Game Theory

Correlated Equilibrium

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Nash equilibrium assumes that players independently randomize their choice of action. In this note, we discuss another solution concept of **correlated equilibrium**, which allows for correlated randomization (Aumann, 1987).

1 Correlated Equilibrium

Example 1. Two partners, denoted 1 and 2, decide whether to work (W) or shirk (S) in a joint project. This game, called a teamwork game, is represented as follows:[1]

	W	S
W	$4,4$	$1,5$
S	$5,1$	$0,0$

Table 1: the teamwork game

There are three Nash equilibria: two pure-strategy Nash equilibria (W,S) and (S,W), and one mixed-strategy Nash equilibrium $(\frac{1}{2}\mathrm{W}\oplus\frac{1}{2}\mathrm{S},\frac{1}{2}\mathrm{W}\oplus\frac{1}{2}\mathrm{S})$, in which each player randomizes the two actions with equal probabilities.

We consider the following "correlation mechanism." There are three possible states $\omega\in\{\mathrm{A},\mathrm{B},\mathrm{C}\}$. When state ω is realized, player 1 learns whether "$\omega=\mathrm{A}$" or "$\omega=\mathrm{B}$ or C" but not whether "$\omega=\mathrm{B}$" or "$\omega=\mathrm{C}$". Similarly, player 2 learns whether "$\omega=\mathrm{A}$ or B" or "$\omega=\mathrm{C}$" but not whether "$\omega=\mathrm{A}$" or "$\omega=\mathrm{B}$". This "knowledge" can be schematized as follows:

$$\text{Player 1: } \{\mathrm{A}\},\{\mathrm{B},\mathrm{C}\},$$
$$\text{Player 2: } \{\mathrm{A},\mathrm{B}\},\{\mathrm{C}\}.$$

Here, a player learns which set state ω is in but does not learn which element state ω is (if there are multiple states in the set).

Suppose that each $\omega\in\{\mathrm{A},\mathrm{B},\mathrm{C}\}$ is realized with probability $\frac{1}{3}$. Then, player 1 will observe $\{\mathrm{A}\}$ with probability $\frac{1}{3}$ and $\{\mathrm{B},\mathrm{C}\}$ with probability $\frac{2}{3}$, while player 2 will observe $\{\mathrm{A},\mathrm{B}\}$ with probability $\frac{2}{3}$ and $\{\mathrm{C}\}$ with probability $\frac{1}{3}$.

Suppose that players have agreed on the following strategy profile (before observing any

[1] This teamwork game is effectively the same as Aumann's (1974) game.

Recap (Partition): A **partition** P of a set $X \neq \varnothing$ is a family of subsets of the set X such that:

- $\varnothing \notin P$.
- $\bigcup_{A \in P} A = X$.
- $A \cap B = \varnothing$ if $A, B \in P$ and $A \neq B$.

information): player i plays action a_i such that

$$a_1 = \begin{cases} \text{S} & \text{if player 1 observes } \{\text{A}\} \\ \text{W} & \text{if player 1 observes } \{\text{B}, \text{C}\}, \end{cases} \qquad a_2 = \begin{cases} \text{W} & \text{if player 2 observes } \{\text{A}, \text{B}\} \\ \text{S} & \text{if player 2 observes } \{\text{C}\}. \end{cases}$$

Are they willing to follow the agreement? First, suppose that player 1 (he) observes $\{\text{A}\}$. Then, he believes that player 2 (she) will play $a_2 = \text{W}$ if she obeys the agreement. Hence, he has no incentive to deviate. Second, suppose that he observes $\{\text{B}, \text{C}\}$. Then, he believes that she will play $a_2 = \text{W}, \text{S}$ with equal probabilities if she obeys the agreement. Hence, his payoff is $\frac{5}{2}$ if he plays $a_1 = \text{W}$ but still is $\frac{5}{2}$ if he plays $a_1 = \text{S}$. He has no (strict) incentive to deviate. That is, he is willing to obey the agreement. Similarly, player 2 is willing to obey the agreement. This is called a **correlated equilibrium**. The corresponding payoff profile is $\frac{1}{3}(4,4) + \frac{1}{3}(1,5) + \frac{1}{3}(5,1) = (\frac{10}{3}, \frac{10}{3})$, which Pareto-dominates the mixed-strategy Nash equilibrium payoff profile $(\frac{5}{2}, \frac{5}{2})$. □

1.1 Correlated Equilibrium

Correlated Mechanism

Definition 1. In a finite normal-form game G, a **correlation mechanism** is a triple (Ω, π, H) such that:

1. Ω is a finite set of states.
2. $\pi \in \Delta(\Omega)$ is a full-support prior.[2]
3. $H = (H_i)_i$ is the profile of player i's partition H_i of Ω.
   - $H_i(\omega)$ is the (unique) element $h_i \in H_i$ such that $\omega \in h_i$.
   - H_i is called player i's **information partition** and $h_i \in H_i$ his **information set**.

