OVERLAPPING GENERATIONS MODELS AND LIFE CYCLE

An important dimension in which agents differ is age

• In each moment of time, young and old individuals coexist

• These agents have different income levels and incentives to save

• *Life cycle theory*: given a profile of income throughout their life, individuals smooth consumption by saving and spending at different ages

Basic Model

Individuals live for two periods: young and old (retired)

Exogenous profile of labor productivity for each agent i,

$$\lambda_t^{i,1} = 1 \qquad \lambda_{t+1}^{i,2} = 0$$

There is no uncertainty in the model

We assume that in each period a continuum of identical young people is born in $[0,1]$

There is no population growth

All agents are *ex-ante* equal and start with zero assets

Young agent problem

$$\max_{\left\{c_t^1, c_{t+1}^2, a_{t+1}^2\right\}} u\left(c_t^1\right) + \beta u\left(c_{t+1}^2\right)$$

$$\begin{aligned} s.t. \qquad c_t^1 + a_{t+1}^2 &= w_t \\ c_{t+1}^2 &= R_{t+1} a_{t+1}^2 \end{aligned}$$

There are no credit restrictions; nor are Ponzi schemes possible in finite horizon

In this model, young workers save to finance their retirement

We impose the transversality condition $a_{t+2}^3 = 0$; note that the old can transform non-depreciated capital into consumer goods

The representative firm combines capital and labor to produce the unique good, according to

$$Y_t = F\left(K_t, L_t\right) = F\left(K_t, 1\right) = f\left(K_t\right)$$

where F has constant returns to scale and the other usual properties

The stock of capital is equal to the assets of the retirees

$$K_t = \int_0^1 a_t^2 di = a_t^2$$

or else

$$K_{t+1} = \int_0^1 a_{t+1}^2 di = a_{t+1}^2$$

tomorrow's capital is equal to the savings of young people

`Definition of Equilibrium`

A Sequential Competitive Equilibrium for this economy is a set of sequences for individual quantities c_t^1, c_t^2, a_t^2, aggregate quantitiesY_t, K_t and prices w_t, R_t such that

i) In each period t, given w_t and R_t, the values c_t^1, c_{t+1}^2, a_{t+1}^2 solves the problem of the young agent:

$$\max \qquad u\left(c_t^1\right) + \beta u\left(c_{t+1}^2\right)$$

$$\begin{aligned} s.t. \qquad c_t^1 + a_{t+1}^2 &= w_t \\ c_{t+1}^2 &= R_{t+1} a_{t+1}^2 \end{aligned}$$

ii) In the initial period, given a_0^2 and R_0, the value c_0^2 satisfies the condition for the retired agent

$$c_0^2 = R_0 a_0^2$$

iii) In each period t, given w_t and R_t, the values Y_t and K_t solve the firm's problem:

$$\max \qquad Y_t - w_t - \left[R_t - (1-\delta)\right] K_t$$

$$s.t. \qquad Y_t = f\left(K_t\right)$$

and the benefits are zero

iv) In each period t, markets clear:

$$Y_t = c_t^1 + c_t^2 + K_{t+1} - (1-\delta) K_t$$

$$K_t = a_t^2$$

`First Order Conditions`

Solving the problem of the young worker, we obtain Euler's equation

$$u'\left(c_t^1\right) = \beta R_{t+1} u'\left(c_{t+1}^2\right)$$

We can also combine the budget constraints as a single intertemporal constraint in present value:

$$c_t^1 + \frac{c_{t+1}^2}{R_{t+1}} = w_t$$

These two equations implicitly define the consumption functions for young and old agent:

$$c_t^1 = c_t^1\left(w_t, R_{t+1}\right) \qquad c_{t+1}^2 = c_{t+1}^2\left(w_t, R_{t+1}\right)$$

from where we can also get the savings function

$$a_{t+1}^2 = w_t - c_t^1\left(w_t, R_{t+1}\right) = a_{t+1}^2\left(w_t, R_{t+1}\right)$$

From the problem of the firm, we obtain equilibrium prices

$$R_t = f'\left(K_t\right) + (1 - \delta)$$

$$w_t = f\left(K_t\right) - f'\left(K_t\right) K_t$$

Replacing in the saving function, it implicitly defines

$$a_{t+1}^2 = a_{t+1}^2\left(w_t\left(K_t\right), R_{t+1}\left(K_{t+1}\right)\right) \equiv S\left(K_t, K_{t+1}\right)$$

