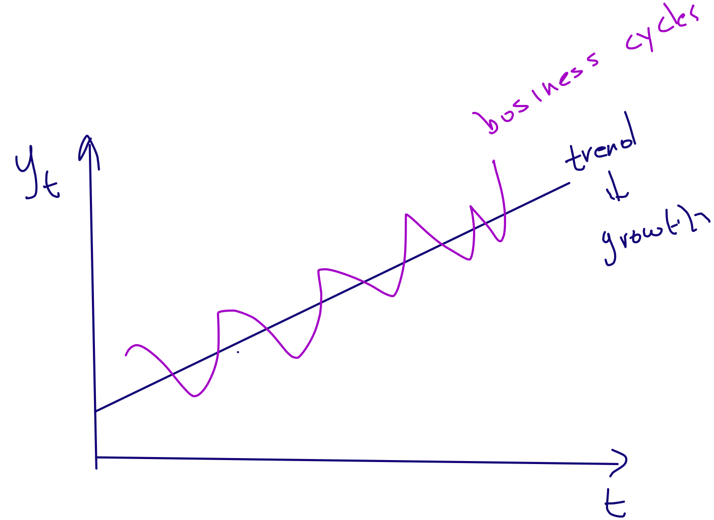
NEOCLASSICAL GROWTH MODEL WITH UNCERTAINTY



Adding uncertainty to the neoclassical growth model:

supply

- Technology shocks

$$Y_t = \theta_t F(K_t, L_t)$$

where θ_t is a random variable

demand

- Shocks to preferences (ex: β_t stochastic discount factor)

$\beta_t \uparrow$

$C_t \downarrow$

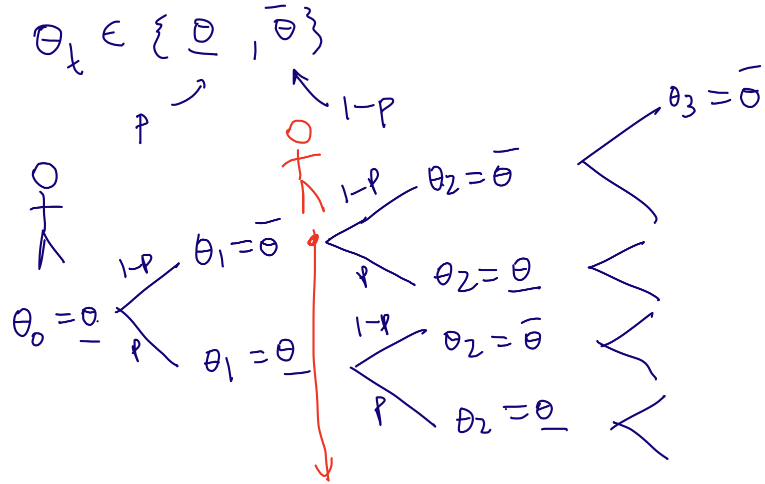
$C_{t+1} \uparrow$

- Policy shocks (ex: g_t o M_t random)

Uncertainty affects the behavior of consumers:

- Agents know current and past realizations (history) of the shocks - but not future realizations
- However, they know the stochastic process
- Agents choose contingent plans (that depend on the history of the shocks) for each variable
- Objective: maximize expected utility

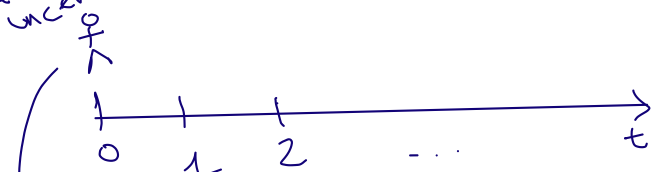
$$E_0 \sum_{t=0}^{\infty} \beta^t u(c_t)$$



date t

Agent knows $\underbrace{\{\theta_0 = \underline{\theta}, \theta_1 = \bar{\theta}\}}_{\text{history up to date } t}$

without uncertainty



Choice $\rightarrow$ sequence $\{c_t\}_{t=0}^{\infty}$

example above:

$$Z \to \theta \quad \text{(one-dimensional)}$$

$$Z \to \{\underline{\theta}, \bar{\theta}\} \qquad \theta_t \in \underbrace{\{\underline{\theta}, \bar{\theta}\}}_{\Theta}$$