Example 2. The correlation mechanism of Example 1 is such that:

1. $\Omega = \{\text{A}, \text{B}, \text{C}\}$.
2. $\pi(\omega) = \frac{1}{3}$ for each $\omega \in \Omega$.

[2] That is, $\pi(\omega) > 0$ for each $\omega \in \Omega$. This full-support assumption is without loss of generality.

3. $H_1 = \{\{\mathrm{A}\}, \{\mathrm{B}, \mathrm{C}\}\}$ and $H_2 = \{\{\mathrm{A}, \mathrm{B}\}, \{\mathrm{C}\}\}$.

For example, if $\omega = \mathrm{A}$, $H_1(\omega) = \{\mathrm{A}\}$ and $H_2(\omega) = \{\mathrm{A}, \mathrm{B}\}$. $\square$

Correlated Strategies In a finite normal-form game G without a correlation mechanism, players do not have any information about the game (other than the setting itself), but now since they may learn some information, their choice of action should be conditioned on the information they learn.

Definition 2. Let G be a finite normal-form game and let (Ω, π, H) be a correlation mechanism. Player i's **(correlation) strategy** is a function $f_i : \Omega \to A_i$ such that for each $\omega, \omega' \in \Omega$,[3]

$$H_i(\omega) = H_i(\omega') \quad \Longrightarrow \quad f_i(\omega) = f_i(\omega').$$

Let $F_i(\Omega, H_i)$ be the set of all player i's strategies. A **(correlation) strategy profile** $f = (f_i)_i$ is the profile consisting of each player's strategy $f_i \in F_i(\Omega, H_i)$. Let $F(\Omega, H)$ be the set of all the strategy profiles.

Remark 1. Player i's strategy does not allow for randomization except through observations of an information set h_i. This is without loss of generality, if we enlarge the state space Ω.

We illustrate this idea, using Example 1. We will attempt to let player 1 randomize actions W and S with equal probabilities when he observes $\{\mathrm{B}, \mathrm{C}\}$. Enlarge the state space Ω to $\Omega' = \{\mathrm{A}_H, \mathrm{A}_T, \mathrm{B}_H, \mathrm{B}_T, \mathrm{C}_H, \mathrm{C}_T\}$. When a state $\omega' \in \Omega'$ is realized, player 1 observes the same information as in the original setting (i.e., $\{\mathrm{A}\}$ or $\{\mathrm{B}, \mathrm{C}\}$), plus the result of a coin toss (heads or tails). Player 1's information partition is as follows:

$$\text{Player 1: } \{\mathrm{A}_H\}, \{\mathrm{A}_T\}, \{\mathrm{B}_H, \mathrm{C}_H\}, \{\mathrm{B}_T, \mathrm{C}_T\},$$

where subscripts H and T denote heads and tails of the coin toss respectively. Consider his strategy f_1 such that $f_1(\mathrm{A}_H) = f_1(\mathrm{A}_T) = \mathrm{S}$, $f_1(\mathrm{B}_H) = f_1(\mathrm{C}_H) = \mathrm{W}$, and $f_1(\mathrm{B}_T) = f_1(\mathrm{C}_T) = \mathrm{S}$. If each $\omega' \in \Omega'$ is realized with equal probability $\frac{1}{6}$ then this strategy f_1 randomizes actions W and S with equal probabilities when he observes $\{\mathrm{B}, \mathrm{C}\}$ in the original setting. $\square$

Correlated Equilibrium

Definition 3. In a finite normal-form game G, a **correlated equilibrium** is a pair $(\Omega, \pi, H; f)$ of a correlation mechanism (Ω, π, H) and a strategy profile $f \in F(\Omega, H)$ such that for each $i \in I$ and each $f_i' \in F_i(\Omega, H_i)$,

$$\sum_{\omega \in \Omega} \pi(\omega) u_i(f_i(\omega), f_{-i}(\omega)) \geq \sum_{\omega \in \Omega} \pi(\omega) u_i(f_i'(\omega), f_{-i}(\omega)). \tag{1}$$

[3] That is, her strategy f_i must be constant over each $h_i \in H_i$. In terms of measure theory, it must be measurable with respect to the sigma-algebra generated by the partition H_i.

