\theta), \theta)]$.

□

Lemma 6. *For every $u \in \mathcal{U}^*$ such that $u_f > \underline{u}^*_f$ for all $f \in \mathcal{F}$, there exist vectors $\{u^f\}_{f\in\mathcal{F}} \subseteq \mathcal{U}^*$ such that for every $f \in \mathcal{F}$, $\underline{u}^*_f < u^f_f < u_f$. Moreover, if $|\mathcal{F}| \geq 2$, then for all distinct $f, f' \in \mathcal{F}$, $u^f_f < u^{f'}_f$.*

Proof of Lemma 6. This lemma states that $\mathcal{U}^*$ satisfies the NEU condition (Abreu et al. 1994). The intuition is straightforward: Consider any stage-game matching m. If we modify the matching by requiring firm f to pay higher wages to its employees, f becomes strictly worse off, while all other firms $f' \neq f$ retain their previous payoffs. This logic allows us to construct u^f from u for each $f \in \mathcal{F}$. The detailed arguments are as follows.

Fix stage-game matching $m = (\phi, p)$ and state θ. For every $f \in \mathcal{F}$, define a matching $m^f = (\phi^f, p^f)$ by $\phi^f = \phi$, and

$$p^f_{f'w} = \begin{cases} p_{f'w} + 1 & \text{if } f' = f \text{ and } w \in \phi^f, \\ p_{f'w} & \text{otherwise.} \end{cases}$$

Clearly, $v_w(m^f, \theta) = v_w(m, \theta)$ for every $w \in \mathcal{W} \backslash \phi^f$, and $v_w(m^f, \theta) > v_w(m, \theta)$ for every $w \in \phi^f$, so m^f is individually rational for all workers whenever m is—i.e., $m \in M^\circ(\theta)$ implies $m^f \in M^\circ(\theta)$. Moreover, by the construction of $\{m^f\}_{f \in \mathcal{F}}$, we have $u_f(m^f, \theta) \leq u_f(m, \theta)$ for every $f \in \mathcal{F}$, with the inequality being strict if $\phi(f) \neq \emptyset$. In the following, we write $\zeta^f : m \mapsto m^f$, which is a mapping that "pushes" every matching m to a new one that makes firm f worse off while keeping all other firms indifferent.

For an arbitrary vector $u \in \mathcal{U}^*$, there exists a collection $(\lambda(\theta))_{\theta \in \Theta}$, where $\lambda(\theta) \in \Delta(M^\circ(\theta))$ for every $\theta \in \Theta$, such that $u_f = \mathbb{E}_\pi[\mathbb{E}_{\lambda(\theta)}[u_f(m, \theta)]] > \underline{u}^*_f$ for every f. Now for every firm $f \in \mathcal{F}$, we perturb the payoff profile u using the mapping ζ^f and a small positive number ϵ by letting $u^f = \epsilon \mathbb{E}_\pi[\mathbb{E}_{\lambda(\theta)}[u(\zeta^f(m), \theta)]] + (1 - \epsilon) u$.

First observe that because $u_f > \underline{u}^*_f \geq 0$, there exist $\theta \in \Theta$ and $m = (\phi, p) \in M^\circ(\theta)$ such that $\lambda(\theta)[m] > 0$ and $\phi(f) \neq \emptyset$; by the construction of $\{\zeta^f(m)\}_{f \in \mathcal{F}}$, $u_f(\zeta^f(m), \theta) < u_f(m, \theta)$, so $u^f_f < u_f$. Moreover, if $|\mathcal{F}| \geq 2$ and $f' \neq f$, we have $u^f_f < u^{f'}_f$ since $u_f(m, \theta) = u_f(\zeta^{f'}(m), \theta)$. Second, every u^f can be written as $u^f = \mathbb{E}_\pi[\mathbb{E}_{\lambda^f(\theta)}[u_f(m, \theta)]]$, where

$$\lambda^f(\theta) = \epsilon \lambda(\theta) \circ (\zeta^f)^{-1} + (1 - \epsilon) \lambda(\theta) \quad f \in \mathcal{F}. \tag{7}$$

Note that the support of every $\lambda^f(\theta)$ is also bounded. Finally, for small enough $\epsilon > 0$, $u^f_f > \underline{u}^*_f$. Therefore, there must exist $\{u^f\}_{f \in \mathcal{F}} \subseteq \mathcal{U}^*$ such that $\underline{u}^*_f < u^f_f < u_f$ for all $f \in \mathcal{F}$, and for every distinct $f' \neq f$, we have $u^f_f < u^{f'}_f$. $\square$

Proof of Proposition 1(i). Fix $u \in \mathcal{U}^*$. We first consider the more general case where $|\mathcal{F}| \geq 2$. Let $(\lambda(\theta))_{\theta\in\Theta}$ be a tuple of lotteries such that $\lambda(\theta) \in \Delta(M^\circ(\theta))$ for every $\theta \in \Theta$, and $u = \mathbb{E}_\pi[\mathbb{E}_{\lambda(\theta)}[u(m,\theta)]]$. Let $\{\underline{m}_f(\theta)\}_{\theta\in\Theta, f\in\mathcal{F}}$ be the minmax stage-game matchings constructed in Lemma 5. By Lemma 5(ii),

$$u_f(\underline{m}_f(\theta), \theta) \leq \underline{u}_f(\theta) \quad \forall (f,\theta) \in \mathcal{F} \times \Theta. \tag{8}$$

By Lemma 6, there exist $\{u^f\}_{f\in\mathcal{F}} \subseteq \mathcal{U}^*$ such that $u^f_f < u_f$ and $u^f_f < u^{f'}_f$ for all distinct $f, f' \in \mathcal{F}$. For every $f \in \mathcal{F}$, let $(\lambda^f(\theta))_{\theta\in\Theta}$ be the tuple of lotteries that satisfies $\lambda^f(\theta) \in \Delta(M^\circ(\theta))$ for every $\theta \in \Theta$, and $u^f = \mathbb{E}_\pi[\mathbb{E}_{\lambda^f(\theta)}[u(m,\theta)]]$. By Carathéodory's theorem, it is without loss to assume every $\lambda^f(\theta)$ has bounded support. Define

$$Z_f(\theta) \equiv \inf_{m\in\operatorname{supp}(\lambda^f(\theta))} u_f(m,\theta). \tag{9}$$

At any θ, let $\overline{u}_f(\theta)$ denote a firm f's maximum payoff from any matching in $M^\circ(\theta)$ as defined in (2). Note that $\overline{u}_f(\theta)$ is an upper bound for the payoff f can obtain in any feasible deviation from matchings in $M^\circ(\theta)$. Let L be an integer greater than

$$\max_{f\in\mathcal{F},\,\theta\in\Theta} \frac{\overline{u}_f(\theta) - Z_f(\theta)}{u^f_f - \underline{u}^*_f}.$$

Consider a matching process with the following three phases:

(I) If past realizations from $\lambda(\cdot)$ were followed: Match according to $\lambda(\cdot)$;

(II) If f deviates: For the next L periods, match according to $\underline{m}_f(\cdot)$;

(III) If $\underline{m}_f(\cdot)$ was followed for L periods: Match according to the realization from $\lambda^f(\cdot)$ until a firm deviates.

Note that if firm f' deviates from (II) or (III), the process restarts (II) with f replaced by f'. All deviations by individual workers are ignored.

