$v_1 \equiv \sum_{t=1}^{\infty} \beta^t u(c_t)$

$\hookrightarrow v(k_1)$

$v_0 \equiv \sum_{t=0}^{\infty} \beta^t u(c_t)$

$\hookrightarrow v(k_0)$

$0 \rightarrow \{c_t\}_{t=0}^{\infty} \quad \{k_{t+1}\}_{t=0}^{\infty}$ sequential

0 1 2 3 t

k_0 k_1 k_2 k_3 → state variable

$k_1(k_0)$

RECURSIVE REPRESENTATION AND DYNAMIC PROGRAMMING

Within the Neoclassical Growth Model, we can formulate the social planner problem in a recursive language:

- Given a set of *state variables* today, the social planner chooses decision rules which determine the state of the economy tomorrow

- These decision rules determine the *value* (or lifetime utility) of starting in a given state today

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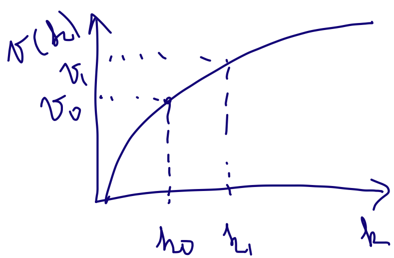
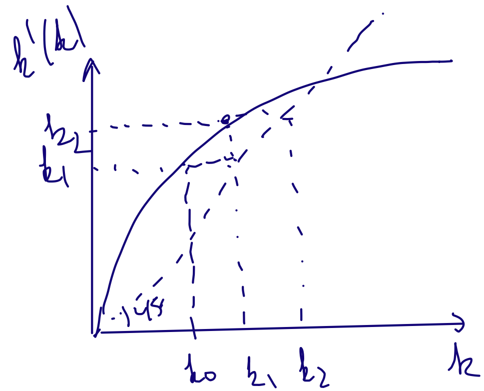
decision rule $k'(k)$ $\quad k' = g(k)$

$\hookrightarrow c(k)$



k_0 given



k_0 given

$k_1 = k'(k_0)$

$v_0 = v(k_0)$ $\quad$ $k_2 = k'(k_1) = k'(k'(k_0))$

$v_1 = v(k_1) = v(k'(k_0))$ ... $\{k_t\}_{t=0}^{\infty}$

In each period t, we can decompose lifetime utility

$$\underbrace{\sum_{s=0}^{\infty} \beta^s u(c_{t+s})}_{v_t} = u(c_t) + \beta \underbrace{\sum_{s=0}^{\infty} \beta^s u(c_{t+1+s})}_{v_{t+1}}$$

The only thing that differentiates the period t from $t+1$ is the stock of capital

If the problem is recursive, we can write

$$v(k_t) = u(c(k_t)) + \beta v\left(k'(k_t)\right)$$

where $c(k_t)$ and $k'(k_t)$ are optimal decision rules for the consumption and tomorrow's capital which depend solely on the state of the economy (today's capital k_t)

$$v_t \equiv \sum_{s=t}^{\infty} \beta^{t-s} u(c_s)$$

$$= \sum_{s=0}^{\infty} \beta^s u(c_{t+s})$$

$$v_0 = \sum_{s=0}^{\infty} \beta^s u(c_s)$$

$$v_t = u(c_t) + \sum_{s=1}^{\infty} \beta^s u(c_{t+s})$$

$$= u(c_t) + \beta \underbrace{\sum_{s=0}^{\infty} \beta^s u(c_{t+1+s})}_{v_{t+1}}$$

$$v_t = u(c_t) + \beta v_{t+1}$$

Recursive Social Planner's Problem

The social planner chooses functions $v(k)$, $c(k)$, $i(k)$, $k'(k)$ which solve the Bellman equation:

$$v(k) = \max_{c,i,k'} \left\{ u(c) + \beta v\left(k'\right) \right\}$$

$$\begin{aligned} s.t. \quad c + i &= f(k) \\ k' &= (1-\delta) k + i \end{aligned}$$

for all $k > 0$ $\qquad c, k' \geq 0$

This is a *functional* equation in v

(for simplicity, we are assuming no population growth nor technological change)

$v(k) = k^2 - 1$ $\times$

$v(k) = 3k + 2$ $\times$

$v(k) = \log(k)$ $\checkmark$

given k_0

$k_1 = k'(k_0)$ . . . . . $\{k_{t+1}\}_{t=0}^{\infty}$

$c = f(k) + (1-\delta)k - k'$

$$v(k) = \max_{k'} \left\{ u\left[f(k) + (1-\delta)k - k'\right] + \beta v(k') \right\}$$

