Analytical solution of the Heckscher-Ohlin model

Tiago Tavares*
International Trade I at ITAM

September 11, 2017

1 Heckscher-Ohlin: analytical solution

Preferences Preferences for both $j = \{home, foreign\}$ are given by the following function

$$u(x_1^j, x_2^j) = \alpha \log x_1^j + (1-\alpha) \log x_2^j$$

with the respective budget constraint

$$p_1 x_1^j + p_2 x_2^j = w^j L^j + r^j K^j$$

Technology The two final goods, $i = 1, 2$, in this economy are characterized by the following technology:

$$y_i^j = z_i \left(k_i^j\right)^{\beta_i} \left(l_i^j\right)^{1-\beta_i}$$

Endowments Without loss of generality lets assume that

$$K^h / L^h > K^f / L^f$$

that is, home is relatively abundant in capital.

*Email: Please contact me to tgstavares@gmail.com if you find any errors in the text.

1.1 Autarky

Consumer maximization Consumers solve the following problem

$$\max_{x_1,x_2}\{\alpha\log x_1+(1-\alpha)\log x_2,\ \ st\ \ p_1x_1+p_2x_2=wL+rK\}$$

implying:

$$\frac{\alpha}{1-\alpha}=\frac{p_1x_1}{p_2x_2}$$

Firms minimization Firms minimize costs

$$\min_{l_i,k_i}\left\{wl_i+rk_i,\ \ st\ \ y_i\leq z_i\left(k_i\right)^{\beta_i}\left(l_i\right)^{1-\beta_i}\right\}$$

implying:

$$\frac{1-\beta_i}{\beta_i}=\frac{wl_i}{rk_i} \tag{1}$$

Market clearing As usually $c_i=y_i$, and with (1) implies the following:

$$\begin{aligned} x_i &= y_i=z_i\left(k_i\right)^{\beta_i}\left(l_i\right)^{1-\beta_i}\\ \Rightarrow\ \ x_i &= z_il_i\left(\frac{\beta_i}{1-\beta_i}\frac{w}{r}\right)^{\beta_i}\end{aligned}$$

Because in equilibrium firms make zero profits, that is, $p_iz_i\left(k_i\right)^{\beta_i}\left(l_i\right)^{1-\beta_i}=wl_i+rk_i$, we have:

$$\begin{aligned} p_i &= \frac{rk_i+wl_i}{z_i\left(k_i\right)^{\beta_i}\left(l_i\right)^{1-\beta_i}}\\ \Rightarrow\ \ p_i &= \frac{rk_i\frac{1}{\beta_i}}{z_il_i\left(\frac{\beta_i}{1-\beta_i}\frac{w}{r}\right)^{\beta_i}}\\ \Rightarrow\ \ p_i &= \frac{rk_i\frac{1}{\beta_i}}{z_i\frac{1-\beta_i}{\beta_i}\frac{r}{w}k_i\left(\frac{\beta_i}{1-\beta_i}\frac{w}{r}\right)^{\beta_i}}\\ \Rightarrow\ \ p_i &= \frac{r^{\beta_i}w^{1-\beta_i}}{z_i\left(1-\beta_i\right)^{1-\beta_i}\beta_i^{\beta_i}} \end{aligned} \tag{2}$$

Using the foc for the consumer (λ is a Lagrangian multiplier):

$$\begin{aligned}
&\alpha = \lambda p_1 x_1 \\
\Rightarrow &\alpha = \lambda \left(w l_1 + r k_1\right) \\
\Rightarrow &\alpha = \lambda \left(w l_1 + \frac{\beta_1}{1-\beta_1} w l_1\right) \\
\Rightarrow &\alpha \left(1-\beta_1\right) = \lambda w l_1
\end{aligned}$$

and, using similar calculations:

$$\left(1-\alpha\right)\left(1-\beta_2\right) = \lambda w l_2$$

So, dividing the ratio of the previous 2 equations gives:

$$\frac{l_1}{l_2} = \frac{\alpha\left(1-\beta_1\right)}{\left(1-\alpha\right)\left(1-\beta_2\right)}$$

which together with the fact that $L = l_1 + l_2$ yields:

$$\begin{aligned}
l_1 &= \frac{\alpha\left(1-\beta_1\right)}{\left(1-\alpha\right)\left(1-\beta_2\right)+\alpha\left(1-\beta_1\right)} L \qquad (3) \\
l_2 &= \frac{\left(1-\alpha\right)\left(1-\beta_2\right)}{\left(1-\alpha\right)\left(1-\beta_2\right)+\alpha\left(1-\beta_1\right)} L
\end{aligned}$$