By Lemma 4, for any stage-game matching that can result from a firm's deviation, we can uniquely identify the firm, so the transition above is well-defined. All stage-game matchings in this matching process are individually rational for workers. It remains to check that no firm has profitable one-shot deviations.

(I) f has no profitable one-shot deviation if

$$(1-\delta)u_f(m,\theta)+\delta u_f \geq (1-\delta)\overline{u}_f(\theta)+\delta(1-\delta^L)\sum_{\theta'\in\Theta}\pi(\theta')u_f(\underline{m}_f(\theta'),\theta')+\delta^{L+1}u_f^f.$$

Since $u_f > u_f^f$ by construction, f has no profitable one-shot deviation for δ high enough.

(II) Consider two cases.

Case a: $f' \neq f$. Without deviation, f' gets

$$(1-\delta)u_{f'}(\underline{m}_f(\theta),\theta)+\delta(1-\delta^{L-\tau-1})\sum_{\theta'\in\Theta}\pi(\theta')u_{f'}(\underline{m}_f(\theta'),\theta')+\delta^{L-\tau}u_{f'}^f. \tag{10}$$

By deviating, f' gets less than

$$(1-\delta)\overline{u}_{f'}(\theta)+\delta(1-\delta^L)\sum_{\theta'\in\Theta}\pi(\theta')u_{f'}(\underline{m}_{f'}(\theta'),\theta')+\delta^{L+1}u_{f'}^{f'}. \tag{11}$$

As $\delta \to 1$, (10) converges to $u_{f'}^f$ and (11) to $u_{f'}^{f'}$. Because $u_{f'}^f > u_{f'}^{f'}$ by construction, no one-shot deviation is profitable for high δ.

Case b: $f' = f$. Without deviation, f gets

$$(1-\delta)u_f(\underline{m}_f(\theta),\theta)+\delta(1-\delta^{L-\tau-1})\sum_{\theta'\in\Theta}\pi(\theta')u_f(\underline{m}_f(\theta'),\theta')+\delta^{L-\tau}u_f^f. \tag{12}$$

With deviation, f gets less than

$$(1-\delta)\underline{u}_f(\theta)+\delta(1-\delta^L)\sum_{\theta'\in\Theta}\pi(\theta')u_f(\underline{m}_f(\theta'),\theta')+\delta^{L+1}u_f^f. \tag{13}$$

Since $\sum_{\theta'\in\Theta}\pi(\theta')u_f(\underline{m}_f(\theta'),\theta') \leq \underline{u}_f^* < u_f^f$, expression (12) is increasing in τ. Hence, it suffices to prove that expression (12) is weakly greater than expression (13) when $\tau = 0$.

When $\tau = 0$, subtracting (13) from (12) yields

$$(1-\delta)\left[u_f(\underline{m}_f(\theta),\theta) - \underline{u}_f(\theta)\right] - (1-\delta)\delta^L \sum_{\theta'\in\Theta} \pi(\theta')u_f(\underline{m}_f(\theta'),\theta') + (1-\delta)\delta^L u_f^f.$$

Multiplying the expression above by the positive factor $\frac{1}{1-\delta}$, it suffices to establish the following inequality

$$\left[u_f(\underline{m}_f(\theta),\theta) - \underline{u}_f(\theta)\right] - \delta^L \sum_{\theta'\in\Theta} \pi(\theta')u_f(\underline{m}_f(\theta'),\theta') + \delta^L u_f^f \geq 0. \tag{14}$$

By Lemma 5(iii),

$$u_f(\underline{m}_f(\theta),\theta) - \underline{u}_f(\theta) = \sum_{\theta'\in\Theta} \pi(\theta')u_f(\underline{m}_f(\theta'),\theta') - \underline{u}_f^*. \tag{15}$$

Substituting (15) into (14), we need to show that

$$(1-\delta^L) \sum_{\theta'\in\Theta} \pi(\theta')u_f(\underline{m}_f(\theta'),\theta') + \delta^L u_f^f \geq \underline{u}_f^*.$$

By construction, $u_f^f > \underline{u}_f^*$, so the inequality holds for δ high enough.

<u>(III)</u> Consider two cases.

<u>Case a:</u> $f' \neq f$. The argument for (I) holds analogously here, once we replace f with f' and u_f with $u_{f'}^f$.

<u>Case b:</u> $f' = f$. There is no profitable one-shot deviation for f' if

$$(1-\delta)u_{f'}(m,\theta) + \delta u_{f'}^{f'} \geq (1-\delta)\overline{u}_{f'}(\theta) + \delta(1-\delta^L) \sum_{\theta'\in\Theta} \pi(\theta')u_{f'}(\underline{m}_{f'}(\theta'),\theta') + \delta^{L+1} u_{f'}^{f'}.$$

This is equivalent to

$$\delta \frac{1-\delta^L}{1-\delta}\left[u_{f'}^{f'} - \sum_{\theta'\in\Theta} \pi(\theta')u_{f'}(\underline{m}_{f'}(\theta'),\theta')\right] \geq \overline{u}_{f'}(\theta) - u_{f'}(m,\theta).$$

Because $\frac{1-\delta^L}{1-\delta} \to L$ as $\delta \to 1$, the LHS converges to

$$\begin{aligned}
& L\left[u_{f'}^{f'} - \sum_{\theta' \in \Theta} \pi(\theta') u_{f'}(\underline{m}_{f'}(\theta'), \theta')\right] \\
> \; & \frac{\overline{u}_{f'}(\theta) - Z_{f'}(\theta)}{u_{f'}^{f'} - \underline{u}_{f'}^{*}} \left[u_{f'}^{f'} - \sum_{\theta' \in \Theta} \pi(\theta') u_{f'}(\underline{m}_{f'}(\theta'), \theta')\right] \qquad (16) \\
\geq \; & \overline{u}_{f'}(\theta) - u_{f'}(m, \theta), \qquad (17)
\end{aligned}$$

where (16) follows from the definition of L, and (17) follows from (8), (9), and $m \in \text{supp}(\lambda^f(\theta))$. Thus, no deviation is profitable for δ high enough.

We have proved the statement for the case $|\mathcal{F}| \geq 2$. When $\mathcal{F} = \{f\}$, we do not have a distinct $f' \neq f$ such that $u_f^f < u_f^{f'}$. However, this condition is only used when showing that no profitable and feasible deviation exists for firms other than f from the matching process constructed in the general case. Therefore, the same proof is valid for the case $|\mathcal{F}| = 1$. $\square$

Before proceeding with the proof of part (ii) of Proposition 1, we first establish a lemma that follows from the assumption of flexible wages.