Stokey-Lucas

The *Principle of Optimality* guarantees that the solution to this problem is equivalent to solving the sequential problem:

$$c_t = c(k_t) \qquad k_{t+1} = k'(k_t)$$

plus

$$v(k_0) = \sum_{t=0}^{\infty} \beta^t u(c_t)$$

Starting from a given k_0 , we can build recursively

$$k_1 = k'(k_0) \qquad k_2 = k'(k_1) = k'\left(k'(k_0)\right) \qquad \ldots$$

Dynamic Programming Overview

Define the following notation:

- state x: column vector with n components

  state vector

  $n = 1$, $x = k$

- state space X : subset of R^n

  $x \in X$

  $X = R_+ = [0, k_{max}]$

- control y: column vector with n components

  $y = k'$

- return function $F : X \times X \rightarrow R$
- correspondence $\Omega : X \rightarrow X$

We define the value function $v : X \rightarrow R$ as the solution to the Bellman equation:

$$v(x) = \max_y \quad \{F(x, y) + \beta v(y)\}$$

$$s.t. \quad y \in \Omega(x)$$

$\forall x \in X$

Also, we define the decision rule $g : X \rightarrow X$:

$k'(k)$

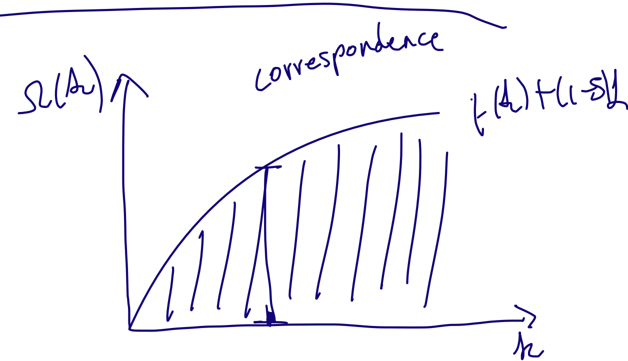
$$g(x) = \arg\max_y \quad \{F(x, y) + \beta v(y)\}$$

$$s.t. \quad y \in \Omega(x)$$

$\forall x \in X$, such that: $v(x) = F(x, g(x)) + \beta v(g(x))$

$$F(k, k') = u[\underbrace{f(k) + (1-\delta)k - k'}_{c}]$$

$$\Omega(k) = [0, f(k) + (1-\delta)k]$$



$k' \in [0, f(k)+(1-\delta)k]$

$k' \geq 0$

$c \geq 0 \Rightarrow k' \leq f(k)+(1-\delta)k$

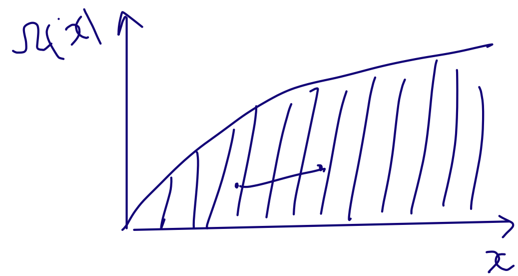
$x_1 \in X, \; x_2 \in X \quad \alpha x_1 + (1-\alpha) x_2 \in X$

Suppose that:

(i) X is a convex set — $X = [0, k_{max}]$

(ii) $\Omega(x)$ is compact and nonempty, $\forall x \in X$ — $\Omega(x) = [0, f(x)+(1-\delta)x]$

(iii) Ω is convex and continuous



(iv) F is bounded and continuous — $F(x,y) = u[f(x)+(1-\delta)x - y]$

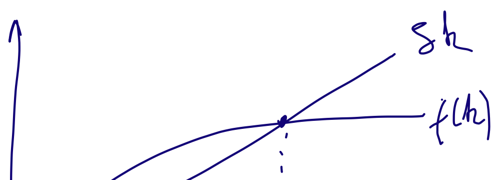
(v) $\beta < 1$

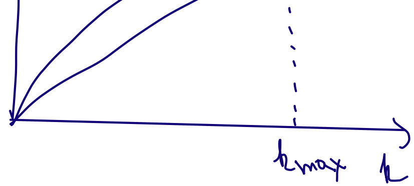
In many applications, including different versions of the Neoclassical Growth Model, restrictions on technology and preferences ensure that these conditions are met

k_{max}: maximum sustainable level of k

lose δk

gain $i = f(k) - \cancel{c}$





We now define the operator $T : B(X) \to B(X)$, where $B(X)$ is a set of bounded functions in X, as:

$$
\begin{cases}
T[f(x)] = \max_{y} \quad \{F(x,y) + \beta f(y)\} \\
\qquad\qquad s.t. \quad y \in \Omega(x)
\end{cases}
$$

