Estimación puntual
↳ *Máxima verosimilitud*

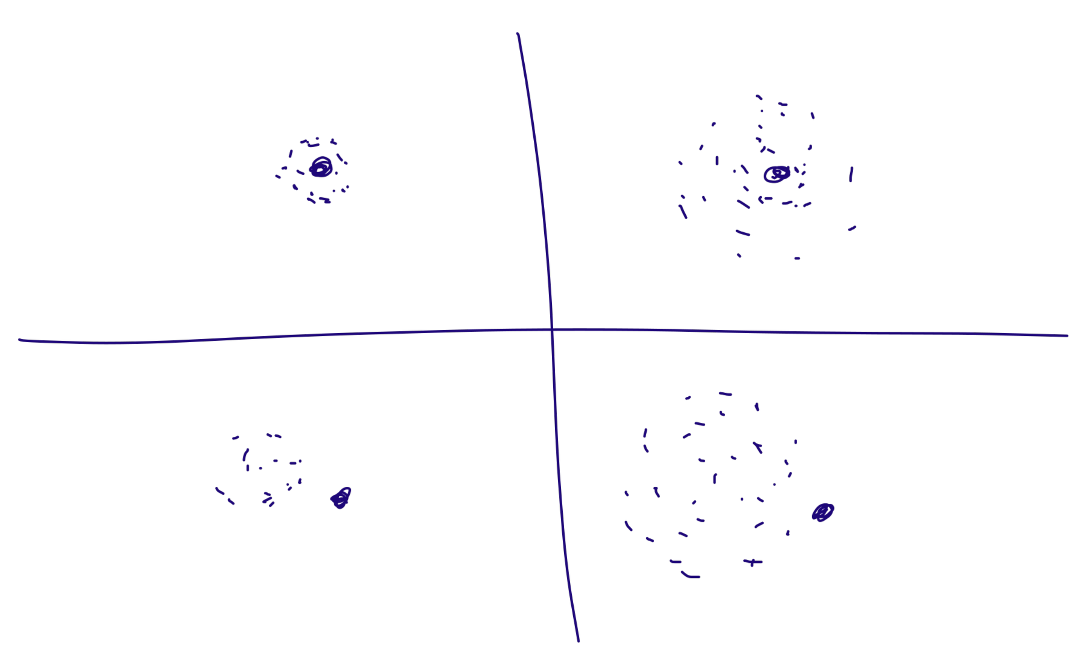
$$\mathcal{L}(\theta;\underline{x}) = \prod_{i=1}^{n} f_{X_i}(x_i;\theta) = f_{\underline{X}}(\underline{x};\theta)$$

$$\hat{\theta}_{MV} = \max_{\theta}\{\mathcal{L}(\theta;\underline{x})\}$$

$$\ell(\theta;\underline{x}) = \log(\mathcal{L}(\theta;\underline{x}))$$

log: monótona creciente
$\log(ab) = \log(a) + \log(b)$

- Idealmente querríamos que
  1) $E[\hat{\theta}] = \theta$
  2) $V(\hat{\theta})$ sea pequeña
     - decrece cuando n crece
     - Si tenemos $\hat{\theta}$ y $\tilde{\theta}$ dos estimadores, preferimos el que tenga menor Var.
  3) $\hat{\theta} \xrightarrow[n\to\infty]{} \theta$