Lemma 7. *Fix a matching process μ. Firm f has a feasible and profitable deviation plan if and only if f has a profitable deviation plan (d, η) such that at every ex post history $h = (\overline{h}, \theta, \gamma)$,*

$$v_w\Big(\Big[\mu, (f, d, \eta)\Big](h),\, \theta\Big) \geq v_w\Big(\mu(h), \theta\Big) \quad \forall w \in d(h).$$

Proof of Lemma 7. The "only if" part is trivial. To prove the "if" part, we only need to construct from (d, η) a deviation plan (d', η') for firm f that is both feasible and profitable. Since (d, η) is profitable, there exists an ex post history h' such that $U_f\big(h' \,\big|\, [\mu, (f, d, \eta)]\big) > U_f(h'|\mu)$. We now construct the deviation plan (d', η') as follows: For any ex post history h, let $d'(h) = d(h)$ and, when $d(h) \neq \emptyset$,

$$\eta'_w(h) = \eta_w(h) + \frac{1}{2\,|d'(h)|} \left[U_f\big(h' \,\big|\, [\mu, (f, d, \eta)]\big) - U_f(h'|\mu)\right] \quad \forall w \in d'(h).$$

The deviation plan (d', η') is feasible because for every ex post history $h = (\overline{h}, \theta, \gamma)$ and every

$w \in d'(h)$,

$$v_w\Big(\Big[\mu, (f, d', \eta')\Big](h),\, \theta\Big) > v_w\Big(\Big[\mu, (f, d, \eta)\Big](h),\, \theta\Big) \geq v_w\Big(\mu(h), \theta\Big).$$

To see that the deviation plan (d', η') is also profitable, note that at ex post history h', the continuation payoff of firm f from the manipulated matching process satisfies

$$\begin{aligned} &U_f\big(h' \,\big|\, [\mu, (f, d', \eta')]\big) \geq U_f\big(h' \,\big|\, [\mu, (f, d, \eta)]\big) - \frac{1}{2}\left[U_f\big(h' \,\big|\, [\mu, (f, d, \eta)]\big) - U_f(h'|\mu)\right] \\ &= \frac{1}{2}\left[U_f\big(h' \,\big|\, [\mu, (f, d, \eta)]\big) + U_f(h'|\mu)\right] > \frac{1}{2}\left[U_f(h'|\mu) + U_f(h'|\mu)\right] = U_f(h'|\mu), \end{aligned}$$

where the strict inequality comes from the fact that $U_f\big(h' \,\big|\, [\mu, (f, d, \eta)]\big) > U_f(h'|\mu)$. $\square$

Proof of Proposition 1(ii). By the definition of self-enforcing matching process, at every ex post history $h = (\overline{h}, \theta, \gamma)$, all workers must receive weakly positive payoffs, i.e., $v_w(\mu(h), \theta) \geq 0$ for every $w \in \mathcal{W}$. This means $\mu(h) \in M^\circ(\theta)$. Taking expectations over γ and θ, we have $\big(U_f(\overline{h} \,|\, \mu)\big)_{f \in \mathcal{F}} \in \mathcal{U}^*$ for any ex ante history $\overline{h} \in \overline{\mathcal{H}}$.

For each $f \in \mathcal{F}$, the intuition for $U_f(\overline{h} \,|\, \mu) \geq \underline{u}_f^*$ is straightforward: By definition, every firm f can obtain $\underline{u}_f(\theta)$ by deviating with certain workers. Taking an expectation over θ implies that each firm f can secure a continuation payoff of $\underline{u}_f^*$ at every ex ante history.

More rigorously, suppose by contradiction that μ is self-enforcing and $U_f(\overline{h} \,|\, \mu) < \underline{u}_f^*$ for some $f \in \mathcal{F}$ and ex ante history $\overline{h}$. Consider the following deviation plan (d, η) from μ: For all ex post histories that precede h, the deviation plan is consistent with the matching process μ. For any $h' = (\overline{h}', \theta', \gamma') \in \mathcal{H}$ that succeeds $\overline{h}$, let

$$d(h') = \operatorname*{arg\,max}_{A \subseteq \mathcal{W} \setminus \bigcup_{f' \in \mathcal{F}} \mu(f'|h')} s(f, A, \theta'),$$

and

$$\eta_w(h') = \begin{cases} -\widetilde{v}_w(f, d(h')) & \text{if } w \in d(h'); \\ 0 & \text{otherwise.} \end{cases}$$

Note that by construction, $s(f, d(h'), \theta') \geq \underline{u}_f(\theta')$ because $\bigcup_{f' \in \mathcal{F}} \mu(f'|h) \leq Q$.

At any history h' that succeeds h, any $w \in d(h')$ is unmatched under $\mu(h')$ by construction, which means $v_w(\mu(h'), \theta') = 0$. Meanwhile, according to the deviation plan,

$$v_w\Big(\Big[\mu, (f, d, \eta)\Big](h'), \theta'\Big) = \widetilde{v}_w(f, d(h')) + \eta_w(h') = \widetilde{v}_w(f, d(h')) - \widetilde{v}_w(f, d(h')) = 0$$

for all $w \in d(h')$. So at every ex post history h', every worker in $d(h')$ finds herself weakly better off by joining the deviation.

To see that (d, η) is profitable, observe that at every ex post history h' that succeeds $\overline{h}$, firm f's stage-game payoff from the manipulated static matching is

$$u_f\Big(\Big[\mu, (f, d, \eta)\Big](h'), \theta'\Big) = s(f, d(h'), \theta') \geq \underline{u}_f(\theta').$$

Since this is true for every ex post history h' after $\overline{h}$, f's continuation payoff at $\overline{h}$ from the deviation plan is $U_f(\overline{h} \,|\, [\mu, (f, d, \eta)]) \geq \mathbb{E}_\pi[\underline{u}_f(\theta)] = \underline{u}_f^* > U_f(h \,|\, \mu)$, where the last inequality is by assumption. Hence, the deviation plan (d, η) is profitable for firm f. In view of Lemma 7, firm f must have a deviation plan that is both feasible and profitable, which contradicts the assumption that μ is self-enforcing. Therefore, $U_f(\overline{h} \,|\, \mu) \geq \underline{u}_f^*$ for all $f \in \mathcal{F}$. $\square$

A.2.2 Proof of Lemma 2

For every $f \in \mathcal{F}$, let $\mathcal{O}(f) = \{o \in \mathcal{O} : o(f) = 1\}$ denote the set of orderings in which f is ranked first. Recall that for every $o \in \mathcal{O}$ and $\theta \in \Theta$, matching $\widehat{m}(\theta, o)$ is produced by serial dictatorship (Definition 3). It is straightforward that

$$u_f(\widehat{m}(\theta, o), \theta) = \overline{u}_f(\theta). \tag{18}$$

Additionally, in serial dictatorship, for each $\widehat{m}(\theta, o)$ and $f \in \mathcal{F}$,

$$\begin{aligned} u_f(\widehat{m}(\theta, o), \theta) &= \widetilde{u}_f(\widehat{W}_{o(f)}, \theta) + \sum_{w \in \widehat{W}_{o(f)}} \widetilde{v}_w(f, \widehat{W}_{o(f)} \backslash \{w\}, \theta) \\ &= \max_{W' \subseteq \mathcal{W} \backslash \cup_{1 \leq i < o(f)} \widehat{W}_i, |W'| \leq q_f} s(f, W', \theta). \end{aligned}$$

Since every $o \in \mathcal{O}$ satisfies $\left|\cup_{1\le i<o(f)} \widehat{W}_i\right| \le Q$,

$$u_f(\widehat{m}(\theta, o), \theta) \ge \min_{W\subseteq \mathcal{W}, |W|\le Q} \; \max_{W'\subseteq \mathcal{W}\setminus W, |W'|\le q_f} s(f, W', \theta) = \underline{u}_f(\theta) \quad \forall \theta \in \Theta, f \in \mathcal{F}, o \in \mathcal{O}. \tag{19}$$

For every $f \in \mathcal{F}$,

$$\frac{1}{|\mathcal{O}|}\sum_{o\in\mathcal{O}} \mathbb{E}_\pi\Big[u_f\big(\widehat{m}(\theta, o), \theta\big)\Big] = \sum_{\theta\in\Theta} \pi(\theta) \sum_{o\in\mathcal{O}} \frac{1}{|\mathcal{O}|}\Big[u_f\big(\widehat{m}(\theta, o), \theta\big)\Big] > \sum_{\theta\in\Theta} \pi(\theta)\, \underline{u}_f(\theta) = \underline{u}_f^*,$$

where the strict inequality follows from (18), (19), and Assumption 2.

