CPT Lecture Notes 6: Support functions

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Support functions for maximization problems

For any nonempty $S \subseteq \mathbb{R}^n$, let

$$b(S) = \{p \in \mathbb{R}^n : \max_{x \in S} p \cdot x \text{ is well-defined}\}$$

and let

$$\sigma_S(p) = \max_{x \in S} p \cdot x$$

for any $p \in b(S)$. Hence $\sigma_S : b(S) \to \mathbb{R}$.

The set $b(S)$ is the barrier cone of S and the function σ_S is the support function of S.

Note that if $p \in b(S)$ and $\lambda \in \mathbb{R}_+$, then $\lambda p \in b(S)$ as well. Hence $b(S)$ is a cone and in particular $0 \in b(S)$.

If $S = \{x \in \mathbb{R}^2 : \|x\| \leq 1\}$, then $b(S) = \mathbb{R}^2$.

If $S = \{x \in \mathbb{R}^2 : \|x\| < 1\}$, then $b(S) = \{0\}$.

$b(S)$ need not be convex even if S is. For instance if $S = \{x \in \mathbb{R}^2 : \|x\| < 1\} \cup \{(1,0),(0,1)\}$, then $b(S) = \{x \in \mathbb{R}^2 : x_1 \geq 0 \text{ and } x_2 = 0\} \cup \{x \in \mathbb{R}^2 : x_1 = 0 \text{ and } x_2 \geq 0\}$.

Theorem: Let $S \subseteq \mathbb{R}^n$ be nonempty.

1. If x_1 solves $\max_{x \in S} p_1 \cdot x$ and x_2 solves $\max_{x \in S} p_2 \cdot x$, then $(x_1 - x_2) \cdot (p_1 - p_2) \geq 0$.

2. $\sigma_S(tp) = t\sigma_S(p)$ for every $p \in b(S)$ and $t \geq 0$.

3. If $b(S)$ is convex, then σ_S is convex.

4. Suppose $S \subseteq \mathbb{R}^n$ is closed and convex, and $p \in b(S)$. x_P solves $\max_{x \in S} p \cdot x$ iff x_p is a subgradient of σ_S at p.

5. Suppose $S \subseteq \mathbb{R}^n$ is closed and convex, $p \in intb(S)$ and σ_S is differentiable at p. Then, x_p solves $\max_{x \in S} p \cdot x$ iff $x_p = \nabla \sigma_S(p)$.

Proof: Let $S \subseteq \mathbb{R}^n$ be nonempty.

TS1: If x_1 solves $\max_{x \in S} p_1 \cdot x$ and x_2 solves $\max_{x \in S} p_2 \cdot x$, then $(x_1 - x_2) \cdot (p_1 - p_2) \geq 0$.

Suppose x_i solves $\max_{x \in S} p_i \cdot x$ for $i = 1, 2$.

Then $p_1 \cdot x_1 \geq p_1 \cdot x_2$ and $p_2 \cdot x_2 \geq p_2 \cdot x_1$.

Manipulating, we get $(x_1 - x_2) \cdot (p_1 - p_2) \geq 0$.

TS2: $\sigma_S(tp) = t\sigma_S(p)$ for every $p \in b(S)$ and $t \geq 0$.

Let $p \in b(S)$ and $t \geq 0$.

Then $tp \in b(S)$ since $b(S)$ is a cone.

Furthermore $\sigma_S(tp) = \max_{x \in S}(tp) \cdot x = t \max_{x \in S} p \cdot x = t\sigma_S(p)$.

TS3: If $b(S)$ is convex, then σ_S is convex.

Suppose $b(S)$ is convex.

Pick $p, q \in b(S)$ and $t \in [0, 1]$. Let $r = tp + (1 - t)q$. Since $b(S)$ is convex, $r \in b(S)$ as well.

Take any $x \in S$.

Note $\sigma_S(p) \geq p \cdot x$ and $\sigma_S(q) \geq q \cdot x$.

Hence $t\sigma_S(p) + (1 - t)\sigma_S(q) \geq r \cdot x$.

Since x is arbitrary, $\sigma_S(r) \leq t\sigma_S(p) + (1 - t)\sigma_S(q)$ and σ_S is convex.

TS4: Suppose $S \subseteq \mathbb{R}^n$ is closed and convex, and $p \in b(S)$. x_P solves $\max_{x \in S} p \cdot x$ iff x_p is a subgradient of σ_S at p.

Suppose $S \subseteq \mathbb{R}^n$ is closed and convex, and $p \in b(S)$.

Suppose x_p solves $\max_{x \in S} p \cdot x$. Hence $\sigma_S(p) = p \cdot x_p$. If $q \in b(S)$, then $\sigma_S(q) \geq q \cdot x_p$. Therefore $\sigma_S(q) \geq q \cdot x_p + \sigma_S(p) - p \cdot x_p$, i.e., $\sigma_S(q) \geq \sigma_S(p) + x_P \cdot (q - p)$. It follows that x_p is a subgradient of σ_S at p.