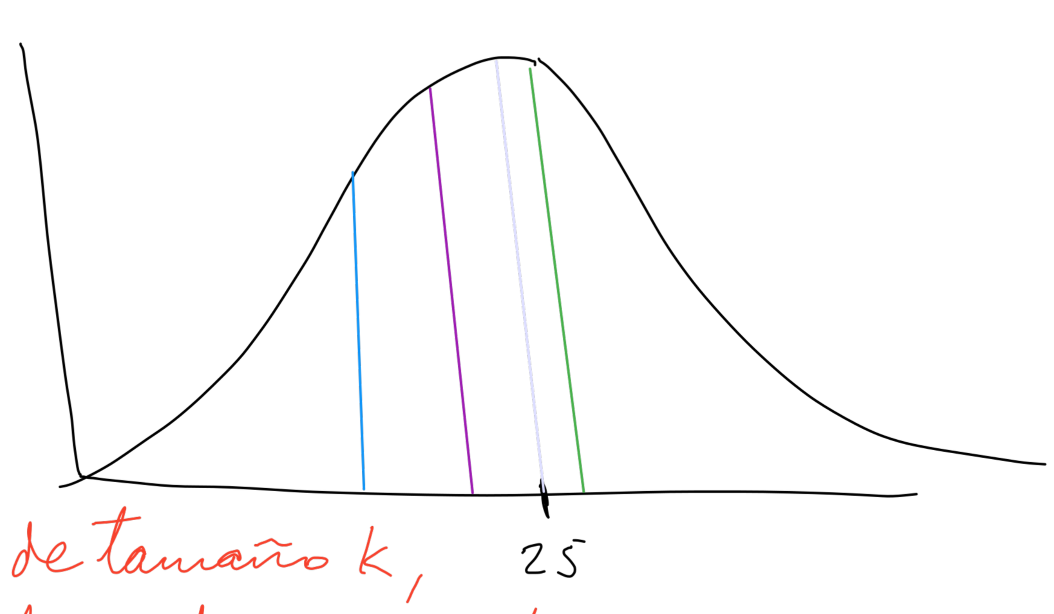


Ejemplo (en R)

- Sea una población distribuida $N(25, 100)$
- Suponer que NO conocemos los parámetros poblacionales
- Podemos obtener k muestras independientes, cada una de tamaño n.

$X_{1,1}$	$X_{2,1}$	...	$X_{k,1}$
$X_{1,2}$	$X_{2,2}$		$X_{k,2}$
$\vdots$	$\vdots$		$\vdots$
$X_{1,n}$	$X_{2,n}$		$X_{k,n}$
$\bar{X}_1$	$\bar{X}_2$		$\bar{X}_k$

Vector de tamaño k, cada elemento representa la media muestral de una muestra diferente.

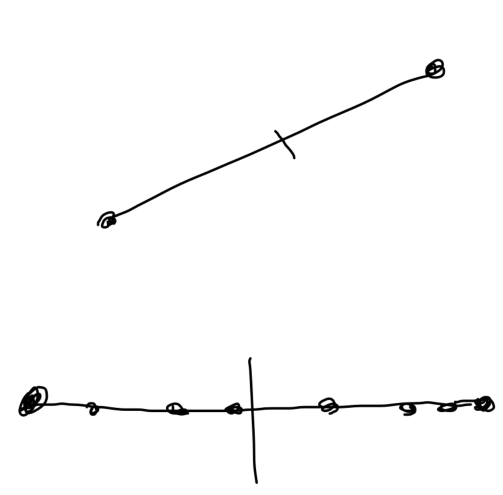


$\bar{X}_1 = \frac{1}{n}\sum_{i=1}^{n} X_{1,i}$

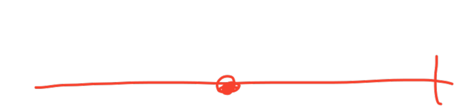
$\bar{X}_2 = \frac{1}{n}\sum X_{2,i}$

$\bar{X}_k = \frac{1}{n}\sum X_{k,i}$

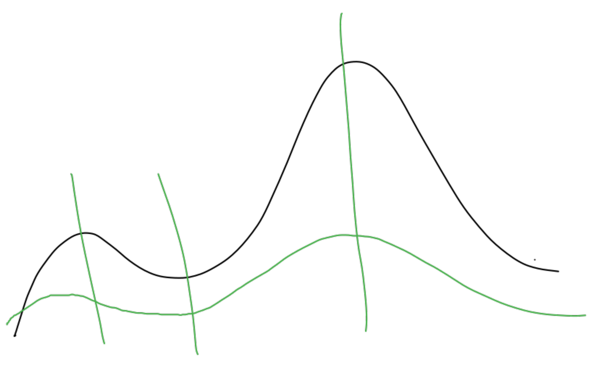
$\hat{\mu}_1 = \bar{X}$, $V(\bar{X}) = \frac{\sigma^2}{n}$
- $E[\hat{\mu}_1] = \mu$ ✓
- $V(\hat{\mu}_1)$ decrece si n crece ✓
- $\hat{\mu}_1 \xrightarrow[n\to\infty]{} \mu$ ✓



$\hat{\mu}_2 = \frac{X_{(1)} + X_{(n)}}{2}$, $V(\bar{X}) = \frac{\sigma^2}{2}$
- $E[\hat{\mu}_2] = \mu$ ✓
- $V(\hat{\mu}_2)$ no decrece ?
- $\hat{\mu}_2 \rightarrow \mu$?