$f(x) = 3x^2 - 2x \qquad \underbrace{x \in [0, 10]}_{X}$

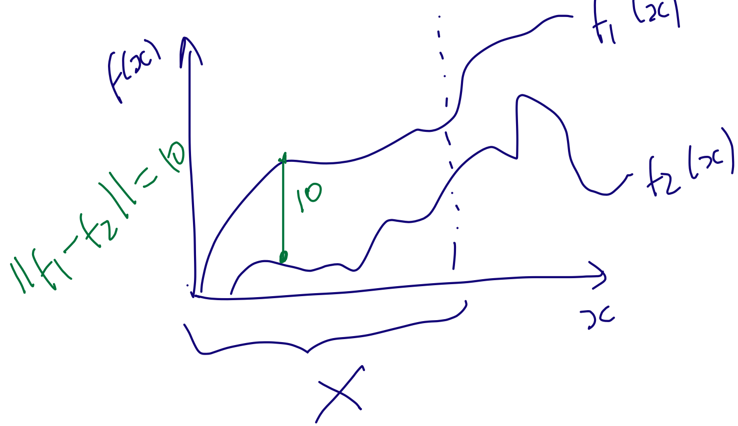
$Tf(x) = 2\bar{x} - 4$

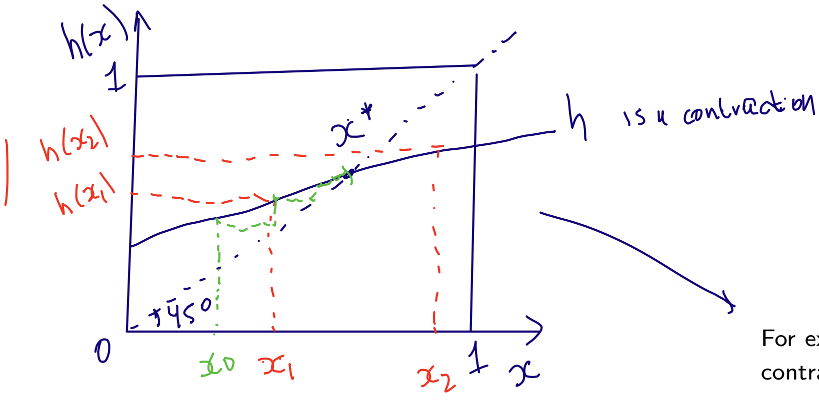
v, the solution to the Bellman equation is such that $Tv = v$

T is a functional operator over the metric space $B(X)$, with the norm

$$
\underbrace{\|f_1 - f_2\| = \sup_{x \in X} |f_1(x) - f_2(x)|}
$$

By construction, the value function v that solves the Bellman equation is a fixed point of the operator T $(v = Tv)$





$\Rightarrow$ guess x_0

$x_1 = h(x_0)$

$x_2 = h(x_1)$

$\dots \quad \{x_i\}_{i=0}^{\infty} \rightarrow x^*$

Result 1: (*Contraction mapping*)

If conditions (i)-(v) are met, then the operator T is a *contraction* with module β; in other words

$$\|Tf_1 - Tf_2\| \leq \beta \|f_1 - f_2\| \qquad , \forall f_1, f_2 \in B(X)$$

In simple terms, a contraction is a function (or operator) that shortens distances

For example, the real function $h : [0,1] \to [0,1]$ is a contraction if

$$|h(x) - h(y)| \leq \kappa |x - y|$$

that is, if the slope of the function is less than a constant $\kappa < 1$ called contraction module

As we can see clearly in this example, a contraction has a single fixed point $x^* = h(x^*)$, which can be reached iteratively from any $x^0 \in [0,1]$ and calculating $x^{n+1} = h(x^n)$

The *Banach Theorem* generalizes this result to complete metric spaces, such as the one we are analyzing