Coffee Break ☕. If you have never seen this kind of setting (e.g., information partitions), you might be being puzzled. No worries! What is modeled here is actually simple. I would recommend you to compare the formal definitions with the example, again and again; you would be get used to the modeling soon. □

Ex-Ante vs Interim Perspectives In Definition 3, player i's payoff maximization (1) is formulated in term of the **ex-ante** perspective—that is, before each player learns his information $H_i(\omega)$. In Example 1, however, we have taken the **interim** perspective—that is, after each player learned his information (but does not know the other players's information). It turns out that these two formulations are equivalent.

Theorem 1. *Let G be a finite normal-form game, and let (Ω, π, H) be a correlation mechanism. Let $f \in F(\Omega, H)$ be a strategy profile. For each $f_i' \in F_i(\Omega, H_i)$,*

$$\sum_{\omega \in \Omega} \pi(\omega) u_i(f_i(\omega), f_{-i}(\omega)) \geq \sum_{\omega \in \Omega} \pi(\omega) u_i(f_i'(\omega), f_{-i}(\omega)) \tag{2}$$

if and only if for each $h_i \in H_i$ and each $a_i' \in A_i$,

$$\sum_{\omega \in h_i} \pi(\omega \mid h_i) u_i(f_i(\omega), f_{-i}(\omega)) \geq \sum_{\omega \in h_i} \pi(\omega \mid h_i) u_i(a_i', f_{-i}(\omega)), \tag{3}$$

where $\pi(\omega \mid h_i)$ is player i's posterior based on information set h_i:

$$\pi(\omega \mid h_i) = \begin{cases} \frac{\pi(\omega)}{\pi(h_i)} & \text{if } \omega \in h_i \\ 0 & \text{if } \omega \notin h_i. \end{cases}$$

Proof. See Appendix A. ■

1.2 Revelation Principle

A correlated mechanism specifies which information each player observes with what probability. It does not specify the resulting distribution of action profiles itself. Each player's strategy f_i convert information into action. Combining them, we can define the resulting distribution of action profiles.

Example 3. Example 1 assumes that each state $\omega \in \{\mathrm{A}, \mathrm{B}, \mathrm{C}\}$ is drawn with equal probability $\frac{1}{3}$. It considers the strategies f_1, f_2 such that $f_1(\{\mathrm{A}\}) = f_2(\{\mathrm{C}\}) = \mathrm{S}$ and $f_1(\{\mathrm{B}, \mathrm{C}\}) = f_2(\{\mathrm{A}, \mathrm{B}\}) = \mathrm{W}$. Hence, the resulting distribution of action profiles, denoted $\mu \in \Delta(A)$, is such that $\mu(\mathrm{W}, \mathrm{W}) = \mu(\mathrm{W}, \mathrm{S}) = \mu(\mathrm{S}, \mathrm{W}) = \frac{1}{3}$ and $\mu(\mathrm{S}, \mathrm{S}) = 0$. □

It is straightforward to generalize this example.

Definition 4. In a finite normal-form game G, we say that a distribution $\mu \in \Delta(A)$ is a **correlated equilibrium action distribution** if there exists some correlated equilibrium, denoted $(\Omega, \pi, H; f)$, such that for each $a \in A$,

$$\mu(a) = \sum_{\omega : f(\omega) = a} \pi(\omega).$$

Direct Correlated Mechanism It sounds a daunting task to find *all* correlated equilibrium action distributions, because a state space Ω can be taken arbitrarily. However, we note that when we define a correlated equilibrium, the end-result that players would care about is only the induced action distribution. This is because players do not care about a state ω itself, aside from its indirect effect on the induced distribution over action profiles.