Using similar calculations for capital:

$$\begin{aligned}
k_1 &= \frac{\alpha\beta_1}{\left(1-\alpha\right)\beta_2+\alpha\beta_1} K \qquad (4) \\
k_2 &= \frac{\left(1-\alpha\right)\beta_2}{\left(1-\alpha\right)\beta_2+\alpha\beta_1} K
\end{aligned}$$

Dividing (3) on (4) and rearranging:

$$\begin{aligned}
\frac{l_1}{k_1} =& \frac{\frac{\alpha(1-\beta_1)}{(1-\alpha)(1-\beta_2)+\alpha(1-\beta_1)}}{\frac{\alpha\beta_1}{(1-\alpha)\beta_2+\alpha\beta_1}}\frac{L}{K} \\
\Rightarrow \frac{1-\beta_1}{\beta_1}\frac{r}{w} =& \frac{\frac{\alpha(1-\beta_1)}{(1-\alpha)(1-\beta_2)+\alpha(1-\beta_1)}}{\frac{\alpha\beta_1}{(1-\alpha)\beta_2+\alpha\beta_1}}\frac{L}{K} \\
\Rightarrow \qquad \frac{L}{K} =& \frac{(1-\alpha)(1-\beta_2)+\alpha(1-\beta_1)}{(1-\alpha)\beta_2+\alpha\beta_1}\frac{r}{w} \qquad (5)
\end{aligned}$$

Thus, in a labor abundant country capital is relatively more expensive as we would expect. Finally, consumption is given by:

$$\begin{aligned}
x_1 =& z_1 l_1 \left(\frac{\beta_1}{1-\beta_1}\frac{w}{r}\right)^{\beta_1} \\
\Rightarrow x_1 =& z_1 \frac{\alpha(1-\beta_1)}{(1-\alpha)(1-\beta_2)+\alpha(1-\beta_1)} L \left(\frac{\beta_1}{1-\beta_1}\frac{w}{r}\right)^{\beta_1} \\
\Rightarrow x_1 =& z_1 \frac{\alpha(1-\beta_1)}{(1-\alpha)(1-\beta_2)+\alpha(1-\beta_1)} L \left(\frac{\beta_1}{1-\beta_1}\frac{(1-\alpha)(1-\beta_2)+\alpha(1-\beta_1)}{(1-\alpha)\beta_2+\alpha\beta_1}\frac{K}{L}\right)^{\beta_1} \\
\Rightarrow x_1 =& z_1 \frac{\alpha\beta_1^{\beta_1}(1-\beta_1)^{1-\beta_1}}{((1-\alpha)\beta_2+\alpha\beta_1)^{\beta_1}((1-\alpha)(1-\beta_2)+\alpha(1-\beta_1))^{1-\beta_1}} L^{1-\beta_1}K^{\beta_1}
\end{aligned}$$

1.2 Free trade equilibrium

We'll stay in the case where both countries produce both goods. Note that now the goods market clearing conditions are the following:

$$\begin{aligned}
x_1^h + x_1^f &= y_1^h + y_1^f \\
x_2^h + x_2^f &= y_2^h + y_2^f
\end{aligned}$$

Firms maximization Firms maximize profits

$$\max_{l_i,k_i}\left\{p_i z_i \left(k_i^j\right)^{\beta_i}\left(l_i^j\right)^{1-\beta_i} - w^j l_i^j - r^j k_i^j\right\}$$

implying:

$$\begin{aligned}
w^j &= (1-\beta_i)\, p_i z_i \left(k_i^j\right)^{\beta_i}\left(l_i^j\right)^{-\beta_i} \qquad (6) \\
r^j &= \beta_i p_i z_i \left(k_i^j\right)^{\beta_i - 1}\left(l_i^j\right)^{1-\beta_i} \qquad (7)
\end{aligned}$$

