CPT Lecture Notes 8: Preferences and utility representation

Levent Ülkü

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Binary relations, preferences, maximization

Let X be a set and $X^2 = \{(x, y\} : x, y \in X\}$.

A binary relation on X is any subset of X^2. We will typically denote binary relations by P, R, I or $\succsim, \succ, \sim$.

Let P be a binary relation. Instead of $(x, y) \in P$, we write xPy. Similary, instead of $(x, y) \notin P$, we write $x \not P y$.

Strict preference

Let P be a binary relation. Suppose we would like xPy to mean x is strictly better than y. Here are various properties which we might like to impose on P.

irreflexivity: xPx for no $x \in X$.
asymmetry: for any $x, y \in X$, if xPy, then $y \not{P} x$.
acyclicity: for every n and for every $x_1, ..., x_n \in X$, if $x_i P x_{i+1}$ for every $i < n$, then $x_n \not{P} x_1$.
transitivity: if xPy and yPz, then xPz.
negative transitivity: if xPy and $z \in X$, then either xPz or zPy.
connectedness: if $x \neq y$, then either xPy or yPx.

P is a (strict) *linear order* if it is connected, asymmetric and negatively transitive.
P is a (strict) *weak order* if it is asymmetric and negatively transitive.
P is a (strict) *partial order* if it is irreflexive and transitive.

Homework: Suppose there exists a function $u : X \to \mathbb{R}$ such that $xPy \Leftrightarrow u(x) > u(y)$. Show that P is a (strict) weak order. Now suppose u is injective: $x \neq y \Rightarrow u(x) \neq u(y)$. Show that P is a (strict) linear order.

Homework: Are the following statements true or false? Asymmetry implies irreflexivity but not vice versa. Acyclicity implies asymmetry but not vice versa. Transitivity implies acyclicity but not vice versa. Negative transitivity implies transitivity but not vice versa.

Weak preference

Let R be a binary relation. Suppose we would like xRy to mean x is at least as good as y. Here are various properties which we might like to impose on R.

reflexivity: xRx for all $x \in X$.
completeness: for all $x, y \in X$, xRy or yRx.
transitivity: if xRy and yRz, then xRz.
antisymmetry: if xRy and yRx, then $x = y$.

R is a (weak) *linear order* if it is antisymmetric, complete and transitive.

R is a (weak) *weak order* if it is complete and transitive.

R is a (weak) *partial order* if it is reflexive and transitive.

Homework: Suppose there exists a function $u : X \to \mathbb{R}$ such that $xRy \Leftrightarrow u(x) \geq u(y)$. Show that R is a (weak) weak order. Now suppose u is injective: $x \neq y \Rightarrow u(x) \neq u(y)$. Show that R is a (weak) linear order.

Homework: Is the following statement true or false? Completeness implies reflexivity.

Indifference

Let I be a binary relation. Suppose we would like xIy to mean that x and y are equally good. Here are some properties which we might like to impose on I.

reflexivity: xIx for every $x \in X$.
symmetry: if xIy, then yIx.
transitivity: if xIy and yIz, then xIz.

Call I an *equivalence class* if it is reflexive, symmetric and transitive.

Homework: Is the following statement true or false? Transitivity and symmetry together imply reflexivity.

Homework: Suppose there exists a function $u : X \to \mathbb{R}$ such that $xIy \Leftrightarrow u(x) = u(y)$. Show that I is an equivalence class.

We can conceptualize preferences by taking the strict preference as the fundamental, or alternatively by taking the weak preference as the fundamental. The following two results indicate that there is a sense in which these two means lead to the same end.

Proposition: Suppose that P is a (strict) weak order and that binary relations R and I are defined, using P, as follows:

$$\begin{aligned} xRy &\Leftrightarrow y \not P x \\ xIy &\Leftrightarrow x \not P y \text{ and } y \not P x \end{aligned}$$

Then R is a (weak) weak order and I is an equivalence class. If, in particular, P is a (strict) linear order, then R is a (weak) linear order.