Example 4. Consider the normal-form game of Example 1. Suppose that there is a **direct** correlation mechanism $\mu \in \Delta(A)$. We assume that if $a = (a_1, a_2)$ is drawn, players $1, 2$ are "recommend" to play a_1, a_2 respectively (but neither knows the recommendation to the other player). Hence, if player 1 is recommended to play a_1 then he knows that either (a_1, W) or (a_1, S) is drawn; by Bayes's Theorem, he believes that the probability that player 2 is recommended to play a_2 is $\frac{\mu(a_1, a_2)}{\mu(a_1, \mathrm{W}) + \mu(a_1, \mathrm{S})}$. The same applies to player 2.

	W	S
W	$\mu(\mathrm{W}, \mathrm{W})$	$\mu(\mathrm{W}, \mathrm{S})$
S	$\mu(\mathrm{S}, \mathrm{W})$	$\mu(\mathrm{S}, \mathrm{S})$

Table 2: a recommendation distribution

Let $\mu(\mathrm{W}, \mathrm{W}) = \mu(\mathrm{W}, \mathrm{S}) = \mu(\mathrm{S}, \mathrm{W}) = \frac{1}{3}$ and $\mu(\mathrm{S}, \mathrm{S}) = 0$. Since both players are willing to obey the recommendations (why?), this play should constitutes a "correlated equilibrium," which yields a payoff profile $\frac{1}{3}(4, 4) + \frac{1}{3}(1, 5) + \frac{1}{3}(5, 1) = (\frac{10}{3}, \frac{10}{3})$. $\square$

In a direct correlation mechanism, as we shall see below, each player will receive a recommendation of action that he should play, rather than an arbitrary information set.

Definition 5. In a finite normal-form game G, a **direct correlation mechanism** is a correlation mechanism $(A, \mu, (A_i)_i)$ such that:

1. A is the set of action profiles.
2. $\mu \in \Delta(A)$ is a distribution.[4]
3. $A_i = \{\{a_i\} \times A_{-i}\}_{a_i \in A_i}$ by abuse of notation.

For each $a \in A$, let $A_i(a)$ denote the (unique) element $\{a_i\} \times A_{-i}$, denoted a_i by abuse of notation.

[4] We no longer assume the full-support. That is, there may exist some $a \in A$ such that $\mu(a) = 0$.

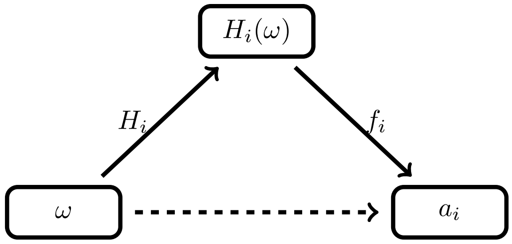

Figure 1: the revelation principle

Direct Correlated Equilibrium

Definition 6. In a finite normal-form game G with a direct correlation mechanism $(A, \mu, (A_i)_i)$, we say that $\mu \in \Delta(A)$ is a **(direct) correlated equilibrium** if for each $i \in I$ and each $a_i, a_i' \in A_i$,

$$\sum_{a_{-i}} \mu(a_{-i} \mid a_i) u_i(a_i, a_{-i}) \geq \sum_{a_{-i}} \mu(a_{-i} \mid a_i) u_i(a_i', a_{-i}). \tag{4}$$

where $\mu(a_{-i} \mid a_i)$ is player i's posterior when she is recommended to play action a_i:

$$\mu(a_{-i} \mid a_i) = \begin{cases} \frac{\mu(a_i, a_{-i})}{\sum_{a'_{-i}} \mu(a_i, a'_{-i})} & \text{if } \sum_{a'_{-i}} \mu(a_i, a'_{-i}) > 0 \\ 0 & \text{if } \sum_{a'_{-i}} \mu(a_i, a'_{-i}) = 0. \end{cases}$$

The left-hand side of inequality (4) is player i's payoff when he "obeys" the recommendation to play action a_i, while the right-hand side is his player when he "deviates" to playing action a_i'. Hence, the correlated equilibrium is such that every player is willing to obey the recommendation. Such a property is called the obedience or incentive compatibility in other contexts.

Revelation Principle The following result—in the spirit of the revelation principle in mechanism design—claims that it suffices to consider Definition 6, not Definition 3, to find correlated equilibrium action distributions. The idea is illustrated in Figure 1. The solid arrows indicate how players choose their actions under an (indirect) correlation mechanism. The dashed arrow is a "composite" of the two solid arrows. The revelation principle-like result claims that these two are, in effect, the same.

