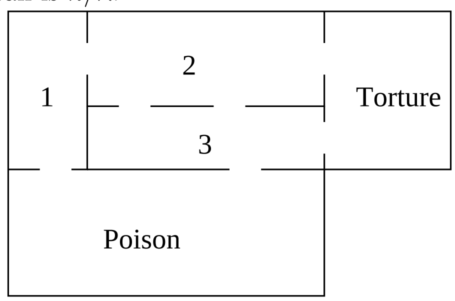
AMS 311 Joe Mitchell

Computing Expectations and Probabilities by Conditioning

Rat-in-Maze. Consider the maze shown below. There are three cells (Cell 1, Cell 2, and Cell 3) and two deadly (quite permanent) outcomes (Death By Poison, and the dreaded Death By Torture). A rat is initially placed in Cell 1. When the rat enters Cell i, he wanders around within the cell for X_i minutes, where X_i is Gamma(4,3i), and then exits the cell by picking one of the doors at random (e.g., if there are 3 doors, he picks each with probability 1/3). (For example, the time (in minutes) he wanders around in Cell 2 before leaving is a random variable with a Gamma(4,6) distribution.) Note: the density of a Gamma(n, λ) random variable is given by $f(x) = \frac{\lambda e^{-\lambda x}(\lambda x)^{n-1}}{(n-1)!}$, if $x \geq 0$, and $f(x) = 0$ otherwise; the mean is n/λ.



(a). *Find the probability that the rat dies by torture. (Recall that he starts in Cell 1.)*
Let $p_i = P$(the rat dies by torture | the rat starts in Cell i). Then, conditioning on the first door chosen, we get:

$$\begin{aligned} p_1 &= (1/2)0 + (1/2)p_2 \\ p_2 &= (1/4)p_1 + (1/4)1 + (2/4)p_3 \\ p_3 &= (1/4)1 + (1/4)0 + (2/4)p_2 \end{aligned}$$

Solve, getting $p_1 = 0.3, p_2 = 0.6, p_3 = 0.55$; the desired quantity is $p_1 = 0.3$, since we were told that the rat starts in fact in Cell 1.

(b). *Find the probability that the rat visits Cell 3 before he dies. (Recall that he starts in Cell 1.)*
Let $q_i = P($ the rat visits Cell 3 before he dies | the rat starts in Cell i). Then, conditioning on the first door chosen, we get:

$$\begin{aligned} q_1 &= (1/2)0 + (1/2)q_2 \\ q_2 &= (1/4)q_1 + (1/4)0 + (2/4)q_3 \\ q_3 &= 1 \end{aligned}$$

Solve, getting $q_1 = (2/7), q_2 = (4/7), q_3 = 1$; the desired quantity is $q_1 = (2/7)$, since we were told that the rat starts in fact in Cell 1.

(c). *What is the expected number of minutes that the rat lives?*
Let Y be the number of minutes the rat lives in total. Let $t_i = E(Y \mid$ the rat starts in Cell i). We want to compute t_1. Note that the rat wanders in Cell i for an expected amount of time $(4/(3i))$ minutes, since the expected value of a Gamma(n, λ) is (n/λ). Then, conditioning on the first door chosen, we get:

$$\begin{aligned} t_1 &= (4/(3 \cdot 1)) + (1/2)(0) + (1/2)(t_2) \\ t_2 &= (4/(3 \cdot 2)) + (1/4)(t_1) + (1/4)(0) + (2/4)t_3 \\ t_3 &= (4/(3 \cdot 3)) + (1/4)(0) + (1/4)(0) + (2/4)t_2 \end{aligned}$$

Solve these 3 equations in 3 unknowns. The desired quantity is $t_1 = 104/45$.

Gambler's Ruin. A gambler has \$3 in his pocket when he walks into a casino. He plays a game in which the probability he gains \$1 is 1/3, the probability that he gains \$2 is 1/6, and the probability that he loses \$1 is 1/2. He decides ahead of time that he will stop playing as soon as he has enough money to buy a \$5 toy for his daughter. Of course, he also must stop playing if he goes broke!

For each of the following quantities, show exactly how you would compute it. *Define precisely any quantities you use!*. You need not solve systems of equations, but you must be very explicit about exactly how you would obtain the final numerical answer.

(a). *The probability that he goes broke.*

Let $p_i = P(\text{he goes broke} \mid \text{he starts with } i \text{ dollars})$. Condition on the result of the first play:

$$\begin{aligned}
p_1 &= \frac{1}{3}p_2 + \frac{1}{6}p_3 + \frac{1}{2}\cdot 1 \\
p_2 &= \frac{1}{3}p_3 + \frac{1}{6}p_4 + \frac{1}{2}p_1 \\
p_3 &= \frac{1}{3}p_4 + \frac{1}{6}\cdot 0 + \frac{1}{2}p_2 \\
p_4 &= \frac{1}{3}\cdot 0 + \frac{1}{6}\cdot 0 + \frac{1}{2}p_3
\end{aligned}$$

Now, solve the system of four equations in four unknowns; the final answer is p_3.