Proposition: Suppose that R is a (weak) weak order and that binary relations P and I are defined, using R, as follows:

$$\begin{aligned} xPy &\Leftrightarrow xRy \text{ and } y \not R x \\ xIy &\Leftrightarrow xRy \text{ and } yRx \end{aligned}$$

Then P is a (strict) weak order and I is an equivalence class. If, in particular, R is a (weak) linear order, then P is a (strict) linear order.

Homework: Prove the propositions.

Homework: Suppose P is a (strict) partial order and define R and I using P as in Proposition 1. Investigate the properties of R and I.

Homework: Suppose P is an acyclic binary relation which gives a decision maker's strict preferences. The *transitive closure* of P is a binary relation P^T such that xP^Ty if for some n and $z_1, ..., z_n \in X$ such that $x = z_1P...Pz_n = y$. Show that P^T is the smallest transitive binary relation containing P. In other words, show that (1) P^T is transitive, (2) if xPy, then xP^Ty (hence $P \subseteq P^T$), and (3) For every transitive T such that $P \subseteq T$, $P^T \subseteq T$ as well.

Homework: Suppose that for some n and some $u_i : X \to \mathbb{R}$, $i = 1, ..., n$, $xPy \Leftrightarrow u_i(x) > u_i(y)$ for every $i = 1, ..., n$. Investigate the properties of P. Define R and I using P as in Proposition 1 and investigate their properties.

Homework: Suppose that for some $u : X \to \mathbb{R}$ and $\varepsilon > 0$, $xPy \Leftrightarrow u(x) > u(y) + \varepsilon$. Investigate the properties of P. Define I using P as in Proposition 1 and investigate its properties as well.

Homework: Do we want I to be transitive? Do we want P to be transitive?

Homework: How would you model incomparability using a binary relation? (Formalize the following attitude towards x and y: the DM cannot compare x and y.) What properties do you think should the incomparability relation have?

Homework: How would you model similarity using a binary relation? (Formalize the following attitude towards x and y: x and y are similar.) What properties do you think should the similarity relation have?

Homework: How would you model complementarity using a binary relation? (For instance, x=cookie complements y=tea.) What properties do you think should the complementarity relation have?

Homework: In modeling preferences using binary relations, are we ruling out some interesting attitudes a decision maker may have towards alternatives? (Read Lecture 1 in Rubinstein.)

Maximization

Let $2^X = \{A \subseteq X : A \neq \varnothing\}$. Any member of 2^X is a *menu*. For any binary relation T and any menu A, let

$$\begin{aligned} m(A, T) &= \{x \in A : yTx \text{ for no } y \in A\}, \text{ and} \\ M(A, T) &= \{x \in A : xTy \text{ for all } y \in A\}. \end{aligned}$$

In general, these sets may be empty and they need not coincide.

Proposition: Suppose X is finite. P is acyclic if and only $m(A, P) \neq \varnothing$ for all A.

Proposition: Suppose R is a (weak) weak order and define P using R as follows: $xPy \Leftrightarrow xRy$ and $y \not{R} x$. Then $m(A, P) = M(A, R)$ for all A. If R is a (weak) linear order, then $M(A, R)$ is a singleton.

Homework: Prove the propositions.

It is standard to use $m(A, T)$ to denote undominated alternatives according to a strict preference T. In this case T is often assumed to be acyclic. $M(A, T)$ gives the set of maximal alternatives in A and it is often used when T is a (weak) weak order.

Homework: The sets $m(A, T)$ and $M(A, T)$ are meant to capture potentially different conceptualizations of *best* alternatives in menu $A \subseteq X$ according to binary relation T on X. How would you describe alternatives in A which are *good enough* according to T?

Homework: Suppose $M(A, R) \neq \varnothing$ for all A. What does this imply about R? Is R reflexive, complete, transitive?