T is a contraction

$$v_0(x) = 0 \quad \forall x \in X$$

$$v_1 = T v_0(x) = 3x^2 - 2$$

$$v_2 = T v_1(x) = 4x - 2$$

$$\cdots \quad \{v_i\}_{i=0}^{\infty} \rightarrow v$$

Corollary *(Existence and uniqueness of the value function):*

> If conditions (i)-(v) are met, the value function v exists and is unique. Additionally, starting from *any* $v^0 \in B(X)$, the sequence v^n constructed as:
>
> $$\begin{array}{rcl} v^1 & = & Tv^0 \\ v^2 & = & Tv^1 = T^2 v^0 \end{array}$$
>
> ....
>
> converges to the function v, the only fixed point of the operator T

This corollary is the basis of the numerical iteration method of the value function

The previous corollary implies that if the operator T preserves a certain property (for example, continuity), the value function will also have this property. With that argument, we can demonstrate:

Corollary 2: (*Properties of the value function*)

> If conditions (i)-(v) are met, plus (vi) F is strictly concave, then the value function v is continuous, bounded and strictly concave

For instance, in the case of concavity, we only need to show that f weakly concave implies that Tf is strictly concave

$\wedge$

Let $x_1, x_2 \in X$, $0 < \alpha < 1$, and $\hat{x} = \alpha x_1 + (1-\alpha) x_2$; also let $y_1 = g(x_1)$ and $y_2 = g(x_2)$ attain the maximum for x_1, x_2, and define $\hat{y} = \alpha y_1 + (1-\alpha) y_2$

Then, by strict concavity of F and weak concavity of f,

$$F(\hat{x}, \hat{y}) + \beta f(\hat{y}) > \alpha [F(x_1, y_1) + \beta f(y_1)]$$

$$+ (1-\alpha) [F(x_2, y_2) + \beta f(y_2)]$$

$$= \alpha Tf(x_1) + (1-\alpha) Tf(x_2)$$

But, by definition,

$$Tf(\hat{x}) \geq F(\hat{x}, \hat{y}) + \beta f(\hat{y})$$

so the desired result follows

$$Tf(\hat{x}) > \alpha Tf(x_1) + (1-\alpha) Tf(x_2)$$

$k \to x_1$

$g(k) = k' \to y_1$

(vi) F is strictly concave

f is weakly concave

$$F(\hat{x}, \hat{y}) > \alpha F(x_1, y_1) + (1-\alpha) F(x_2, y_2)$$

$$f(\hat{y}) \geq \alpha f(\hat{y}_1) + (1-\alpha) f(\hat{y}_2)$$

$$\underbrace{F(\hat{x}, \hat{y}) + \beta f(\hat{y})} > \alpha [F(x_1, y_1) + \beta f(y_1)]$$

$$+ (1-\alpha) [F(x_2, y_2) + \beta f(y_2)]$$

$$T[f(x)] = \max_y [F(x, y) + \beta f(y)]$$

s.t $y \in \Omega(x)$

$$Tf(\hat{x}) \geq F(\hat{x}, \hat{y}) + \beta f(\hat{y})$$

$$Tf(\alpha x_1 + (1-\alpha) x_2) > \alpha [F(x_1, y_1) + \beta f(y_1)]$$

What about the properties of the optimal policy rule $g(x)$?

Result 2: (*Properties of the optimal policy rule*)

$y = g(x)$

> If conditions (i)-(vi) are met, the optimal decision rule g exists and is unique; plus, $g(x)$ is continuous in x

The existence of $g : X \to X$ follows form *Weierstrass Theorem*, once it has been proven that the value function v is continuous and bounded

... the uniqueness of g comes from the strict concavity of v

... the continuity of g is an application of the *Maximum Theorem*

$+ (1-\alpha)\left[F(x_2, y_2) + \beta f(y_2)\right]$

$Tf(\alpha x_1 + (1-\alpha) x_2) > \alpha\, Tf(x_1) + (1-\alpha)\, Tf(x_2)$

$\Rightarrow$ Tf is strictly concave

Finally, we want to know in which cases the value function is differentiable, which would allow us working with the first order conditions

Result 3: (*Differentiability of the value function*)