A.3 Example: How to Minmax

We extend Example 1 to illustrate the complications in minmaxing firms when $\pi \in \Delta(\Theta)$ is non-degenerate. As in Example 1, suppose there is a single firm f, and three workers $\mathcal{W} = \{w_1, w_2, w_3\}$ enter the market in each period. Workers' payoffs depend solely on their wages, and the firm can hire at most two workers. However, unlike in Example 1, suppose the firm's revenue function depends on the state of the world $\theta \in \Theta = \{G, B\}$. If $\theta = G$, each employee generates a revenue of 2 for the firm; if $\theta = B$, workers are unproductive and the firm earns 0 revenue regardless of its hiring decision. The two states are equally likely. Using Lemma 1, the firm's minmax payoffs in the two states are $\underline{u}_f(G) = 2$ and $\underline{u}_f(B) = 0$, so its expected minmax payoff is $\underline{u}_f^* \equiv \mathbb{E}_\theta[\underline{u}_f(\theta)] = 1$. Note also that f's highest possible expected stage-game payoff is 2.

Suppose we aim to sustain a payoff $u_f \in (1, 2]$ for firm f. If f deviates at any history, it is minmaxed by the workers for the next L periods, after which the market returns to giving f the payoff u_f. In each of the L minmax periods, suppose that f is minmaxed by some matching $\underline{m}_f(G)$ if $\theta = G$ is realized, and $\underline{m}_f(B)$ if $\theta = B$. Using standard arguments, we know that out of the L minmax periods, f is most tempted to deviate in the first period, so we can without loss focus on checking its incentive constraints in the first of these L minmax periods.

A Naive Way to Minmax. In state G, as explained in Example 1, one natural candidate for $\underline{m}_f(G)$ is to let f hire two workers, each at a wage of 2, which leads to $u_f(\underline{m}_f(G), G) = 0$.

This ensures that f's best deviation yields its minmax payoff $\underline{u}_f(G) = 2$. In state B, we can set $\underline{m}_f(B)$ such that f is shut down and all workers are unemployed. This ensures that $u_f(\underline{m}_f(B), B) = \underline{u}_f(B) = 0$.

Why f Has Incentive to Deviate. However, simply playing $\underline{m}_f(G)$ in state G and $\underline{m}_f(B)$ in state B creates an incentive for f to deviate when $\theta = G$. To see why, note that once $\theta = G$ is realized, the benefit to f from deviating from $\underline{m}_f(G)$ is an immediate gain of $\underline{u}_f(G) - u_f(\underline{m}_f(G), G) = 2$. The cost of this deviation is that the minmax phase restarts, so in period $L+1$, instead of receiving $u_f \in (1, 2]$, it will receive $\mathbb{E}_\theta[u_f(\underline{m}_f(\theta), \theta)] = 0$. Since $\delta < 1$, the discounted future loss is less than the immediate gain, so the matching process cannot be self-enforcing.

How to Properly Adjust Minmax Matchings. One way to eliminate f's incentive to deviate in state G is to calibrate the transfers in the minmax matchings $\underline{m}_f(G)$ and $\underline{m}_f(B)$, so that the firm's gain from deviation during the minmax phase is equalized across states; that is,

$$\underline{u}_f(G) - u_f(\underline{m}_f(G), G) = \underline{u}_f(B) - u_f(\underline{m}_f(B), B).$$

To achieve this, we let f hire two workers each at a wage of 2 in $\underline{m}_f(G)$, and one worker at a wage of 2 in $\underline{m}_f(B)$. The expected stage-game payoff in the minmax phase now becomes $\mathbb{E}_\theta[u_f(\underline{m}_f(\theta), \theta)] = -1$. In this case, the benefit of deviating in either state is equal to its ex ante expectation

$$\underline{u}_f(G) - u_f(\underline{m}_f(G), G) = \underline{u}_f(B) - u_f(\underline{m}_f(B), B) = \underline{u}_f^* - \mathbb{E}_\theta[u_f(\underline{m}_f(\theta), \theta)] = 2,$$

while the cost is a loss of

$$u_f - \mathbb{E}_\theta[u_f(\underline{m}_f(\theta), \theta)] > 2$$

in the $(L+1)$-th period following the deviation. Now for any given L, the cost outweighs the benefit as long as δ is sufficiently close to 1, and f has no incentive to deviate.

A.4 List of Notations

We list our notations in the order as they first appear in the manuscript.

$f \in \mathcal{F}$	a firm
q_f	number of hiring slots of firm f
$w \in \mathcal{W}$	a worker
$\theta \in \Theta$	a state of the world
$\pi \in \Delta(\Theta)$	distribution of states of the world
$\phi_w \in \Phi_w$	work environment of worker w, $\Phi_w = \left(\mathcal{F} \times 2^{\mathcal{W} \setminus w}\right) \cup \{(\emptyset, \emptyset)\}$
$\widetilde{u}_f : 2^{\mathcal{W}} \times \Theta \to \mathbb{R}$	firm f's stage-game revenue over workers and states
$u_f(m, \theta)$	firm f's stage-game payoff (net transfers)
$\widetilde{v}_w : \Phi_w \times \Theta \to \mathbb{R}$	worker w's payoff function over work environments and states
$v_m(m, \theta)$	worker w's stage-game payoff (net transfers)
$m = (\phi, p)$	stage-game matching that consists of assignment ϕ and wage vector p
$\gamma \in \Gamma = [0, 1]$	public randomization
$\overline{h} = (\theta_\tau, \gamma_\tau, m_\tau)_{\tau=0}^{t-1}$	a t-period ex ante history that specifies a sequence of past realizations of the state, public correlation device, and stage-game matching up to period $t-1$
$\overline{\mathcal{H}}_t$	the set of all t-period ex ante histories
$\overline{\mathcal{H}}_0 = \{\emptyset\}$	the singleton set comprising the null history
$\overline{\mathcal{H}} \equiv \cup_{t=0}^{\infty} \overline{\mathcal{H}_t}$	the set of all ex ante histories
$\mathcal{H}_t \equiv \overline{\mathcal{H}_t} \times \Theta \times \Gamma$	the set of t-period ex post histories
$\mathcal{H} \equiv \overline{\mathcal{H}} \times \Theta \times \Gamma$	the set of all ex post histories
$\mu : \mathcal{H} \to M$	matching process that specifies a stage-game matching for every ex post history
h_∞	an outcome of the repeated matching game
$\overline{\mathcal{H}}_\infty = (\Theta \times \Gamma \times M)^\infty$	set of outcomes h_∞ of the repeated matching game
$U_f(\widehat{h} \vert \mu)$	the continuation payoff firm f obtains from matching process μ following history $\widehat{h}$
$M^\circ(\theta)$	the set of stage-game matchings that are individually rational for workers at state θ

(d, η)	deviation plan
$D_f(m, \theta)$	set of feasible stage-game deviations for f at state θ
$\underline{u}_f(\theta)$	firm f's minmax payoff at state θ
$s(f, W, \theta)$	total surplus of coalition (f, W) at state θ
Q	the sum of all firms' hiring quotas
$u(m, \theta)$	firms' payoff profile under matching m at state θ
$\mathcal{U}(\theta)$	convex hull of payoff profiles
$\mathcal{U}^*$	convex hull of expected payoffs
$\underline{u}_f^*$	expected minmax payoff over states of the world
$o : \mathcal{F} \to \{1, \ldots, \lvert\mathcal{F}\rvert\}$	an ordering over firms
$\mathcal{O}$	the set of all orderings o
$\overline{u}_f(\theta)$	firm f's maximum feasible payoff at state θ

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B Online Appendix

B.1 Relaxing Assumption 2

In this section, we explain how the existence result Proposition 2 can be generalized when Assumption 2 is not satisfied. As an overview, we first generalize Proposition 1 to Proposition 1^*. Next, we extend Lemma 2 to Lemma 2^*. Combining Lemma 2^* and Proposition 1^* allows us to establish the existence result without relying on Assumption 2, as presented in Proposition 2^*. The proofs are presented at the end of this section.