Theorem 2. *In a finite normal-form game G, $\mu \in \Delta(A)$ is a correlated equilibrium action distribution if and only if μ is a direct correlated equilibrium.*

Proof. See Appendix A. ■

Remark 2. Correlated equilibrium is often defined in the fashion of Definition 6, without mentioning the direct correlation mechanism. □

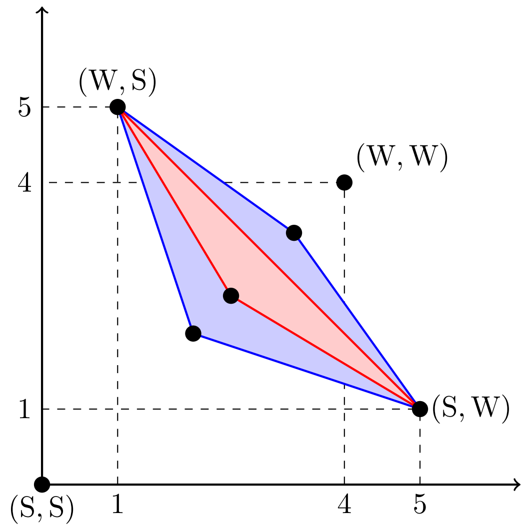

Figure 2: equilibrium payoffs for Example 1

1.3 Properties of Correlated Equilibria

Correlated Equilibrium has desirable properties, which we will now discuss. We analyze a simple example and then generalize the example.

Example 5. Using Example 1, we find correlated equilibria. For example, we have the greatest (symmetric) correlated equilibrium payoff $\frac{10}{3}$, which we have found in Example 1, and the smallest (symmetric) correlated equilibrium payoff 2. Along this line of analysis, we can characterize the correlated equilibria, drawing the corresponding payoff set in blue in Figure 2. Note that the correlated equilibrium payoff set is convex.

Next, we compare the correlated equilibria with the Nash equilibria. Recall that there are three Nash equilibria: (W, S), (S, W), and $(\frac{1}{2}\mathrm{W} \oplus \frac{1}{2}\mathrm{S}, \frac{1}{2}\mathrm{W} \oplus \frac{1}{2}\mathrm{S})$ with respective payoff profiles $(1, 5)$, $(5, 1)$, and $(\frac{5}{2}, \frac{5}{2})$. We draw the convex hull of the Nash equilibrium payoff set in red in Figure 2. Note that the correlated equilibrium payoff set contains the convex hull of the Nash equilibrium payoff set. □

Convexity The set of correlated equilibrium action distributions is convex.

Theorem 3. *In a finite normal-form game G, the set of correlated equilibrium action distributions is convex.*

Proof. Take any two correlated equilibrium action distributions μ, μ', which we identify with direct correlated equilibria, still denoted μ, μ', respectively. Consider a biased coin that lands heads with probability λ and tails with probability $1 - \lambda$. All the players observe the result of the coin toss. Assume that they play the correlated equilibrium μ if the coin lands heads and the correlated equilibrium μ' if the coin lands tails. Then, this mixture $\lambda\mu + (1 - \lambda)\mu'$ is still a correlated equilibrium. ■

Correlated Equilibrium versus Nash Equilibrium Next, we compare correlated equilibria with Nash equilibria.

Theorem 4. *In a finite normal-form game G, the convex hull of Nash equilibrium action distributions is a correlated equilibrium action distribution.*

Proof. By Theorems 2 and 3, it suffices to show that every Nash equilibrium σ^* is a (direct) correlated equilibrium with the direct mechanism $(A, \mu, (A_i)_i)$, where a distribution $\mu(a) = \prod_{i \in I} \sigma^*(a_i)$. To this end, take any action a_i such that $\mu_i(a_i) \equiv \sum_{a_{-i}} \mu(a_i, a_{-i}) > 0$. Then, for each $a_i' \in A_i$, it must be that:

$$\sum_{a_{-i}} \mu(a_{-i} \mid a_i) u_i(a_i', a_{-i}) = \sum_{a_{-i}} \prod_{j \neq i} \sigma_j^*(a_j) u_i(a_i', a_{-i}) = u_i(a_i', \sigma_{-i}^*).$$

Hence,

$$\sum_{a_{-i}} \mu(a_{-i} \mid a_i) u_i(a_i, a_{-i}) = u_i(a_i, \sigma_{-i}^*) \geq u_i(a_i', \sigma_{-i}^*) = \sum_{a_{-i}} \mu(a_{-i} \mid a_i) u_i(a_i', a_{-i}),$$

which is desired. ■

2 Correlated Equilibrium as Bayesian Rationality*

In the Bayesian view of decision-making under uncertainty, an agent first forms a belief about a (relevant) state of interest and then optimizes his action. In the context of a game, the (relevant) state for each player is about how the other players in the game will play. If an agent were given this information, he would know the best course of action for himself.