As in (2), zero profits also imply that:

$$p_i = \frac{\left(r^j\right)^{\beta_i}\left(w^j\right)^{1-\beta_i}}{z_i\left(1-\beta_i\right)^{1-\beta_i}\beta_i^{\beta_i}}$$

Therefore

$$\begin{aligned}1 = & \left(\frac{r^h}{r^f}\right)^{\beta_1}\left(\frac{w^h}{w^f}\right)^{1-\beta_1} \\ 1 = & \left(\frac{r^h}{r^f}\right)^{\beta_2}\left(\frac{w^h}{w^f}\right)^{1-\beta_2}\end{aligned}$$

But this implies:

$$\begin{aligned}w^1 &= w^2 \\ r^1 &= r^2\end{aligned}$$

But this is just an implication of the **factor price equalization theorem (FPE)**!

Market clearing Now, rearranging (7):

$$\begin{aligned} r^j k_i^j &= \beta_i p_i z_i \left(k_i^j\right)^{\beta_i}\left(l_i^j\right)^{1-\beta_i} \\ \Rightarrow \quad \tfrac{r^j k_i^j}{\beta_i} &= p_i y_i^j \\ \tfrac{w^j l_i^j}{1-\beta_i} &= p_i y_i^j \end{aligned}$$

And summing up over j together with FPE:

$$\begin{aligned} p_i\left(y_i^h + y_i^f\right) &= \frac{r}{\beta_i}\left(k_i^h + k_i^f\right) \\ p_i\left(y_i^h + y_i^f\right) &= \frac{w}{1-\beta_i}\left(l_i^h + l_i^f\right) \end{aligned}$$

Note that demand for goods 1 and market clearing implies:

$$\begin{aligned}
p_1 x_1^h &= \alpha \left(wL^h + rK^h\right) \\
\Rightarrow p_1 \left(x_1^h + x_1^f\right) &= \alpha \left(w \left(L^h + L^f\right) + r \left(K^h + K^f\right)\right) \\
\Rightarrow \frac{r}{\beta_1} \left(k_1^h + k_1^f\right) &= \alpha \left(w \left(L^h + L^f\right) + r \left(K^h + K^f\right)\right) \\
\Rightarrow \;\; r \left(k_1^h + k_1^f\right) &= \beta_1 \alpha \left(w \left(L^h + L^f\right) + r \left(K^h + K^f\right)\right) \qquad (8)
\end{aligned}$$

and similarly

$$r \left(k_2^h + k_2^f\right) = \beta_2 (1-\alpha) \left(w \left(L^h + L^f\right) + r \left(K^h + K^f\right)\right) \qquad (9)$$

Summing up (8) and (9):

$$\begin{aligned}
r \left(k_2^h + k_2^f + k_1^h + k_1^f\right) &= \left(\beta_1 \alpha + \beta_2 (1-\alpha)\right) \left(w \left(L^h + L^f\right) + r \left(K^h + K^f\right)\right) \\
\Rightarrow \qquad r \left(K^h + K^f\right) &= \left(\beta_1 \alpha + \beta_2 (1-\alpha)\right) \left(w \left(L^h + L^f\right) + r \left(K^h + K^f\right)\right) \\
\Rightarrow r \left(K^h + K^f\right) \left(1 - \beta_1 \alpha - \beta_2 (1-\alpha)\right) &= \left(\beta_1 \alpha + \beta_2 (1-\alpha)\right) w \left(L^h + L^f\right) \\
\Rightarrow \qquad \frac{L^h + L^f}{K^h + K^f} &= \frac{r}{w} \frac{\left(1 - \beta_1 \alpha - \beta_2 (1-\alpha)\right)}{\left(\beta_1 \alpha + \beta_2 (1-\alpha)\right)} \\
\Rightarrow \qquad \frac{L^h + L^f}{K^h + K^f} &= \frac{r}{w} \frac{\left(\alpha + (1-\alpha) - \beta_1 \alpha - \beta_2 (1-\alpha)\right)}{\left(\beta_1 \alpha + \beta_2 (1-\alpha)\right)} \\
\Rightarrow \qquad \frac{L^h + L^f}{K^h + K^f} &= \frac{r}{w} \frac{\left((1-\alpha)(1-\beta_2) + \alpha(1-\beta_1)\right)}{\left(\beta_2 (1-\alpha) + \beta_1 \alpha\right)} \qquad (10)
\end{aligned}$$