> If conditions (i)-(v) are met and (vii) F is continuously differentiable, then, for each $x^0 \in int(X)$ with $g(x^0) \in int(\Omega(x^0))$, the value function v is continuously differentiable in x^0 and its derivatives can be found according to:
>
> $$\frac{\partial v}{\partial x_i}\left(x^0\right) = \frac{\partial F}{\partial x_i}\left(x^0, \underbrace{g\left(x^0\right)}_{y^0}\right)$$

envelope theorem

Benveniste and Scheinkman propose this set of conditions

Neoclassical growth model

First Order Conditions

Coming back to the recursive social planner's problem, we have:

$$v(k) = \max_{k'} \left\{ \underbrace{u\left[f(k) + (1-\delta)k - k'\right] + \beta v\left(k'\right)}\right\}$$

$$s.t. \quad k' \in [0, f(k) + (1-\delta)k]$$

a particular case of the problem described earlier with

$c \geq 0$

$$x = k \qquad y = k' \qquad X = [0, k_{\max}]$$

$$F(x,y) = u[f(x) + (1-\delta)x - y] \qquad \Omega(x) = [0, f(x) + (1-\delta)x]$$

where $k_{\max}$ is the maximum sustainable level of capital that satisfies $f(k_{\max}) = \delta k_{\max}$

$$c = f(k) - i = f(k) + (1-\delta)k - k'$$

F.O.C.

$$\frac{\partial}{\partial k'} = -u'\left(f(k) + (1-\delta)k - k'\right) + \beta \underbrace{v'(k')} = 0$$

$$v'(k) = u'\left[f(k) + (1-\delta)k - k'(k)\right] \times \left[f'(k) + (1-\delta)\right]$$

$$v'(k') = u'\left[f(k') + (1-\delta)k' - k'(k'(k))\right] \times \left[f'(k') + (1-\delta)\right]$$

$$u'\left[f(k) + (1-\delta)k - k'\right] = \beta\left\{u'\left[f(k') + (1-\delta)k' - k''\right] \times \left(f'(k') + (1-\delta)\right)\right\}$$

(k_{t+1}) $\quad k' = k'(k)$

(k_{t+2}) $\quad k'' = k'(k') = k'(k'(k))$

Solving this problem, the first order conditions for an internal solution are:

$$\frac{\partial}{\partial k'}\left(k, k'\right) = -u'\left[f\left(k\right) + \left(1-\delta\right)k - k'\right] + \beta v'\left(k'\right) = 0$$

but, using Benveniste-Scheinkman:

$$v'\left(k'\right) = u'\left[f\left(k'\right) + \left(1-\delta\right)k' - k''\right]\left(f'\left(k'\right) + \left(1-\delta\right)\right)$$

Replacing and simplifying, we get Euler's equation:

$$\frac{u'\left[f\left(k\right) + \left(1-\delta\right)k - k'\right]}{\beta u'\left[f\left(k'\right) + \left(1-\delta\right)k' - k''\right]} = f'\left(k'\right) + \left(1-\delta\right)$$

Notice the similarity with the Euler equation obtained from the sequential problem

Stationary Equilibrium (steady state)

In recursive language, a steady equilibrium is a value for k^* such that:

$$k^* = k'(k^*) \quad \longrightarrow \quad k_{t+1} = k_t = k^*$$

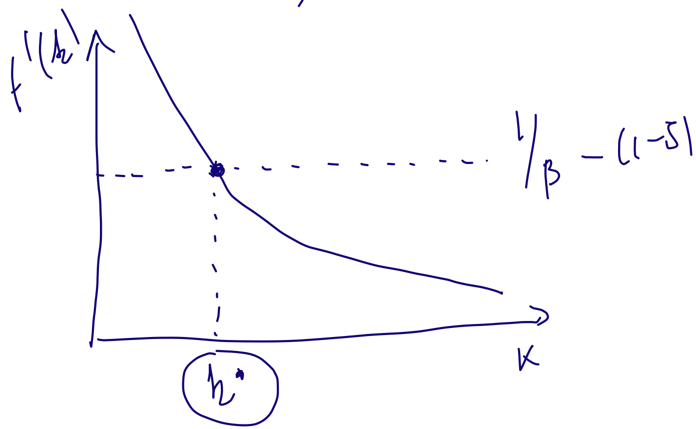
where k' is the optimal decision rule of the social planner

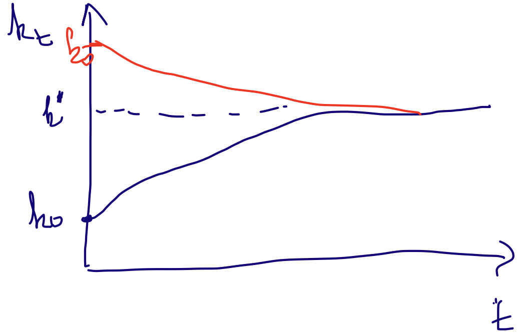
$$\frac{u'\cancel{[f(k^*) + (1-\delta)k^* - k^*]}}{\beta\, u'\cancel{[f(k^*) + (1-\delta)k^* - k^*]}} = f'(k^*) + (1-\delta)$$