If it is common knowledge that every player is rational at every state, then what does play correspond to? Aumann (1987) shows that the action distribution induced by such play is given by some correlated equilibrium.

Consider a normal-form game G. Let Ω be a finite set of states of the world with typical element ω. The state of the world ω encodes all possible information about the relevant uncertainty, which, in this case, is how players play. Thus if the players knew a state ω, they would know exactly what will happen. Assume that an information partition H_i is given to each player i.

Common Prior Assumption Suppose that player i has his own prior $\pi^i \in \Delta(\Omega)$. To form his (posterior) belief, he does Bayesian inference based on the information that he observes.

Definition 7. Let G be a normal-form game and let Ω be the set of states of the world. Suppose that player i has a prior $\pi^i \in \Delta(\Omega)$. There is a **common prior** $\pi \in \Delta(\Omega)$ if $\pi = \pi^i$ for each $i \in I$.

The **Common Prior Assumption** (CPA) is the assumption that admits the existence of a common prior.[5] The CPA does **not** assume that all players must have the same information. Player i's subjective belief about a state ω is his posterior (after Bayesian updating) based on the information that he has. Therefore, different players may have different subjective beliefs if they have different information. The CPA rather says that differences in subjective beliefs arise **only** from differences in information, rather than a priori differences in beliefs.

Remark 3. The CPA has been pervasive in economic models, but it is actually known that the CPA has very strong implications on economic modeling. We will revisit the CPA in lectures on games of incomplete information. □

Correlated Equilibrium as Bayesian Rationality

Definition 8. Let G be a normal-form game. Let Ω be the finite set of states of the world with a common prior π, and let H_i be player i's information partition for each $i \in I$. For each $i \in I$, let $f_i : \Omega \to A_i$ be a function such that for each $\omega, \omega' \in \Omega$,

$$H_i(\omega) = H_i(\omega') \quad \Longrightarrow \quad f_i(\omega) = f_i(\omega'),$$

and let $f(\omega) = (f_i(\omega))_i$. Player i is **Bayesian rational** at $\omega \in \Omega$ if for each $a_i \in A_i$,

$$\sum_{\omega' \in H_i(\omega)} \pi(\omega' \mid H_i(\omega)) u_i(f_i(\omega'), f_{-i}(\omega')) \geq \sum_{\omega' \in H_i(\omega)} \pi(\omega' \mid H_i(\omega)) u_i(a_i, f_{-i}(\omega')).$$

Theorem 5. *Let G be a normal-form game. Let Ω be the finite set of states of the world with a common prior π, and let H_i be player i's information partition for each $i \in I$. Suppose that (it is common knowledge that) every player is Bayesian rational at each $\omega \in \Omega$. Let $\mu \in \Delta(A)$ be the action distribution such that for each $a \in A$,*

$$\mu(a) = \sum_{\omega : f(\omega) = a} \pi(\omega).$$

Then, μ is a correlated equilibrium action distribution.

Proof. Interpret (Ω, π, H) as a correlation mechanism. Since all players are Bayesian rational at each ω, f is a correlated equilibrium with respect to (Ω, π, H). Hence, μ is a correlated equilibrium action distribution. ■

The proof is simple, but the implication is far-reaching. One conclusion that we can draw from the above result is that if an action distribution μ is not a correlated equilibrium action distribution but is the action distribution induced by such an uncertainty model, then either the players must have started out with uncommon priors or there must be at least one state

[5] The CPA is sometimes called the Harsanyi doctrine, named after Harsanyi (1967, 1968a,b).

ω at which at least one player is not acting as a Bayesian rational agent (that is, the common knowledge of Bayes rationality is violated).

