Notes on Continuous Distributions

Summary

The following table compares discrete and continuous distributions. There are also distributions that are not purely discrete nor purely continuous; see, e.g., Example 9a, page 184 of Ross (7th).

Discrete r.v.	Continuous r.v.
probability mass function (pmf), $p(x) = P(X = x)$	probability density function (pdf), $f(x)$
	$P(x < X < x + \Delta x) \approx f(x) \cdot \Delta x$
	$P(X = x) = 0$, for any x
$0 \leq p(x_i) \leq 1$	$f(x) \geq 0$
possible values $X \in \{x_1, x_2, \ldots\}$	$X \in$ a union of intervals
$\sum_{x_i} p(x_i) = 1$	$\int_{-\infty}^{\infty} f(x)dx = 1$
$E(X) = \sum_{x_i} x_i \cdot p(x_i)$	$E(X) = \int_{-\infty}^{\infty} x \cdot f(x)dx$
$E(g(X)) = \sum_{x_i} g(x_i) \cdot p(x_i)$	$E(g(X)) = \int_{-\infty}^{\infty} g(x) \cdot f(x)dx$
$F(x) = P(X \leq x) = \sum_{x_i : x_i \leq x} p(x_i)$	$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(u)du$
$F(x)$ is staircase, jump by $p(x_i)$ at x_i	$F(x)$ continuous, $f(x) = F'(x)$

Examples

Example: Let X denote the length (in meters) of one side of a square sheet of plywood. Assume that the density function of X is given by

$$f(x) = \begin{cases} x & \text{if } 0 \leq x < 1 \\ 1/2 & \text{if } 1 \leq x < 2; \\ 0 & \text{otherwise.} \end{cases}$$

(a). On average, what fraction of squares of plywood have side length longer than 75cm?

$$P(X > 0.75) = \int_{0.75}^{\infty} f(x)dx = \int_{0.75}^{1} xdx + \int_{1}^{2} \frac{1}{2}dx = \frac{23}{32}$$

(or write $P(X > 0.75) = 1 - \int_0^{0.75} xdx$)

(b). Let A denote the area (in square meters) of the sheet of plywood. Compute $var(A)$.

The area is given by $A = X^2$, since it is an X-by-X square. Thus, we compute:

$$var(X^2) = E(X^4) - [E(X^2)]^2 = \int_0^1 x^4 \cdot xdx + \int_1^2 x^4 \cdot \frac{1}{2}dx - \left[\int_0^1 x^2 \cdot xdx + \int_1^2 x^2 \cdot \frac{1}{2}dx\right]^2$$

(c). Let $Y = 5X$. Compute the cdf, $F_Y(y)$, of Y.

(First, note that since X is always in [0,1) or [1,2), then $Y = 5X$ is always in [0,5) or [5,10).) We want to compute $F_Y(y) = P(Y \leq y) = P(5X \leq y) = P(X \leq \frac{y}{5}) = F_X(\frac{y}{5})$, which requires looking at cases:

$$F_Y(y) = \begin{cases} 0 & \text{if } \frac{y}{5} \leq 0\text{; i.e., if } y \leq 0 \\ \int_0^{y/5} xdx = \frac{y^2}{50} & \text{if } 0 \leq \frac{y}{5} \leq 1\text{; i.e., if } 0 \leq y \leq 5 \\ \int_0^1 xdx + \int_1^{y/5} \frac{1}{2}dx = \frac{y}{10} & \text{if } 1 \leq \frac{y}{5} \leq 2\text{; i.e., if } 5 \leq y \leq 10 \\ 1 & \text{if } \frac{y}{5} \geq 2\text{; i.e., if } y \geq 10 \end{cases}$$

Example: Let X denote the lifetime of a radio, in years, manufactured by a certain company. The density function of X is given by

$$f(x) = \begin{cases} \frac{1}{15}e^{-x/15} & \text{if } 0 \leq x < \infty \\ 0 & \text{otherwise.} \end{cases}$$

(a). What is the mean lifetime of a radio?