Utility representation of preferences

Basic definitions

Let X be a set. A utility function (on X) is a map $u : X \to \mathbb{R}$. A utility function u represents a binary relation P (on X) if for every $x, y \in X$, $xPy \Leftrightarrow u(x) > u(y)$. If this is the case, we say that P admits a utility representation. It is straightforward to see that if u represents P and $f : \mathbb{R} \to \mathbb{R}$ is strictly increasing, then $f \circ u$ represents P as well. (Right?)

Recall: (1) P is a weak order if it is asymmetric (xPy and yPx for no x, y) and negatively transitive (if xPy, then xPz or zPy for every z). (2) P is a linear order if it is asymmetric, negatively transitive and connected (xPy or yPx for every distinct x and y).

Results

Theorem 1: Let P be a binary relation on a countable set X. P admits a utility representation if and only if P is a weak order.

In general, there exist weak orders (even linear orders) which do not have utility representations.

Example: Let $X = [0,1]^2$ and consider the binary relation P on X^2 defined by $(a,b)P(a',b')$ iff $a > a'$ or $a = a$ and $b > b'$. This is the lexicographic order and it is a linear order. (Why?) If u represents P, then for every $a \in [0,1]$, there exists a rational number $f(a)$ such that $u(a,1) > f(a) > u(a,0)$. If $a' > a$, then $f(a') > u(a',0) > u(a,1) > f(a)$, hence $f : [0,1] \to \mathbb{Q}$ is strictly increasing. This implies $[0,1]$ is countable, which it very much isn't. Hence in general weak orders (even linear orders) may not admit utility representations.

Let P be a binary relation on X. A set $A \subseteq X$ is P-dense if for every $x, y \in X \backslash A$ such that xPy, there exists some $z \in A$ such that xPz and zPy. Note that X is a P-dense set. (Why?)

Theorem 2: Let P be a linear order on an arbitrary set X. P admits a utility representation if and only if X contains a countable P-dense subset.

Let P be a binary relation on an arbitrary set X. Let $x^* = \{a \in X : x \not P a \text{ and } a \not P x\}$ and $X^* = \{x^* : x \in X\}$. Define a binary relation P^* on X^* as follows: $x^* P^* y^*$ iff xPy.

Note that if P is a weak order on X, then P^* is a linear order on X^*. (Why?)

Theorem 3: A binary relation P on an arbitrary set X admits a utility representation if and only if P is a weak order on X and X^* contains a countable P^*-dense subset.

Proofs

Proof of Theorem 1: In one direction, suppose u represents P. If xPy, then $u(x) > u(y)$. Hence $u(y) \not> u(x)$ and $y\not{P}x$. This establishes asymmetry of P. If xPy and $x\not{P}z$, then $u(z) \geq u(x) > u(y)$. Hence $u(z) > u(y)$ and zPy. This establishes negative transitivity of P.

In the other direction suppose that P is a weak order on a countable set X. Note that P is transitive and irreflexive as well. (Right?) There are two cases.

Case 1: X is finite. Write $X = \{x_1, ..., x_n\}$. Let

$$\begin{aligned} r_{ij} &= 1 \text{ if } x_i P x_j, \\ r_{ij} &= 0 \text{ otherwise.} \end{aligned}$$

Set $u(x_i) = \sum_{j=1}^{n} r_{ij}$ for every i. Then u is a real-valued function on X. Suppose $x_i P x_j$. Since P is transitive, if $x_j P x_k$, then $x_i P x_k$. Furthermore, by irreflexivity, $x_j \not P x_j$. Hence $u(x_i) > u(x_j)$. Now suppose $x_i \not P x_j$. If $x_j \not P x_k$, then $x_i \not P x_k$ by negative transitivity. (Why?) Hence if $x_i P x_k$, then $x_j P x_k$ and $u(x_i) \leq u(x_j)$. It follows that if $u(x_i) > u(x_j)$, then $x_i P x_j$. We conclude that u represents P.