Suppose Assumption 2 does not hold, so that there exists some firm f such that for every $\theta \in \Theta$ with $\pi(\theta) > 0$, we have $\overline{u}_f(\theta) = \underline{u}_f(\theta)$. In this case, other players' matching decisions have *no impact* on the maximum static payoff firm f can derive, since it can always turn to the unmatched workers and extract their surpluses. This means that future punishments cannot affect the matching behavior of f via dynamic incentives, nor does f find it beneficial to participate in any punishment scheme of other firms. Therefore, we can treat such a firm as "inactive" in our analysis, assign it the maximum payoff it can receive in every period, and ignore it for the rest of our analysis. An iteration is needed to identify all such firms that cannot be incentivized dynamically.

Formally, for a set of firms $\mathcal{F}' \subseteq \mathcal{F}$, denote by $Q(\mathcal{F}') \equiv \sum_{f \in \mathcal{F}'} q_f$ the total hiring capacity of $\mathcal{F}'$. When all firms in $\mathcal{F} \backslash \mathcal{F}'$ are inactive, the effective minmax payoff of a firm $f \in \mathcal{F}'$ at θ is

$$\underline{u}_f(\mathcal{F}', \theta) \equiv \min_{W' \subseteq \mathcal{W}, |W'| \leq Q(\mathcal{F}')} \max_{W \subseteq \mathcal{W} \backslash W', |W| \leq q_f} s(f, W, \theta).$$

Using a similar argument as in Lemma 1, we can show that this is exactly firm f's payoff from "best responding" to the worst punishment by firms in $\mathcal{F}'$, while firms in $\mathcal{F} \backslash \mathcal{F}'$ leave no surplus to their employees. Note that for each θ, the value $\underline{u}_f(\mathcal{F}', \theta)$ weakly increases as $\mathcal{F}'$ becomes smaller.

Definition 4. A hierarchical partition $\mathcal{P} \equiv \{\mathcal{P}_1, \ldots, \mathcal{P}_N, \mathcal{R}\}$ over firms $\mathcal{F}$ is induced by the following procedure. Initialize $\mathcal{P}_0 \equiv \emptyset$. For $n \geq 1$:

- If $\left\{f \in \mathcal{F} \backslash \bigcup_{k=0}^{n-1} \mathcal{P}_k : \overline{u}_f(\theta) = \underline{u}_f(\mathcal{F} \backslash \bigcup_{k=0}^{n-1} \mathcal{P}_k, \theta) \ \forall \theta\right\} \neq \emptyset$, let this set be $\mathcal{P}_n$. Assign

$n = n+1$ and continue;

- If $\left\{f \in \mathcal{F} \backslash \bigcup_{k=0}^{n-1} \mathcal{P}_k : \overline{u}_f(\theta) = \underline{u}_f(\mathcal{F} \backslash \bigcup_{k=0}^{n-1} \mathcal{P}_k, \theta)\ \forall \theta\right\} = \emptyset$, let $\mathcal{R} = \mathcal{F} \backslash \bigcup_{k=0}^{n-1} \mathcal{P}_k$ and stop.

Intuitively, each $\mathcal{P}_n$ consists of firms that cannot be punished in the matching process without cooperation from those in $\bigcup_{k=0}^{n-1} \mathcal{P}_k$. If $\mathcal{R} \neq \emptyset$, by construction, $\overline{u}_f(\theta) > \underline{u}_f(\mathcal{R}, \theta)$ for every $f \in \mathcal{R}$ and $\theta \in \Theta$. Let

$$\overline{u}_f^* \equiv \mathbb{E}_\pi[\overline{u}_f(\theta)] \quad \forall f \in \mathcal{F} \backslash \mathcal{R},$$

and

$$\underline{u}_f^*(\mathcal{R}) \equiv \mathbb{E}_\pi[\underline{u}_f(\mathcal{R}, \theta)] \quad \forall f \in \mathcal{R}.$$

A generalized version of Proposition 1 can be stated as follows.

Proposition 1*. (i) If $u \in \mathcal{U}^*$ satisfies $u_f = \overline{u}_f^*$ for all $f \in \mathcal{F} \backslash \mathcal{R}$ and $u_f > \underline{u}_f^*(\mathcal{R})$ for all $f \in \mathcal{R}$, then there is a $\underline{\delta} \in (0,1)$ such that for every $\delta \in (\underline{\delta}, 1)$, there exists a self-enforcing matching process with firms' continuation payoffs u at the beginning of period 0. (ii) Suppose μ is a self-enforcing matching process for a given $\delta \in (0,1)$. For every ex ante history $\overline{h} \in \overline{\mathcal{H}}$, firms' continuation payoff profile satisfies $\left(U_f(\overline{h} \,|\, \mu)\right)_{f \in \mathcal{F}} \in \mathcal{U}^*$, $U_f(\overline{h} \,|\, \mu) = \overline{u}_f^*$ for every $f \in \mathcal{F} \backslash \mathcal{R}$, and $U_f(\overline{h} \,|\, \mu) \geq \underline{u}_f^*(\mathcal{R})$ for every $f \in \mathcal{R}$.

The following is an immediate corollary of Proposition 1*. It states that if no firms can be dynamically incentivized, then a self-enforcing matching process always exists, and all such processes result in a unique profile of continuation payoffs for the firms.

Corollary 1. *Suppose $\mathcal{R} = \emptyset$. A self-enforcing matching process exists for all $0 \leq \delta < 1$; furthermore, for every self-enforcing matching process, each firm f's continuation payoff is equal to $\overline{u}_f^*$ at all ex ante histories.*

The existence of a self-enforcing matching process in this special case follows from the observation that, in the proof of Proposition 1*(i), the condition on δ is invoked only for firms in $\mathcal{R}$. For the general case of existence, we next define a random serial dictatorship with respect to the hierarchical partition $\mathcal{P} = \{\mathcal{P}_1, \ldots, \mathcal{P}_N, \mathcal{R}\}$. To do this, we first introduce a subset of orderings $\mathcal{O}_{\mathcal{P}} \subseteq \mathcal{O}$ that contains those that (i) give firms in $\mathcal{R}$ the highest priorities and (ii) rank firms in $\mathcal{P}_n$ higher than those in $\mathcal{P}_k$ if $n > k$. That is,

$$\mathcal{O}_{\mathcal{P}} \equiv \left\{ o \in \mathcal{O} : \begin{array}{c} o(\mathcal{R}) = \{1, 2, \ldots, |\mathcal{R}|\}, \text{ and} \\ \text{if } f \in \mathcal{P}_k,\ f' \in \mathcal{P}_n, \text{ and } k < n, \text{ then } o(f) > o(f') \end{array} \right\}.$$

The stage-game matching $\widehat{m}(\theta, o)$ induced by a serial dictatorship according to o is defined as in Definition 3. By using a random serial dictatorship restricted to $\mathcal{O}_{\mathcal{P}}$, the following lemma generalizes Lemma 2 and shows that Proposition 1*(i) is not vacuously true.