$\hat{\mu}_3 = \sqrt[n]{X_1 \cdot X_2 \cdots X_n}$
- $E[\hat{\mu}_3] = \mu$ ✗
- $V(\hat{\mu}_3)$ decrece si n crece ✓
- $\hat{\mu}_3 \xrightarrow[n\to\infty]{} \mu$ ✗



Ejemplo 1 $X_1, \dots, X_n \overset{iid}{\sim} \text{Poisson}(\lambda)$ $\rightarrow E[X_i] = \lambda$

Calcular el EMV de λ. $\bar{X} = \hat{\lambda}_{MV}$

$$f(x;\lambda) = \frac{\lambda^x e^{-\lambda}}{x!} I_{\mathbb{Z}_0^+}(x)$$

$$\mathcal{L}(\lambda;\underline{x}) = \prod_{i=1}^n f_{X_i}(x_i;\lambda) = \prod_{i=1}^n \frac{\lambda^{x_i} e^{-\lambda}}{x_i!} I_{\mathbb{Z}_0^+}(x_i)$$

$$= \lambda^{\sum x_i} e^{-n\lambda} \prod_{i=1}^n \frac{1}{x_i!} \boxed{I_{\mathbb{Z}_0^+}(x_i)}$$

$\log(I(x_1)\cdot I(x_2)\cdots I(x_n))$
$= \log(1\cdot 1\cdots 1)$
$= \log(1) = 0$

$$\Rightarrow \ell(\lambda;\underline{x}) = \left(\sum x_i\right)\log(\lambda) - n\lambda + \log\left(\prod_{i=1}^n \frac{1}{x_i}\right) + 0$$

$$I_D(x) = \begin{cases} 1 & \text{si } x \in D \\ 0 & \text{si } x \in D \end{cases}$$

$$\Rightarrow \frac{\partial \ell}{\partial \lambda} = \frac{\sum x_i}{\lambda} - n + 0 \equiv 0$$

$$\iff \frac{\sum x_i}{\lambda} = n \iff \lambda = \frac{1}{n}\sum x_i$$

$$\Rightarrow \frac{\partial^2 \ell}{\partial \lambda^2} = -\frac{\sum x_i}{\lambda^2} < 0 \quad \therefore \text{ es un máximo}$$

$\therefore \hat{\lambda} = \frac{1}{n}\sum X_i = \bar{X}$ es el EMV de λ

Ejemplo 2.

$X_1, \dots, X_n \overset{iid}{\sim} Exp(\lambda)$. ¿Cuál es el EMV de λ?

$$f(x;\lambda) = \lambda e^{-\lambda x} I_{\mathbb{R}_0^+}(x) \qquad \lambda > 0$$

$$E[X_i] = \frac{1}{\lambda} \Rightarrow \lambda = \frac{1}{E[X]}$$

$$\hat{\lambda} = \frac{1}{\bar{X}}$$

$$\Rightarrow \mathcal{L}(\lambda;\underline{x}) = \prod_{i=1}^n \lambda e^{-\lambda x_i} I_{\mathbb{R}_0^+}(x_i) = \lambda^n e^{-\lambda \sum x_i} I_{\mathbb{R}^+}(\lambda)$$

$$e^{-\lambda x_1 - \lambda x_2 - \cdots - \lambda x_n}$$
$$e^{-\lambda(x_1 + x_2 + \cdots + x_n)}$$
$$e^{-\lambda \sum x_i}$$

$$\Rightarrow \ell(\lambda;\underline{x}) = n\log(\lambda) - \lambda \sum x_i$$

$$\Rightarrow \frac{\partial \ell}{\partial \lambda} = \frac{n}{\lambda} - \sum x_i \equiv 0$$

$$\iff \frac{n}{\lambda} = \sum x_i$$

$$\iff \frac{n}{\sum x_i} = \lambda$$

$$\iff \frac{1}{\bar{x}} = \lambda \qquad \therefore \hat{\lambda}_{MV} = \frac{1}{\bar{X}}$$

¿Qué ocurre si queremos estimar más de un parámetro a la vez?