Case 2: X is infinite. Write $X = \{x_1, x_2, ...\}$. Set

$$u(x_i) = \sum_{j=1}^{\infty} \frac{r_{ij}}{2^j}.$$

Then $u(x_i) < \infty$ for every i and u is real-valued. Suppose $x_i P x_j$. If $x_j P x_k$, then $x_i P x_k$. Furthermore, by irreflexivity, $x_j \not{P} x_j$. Hence

$$u(x_i) \geq u(x_j) + \frac{1}{2^j} > u(x_j).$$

Now suppose $x_i \not{P} x_j$. If $x_j \not{P} x_k$, then $x_i \not{P} x_k$ by negative transitivity. Hence if $x_i P x_k$, then $x_j P x_k$ and $u(x_i) \leq u(x_j)$. We conclude that u represents P. ■

In order to prove Theorem 2, we need some additional jargon and an intermediate lemma. Given P, say that the pair (a, b) is a gap if aPb but there exists no c such that aPc and cPb. Let $G_1 = \{a : (a, b)$ is a gap for some $b\}$, $G_2 = \{b : (a, b)$ if a gap for some $a\}$ and $G = G_1 \cup G_2$.

Lemma: Suppose P is a linear order on an arbitrary set X. G is countable if one of the following two conditions holds: (1) X contains a countable P-dense subset. (2) P admits a utility representation.

Proof of Lemma: Suppose P is a linear order on an arbitrary set X.

Step 1: For every $a \in G_1$, there exists a unique $b \in X$ such that (a, b) is a gap. (Why?) Furthermore if $a \neq a'$ and (a, b) and (a', b') are gaps, then $b \neq b'$. (Why?) Similarly, for every $b \in G_2$, there exists a unique $a \in X$ such that (a, b) is a gap. (Why?) Furthermore if $b \neq b'$ and (a, b) and (a', b') are gaps, then $a \neq a'$. (Why?)

Step 2: We will show that (1) implies G is countable. Suppose B is a countable P-dense subset of X. If $a \in G_1 \backslash B$ and (a, b) is a gap, then $b \in B$. (Why?) Since such b is unique by Step 1, we can define a function $\beta : G_1 \backslash B \to B$ such that $(a, \beta(a))$ is a gap for every $a \in G_1 \backslash B$. Again by Step 1 β is an injection. Therefore $G_1 \backslash B$ is countable. Similarly $G_2 \backslash B$ is also countable. It follows that $G = (G_1 \backslash B) \cup (G_2 \backslash B) \cup (B \cap G)$ is also countable, as desired.

Step 3: We will show that (2) implies G is countable. Suppose that u represents P. For every $a \in G_1$, let $\beta(a)$ be the unique alternative in X such that $(a, \beta(a))$ is a gap. Let $r : G_1 \to \mathbb{Q}$ such that $u(a) > r(a) > u(\beta(a))$ for all $a \in G_1$. Then r is an injection: $a \neq a'$ implies $r(a) \neq r(a')$. Hence G_1 is countable. Similarly it can be shown that G_2 is also countable. Hence $G = G_1 \cup G_2$ is countable. ■

Proof of Theorem 2: Suppose P is a linear order on an arbitrary set X.

In one direction, suppose that X contains a countable P-dense subset B. By Lemma, then, G is countable. Let $A = G \cup B$. Then A is countable. Write $A = \{x_1, x_2, ...\}$. Set $r(x, y) = 1$ if xPy and $r(x, y) = 0$ if $x\not{P}y$. Let

$$u(x) = \sum_{j=1}^{\infty} \frac{r(x, x_j)}{2^j} < \infty.$$

We will show that u represents P. If $a \not P b$, then, by negative transitivity, for every x_j such that aPx_j, bPx_j as well, giving $u(a) \leq u(b)$. Suppose now that aPb. By transitivity, if bPx_j, then aPx_j as well, giving $u(a) \geq u(b)$. We need to establish that there exists j such that aPx_j, but $b \not P x_j$. If $b \in A$, then $b = x_j$ for some j and we are done. Suppose $b \notin A$. Then $b \notin G$ and (a, b) is not a gap: aPc and cPb for some $c \in X$. If $c \in A$, again, we are done. Suppose $c \notin A$. Note, now, that $b, c \notin B$, and by P-denseness, cPd and dPb for some $d \in B$. By asymmetry, $b \not P d$. Let $d = x_j$. then $u(a) \geq u(b) + \frac{1}{2^j} > u(b)$, as we needed to show.