Lemma 2*. For every firm $f \in \mathcal{R}$,

$$\frac{1}{|\mathcal{O}_{\mathcal{P}}|} \sum_{o \in \mathcal{O}_{\mathcal{P}}} \mathbb{E}_\pi\Big[u_f\big(\widehat{m}(\theta, o), \theta\big)\Big] > \underline{u}^*_f(\mathcal{R}).$$

For every firm $f \in \mathcal{F} \backslash \mathcal{R}$,

$$\frac{1}{|\mathcal{O}_{\mathcal{P}}|} \sum_{o \in \mathcal{O}_{\mathcal{P}}} \mathbb{E}_\pi\Big[u_f\big(\widehat{m}(\theta, o), \theta\big)\Big] = \overline{u}^*_f.$$

In view of Lemma 2* and Proposition 1*, we can establish Proposition 2 without Assumption 2.

Proposition 2*. When firms are sufficiently patient, there exists a self-enforcing matching process in which players match according to the outcome of an RSD in every period on the path.

B.1.1 Proof of Proposition 1*

For part (i), let $(\lambda(\theta))_{\theta \in \Theta}$ be the tuple of lotteries such that $\lambda(\theta) \in \Delta(M^\circ(\theta))$ for every $\theta \in \Theta$, and $u = \mathbb{E}_\pi[\mathbb{E}_{\lambda(\theta)}[u(m, \theta)]]$. There are three cases to consider.

<u>Case 1:</u> $\mathcal{R} = \emptyset$. If a firm f receives $u_f = \overline{u}^*_f$ on average, it is necessary that this firm receives the highest possible payoff $\overline{u}_f(\theta)$ at every realization of θ. This in turn implies that, for each θ, $\lambda(\theta)$ only assigns positive probability to stage-game matchings that are *stable* in a static sense. Therefore, a matching process that recommends according to $(\lambda(\theta))_{\theta \in \Theta}$ in every period

is self-enforcing.

Case 2: $|\mathcal{R}| = 1$. Let f denote the single firm in $\mathcal{R}$, and let $(\underline{m}_f(\theta))_{\theta\in\Theta}$ be the matchings that give f the most severe punishment by f itself, while all other firms receive the highest possible payoff $\overline{u}_f(\theta)$. Consider the following matching process:

(I) Match according to $\lambda(\cdot)$ if $\lambda(\cdot)$ was followed in the last period or $\underline{m}_f(\cdot)$ was followed for L periods;

(II) If firm f deviates from (I), match according to $\underline{m}_f(\cdot)$ for L periods.

If firm f deviates from (II), restart (II).

It is easy to check that when L is sufficiently large, firm f has no incentive to deviate in either phase, since $\delta \to 1$. All other firms have no incentive to deviate, since they already receive maximum stage-game payoff in every period.

Case 3: $|\mathcal{R}| \geq 2$. The proof for this case essentially follows the one for Proposition 1 with proper adjustments.

For part (ii), by construction, any firm $f \in \mathcal{P}_1$ can secure a stage-game payoff $\overline{u}_f(\theta)$ by deviation at every θ regardless of the stage-game matching. Taking expectation yields $U_f(\overline{h}\,|\,\mu) = \overline{u}_f^*$ for every $f \in \mathcal{P}_1$.

Suppose $U_{f'}(\overline{h}\,|\,\mu) = \overline{u}_{f'}^*$ for every $f' \in \bigcup_{k=1}^n \mathcal{P}_k$ with $n < N$. Then all firms in $\bigcup_{k=1}^n \mathcal{P}_k$ offer zero wages to their employees. By construction, each firm $f \in \mathcal{P}_{n+1}$ can secure a stage-game payoff $\overline{u}_f(\theta)$ by deviation at every θ regardless of how firms in $\mathcal{F}\setminus \bigcup_{k=1}^n \mathcal{P}_k$ are matched in each period. Taking expectation yields $U_f(\overline{h}\,|\,\mu) = \overline{u}_f^*$ for every $f \in \mathcal{P}_{n+1}$. By induction, the equality holds for all $f \in \mathcal{F}\setminus\mathcal{R}$.

By definition of the effective minmax payoff, every firm $f \in \mathcal{R}$ can secure $\underline{u}_f(\mathcal{R}, \theta)$ in each period by deviating with workers. Taking expectation over θ yields $U_f(\overline{h}\,|\,\mu) \geq \underline{u}_f^*(\mathcal{R})$ for these firms. Rigorous proof can be adapted from that of Proposition 1(ii) in Appendix A.2.1.

B.1.2 Proof of Lemma 2*

The proof of the first statement follows that of Lemma 2.

For the second statement, take any $f \in \mathcal{P}_n$, $n = 1, 2, \ldots, N$. By definition, for every

$o \in \mathcal{O}_{\mathcal{P}}$, we have $\left| W^{\#}_{o(f)} \right| < Q(\mathcal{F} \backslash \bigcup_{k=0}^{n-1} \mathcal{P}_k)$. This means

$$\begin{aligned} u_f\big(\widehat{m}(\theta, o), \theta\big) &= \max_{W \subseteq \mathcal{W} \backslash W^{\#}_{o(f)}, |W| \le q_f} s(f, W, \theta) \\ &\ge \min_{W' \subseteq \mathcal{W}, |W'| \le Q(\mathcal{F} \backslash \bigcup_{k=0}^{n-1} \mathcal{P}_k)} \max_{W \subseteq \mathcal{W} \backslash W', |W| \le q_f} s(f, W, \theta) \\ &= \overline{u}_f(\theta), \quad \forall \theta \in \Theta, \end{aligned}$$

where the last equality comes from the definition of $\mathcal{P}_n$. Taking expectation over Θ gives $\mathbb{E}_\pi\big[u_f\big(\widehat{m}(\theta, o), \theta\big)\big] = \overline{u}^*_f$, which suffices for the second statement to hold.

B.2 Multi-Firm Deviation Plans

We now generalize our model to study deviation plans that involve more than one firm. As discussed in the main text, we assume that whenever a deviation from the process happens (i.e., the realized stage-game matching differs from the default specified by the matching process in some period), the set of firms in the deviation plan can be identified and recorded. Formally, a t-period ex ante history is defined as $\overline{h} = (\theta_\tau, \gamma_\tau, m_\tau, F_\tau)_{\tau=0}^{t-1}$, where $F_\tau \subseteq \mathcal{F}$ records all the firms in the deviation plan (if any) responsible for the blocking in period τ, while an empty set $\emptyset$ indicates the realized stage-game matching m_τ follows the matching process being studied. As before, $\overline{\mathcal{H}}_t$ denotes the set of all t-period ex ante histories, $\overline{\mathcal{H}} \equiv \bigcup_{t=0}^{\infty} \overline{\mathcal{H}}_t$ the set of all ex ante histories, $\mathcal{H}_t \equiv \overline{\mathcal{H}}_t \times \Theta \times \Gamma$ the set of t-period ex post histories, and $\mathcal{H} \equiv \overline{\mathcal{H}} \times \Theta \times \Gamma$ the set of all ex post histories. A matching process is then $\mu : \mathcal{H} \to M$.