In equilibrium, we obtain

$$K_{t+1} = S\left(K_t, K_{t+1}\right)$$

a first order difference equation in K_t

This equation characterizes the dynamics of aggregate capital in the transition; depending on the utility and production functions, it can be very complicated

Stationary Equilibrium and Dynamic Efficiency

We will focus now on a stationary equilibrium, in which aggregate quantities and prices are constant

This also implies that the consumption levels for each age are stationary

$$c_t^1 = c_{t+1}^1 = c^1 \qquad \text{, etc.}$$

but does NOT imply that consumption is independent of age

$$c^1 \neq c^2 \qquad \text{, in general}$$

In a stationary equilibrium individual agents do not have a constant behavior since their decisions depend on age

... even though at the aggregate level the economy is stationary

In a stationary equilibrium, the Euler equation

$$\frac{u'\left(c^1\right)}{\beta u'\left(c^2\right)} = R\left(K^*\right)$$

and the intertemporal budget constraint

$$c^1 + \frac{c^2}{R\left(K^*\right)} = w\left(K^*\right)$$

hold, where $w\left(K^*\right)$ and $R\left(K^*\right)$ are equilibrium prices

We can derive in a similar way the consumption of each type of agent and the savings of the economy as a functions of aggregate capital

$$a^2 = S\left(K^*, K^*\right)$$

The value of aggregate capital in a stationary equilibrium solves the equation $S\left(K^*, K^*\right) = K^*$

Proposition: *All stationary equilibria in which* $R(K^*) > 1$ *are efficient*

Let (c^1, c^2 y K^*) be an stationary equilibrium. We can verify that it satisfies the first order conditions of the problem of a social planner

$$\max \qquad \sum_{t=0}^{\infty} \gamma^t \left[u\left(c_t^1\right) + \beta u\left(c_{t+1}^2\right)\right]$$

$$s.t. \qquad c_t^1 + c_t^2 = f\left(K_t\right) + (1-\delta) K_t - K_{t+1}$$

which discounts the weight of future generations at the rate γ

The solution to this problem is the Pareto optimal

The first order conditions of this problem are

$$\frac{u'\left(c_t^1\right)}{\beta u'\left(c_{t+1}^2\right)} = f'\left(K_{t+1}\right) + (1-\delta)$$

and

$$\frac{u'\left(c_t^1\right)}{\beta u'\left(c_t^2\right)} = \frac{1}{\gamma}$$

also

$$c_t^1 + c_t^2 = f\left(K_t\right) + (1-\delta) K_t - K_{t+1}$$

We can easily show that the stationary equilibrium satisfies these conditions with the discount rate $\gamma = R(K^*)^{-1}$

Note that we need $R(K^*) > 1$ such that $\gamma < 1$ and the problem of the social planner is well defined ■

On the contrary, if $R(K^*) < 1$ the equilibrium *is dynamically inefficient*

- $R(K^*) < 1$ implies that $f'(K^*) < \delta$, then there is over-accumulation of capital above its efficient limit

- Despite having a negative return, young agents save even more because they need to have an income for retirement

- In this context, public intervention through a social security system can improve welfare

`An Illustrative Example`

Simple case with log utility and Cobb-Douglas production function

$$u(c_t) = \log(c_t) \qquad f(K_t) = K_t^\alpha$$

From the problem of the young agent we obtain the Euler equation

$$c_{t+1}^2 = \beta R_{t+1} c_t^1$$

from where

$$c_t^1 = \frac{w_t}{1+\beta} \qquad c_t^2 = \frac{\beta R_t w_{t-1}}{1+\beta}$$

and

$$a_{t+1}^2 = \frac{\beta w_t}{1+\beta}$$

Note that savings does not depend on R_{t+1}; the income and substitution effects of the interest rate cancel out

From the problem of the firm

$$w_t = (1-\alpha) K_t^{\alpha} \qquad R_t = \alpha K_t^{\alpha-1} + (1-\delta)$$