Now suppose that x_p is a subgradient of σ_S at p. Hence $\sigma_S(q) \geq \sigma_S(p) + x_P \cdot (q - p)$ for every $q \in b(S)$. Note that $0 \in b(S)$ and therefore $0 \geq \sigma_S(p) - x_p \cdot p$. Furthermore $2p \in b(S)$ and $\sigma_S(2p) = 2\sigma_S(p)$ and therefore $2\sigma_S(p) \geq \sigma_S(p) + x_p \cdot p$. It follows that $\sigma_S(p) = p \cdot x_p$. It remains to show that $x_p \in S$. If not, by Minkowski's Theorem, there exists some $q \neq 0$ and $x_0 \in S$ such that $q \cdot x_p > q \cdot x_0 \geq q \cdot x$ for every $x \in S$. Then $q \in b(S)$ and $\sigma_S(q) = q \cdot x_0$. It follows that $\sigma_S(p) - p \cdot x_p + q \cdot x_p > \sigma_S(q)$, a contradiction.

TS5: Suppose $S \subseteq \mathbb{R}^n$ is closed and convex, $p \in intb(S)$ and σ_S is differentiable at p. Then, x_p solves $\max_{x \in S} p \cdot x$ iff $x_p = \nabla \sigma_S(p)$.

Follows from the observation that the only subgradient of σ_S at p is $\nabla \sigma_S(p)$ if $p \in intb(S)$ and σ_S is differentiable at p. $\blacksquare$

Application: Profit maximization

Suppose there are n commodities.

A production set is any $Y \subseteq \mathbb{R}^n$.

If $y \in Y$, then y is a feasible production plan.

If $y = (y_1, ..., y_n) \in Y$ and $y_i < 0 < y_j$, then the ith commodity is used as input and the jth commodity is an output.

Depending on the context, various assumptions can be placed on Y, such as closedness, boundedness and convexity. If $0 \in Y$, inactivity is feasible. If $Y = -\mathbb{R}^n_+$, then the only technology available is free disposal. If we would like to assume irreversibility of production, we impose $y = 0$ whenever $y, -y \in Y$.

Example:

Consider a firm producing output y_1 using $n-1$ inputs $y_2, ..., y_n$.

Let $f : \mathbb{R}_+^{n-1} \rightarrow \mathbb{R}$ denote the firm's production function.

The production set is given by
$Y = \{(y_1, ..., y_n) : y_1 \leq f(-y_2, ..., -y_n)\}$.

Let $p \in \mathbb{R}^n$ denote the price vector.

The profit maximization problem is $\max_{y \in Y} p \cdot y$.

The barrier cone $b(Y)$ consists of those price vectors for which the profit maximization problem is well-defined.

The profit function $\pi_Y(p)$ is the support function for this maximization problem, i.e., $\pi_Y(p) = \max_{y \in Y} p \cdot y$ for every $p \in b(Y)$. (I will simply write $\pi(p)$ instead of $\pi_Y(p)$.)

Let $Y \subseteq \mathbb{R}^n$ be nonempty. By Theorem above, it follows that:

1. If $y(p)$ solves $\max_{y \in Y} p \cdot y$ and $y(p')$ solves $\max_{y \in Y} p' \cdot y$, then $(y(p) - y(p')) \cdot (p - p') \geq 0$.

2. $\pi(tp) = t\pi(p)$ for every $p \in b(Y)$ and $t \geq 0$.

3. If $b(Y)$ is convex, then π is convex.

4. Suppose $Y \subseteq \mathbb{R}^n$ is closed and convex, and $p \in b(Y)$. $y(p)$ solves $\max_{y \in Y} p \cdot y$ iff $y(p)$ is a subgradient of π at p.

5. (Hotelling's Lemma) Suppose $Y \subseteq \mathbb{R}^n$ is closed and convex, $p \in intb(Y)$ and π is differentiable at p. $y(p)$ solves $\max_{y \in Y} p \cdot y$ iff $y(p) = \nabla \pi(p)$.

Many results in economics relate suitably defined notions of optimality and efficiency.

For instance, given Y, say that $y \in Y$ is efficient if $y = y'$ whenever $y' \geq y$ and $y' \in Y$.

Proposition: If $p \in \mathbb{R}^n_{++}$ and y^* solves $\max_{y \in Y} p \cdot y$, then y^* is efficient.

Proof: If y is not efficient, then there exists $y' \in Y$ such that $y' \geq y$ and $y'_i > y_i$ for some coordinate i. If in addition p has strictly positive coordinates, then $p \cdot y' - p \cdot y \geq p_i(y'_i - y_i) > 0$. ■

Proposition: Suppose Y is convex and $y^* \in Y$ is efficient. There exists $p \in \mathbb{R}^n_+ \backslash \{0\}$ such that $\pi(p) = p \cdot y^*$.