Recall that a deviation plan by a single firm f is a pair $(d : \mathcal{H} \to 2^{\mathcal{W}}, \eta : \mathcal{H} \to \mathbb{R}^{|\mathcal{W}|})$ such that $|d(h)| \le q_f$ for any h and $\eta_w(h) \ne 0$ only if $w \in d(h)$. For a set of firms $F \in 2^{\mathcal{F}} \backslash \{\emptyset\}$, a joint deviation plan $(F, \{(d_f, \eta_f)\}_{f \in F})$ by firms F is a collection of deviation plans $\{(d_f, \eta_f)\}_{f \in F}$, such that (i) $d_f(h) \cap d_{f'}(h) = \emptyset$ for all $f \ne f'$, $h \in \mathcal{H}$ and (ii) for each $f \in F$, $(d_f(h), \eta_f(h))$ differs from the pair specified by $\mu(h)$ at some $h \in \mathcal{H}$. The following is the multiple-firm counterpart of Assumption 1.

Assumption 3. *Let $[m, (F, \{(\widehat{W}_f, \widehat{p}_f)\}_{f \in F})] \in M$ denote the stage-game matching that is realized after coalitional deviation $(F, \{(\widehat{W}_f, \widehat{p}_f)\}_{f \in F})$ from stage-game matching $m = (\phi, p)$,* *and let (ϕ', p') denote the assignment and wages in $[m, (F, \{(\widehat{W}_f, \widehat{p}_f)\}_{f\in F})]$. We assume that the assignment ϕ' satisfies $\phi'(f) = \widehat{W}_f$ for all $f \in F$ and $\phi'(f') = \phi(f') \setminus \bigcup_{f\in F} \widehat{W}_f$ for every $f' \notin F$; furthermore, the wages satisfy $p'_f = \widehat{p}_f$ for all $f \in F$, while $p'_{f'w} = p_{f'w}$ for every $f' \notin F$ and $w \in \phi'(f')$.*

Given a matching process μ and a joint deviation plan $(F, \{(d_f, \eta_f)\}_{f\in F})$, the manipulated matching process, denoted by $[\mu, (F, \{(d_f, \eta_f)\}_{f\in F})] : \mathcal{H} \to M$, is a matching process defined by

$$\Big[\mu, (F, \{(d_f, \eta_f)\}_{f\in F})\Big](h) \equiv \Big[\mu(h), (F, \{(d_f(h), \eta_f(h))\}_{f\in F})\Big] \quad \forall h \in \mathcal{H}.$$

The joint deviation plan $(F, \{(d_f, \eta_f)\}_{f\in F})$ from μ is *feasible* if for every $f \in F$, at every ex post history $h = \big(\overline{h}, \theta, \gamma, F\big)$ such that $d_f(h) \neq \mu(f|h)$,

$$v_w\Big(\Big[\mu, (F, \{(d_f, \eta_f)\}_{f\in F})\Big](h),\, \theta\Big) > v_w\Big(\mu(h), \theta\Big) \quad \forall w \in d_f(h). \tag{20}$$

Remark. Observe that any worker poached in a multi-firm deviation belongs to the blocking coalition and therefore must be better off than following the recommended stage-game matching. Therefore, if a firm f can poach workers W when jointly deviating with other firms from a stage-game matching m, it can use the same wage offers to poach those workers when it is the only firm that is deviating from m.

Finally, the joint deviation plan $(F, \{(d_f, \eta_f)\}_{f\in F})$ is *profitable* if there exists an ex post history h such that $U_f\big(h \,\big|\, [\mu, (F, \{(d_f, \eta_f)\}_{f\in F})]\big) > U_f(h|\mu)$ for *all* $f \in F$.

A matching process μ is *strongly self-enforcing* if (i) $v_w(\mu(h), \theta) \geq 0$ for every $w \in \mathcal{W}$ at every ex post history $h \in \mathcal{H}$ and (ii) there does not exist a nonempty set of firms that has a feasible and profitable joint deviation plan.

Proposition 2′. When firms are sufficiently patient, there exists a strongly self-enforcing matching process in which players match according to the outcome of an RSD in every period on path.

Proof. For simplicity, we prove the result under Assumption 2, but it can be easily generalized using the arguments in Appendix B.1. Fix an indexing of the finite set of firms $\mathcal{F}$ by the numbers $\{1,2,\ldots,|\mathcal{F}|\}$. Pick $u \in \mathcal{U}^*$. The tuples $(\lambda(\theta))_{\theta\in\Theta}$ and $(\lambda^f(\theta))_{\theta\in\Theta}$ are defined as in the proof of Proposition 1(i). Recall that $\{\underline{m}_f(\theta)\}_{\theta\in\Theta,f\in\mathcal{F}}$ are the minmax stage-game matchings constructed in Lemma 5. Consider a matching process μ with the following three phases:

(I) If past realizations from $\lambda(\cdot)$ were followed: Match according to $\lambda(\cdot)$;

(II) If $\lambda(\cdot)$ was not followed and F is the set of firms responsible for the coalitional deviation, let f be the firm with the lowest index in F: For the next L periods, match according to $\underline{m}_f(\cdot)$.

(III) If $\underline{m}_f(\cdot)$ was followed for L periods: Match according to the realization from $\lambda^f(\cdot)$ until a firm deviates.

If there is a deviation from (II) or (III) and F' is the set of firms responsible for the coalitional deviation, restart (II) with f replaced by f', the firm with the lowest index in F'.

Lemma 8. *If there exists a feasible and profitable joint deviation plan $(F,\{(d_f,\eta_f)\}_{f\in F})$ from μ, then there exists a feasible and profitable deviation plan $(\widehat{d},\widehat{\eta})$ of a single firm $\widehat{f}$ when it is sufficiently patient.*

Proof. Denote by $\widehat{h}=(\bar{h},\theta,\gamma)$ the ex post history such that $[\mu,F,\{(d_f,\eta_f)\}_{f\in F}](\widehat{h})\neq\mu(\widehat{h})$ and $U_f\big(\widehat{h}\,\big|\,[\mu,(F,\{(d_f,\eta_f)\}_{f\in F})]\big)-U_f(\widehat{h}|\mu)>0$ for all $f\in F$; when there are multiple such histories, pick an arbitrary one. Suppose $\widehat{h}$ is a $\widehat{t}$-period ex post history. Let $\widehat{f}$ be the firm with the lowest index in F. Define a deviation plan $(\widehat{d},\widehat{\eta})$ for firm $\widehat{f}$ as follows:

- At $\widehat{h}$, make a feasible deviation that leads to $\widehat{m}\equiv[\mu(\widehat{h}),(\widehat{f},\widehat{W},\widehat{p})]\neq\mu(\widehat{h})$ and triggers phase (II). Note that this can always be achieved by hiring a different group of workers with sufficiently high wages.
- Let $h_F=(\bar{h},(\theta_\tau,\gamma_\tau,m_\tau,F_\tau)_{\tau=\widehat{t}}^{t-1},(\theta_t,\gamma_t))$ be an ex post history with $t>\widehat{t}$ that satisfies:
    1. h_F is generated by following the manipulated process $[\mu,F,\{(d_f,\eta_f)\}_{f\in F}]$ from $\widehat{h}$ given the realizations $(\theta_\tau,\gamma_\tau)_{\tau=\widehat{t}}^{t-1}$;

2. $\mu(h_F)$ is either in phase (II) $\big($i.e., specifying $\underline{m}_{\widehat{f}}(\theta_t)\big)$ or in phase (III) $\big($i.e., randomizing according to $\lambda^{\widehat{f}}(\theta_t)\big)$;

3. $\widehat{f}$ actively carries out a deviation at h_F under the joint deviation plan $(F, \{(d_f, \eta_f)\}_{f\in F})$ $\big($i.e., $(d_{\widehat{f}}(h_F), \eta_{\widehat{f}}(h_F)) \neq \mu(\widehat{f}|h_F)\big)$.