Combining, we obtain the savings function

$$S(K_t) = \frac{\beta (1-\alpha) K_t^{\alpha}}{1+\beta}$$

and the first order difference equation

$$K_{t+1} = \frac{\beta (1-\alpha)}{1+\beta} K_t^{\alpha}$$

that characterizes the dynamics of capital

In this particular case, there is a single stationary equilibrium with positive capital

$$K^* = \left[\frac{\beta (1-\alpha)}{1+\beta}\right]^{\frac{1}{1-\alpha}}$$

in which

$$R(K^*) = \left(\frac{\alpha}{1-\alpha}\right)\left(\frac{1+\beta}{\beta}\right) + (1-\delta)$$

and this equilibrium will be dynamically inefficient if $\left(\frac{\alpha}{1-\alpha}\right)\left(\frac{1+\beta}{\beta}\right) < \delta$

We can also show in this case that starting from any $K_0 > 0$ the aggregate capital converges monotonically to its value of stationary equilibrium (*global stability*)

With other utility and / or production functions:

- There can be multiple stationary equilibria
- Some stationary equilibria can be dynamically efficient, others can not
- Some stationary balances may be stable, others may not
- The transition to one of the stable stationary equilibria can be non-monotonic (oscillatory, even chaotic)

`Parenthesis: Altruism in Dynamic Models`

Consider now altruistic individuals who are concerned not only about their utility, but about future generations (children, etc.).

In particular,

$$U_j = u\left(c_j^1\right) + \beta\left[u\left(c_{j+1}^2\right) + \psi U_{j+2}\right]$$

where U_j is the utility of a young worker born in the period j and $\psi \in (0, \beta]$ measures the degree of *altruism*; then

$$U_j = \sum_{s=0}^{\infty} (\beta\psi)^s \left[u\left(c_{j+2s}^1\right) + \beta u\left(c_{j+2s+1}^2\right)\right]$$

where s indexes the current and future generations (*dynasties*)

This version of the model looks like an infinitely lived agent model!

The problem of a young worker in period zero would be

$$\max \quad \sum_{s=0}^{\infty} (\beta\psi)^s \left[u\left(c_{2s}^1\right) + \beta u\left(c_{2s+1}^2\right)\right]$$

$$\begin{array}{lrcl} s.t. & c_{2s}^1 + a_{2s+1}^2 & = & w_{2s} + R_{2s} a_{2s}^1 \\ & c_{2s+1}^2 + a_{2s+2}^1 & = & R_{2s+1} a_{2s+1}^2 \qquad \forall t \geq 0 \\ & & & a_0^1 \quad \text{given} \end{array}$$

where a_{2s+2}^1 are *bequests* left to the next generation (perhaps negative), then

$$K_t = a_t^1 + a_t^2$$

The production side of the model is the same

The Euler equations obtained from the dynastic model are

$$\frac{u'\left(c_t^1\right)}{\beta u'\left(c_{t+1}^2\right)} = R_{t+1} \qquad \frac{u'\left(c_{t+1}^2\right)}{\psi u'\left(c_{t+2}^1\right)} = R_{t+2}$$

(reindexing $t = 2s$) and therefore

$$u'\left(c_t^1\right) = \beta\psi R_{t+1} R_{t+2} u'\left(c_{t+2}^1\right)$$

In an stationary equilibrium, $c_t^1 = c_{t+2}^1 = c^1$ implies $R\left(K^*\right) = (\beta\psi)^{-\frac{1}{2}}$

In the dynastic model there is a single interest rate consistent with a stationary equilibrium; its value depends only on the discount factor and the degree of altruism

It is easy to show again that this stationary equilibrium is efficient

Notice finally that if the degree of altruism increases, the interest rate in the stationary equilibrium decreases

- Greater altruism incentivizes young agents to accumulate more capital, not only to consume when they are old but to provide bequests to future generations

- With perfect altruism ($\psi = \beta$), the interest rate is the inverse of the discount factor $R(K^*) = \beta^{-1}$ as in the model with infinite horizon

End of parenthesis

`Recursive Competitive Equilibrium`

The state variables are

- Individual state: age $e = \{1, 2\}$ and assets $a \in (-B, \infty)$

- Aggregate state: distribution or measure $\mu_t(e, a)$ of agents over ages and assets

  This distribution satisfies

$$\lim_{a \to -B} \mu_t(2, a) = 0 \qquad \lim_{a \to \infty} \mu_t(2, a) = 1$$

$$\mu_t(1, a) = \begin{cases} 0 & , \forall a < 0 \\ 1 & , \forall a \geq 0 \end{cases}$$