A Proofs

A.1 Theorem 1

"If" Part Suppose that inequality (3) holds. For each $f'_i \in F_i(\Omega, H_i)$,

$$\sum_{\omega \in h_i} \pi(\omega \mid h_i) u_i(f_i(\omega), f_{-i}(\omega)) \geq \sum_{\omega \in h_i} \pi(\omega \mid h_i) u_i(f'_i(\omega), f_{-i}(\omega)),$$

Taking the summation across all $h_i \in H_i$, we have

$$\sum_{h_i \in H_i} \pi(h_i) \sum_{\omega \in h_i} \pi(\omega \mid h_i) u_i(f_i(\omega), f_{-i}(\omega)) \geq \sum_{h_i \in H_i} \pi(h_i) \sum_{\omega \in h_i} \pi(\omega \mid h_i) u_i(f'_i(\omega), f_{-i}(\omega)),$$

which implies inequality (2).

"Only If" Part Suppose that inequality (2) holds. Suppose, for a contradiction, that there exist some $\hat{h}_i \in H_i$ and some $\hat{a}_i \in A_i$,

$$\sum_{\omega \in \hat{h}_i} \pi(\omega \mid \hat{h}_i) u_i(\hat{a}_i, f_{-i}(\omega)) > \sum_{\omega \in \hat{h}_i} \pi(\omega \mid \hat{h}_i) u_i(f_i(\omega), f_{-i}(\omega)). \tag{5}$$

Let $\hat{f}_i \in F_i(\Omega, H_i)$ be player i's strategy such that

$$\hat{f}_i(\omega) = \begin{cases} f_i(\omega) & \text{if } \omega \notin \hat{h}_i \\ \hat{a}_i & \text{if } \omega \in \hat{h}_i. \end{cases}$$

That is, player i "follows" the original strategy f_i if he learns $h_i \neq \hat{h}_i$ but "deviates" to action $\hat{a}_i$ if he learns $\hat{h}_i$. It is then trivial that for each $h_i \neq \hat{h}_i$,

$$\sum_{\omega \in h_i} \pi(\omega \mid h_i) u_i(\hat{f}_i(\omega), f_{-i}(\omega)) = \sum_{\omega \in h_i} \pi(\omega \mid h_i) u_i(f_i(\omega), f_{-i}(\omega)). \tag{6}$$

Summing inequalities (5) and (6) with weights $\pi(h_i) = \sum_{\omega \in h_i} \pi(\omega)$ for all $h_i \in H_i$, we have

$$\sum_{h_i \in H_i} \pi(h_i) \sum_{\omega \in h_i} \pi(\omega \mid h_i) u_i(\hat{f}_i(\omega), f_{-i}(\omega)) > \sum_{h_i \in H_i} \pi(h_i) \sum_{\omega \in h_i} \pi(\omega \mid h_i) u_i(f_i(\omega), f_{-i}(\omega)).$$

This is equivalent to

$$\sum_{\omega \in \Omega} \pi(\omega) u_i(\hat{f}_i(\omega), f_{-i}(\omega)) > \sum_{\omega \in \Omega} \pi(\omega) u_i(f_i(\omega), f_{-i}(\omega)),$$

but it contradicts inequality (2).

A.2 Theorem 2

We show the "only if" part, since the "if" part is immediate. Since $\mu \in \Delta(A)$ is a correlated equilibrium action distribution, there is a correlated equilibrium $(\Omega, \pi, H; f)$ such that for each $a \in A$,

$$\mu(a) = \sum_{\omega : f(\omega) = a} \pi(\omega). \tag{7}$$

For each $a_i, a_i' \in A_i$, it must be that

$$\sum_{a_{-i}} \mu(a_i, a_{-i}) u_i(a_i', a_{-i}) = \sum_{h_i \in f_i^{-1}(a_i)} \pi(h_i) \sum_{\omega \in \Omega} \pi(\omega \mid h_i) u_i(a_i', f_{-i}(\omega)),$$

where $f_i^{-1}(a_i) = \{h_i \in H_i : f_i(\omega) = a_i \text{ for each } \omega \in h_i\}$. From Definition 3, it follows that for each $h_i \in H_i$ and each $a_i' \in A_i$,

$$\sum_{\omega \in \Omega} \pi(\omega \mid h_i) u_i(f_i(\omega), f_{-i}(\omega)) \geq \sum_{\omega \in \Omega} \pi(\omega \mid h_i) u_i(a_i', f_{-i}(\omega)).$$

From equation (7), it follows that

$$\sum_{a_{-i}} \mu(a_i, a_{-i}) u_i(a_i, a_{-i}) \geq \sum_{a_{-i}} \mu(a_i, a_{-i}) u_i(a_i', a_{-i}).$$

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