Ejemplo $X_1, \dots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$

Obtener $\hat{\mu}_{MV}$ y $\hat{\sigma}^2_{MV}$

$$f(x;\mu,\sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2} I_{\mathbb{R}}(x) \qquad \mu \in \mathbb{R}, \ \sigma^2 \in \mathbb{R}^+$$

$$= (2\pi)^{-\frac{1}{2}} (\sigma^2)^{-\frac{1}{2}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2} I_{\mathbb{R}}(x)$$

$$\mathcal{L}(\mu,\sigma^2;\underline{x}) = \prod_{i=1}^n (2\pi)^{-\frac{1}{2}} (\sigma^2)^{-\frac{1}{2}} e^{-\frac{1}{2\sigma^2}(x_i-\mu)^2} I_{\mathbb{R}}(x_i)$$

$$= (2\pi)^{-\frac{n}{2}} (\sigma^2)^{-\frac{n}{2}} e^{-\frac{1}{2\sigma^2}\sum(x_i-\mu)^2} I_{\mathbb{R}}(\mu) I_{\mathbb{R}^+}(\sigma^2)$$

$$\ell(\mu,\sigma^2;\underline{x}) = -\frac{n}{2}\log(2\pi) - \frac{n}{2}\log(\sigma^2) - \frac{1}{2\sigma^2}\sum(x_i-\mu)^2$$

Caso para μ:

$$\frac{\partial \ell}{\partial \mu} = -\frac{1}{2\sigma^2}\cdot 2\left[\sum(x_i-\mu)\right](-1) = \frac{1}{\sigma^2}\sum(x_i-\mu) \equiv 0$$

$$\iff \sum(x_i-\mu) = 0$$

$$\iff \sum x_i - \sum \mu = 0$$

$$\iff \sum x_i - n\mu = 0$$

$$\iff \sum x_i = n\mu$$

$$\iff \frac{1}{n}\sum x_i = \mu \qquad \therefore \hat{\mu}_{MV} = \bar{X}$$

Caso para σ^2

$$\frac{\partial \ell}{\partial(\sigma^2)} = -\frac{n}{2}\cdot\frac{1}{\sigma^2} - \left[\sum(x_i-\mu)^2\right]\left(-\frac{1}{2(\sigma^2)^2}\right)$$

$$= -\frac{n}{2\sigma^2} + \frac{\sum(x_i-\mu)^2}{2(\sigma^2)^2} \equiv 0$$

$$\iff \frac{\sum(x_i-\mu)^2}{2(\sigma^2)^2} = \frac{n}{2\sigma^2}$$

$$\iff \frac{\sum(x_i-\mu)^2}{\sigma^2} = n$$

$$\iff \sigma^2 = \frac{1}{n}\sum(x_i-\mu)^2$$

$$\therefore \hat{\sigma}^2_{MV} = \frac{1}{n}\sum(x_i-\mu)^2$$

1) Notar que $\hat{\sigma}^2_{MV} = h(\mu)$

2) Si μ es conocida ✓

$\therefore \hat{\sigma}^2_{MV} = \frac{1}{n}\sum(x_i-\mu)^2$

Si μ no es conocida,

$\hat{\sigma}^2_{MV} = \frac{1}{n}\sum(x_i-\hat{\mu})^2$

$= \frac{1}{n}\sum(x_i-\bar{x})^2$

Observación:

$\hat{\sigma}^2_{MV} = \frac{1}{n}\sum(x_i-\bar{x})^2$

$s^2 = \frac{1}{n-1}\sum(x_i-\bar{x})^2$

3) "Propiedad de invarianza de los estimadores": Si $\hat{\theta}$ es el EMV de θ y $g(\cdot)$ es una función cualquiera, entonces el EMV de $g(\theta)$ es $g(\hat{\theta}_{MV})$

Tip: Para casos con dos (o más) parámetros a estimar, siempre uno de los dos estará en función del otro.