Comparing (10) with (5) and because we assume that $K^h/L^h > K^f/L^f$, one can see that r/w under free trade lies within the r/w that would prevail under autarky for both countries. This result could be predicted using both the **Stolper-Samuelson theorem** and the **Heckscher-Ohlin theorem**.

Now, from the factor demand equations and market clearing:

$$\begin{aligned}
& \frac{w}{r}\frac{\beta_i}{(1-\beta_i)}l_i^j = k_i^j \\
\Rightarrow \quad & \frac{w}{r}\left(\frac{\beta_1}{(1-\beta_1)}l_1^j + \frac{\beta_2}{(1-\beta_2)}l_2^j\right) = k_2^j + k_1^j \\
\Rightarrow \quad & \frac{w}{r}\left(\frac{\beta_1}{(1-\beta_1)}\left(L^j - l_2^j\right) + \frac{\beta_2}{(1-\beta_2)}l_2^j\right) = K^j \\
\Rightarrow \quad & \frac{w}{r}\left(\frac{\beta_1}{(1-\beta_1)}L^j + \left(\frac{\beta_2}{(1-\beta_2)} - \frac{\beta_1}{(1-\beta_1)}\right)l_2^j\right) = K^j \\
\Rightarrow \quad & \frac{\beta_2(1-\beta_1) - \beta_1(1-\beta_2)}{(1-\beta_2)(1-\beta_1)}\frac{l_2^j}{L^j} = \frac{K^j}{L^j}\frac{r}{w} - \frac{\beta_1}{(1-\beta_1)} \\
\Rightarrow \quad & \frac{(1-\beta_2)(1-\beta_1)}{\beta_2 - \beta_1}\left(\frac{K^j}{L^j}\frac{r}{w} - \frac{\beta_1}{(1-\beta_1)}\right) = \frac{l_2^j}{L^j}
\end{aligned}$$

Finally, substituting r/w gives:

$$\begin{aligned}
& \frac{l_2^j}{L^j} = \frac{(1-\beta_2)(1-\beta_1)}{\beta_2 - \beta_1}\left(\frac{K^j}{L^j}\frac{L^h + L^f}{K^h + K^f}\frac{\beta_2(1-\alpha) + \beta_1\alpha}{(1-\alpha)(1-\beta_2) + \alpha(1-\beta_1)} - \frac{\beta_1}{(1-\beta_1)}\right) \\
\Rightarrow & \frac{l_2^j}{L^j} = \frac{(1-\beta_2)(1-\beta_1)}{\beta_1 - \beta_2}\left(\frac{\beta_1}{(1-\beta_1)} - \frac{K^j}{L^j}\frac{L^h + L^f}{K^h + K^f}\frac{\beta_2(1-\alpha) + \beta_1\alpha}{(1-\alpha)(1-\beta_2) + \alpha(1-\beta_1)}\right)
\end{aligned}$$

Note that in order for the countries to be in a specialization cone, that is, produce both goods, it must be that $l_2^j/L^j > 0$. Also, suppose that $\beta_1 > \beta_2$, meaning that sector 2 is more intensive in labor. Then a country abundant in labor, uses relatively more labor in that sector - this could be predicted using the **Rybczynsky theorem**! Consumption is given using the foc:

$$\begin{aligned}
& p_2 c_2^j = (1-\alpha)\left(rK^j + wL^j\right) \\
\Rightarrow \quad & c_2^j = (1-\alpha)\left(\frac{r}{p_2}K^j + \frac{w}{p_2}L^j\right)
\end{aligned}$$

The allocation is determined by substitution the equilibrium prices. The same applies to all the remaining variables.