A Recursive Competitive Equilibrium is a set of functions $v(e,a,\mu)$, $c(e,a,\mu)$, $a'(e,a,\mu)$, prices $w(\mu)$ and $R(\mu)$, capital demand $K(\mu)$ and law of motion $\Gamma(\mu)$ such that:

i) For each pair (a,μ), given functions w and Γ, the value function $v(1,a,\mu)$ solves the Bellman equation for the young agent:

$$v(1,a,\mu) = \max_{c,a'} \quad \left\{u(c) + \beta v\left(2,a',\mu'\right)\right\}$$

$$\begin{aligned} s.t. \quad c + a' &= w(\mu) + R(\mu)\, a \\ \mu' &= \Gamma(\mu) \end{aligned}$$

and $c(1,a,\mu)$, $a'(1,a,\mu)$ are optimal decision rules for this problem

ii) For each pair (a,μ), given the function R, the value function $v(2,a,\mu)$ of the retired agent satisfies:

$$v(2,a,\mu) = u(R(\mu)\, a)$$

and we define $c(2,a,\mu) = R(\mu)\, a,\ a'(2,a,\mu) = 0$

iii) For each μ, prices satisfy the marginal conditions of the representative firm:

$$R(\mu) = f'(K(\mu)) + (1-\delta)$$

$$w(\mu) = f(K(\mu)) - f'(K(\mu))\, K(\mu)$$

iv) For each μ, markets clear:

$$f(K(\mu)) = \sum_{e=1}^{2} \int_{-B}^{\infty} \left[c(e,a,\mu) + a'(e,a,\mu) - (1-\delta)a\right] d\mu(e,a)$$

$$K(\mu) = \sum_{e=1}^{2} \int_{-B}^{\infty} a d\mu(e,a)$$

v) For each μ, the law of motion Γ is consistent with the optimal decisions of the agents

`Solving Numerically a Stationary Equilibrium`

In a stationary equilibrium, the aggregate state of the economy reduces to total capital K^*

We need to discretize the space of possible values of individual assets

$$a \in \{a_1, a_2, ..., a_N\}$$

The idea is to iterate on K^* until achieving convergence

For each value of K^*, we solve first the problem of the old agent and we go back to the younger agents (*backwards induction*)

Iterative algorithm:

1. Propose a value for K^* and calculate de corresponding prices $R(K^*)$ and $w(K^*)$

2. Given the prices, calculate the value function of the retired agent $v(2, a_i) = u(R(K^*) a_i)$ for each point $a_i \in \{a_1, a_2, ..., a_N\}$

3. Given the prices and $v(2, a)$, calculate the value function of the young agent $v(1, 0)$ solving the problem

   $$v(1, 0) = \max_{a_j \in \{a_1, a_2, ..., a_N\}} \left\{ u\left(w(K^*) - a_j\right) + \beta v\left(2, a_j\right)\right\}$$

   and store $a'(1) = a_j$

4. Calculate the aggregate capital corresponding to the decision rule found in step 3

   $$\widehat{K} = a'(1)$$

5. Compare K^* and $\widehat{K}$

   - If they are equal (subject to a margin of tolerance), we are done

   - If they are different, go back to step 1 with a new K^*

   $$K^*_{new} = \frac{K^*_{initial} + \widehat{K}}{2}$$

Solving numerically overlapping generations models is simpler than the corresponding models with infinite life horizon

- The reason is that to find the value function of each agent you do not have to find a fixed point in the Bellman equation

The algorithm can be easily adapted to more periods of life

It can also be modified to calculate the *transition* to stationary equilibrium

- Idea: Assume that the stationary equilibrium is reached in T periods and iterate over a vector $\{K_0, K_1, ..., K_T\}$

`Some Implications of the Model and Extensions`

- The age structure and life cycle income profile are key to explain the differences in savings rates between poor and rich (Huggett and Ventura, JME 2000)

- Demographic changes affect the sustainability of the tax and social security system (Auerbach and Kotlikoff, Dynamic Fiscal Policy 1987)

  Generational accounting

- Technological change and its impact on agents' education decisions are consistent with the increase in income inequality in the United States (Heckman, Lochner and Taber, RED 1998)