The mean of X is $E(X) = 1/\lambda = 15$ years, since X is exponential($\lambda = \frac{1}{15}$). (Or simply compute: $E(X) = \int_0^{\infty} x\frac{1}{15}e^{-x/15}dx$)

(b). What is the probability that a radio lasts more than 12 months?

The probability that a radio lasts more than 12 months (1 year) is $P(X > 1) = \int_1^\infty \frac{1}{15} e^{-x/15} dx = e^{-1/15} = .9355$

(c). (5 points) What is the probability that, of eight such radios, at least four last more than 12 months?

Let Y be the number (among the 8) that last 12 months (1 year). Then, Y is Binomial(8,p), where $p = .9355$ was found in part (b). We want

$$P(Y \geq 4) = \sum_{i=4}^{8} \binom{8}{i} p^i (1-p)^{8-i}$$

Example: The donuts you make in your bakery have random sizes. The radius (in inches) of a (circular) donut is a random variable whose density function is given by

$$f(x) = \begin{cases} x^2 & \textit{if } 0 < x < 1; \\ 1 & \textit{if } 1 \leq x < 5/3 \\ 0 & \textit{otherwise.} \end{cases}$$

(a). What fraction of donuts have radius larger than 1/2 inch?

Let X be the radius of a donut, in inches.

$P(X > 1/2) = 1 - P(X < 1/2) = 1 - \int_0^{1/2} x^2 dx = \frac{23}{24}$ (or $P(X > 1/2) = \int_{1/2}^1 x^2 dx + \int_1^{5/3} 1 dx = \frac{23}{24}$)

(b). What is the expected radius, μ, of a donut? What is the probability that a donut has radius equal to μ?

$\mu = E(X) = \int_0^1 x \cdot x^2 dx + \int_1^{5/3} x \cdot 1 dx = \frac{41}{36}$. $P(X = \mu) = 0$ (since, in fact, $P(X = x) = 0$ for *any* number x, for a continuous random variable X).

(c). Compute the variance of the radius, X.

$E(X^2) = \int_0^1 x^2 \cdot x^2 dx + \int_1^{5/3} x^2 \cdot 1 dx = \frac{571}{405}$. Thus, $var(X) = E(X^2) - [E(X)]^2 = \frac{571}{405} - (\frac{41}{36})^2$.

(d). Let Y denote the diameter *of a donut. Compute the cdf, $F_Y(y)$, of Y. Be very explicit! You must show the value of $F_Y(y)$ for* all *values of y; be careful about all cases.*

Now, $Y = 2X$ is the diameter of a donut. First, note that since X is always in (0,5/3), then $Y = 2X$ is always in (0,10/3); thus, we already know that $F_Y(y) = P(Y \leq y) = 0$ for $y \leq 0$ and $F_Y(y) = P(Y \leq y) = 1$ for $y \geq 10/3$. We want to compute $F_Y(y) = P(Y \leq y) = P(2X \leq y) = P(X \leq \frac{y}{2}) = F_X(\frac{y}{2})$, which requires looking at cases:

$$F_Y(y) = \begin{cases} 0 & \text{if } \frac{y}{2} \leq 0 \text{; i.e., if } y \leq 0 \\ \int_0^{y/2} x^2 dx = \frac{y^3}{24} & \text{if } 0 \leq \frac{y}{2} \leq 1 \text{; i.e., if } 0 \leq y \leq 2 \\ \int_0^1 x^2 dx + \int_1^{y/2} 1 dx = \frac{y}{2} - \frac{2}{3} & \text{if } 1 \leq \frac{y}{2} \leq \frac{5}{3} \text{; i.e., if } 2 \leq y \leq \frac{10}{3} \\ 1 & \text{if } \frac{y}{2} \geq \frac{5}{3} \text{; i.e., if } y \geq \frac{10}{3} \end{cases}$$