In the other direction, suppose that u represents P. We will show that X contains a countable P-dense subset. Let G be the set of endpoints of gaps produced by P as before. By Lemma, G is countable. Let J be the set of ordered pairs (r, r') such that (1) $r, r' \in \mathbb{Q}$, (2) $r > u(a) > r'$ for some $a \in X$. For any $(r, r') \in J$, choose $a(r, r')$ such that $r > u(a(r, r')) > r'$ and let $B = \{a(r, r') : (r, r') \in J\}$. Then B is countable. Let $A = B \cup G$. A is also countable. We will show that A is P-dense. Suppose aPb for $a, b \in X \backslash A$. Then (a, b) is not a gap and aPc and cPb for some $c \in X$. Let rationals r and r' be such that $u(a) > r > u(c) > r' > u(b)$. Then $(r, r') \in J$ and $r > u(a(r, r')) > r'$. It follows that $aPa(r, r')$ and $a(r, r')Pb$. We note $a(r, r') \in B$ by construction and this finishes the proof. ■

Proof of Theorem 3: Let P be a binary relation on an arbitrary set X.

In one direction, suppose P is a WO and X^* has a countable P^*-dense subset. Then P^* is a linear order on X^* (Why?) and by Theorem 2, there exists a utility $U : X^* \to \mathbb{R}$ which represents P^*. Set $u(x) = U(x^*)$. Then u represents P. (Why?)

In the other direction, suppose u represents P. Then, clearly, P is a WO and P^* is a LO. Set $U(x^*) = u(x)$. Then U represents P^*. (Why?) By Theorem 2, X^* has a countable P^*-dense subset. ■

Homework: Lemma assumes P is a LO. The proof of Lemma uses the connectedness of P explicitly. Does it rely on asymmetry as well? Transitivity?

Homework: In the last paragraph of the proof of Theorem 2, we showed that $B \cup G$ is P-dense. Is B P-dense? Is G P-dense?

Homework: Suppose $X = [-1, 1]$ and consider P on X given by xPy iff $\{|x| > |y|$ or $[|x| = |y|$ and $x > y]\}$. Does P admit a utility representation?

Homework: Suppose X is finite and P is a binary relation on X. Suppose further that there is a a function $u : X \rightarrow \mathbb{R}$ such that if xPy then $u(x) > u(y)$. What other properties does P necessarily satisfy?

Homework: Suppose X is finite and P is a binary relation on X. Under what conditions on P is there a function $u : X \rightarrow \mathbb{R}$ such that if xPy then $u(x) > u(y)$?

A topological approach

We will follow Lecture 2 in Rubinstein's text. Let R be a complete and transitive binary relation (i.e., a weak order) on $X \subseteq \mathbb{R}^n$. Define, as usual xPy iff xRy and $y \not{R} x$. Recall that such P is asymmetric and negatively transitive (i.e., a weak order in the jargon of the previous section). Endow X with the Euclidean metric. We say that R admits a continuous utility representation if there exists a continuous function $u : X \rightarrow \mathbb{R}$ such that xRy iff $u(x) \geq u(y)$.

R is continuous if for any $x \in X$, the sets $\{y \in X : yRx\}$ and $\{y \in X : xRy\}$ are closed in X. Equivalently, R is continuous if for any sequence $\{y_n\}$ in X which converges to $y \in X$, (i) if $y_n Rx$ for every n, then yRx, (ii) if xRy_n for every n, then xRy.

X is convex if for any $x, y \in X$ and $\alpha \in (0, 1)$, $\alpha x + (1 - \alpha)y \in X$.