For each t-period ex post history h following $\widehat{h}$ such that (i) the realizations $(\theta_\tau, \gamma_\tau)_{\tau=\widehat{t}}^{t-1}$ are the same in h and h_F, and (ii) $\mu(h)$ is in the same phase as $\mu(h_F)$, let $(\widehat{d}(h), \widehat{\eta}(h)) = (d_{\widehat{f}}(h_F), \eta_{\widehat{f}}(h_F))$.

- For any other ex post history h, let $(\widehat{d}(h), \widehat{\eta}(h)) = \mu(\widehat{f}|h)$.

In words, in the constructed single-firm deviation plan, firm $\widehat{f}$ mimics its own deviating behavior in the joint deviation plan only when the manipulated process is either in the punishment phase (II) or in the reward phase (III).

By construction, for every ex post history h such that $(\widehat{d}(h), \widehat{\eta}(h)) \neq \mu(\widehat{f}|h)$, there exists some h_F such that $\mu(h) = \mu(h_F)$ and $(\widehat{d}(h), \widehat{\eta}(h)) = (d_{\widehat{f}}(h_F), \eta_{\widehat{f}}(h_F))$. Since $(F, \{(d_f, \eta_f)\}_{f\in F})$ is feasible, we have

$$\begin{aligned} v_w\Big(\Big[\mu, (\widehat{f}, \widehat{d}, \widehat{\eta})\Big](h),\, \theta\Big) &= v_w\Big(\Big[\mu(h_F), (\widehat{f}, d_{\widehat{f}}(h_F), \eta_{\widehat{f}}(h_F))\Big],\, \theta\Big) \\ &= v_w\Big(\Big[\mu, (F, \{d_f, \eta_f\}_{f\in F})\Big](h_F),\, \theta\Big) \\ &> v_w\Big(\mu(h_F), \theta\Big) \\ &= v_w\Big(\mu(h), \theta\Big) \qquad \forall w \in d_f(h). \end{aligned}$$

Therefore, the deviation plan $(\widehat{d}, \widehat{\eta})$ of firm $\widehat{f}$ is feasible.

The processes $[\mu, (\widehat{f}, \widehat{d}, \widehat{\eta})]$ and $[\mu, (F, \{(d_f, \eta_f)\}_{f\in F})]$ induce two measures over the set of outcomes $\overline{\mathcal{H}}_\infty = (\Theta \times \Gamma \times M)^\infty$, where we suppress the record of deviating firms $F \in 2^{\mathcal{F}}$ as it does not influence continuation payoffs. Note that conditional on each realized sequence of the state and the public randomization device $(\theta_\tau, \gamma_\tau)_{\tau=0}^t$, any process must induce a point mass on some stage-game matching in period t. For $t > \widehat{t}$, fixing the ex post history $\widehat{h}$ and any sequence $(\theta_\tau, \gamma_\tau)_{\tau=\widehat{t}+1}^t$, if firm $\widehat{f}$ is matched to different sets of workers or pays different wage vectors under two manipulated processes, firm $\widehat{f}$ matches according to $\lambda^{\widehat{f}}(\theta_t)$ and the realized γ_t under $[\mu, (\widehat{f}, \widehat{d}, \widehat{\eta})]$, while $\widehat{f}$ matches according to $\underline{m}_{\widehat{f}}(\theta_t)$ or deviates from it under $[\mu, (F, \{(d_f, \eta_f)\}_{f\in F})]$. By the fact that $u_{\widehat{f}}^{\widehat{f}} > \underline{u}_{\widehat{f}}^*$ and parts (i) and (ii) of Lemma 5, we can conclude that in every period $t > \widehat{t}$, firm $\widehat{f}$'s expected payoff averaged over θ_t and γ_t under $[\mu, (\widehat{f}, \widehat{d}, \widehat{\eta})]$ is weakly higher than that under $[\mu, (F, \{(d_f, \eta_f)\}_{f\in F})]$. Thus, we have

$$
\begin{aligned}
U_{\widehat{f}}\big(\widehat{h} \,\big|\, [\mu, (\widehat{f}, \widehat{d}, \widehat{\eta})]\big) - U_{\widehat{f}}(\widehat{h}|\mu) &= \Big[U_{\widehat{f}}\big(\widehat{h} \,\big|\, [\mu, (\widehat{f}, \widehat{d}, \widehat{\eta})]\big) - U_{\widehat{f}}\big(\widehat{h} \,\big|\, [\mu, (F, \{(d_f, \eta_f)\}_{f\in F})]\big)\Big] \\
&\quad + \Big[U_{\widehat{f}}\big(\widehat{h} \,\big|\, [\mu, (F, \{(d_f, \eta_f)\}_{f\in F})]\big) - U_{\widehat{f}}(\widehat{h}|\mu)\Big] \\
&\geq (1-\delta)\Big(u_f(\widehat{m}, \theta_{\widehat{t}}) - u_f\big(\big[\mu, (F, \{d_f, \eta_f\}_{f\in F})\big](\widehat{h}), \theta_{\widehat{t}}\big)\Big) \\
&\quad + \Big[U_{\widehat{f}}\big(\widehat{h} \,\big|\, [\mu, (F, \{(d_f, \eta_f)\}_{f\in F})]\big) - U_{\widehat{f}}(\widehat{h}|\mu)\Big].
\end{aligned}
$$

Since $U_{\widehat{f}}\big(\widehat{h} \,\big|\, [\mu, (F, \{(d_f, \eta_f)\}_{f\in F})]\big) - U_{\widehat{f}}(\widehat{h}|\mu) > 0$ due to the profitability of $(F, \{(d_f, \eta_f)\}_{f\in F})$, when δ is sufficiently close to 1, we have $U_{\widehat{f}}\big(\widehat{h} \,\big|\, [\mu, (\widehat{f}, \widehat{d}, \widehat{\eta})]\big) - U_{\widehat{f}}(\widehat{h}|\mu) > 0$, which means the deviation plan $(\widehat{d}, \widehat{\eta})$ is also profitable for firm $\widehat{f}$. □

Intuitively, under μ, whenever a joint deviation plan $(F, \{(d_f, \eta_f)\}_{f\in F})$ is formed, the firm with the lowest index in F is singled out as the "scapegoat" of the coalition. This firm is punished as if it were the sole deviator, and the punishment scheme restarts if any member of the group fails to follow the matching process. This construction implies that if the scapegoat were to unilaterally implement a deviation plan that replicates its own behavior in the joint deviation, it would be punished less frequently under the manipulated process. As a result, whenever a feasible and profitable joint deviation plan exists, there also exists an alternative feasible and profitable single-firm deviation plan that yields a continuation payoff arbitrarily close to that in the joint deviation, as $\delta \to 1$.

Hence, by Lemma 3, it suffices to rule out feasible and profitable *one-shot* deviations by a single firm. Since the matching process μ constructed above reduces to the one in the proof of Proposition 1 when restricted to deviations by a single firm, and we have shown that no feasible and profitable one-shot deviation exists under such a process, this completes the proof. □