Using the Euler equation:

$$k^* = (f')^{-1}\left(\frac{1}{\beta} - (1-\delta)\right)$$

The Inada conditions over f guarantee the existence of a unique steady state k^*





Moreover, we can prove that the stationary equilibrium is stable: starting from any $k_0 > 0$, the economy converges in the long run to the only steady state k^*

The steps of the proof are the following:

1. Show that the value function of the social planner $v(k)$ is concave (using the operator T) ✓

2. Using the concavity of $v(k)$ and the first order conditions, show that the decision rule $k'(k)$ is increasing ✓

$k_1 > k_2 \quad k'(k_1) > k'(k_2)$

$k \uparrow \Rightarrow k'(k) \uparrow$

F.O.C.

$$\beta v'(k') = u'[f(k) + (1-\delta)k - k']$$

3. The monotonicity of $k'(k)$ implies that the optimal sequence $\{k_0, k_1, ...\}$ is also monotone

4. Show that the optimal sequence $\{k_0, k_1, ...\}$ is bounded $\quad k' \in \Omega(k)$

5. Conclude, using the *Monotone Convergence Theorem*, that $\{k_0, k_1, ...\}$ converges to k^*

k_0 given

- $k_1 = k'(k_0) > k_0$, $k_2 = k'(k_1)$ $\Big\}$ $k_2 > k_1$
- $k_1 = k'(k_0) < k_0$, $k_2 = k'(k_1)$ $\Big\}$ $k_2 < k_1$

by contradiction

$k \uparrow \Rightarrow k' \downarrow$ ✗

$k \uparrow \Rightarrow k' \uparrow$

Recursive Competitive Equilibrium

To move from the problem of the social planner to the recursive equilibrium, we have to distinguish the *individual* state variable k from the *aggregate* state variable K

w(K)
r(K)

• The prices depend of the aggregate capital, not of the individual (*perfect competition*)

• Consumers choose the law of motion for individual capital $k'(k, K)$ → optimal decision rule

... taking as given the law of the motion for aggregate capital $K' = \Gamma(K)$

In equilibrium, both laws of motion must be consistent

A Recursive Competitive Equilibrium is a set of functions $v(k,K)$, $c(k,K)$, $i(k,K)$, $k'(k,K)$, prices $w(K)$ and $r(K)$ and aggregate law of motion $\Gamma(K)$ such that:

consumers' problem

i) For each pair (k,K), given the functions w, r and Γ, the value function $v(k,K)$ solves the Bellman equation:

$$v(k,K) = \max_{c,i,k'} \left\{ u(c) + \beta v\left(k', K'\right) \right\}$$

$$\begin{aligned} s.t. \quad c+i &= w(K) + r(K)\,k \\ k' &= (1-\delta)\,k + i \\ K' &= \Gamma(K) \end{aligned}$$

and $c(k,K)$, $i(k,K)$, $k'(k,K)$ are the optimal decision rules for this problem

ii) For each K, prices satisfy the marginal conditions (from the firm maximization problem):

$$r(K) = f'(K)$$

$$w(K) = f(K) - f'(K)\,K$$

iii) For each K, markets clear:

$k = K$

$$\underline{f(K)} = \underline{c(K,K) + i(K,K)}$$

iv) For each K, the aggregate law of motion is consistent with individual decisions:

$$K' = \underline{\Gamma(K)} = k'(K,K) = k'$$

k_0 given (K_0) $k'(K_0, k_0)$

$K_1 = \Gamma(K_0)$

$K_2 = \Gamma(K_1)$

$\vdots$

$k_1 = k'(k_0, k_0)$

$k_2 = k'(k_1, k_1)$

$\vdots$

Once the recursive equilibrium is solved, starting from a given $k_0 > 0$, we can construct the sequences for the stock of capital:

$$
\begin{aligned}
k_1 &= k'(k_0, k_0) \\
k_2 &= k'(k_1, k_1) = k'\left(k'(k_0, k_0), k'(k_0, k_0)\right) \\
&\quad \text{............................}
\end{aligned}
$$

and for the other variables:

$$
\begin{aligned}
c_t &= c(k_t, k_t) \qquad i_t = i(k_t, k_t) \\
w_t &= w(k_t) \qquad\quad r_t = r(k_t)
\end{aligned}
$$

The *Principle of Optimality* guarantees that those sequences are the same as one would get by solving the sequential competitive equilibrium