Theorem 4: Let $X \subseteq \mathbb{R}^n$ be convex and R be a binary relation on X. R is a continuous weak order if and only if it admits a continuous utility representation.

We will not prove this theorem. (See Rubinstein's text if interested.) Instead we will prove continuous utility representation under an additional condition on preferences.

R is strictly monotone if for every $x, y \in X$, if $x_i \geq y_i$ for all $i = 1, ..., n$ and $x \neq y$, then xPy.

Exercise: Let $e = (1, ..., 1) \in \mathbb{R}^n$. If R is a strictly monotone weak order and α and β are scalars, then

$$\alpha \geq \beta \Leftrightarrow (\alpha e) R (\beta e).$$

Theorem 5: Suppose R is a strictly monotone and continuous weak order on $X = \mathbb{R}^n$. Then R admits a continuous utility representation.

Proof: Define, for any $x \in X$, $B(x) = \{\beta \in \mathbb{R} : (\beta e) R x\}$. Note $B(x)$ is nonempty and bounded from below. (Why?) Let $\alpha(x) = \inf B(x)$. Hence $\alpha(x)$ is a number. Furthermore there exists a sequence $\{\beta_n\}$ in $B(x)$ with limit $\alpha(x)$. Since $\beta_n eRx$, then, $\alpha(x)eRx$ by continuity. On the other hand, for all $\beta \in B(x)$, $\beta \geq \alpha(x)$. In other words, if $\beta < \alpha(x)$, then $xP\beta e$. Take $\alpha_n = \alpha(x) - \frac{1}{n}$, so that $xP\alpha_n e$ for all n and α_n converges to $\alpha(x)$. It follows, again by continuity, that $xR\alpha(x)e$. Letting I denote the indifference of R, we conclude $\alpha(x)eIx$. Furthermore $\alpha(x)$ is the unique number with this property. If γeIx, then $\gamma eI\alpha(x)e$ by transitivity and $\alpha(x) = \gamma$ by strict monotonicity.

Set $u(x) = \alpha(x)$ for all x. This defines a utility function. Suppose $u(x) \geq u(y)$. Then $xIu(x)eRu(y)eIy$, giving xRy. In reverse, if xRy, then $u(x)eRu(y)e$ and consequently $u(x) \geq u(y)$.

All that remains to show is that u is continuous. A function $f : \mathbb{R}^n \to \mathbb{R}$ is continuous iff $f^{-1}((a, b))$ is open for every open interval (a, b). Note $aeIae$ and therefore $u(ae) = \alpha(ae) = a$ for any $a \in \mathbb{R}$. Take any interval (a, b). Note,

$$\begin{aligned}
u^{-1}((a,b)) &= u^{-1}((a,\infty) \cap (-\infty, b)) \\
&= u^{-1}((a,\infty)) \cap u^{-1}((-\infty, b)) \\
&= u^{-1}((u(ae),\infty)) \cap u^{-1}((-\infty, u(be))) \\
&= \{x : xPae\} \cap \{x : xPbe\}.
\end{aligned}$$

Since R is complete and continuous, $\{x : xPae\}$ and $\{x : xPbe\}$ are open. Since the intersection of two open sets are open, so is $u^{-1}((a, b))$, as desired.

Utility-error representations

P satisfies strong intervality if for every $w, x, y, z \in X$, whenever wPx and yPz, either wPz or yPx. P satisfies semitransitivity if for every $w, x, y, z \in X$, whenever xPy and yPz, either xPw or wPz. P is an interval order if it is irreflexive and satisfies strong intervality. P is a semiorder if it is an interval order and satisfies semitransitivity.

Exercise: Every weak order is a semiorder. Every semiorder is an interval order. Every interval order is a partial order.

We first take up the issue of representing interval orders with utilities and errors.