(b). *The expected number of games he plays before he leaves the casino.*

Let $t_i = E(\text{number of games he plays before leaving} \mid \text{he starts with } i \text{ dollars})$. Condition on the result of the first play:

$$\begin{aligned}
t_1 &= \frac{1}{3}(1+t_2) + \frac{1}{6}(1+t_3) + \frac{1}{2}(1+0) \\
t_2 &= \frac{1}{3}(1+t_3) + \frac{1}{6}(1+t_4) + \frac{1}{2}(1+t_1) \\
t_3 &= \frac{1}{3}(1+t_4) + \frac{1}{6}(1+0) + \frac{1}{2}(1+t_2) \\
t_4 &= \frac{1}{3}(1+0) + \frac{1}{6}(1+0) + \frac{1}{2}(1+t_3)
\end{aligned}$$

Now, solve the system of four equations in four unknowns; the final answer is t_3.

Staged Experiments. Suppose that X has density

$$f_X(x) = \begin{cases} xe^{-x} & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

After X is chosen at random from this distribution (which is called a "Gamma(2,1)" distribution), we select Y uniformly at random from the interval $(0, X)$. What is $P(Y \leq 2)$? What is $E(Y)$?

We are given $f_X(x)$. We are also told the conditional desnity, $f_{Y|X}(y|x)$, since *given* $X = x$, Y is uniform on the interval $(0, x)$. This means that for any $x \geq 0$ (we need this, since otherwise $f_X(x) = 0$, and $f_{Y|X}(y|x)$ is undefined),

$$f_{Y|X}(y|x) = \begin{cases} \frac{1}{x-0} & \text{if } 0 < y < x \\ 0 & \text{otherwise} \end{cases}$$

Method 1. Now, by definition, $f_{Y|X}(Y|X) = \frac{f(x,y)}{f_X(x)}$, so the joint density is

$$f(x,y) = f_X(x)\cdot f_{Y|X}(y|x) = \begin{cases} e^{-x} & \text{if } x \geq 0 \text{ and } 0 < y < x \\ 0 & \text{otherwise} \end{cases}$$

Now that we know $f(x,y)$, we can compute anything at all that we could ever want to compute.

In particular,

$$P(Y \le 2) = \int_0^2 \int_y^\infty e^{-x} dx dy = 1 - e^{-2}$$

and

$$E(Y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} y f(x,y) dy dx = \int_0^\infty \int_0^x y \cdot e^{-x} dy dx = 1$$

Method 2. We can use the Law of Total Probability:

$$P(E) = \int_{-\infty}^{\infty} P(E \,|X = x) f_X(x) dx$$

to compute the probability of the event $E = \{Y \le 2\}$ by conditioning on the value of X.

In particular, if we know that $X = x$, for any $x \ge 0$, then we can compute

$$P(Y \le 2 \mid X = x) = \begin{cases} 1 & \text{if } 0 < x < 2 \\ \frac{2}{x} & \text{if } x \ge 2 \end{cases}$$

Thus,

$$P(Y \le 2) = \int_{-\infty}^{\infty} P(Y \le 2 \mid X = x) f_X(x) dx = \int_0^2 1 \cdot x e^{-x} dx + \int_2^\infty \frac{2}{x} x e^{-x} dx = 1 - e^{-2}$$

We can use the Law of Total Expectation:

$$E(Y) = \int_{-\infty}^{\infty} E(Y \mid X = x) f_X(x) dx$$

to compute

$$E(Y) = \int_0^\infty \frac{x}{2} \cdot x e^{-x} dx = 1,$$

where we have used the fact that $E(Y \mid X = x) = x/2$, since the mean of a Uniform$(0, x)$ is $x/2$. (*Given* $X = x$, Y is Uniform$(0, x)$.)

Another Staged Experiment Example. Suppose that X is exponential(5), and then Y is drawn uniformly between -3 and X.

(a). Compute $E(Y)$.

We condition on the value of X:

$$E(Y) = \int_{-\infty}^{\infty} E(Y|X = x) f_X(x) dx = \int_0^\infty \frac{x-3}{2} \cdot 5e^{-5x} dx,$$

where we have used the fact that, given $X = x$, Y is Uniform(-3,x), so $E(Y|X = x) = \frac{x+(-3)}{2}$.

(b). Compute $P(Y > 2)$.

We condition on the value of X:

$$P(Y > 2) = \int_{-\infty}^{\infty} P(Y > 2|X = x) f_X(x) dx = \int_0^\infty P(Y > 2|X = x) \cdot 5e^{-5x} dx = \int_2^\infty \frac{x-2}{x+3} \cdot 5e^{-5x} dx,$$

where we have used the fact that, given $X = x$, Y is Uniform(-3,x), so

$$P(Y > 2|X = x) = \begin{cases} \frac{x-2)}{x-(-3)} & \text{if } x > 2 \\ 0 & \text{if } x < 2 \end{cases}$$