Proof: Fix convex Y and efficient $y^* \in Y$.

Let $B = \{y^*\} + (\mathbb{R}^n_+ \backslash \{0\})$.

Then B is convex and $Y \cap B = \varnothing$.

By the SHT, there exists some $p \in \mathbb{R}^n \backslash \{0\}$ such that $p \cdot y \leq p \cdot b$ for every $y \in Y$ and every $b \in B$.

Let $y^k = y^* + (\frac{1}{k}, ..., \frac{1}{k})$. Hence $y^k \in B$ for every k and $p \cdot y \leq p \cdot y^k$ for every $y \in Y$. Taking the limit, we conclude that $\pi(p) = p \cdot y^*$.

It only remains to show that $p \in \mathbb{R}^n_+$. For any $i = 1, ..., n$, let e_i be the unit vector in $\mathbb{R}^n$ whose ith coordinate is 1 and other coordinates are 0. Then $y^* + e_i \in B$ and therefore $p \cdot (y^* + e_i) \geq p \cdot y^*$. It follows that the ith coordinate of p is nonnegative. Since i is arbitrary, we conclude that $p \in \mathbb{R}^n_{++}$ as desired. ■

Support functions for minimization problems:

Let $x \in S \subseteq \mathbb{R}^n$ and $f : S \to \mathbb{R}$. We say that $p \in \mathbb{R}^n$ is a supergradient of f at x if $f(y) \leq f(x) + p \cdot (y - x)$ for every $y \in S$.

Let $b^-(S) = \{p \in \mathbb{R}^n : \min_{x \in S} p \cdot x \text{ is well-defined}\}$. Let $\tau_S : b^-(S) \to \mathbb{R}$ be given by $\tau_S(p) = \min_{x \in S} p \cdot x$.

Note that $b^-(S) = -b(S) = \{p \in \mathbb{R}^n : -p \in b(S)\}$. Furthermore $\tau_S(p) = -\sigma_S(-p)$.

Theorem: Take a nonempty set $S \subseteq \mathbb{R}^n$.

1. If x_1 solves $\min_{x \in S} p_1 \cdot x$ and x_2 solves $\min_{x \in S} p_2 \cdot x$, then $(x_1 - x_2) \cdot (p_1 - p_2) \leq 0$.

2. $\tau_S(tp) = t\tau_S(p)$ for every $p \in b^-(S)$ and $t \geq 0$.

3. If $b^-(S)$ is convex, then τ_S is concave.

4. Suppose $S \subseteq \mathbb{R}^n$ is closed and convex, and $p \in b^-(S)$. x_P solves $\min_{x \in S} p \cdot x$ iff x_p is a supergradient of τ_S at p.

5. Suppose $S \subseteq \mathbb{R}^n$ is closed and convex, $p \in int b^-(S)$ and τ_S is differentiable at p. Then, x_p solves $\min_{x \in S} p \cdot x$ iff $x_p = \nabla \tau_S(p)$.

Proof: Homework.

Application: Cost minimization

Suppose a firm produces a single output using n inputs:
$Y \subseteq \mathbb{R}_+ \times \mathbb{R}^n_-$.

Define $V(q) = \{x \in \mathbb{R}^n : (q, -x) \in Y\}$ for every output level $q \in \mathbb{R}$.

Let $w \in \mathbb{R}^n$ denote the price vector for inputs.

The cost minimization problem is $\min_{x \in V(q)} w \cdot x$.

The barrier cone $b(V(q))$ consists of those price vectors for which the cost minimization is well-defined for output level q.

The cost function is given by $c(w, q) = \min_{x \in V(q)} w \cdot x$ for every $w \in b(V(q))$. Hence $c(\cdot, q)$ is the support function for $\min_{x \in V(q)} w \cdot x$.

Let $V(q) \subseteq \mathbb{R}^n$ be nonempty. By Theorem above, it follows that:

1. If $x(w)$ solves $\min_{x \in V(q)} w \cdot x$ and $x(w')$ solves $\min_{x \in V(q)} w' \cdot x$, then $(x(w) - x(w')) \cdot (w - w') \leq 0$.

2. $c(tw, q) = tc(w, q)$ for every $w \in b(V(q))$ and $t \geq 0$.

3. If $b(V(q))$ is convex, then $c(\cdot, q)$ is convex.

4. Suppose $V(q)$ is closed and convex, and $w \in b(V(q))$. $x(w)$ solves $\min_{x \in V(q)} w \cdot x$ iff $x(q)$ is a supergradient of $c(\cdot, q)$ at w.

5. (Shephard's Lemma) Suppose $V(q)$ is closed and convex, $w \in intb(V(q))$ and $c(\cdot, q)$ is differentiable at w. $x(w)$ solves $\min_{x \in V(q)} w \cdot x$ iff $x(w) = \nabla_w c(w, q)$.