Theorem 6: Let X be a finite set and P be a binary relation on X. P is an interval order if and only if there exist functions $u : X \to \mathbb{R}$ and $\varepsilon : X \to \mathbb{R}_+$ such that for every $x, y \in X$,

$$xPy \Leftrightarrow u(x) > u(y) + \varepsilon(y).$$

Proof: If $xPy \Leftrightarrow u(x) > u(y) + \varepsilon(y)$ for some $u : X \to \mathbb{R}$ and $\varepsilon : X \to \mathbb{R}_+$, then it is straightforward to show that P is an interval order.
Suppose now that P is an interval order and define I in the usual way: $xIy \Leftrightarrow x \not{P} y$ and $y \not{P} x$. Let $PI = \{(a, b) : aPx \text{ and } xIb \text{ for some } x\}$. Note that since P is irreflexive, I is reflexive.

We will show that PI is a weak order, i.e., asymmetric and negatively transitive. Suppose $xPIy$ and $yPIx$. Then, there exist a and b such that xPa, aIy, yPb and bIx. Since xPa, yPb, and since P is an interval order, xPb or yPa, a contradiction. Hence PI is asymmetric. Suppose now that $xPIy$. Hence there exists a such that xPa and aIy. To establish negative transitivity of PI, we need to show that, for any z, $xPIz$ or $zPIy$. If aIz, $xPIz$. Suppose not: zPa or aPz. If zPa, then $zPIy$, since aIy. If aPz, then, since P is an interval order and xPa, xPz. Since I is reflexive zIz as well, so $xPIz$.
Since X is finite and PI is a weak order, there exists a utility u such that $xPIy$ iff $u(x) > u(y)$. Set $\varepsilon(x) = \max\{u(y) : xIy\} - u(x)$. Note that ε is nonnegative-valued, since xIx. Suppose xPy and $\varepsilon(y) = u(z) - u(y)$. Then yIz and $xPIz$, giving $u(x) > u(z)$. Consequently $u(x) > u(y) + \varepsilon(y)$. Now suppose $u(x) > u(y) + \varepsilon(y)$. Then not xIy by definition of ε. Hence xPy or yPx. If yPx, then, as we have already shown, $u(y) > u(x) + \varepsilon(x)$, a contradiction. This finishes the proof.

Representation of semiorders is slightly more delicate, as it implies constant thresholds across alternatives.

Theorem 7: Let X be a finite set and P be a binary relation on X. P is a semiorder if and only if there exist a scalar $\varepsilon \geq 0$ and a function $u : \mathbb{R} \to \mathbb{R}$ such that for every $x, y \in X$,

$$xPy \Leftrightarrow u(x) > u(y) + \varepsilon.$$

Proof: If P has the representation in the theorem, then it is straightforward to show that P is a semiorder.

In the reverse direction, take any semiorder P on a finite set X.

(A) Define I as before: xIy iff $x \not P y$ and $y \not P x$.

(B) Define R_0 as follows:

$$xR_0y \text{ iff } \begin{cases} \text{for every } a \text{ such that } yPa,\ xPa \text{ as well, and} \\ \text{for every } a \text{ such that } aPx,\ aPy \text{ as well.} \end{cases}$$

It can be shown that R_0 is a weak order. Furthermore, it can be shown that

(B1) if xPy, then xR_0y and $y \not R_0 x$, and

(B2) if xR_0y, yR_0z and xIz, then xIy and yIz.

(C) Let R be a linear order obtained from R_0 by breaking indifferences arbitrarily. Hence
(C1) if xR_0y and $y\not R_0x$, then xRy (and $y\not Rx$), and
(C2) if xRy, then xR_0y.
Then R is related to P in exactly the same way R_0 is related to P:
(C3) if xPy, then xRy (and $y\not Rx$ -this follows since R is a linear order), and
(C4) if xRy, yRz and xIz, then xIy and yIz.

(D) We will show

CLAIM: if P is a semiorder and R is obtained from P as in (C), then there exists $u : X \to \mathbb{R}$ such that
(1) xRy iff $u(x) \geq u(y)$ (and if $x \neq y$, then $u(x) \neq u(y)$),
(2) xPy iff $u(x) > u(y) + 1$, and
(3) $u(x) = u(y) + 1$ for no x and y.

To do so we will use induction on $|X| = k$.

(D1) $k = 1$. Suppose that P is a semiorder on $X = \{x\}$ and R is obtained from P as in (C). The CLAIM follows trivially by setting $u(x) = 0$.

(D2) Take any integer $n \geq 2$. Suppose that the CLAIM holds whenever $k = n - 1$.

(D3) Suppose that $k = n$. Number alternatives so that $X = \{x_1, ..., x_n\}$ and $x_n R x_{n-1} R ... R x_1$. Let $X' = X \backslash \{x_n\}$. Let P' and R' be the restrictions of P and R on X'. By (D2) there exists $u : X' \rightarrow \mathbb{R}$ such that
(D3a) $u(x_{n-1}) > ... > u(x_1)$,
(D3b) for every $i, j = 1, ..., n-1$, $x_i P x_j$ iff $u(x_i) > u(x_j) + 1$, and
(D3c) $u(x_i) - u(x_j) = 1$ for no i and j.
It remains to choose $u(x_n)$ appropriately.
Case 1: $x_n P x_{n-1}$. Choose $u(x_n) > u(x_{n-1}) + 1$ and the CLAIM follows.
Case 2: $x_n \not{P} x_{n-1}$.

Case 2a: $x_n \not{P} x_1$. If $x_1 P x_n$, then $x_1 R x_n$ by (C3), a contradiction. Hence $x_n I x_1$. Furthermore $x_n R x_2$ and $x_2 R x_1$. By (C4) $x_n I x_2$ and $x_2 I x_1$. Similar reasoning gives $x_i I x_j$ for every $i, j = 1, ..., n$. Using (D3b), $x_{n-1} \not{P} x_1$ implies that $u(x_{n-1}) \leq u(x_1) + 1$. Since utility differences are never exactly 1 by (D3c), $u(x_{n-1}) < u(x_1) + 1$. Choose $u(x_n) \in (u(x_{n-1}), u(x_1) + 1)$ and the CLAIM follows.
Case 2b: $x_n P x_1$. Let $i = \max\{j : x_n P x_j\}$. Then $x_n P x_j$ for all $j \in \{1, ..., i\}$ and $x_n \not{P} x_j$ for all $j \in \{i+1, ..., n-1\}$. (Why?) We will show that $x_{n-1} \not{P} x_{i+1}$. If $n-1 = i+1$, this follows from irreflexivity of P. Otherwise, note that $x_{i+1} \not{P} x_n$, since otherwise by (C3) $x_{i+1} R x_n$, a contradiction. Hence $x_n I x_{i+1}$. We also have $x_n R x_{n-1}$ and $x_{n-1} R x_{i+1}$. This gives by (C4) $x_{n-1} I x_{i+1}$, as desired. Hence, by (D3b and D3c) $u(x_{n-1}) < u(x_{i+1}) + 1$. Noting by (D3a) that $u(x_i) + 1 < u(x_{i+1}) + 1$ as well, choose $u(x_n) \in (\max\{u(x_{n-1}), u(x_i) + 1\}, u(x_{i+1}) + 1)$ and the CLAIM follows. The proof is complete.

Example: What if X is not finite? Suppose $X = N \cup \{\pi\}$, where $N = \{0, 1, 2, ...\}$. Consider

$$P = \{(\pi, n) : n \in N\} \cup \{(n, m) \in N^2 : n > m + 1\}.$$

Clearly, P is a semiorder. (Why?) Suppose that there exist $u : X \to \mathbb{R}$ and $\varepsilon \geq 0$ such that xPy iff $u(x) > u(y) + \varepsilon$. Then $\varepsilon > 0$ since P is not a weak order: $3 \not{P} 2 \not{P} 1$ but $3P1$. Now for every $n \in N \backslash \{0\}$, $u(\pi) > u(2n) > u(0) + \varepsilon n$, meaning $u(\pi) = \infty$, a contradiction. Hence P does not admit the representation in the theorem.