$$\theta^4 = (\theta_0, \theta_1, \theta_2, \theta_3, \theta_4)$$

$$\Theta = \{\underline{\theta}, \bar{\theta}\}$$

$\theta_1 = \bar{\theta}, \theta_2 = \bar{\theta}$

$$\Theta^2 = \{(\theta_0, \bar{\theta}, \bar{\theta}), (\theta_0, \bar{\theta}, \underline{\theta}), (\theta_0, \underline{\theta}, \bar{\theta}), (\theta_0, \underline{\theta}, \underline{\theta})\}$$

$$\theta^2 = (\theta_0, \bar{\theta}, \underline{\theta}) \in \Theta^2$$

$$\Theta^1 = \{(\theta_0, \bar{\theta}), (\theta_0, \underline{\theta})\}$$

Notation $\to$ vector

- z : random variable (shock)
- Z : set of possible realizations of z
- $z_t \in Z$: realization of shock in period t
- $z^t = (z_0, z_1, z_2, \ldots z_t)$: history of realizations of the shock up to t

$$z^3 = (z_0, z_1, z_2, z_3)$$

- Z^t : set of all possible stories up to the period t

$$z^t \in Z^t$$

- $Z^t \big| z^{t-s}$: set of all possible stories up to the period t that begin with z^{t-s}

$$Z^t \big| z^{t-s} \subseteq Z^t$$

$$\Theta^2 \Big|_{(\theta_0, \bar{\theta})} = \{(\theta_0, \bar{\theta}, \bar{\theta}), (\theta_0, \bar{\theta}, \underline{\theta})\}$$

our example $(n=2)$

If $Z = (Z_1, Z_2, ..., Z_n)$ is finite, z is a discrete random variable. Then, given $z_0 \in Z$, we define

$\pi : Z^t \longrightarrow [0,1]$

- $\pi\left(z^t\right)$: probability of observing story z^t in period t, with

$$0 \le \pi\left(z^t\right) \le 1 \qquad \sum_{z^t \in Z^t} \pi\left(z^t\right) = 1$$

contingent plan: $x_t : Z^t \longrightarrow \mathbb{R}$

- Expectation: $E_0 x_t\left(z^t\right) = \sum_{z^t \in Z^t} \pi\left(z^t\right) x_t\left(z^t\right)$

- Conditional expectation: $E_{t-s} x_t\left(z^t\right) = \sum_{z^t \in Z^t | z^{t-s}} \frac{\pi(z^t)}{\pi(z^{t-s})} x_t\left(z^t\right)$

If $Z \subseteq R$ is infinite, z is a continuous random variable. Then we will work with a density function $\phi\left(z^t\right)$

$\sum_{z^t \in Z^t | z^{t-s}} \frac{\pi(z^t)}{\pi(z^{t-s})} = 1$

AR(1): $\theta_t = \rho\, \theta_{t-1} + \varepsilon_t$

$\pi\left(z^t\right) \Longrightarrow \phi\left(z^t\right)$

$\sum \longrightarrow \int$

Model with Discrete Technology Shocks

- Stochastic production function:

$$Y_t = e^{z_t} F(K_t, L_t)$$

where F statisfies common assumptions. With $L_t = 1$, we rewrite

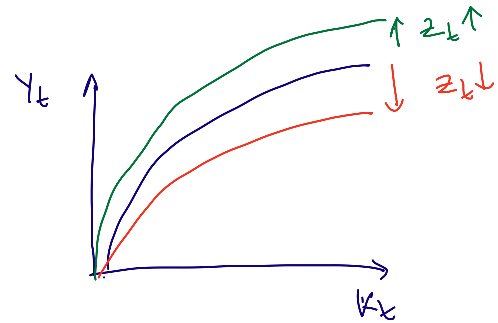
neoclassical growth model

$$Y_t = e^{z_t} F(K_t, 1) = e^{z_t} f(K_t)$$

- Expected utility:

$$E_0 \sum_{t=0}^{\infty} \beta^t u\left[c_t\left(z^t\right)\right] = \sum_{t=0}^{\infty} \sum_{z^t \in Z^t} \beta^t \pi\left(z^t\right) u\left[c_t\left(z^t\right)\right]$$

contingent plan for consumption



Stochastic Competitive General Equilibrium

A stochastic competitive equilibrium for this economy is a set of contingent plans for the quantities $c_t\left(z^t\right)$, $i_t\left(z^t\right)$, $k_{t+1}\left(z^t\right)$, $Y_t\left(z^t\right)$, $K_t\left(z^t\right)$ and prices $w_t\left(z^t\right)$, $r_t\left(z^t\right)$ such that:

i) Given $k_0 > 0$, z_0, $w_t\left(z^t\right)$, $r_t\left(z^t\right)$ and the process for z, the contingent plans $c_t\left(z^t\right)$, $i_t\left(z^t\right)$, and $k_{t+1}\left(z^t\right)$ solve the problem of the household:

$$\max \sum_{t=0}^{\infty} \sum_{z^t \in Z^t} \beta^t \pi\left(z^t\right) u\left[c_t\left(z^t\right)\right]$$

$$s.t. \quad c_t\left(z^t\right)+i_t\left(z^t\right)=w_t\left(z^t\right)+r_t\left(z^t\right) k_t\left(z^{t-1}\right) \qquad \forall z^t, \forall t$$

$$k_{t+1}\left(z^t\right)=(1-\delta)\, k_t\left(z^{t-1}\right)+i_t\left(z^t\right) \qquad \forall z^t, \forall t$$

ii) For each story z^t at each period t, given $w_t\left(z^t\right)$ and $r_t\left(z^t\right)$, the values $Y_t\left(z^t\right)$ and $K_t\left(z^t\right)$ solves the firm problem:

$$\max \quad Y_t\left(z^t\right)-w_t\left(z^t\right)-r_t\left(z^t\right) K_t\left(z^t\right)$$

$$s.t. \quad Y_t\left(z^t\right)=e^{z_t} f\left[K_t\left(z^t\right)\right]$$

iii) For each story z^t at each period t, markets clear:

$$Y_t\left(z^t\right)=c_t\left(z^t\right)+i_t\left(z^t\right)$$

$$K_t\left(z^t\right)=k_t\left(z^{t-1}\right)$$

Social Planner's Problem

Given $k_0 > 0$, z_0 and the stochastic process for z, the social planner chooses contingent plans for the quantities $c_t\left(z^t\right)$, $i_t\left(z^t\right)$ and $k_{t+1}\left(z^t\right)$ solving:

$$\max \quad \sum_{t=0}^{\infty} \sum_{z^t \in Z^t} \beta^t \pi\left(z^t\right) u\left[c_t\left(z^t\right)\right]$$

$$s.t. \quad c_t\left(z^t\right) = e^{z_t} f\left[k_t\left(z^{t-1}\right)\right] - i_t\left(z^t\right) \qquad \forall z^t, \forall t$$

$$k_{t+1}\left(z^t\right) = (1-\delta) k_t\left(z^{t-1}\right) + i_t\left(z^t\right) \qquad \forall z^t, \forall t$$

Even with uncertainty, if there are no distortions or externalities, the Welfare Theorems hold

$$L = \sum_{t=0}^{\infty} \sum_{z^t \in Z^t} \beta^t \pi(z^t)\, u\left[c_t(z_t)\right] - \sum_{t=0}^{\infty} \sum_{z^t \in Z^t} \lambda_{1t}(z^t)\left[c_t(z^t) - e^{z_t} f(k_t(z^{t-1})) + i_t(z^t)\right] - \sum_{t=0}^{\infty} \sum_{z^t \in Z^t} \lambda_{2t}(z^t)\left[k_{t+1}(z^t) - (1-\delta) k_t(z^{t-1}) - i_t(z^t)\right]$$

The Lagrangian for this problem is given by:

$$L = \sum_{t=0}^{\infty} \sum_{z^t \in Z^t} \left\{ \beta^t \pi\left(z^t\right) u\left[c_t\left(z^t\right)\right] \right.$$
$$-\lambda_{1t}\left(z^t\right)\left[c_t\left(z^t\right) + i_t\left(z^t\right) - e^{z_t} f\left[k_t\left(z^{t-1}\right)\right]\right]$$
$$\left. -\lambda_{2t}\left(z^t\right)\left[\underline{k_{t+1}\left(z^t\right)} - (1-\delta) k_t\left(z^{t-1}\right) - i_t(z^t)\right]\right\}$$

with first order conditions:

$$\frac{\partial L}{\partial c_t\left(z^t\right)} = \beta^t \pi\left(z^t\right) u'\left[c_t\left(z^t\right)\right] - \lambda_{1t}\left(z^t\right) = 0$$

$$\frac{\partial L}{\partial i_t\left(z^t\right)} = -\lambda_{1t}\left(z^t\right) + \lambda_{2t}\left(z^t\right) = 0$$

$$\frac{\partial L}{\partial k_{t+1}\left(z^t\right)} = \sum_{z^{t+1} \in Z^{t+1}|z^t} \lambda_{1t+1}\left(z^{t+1}\right) e^{z_{t+1}} f'\left[k_{t+1}\left(z^t\right)\right]$$
$$-\lambda_{2t}\left(z^t\right) + \sum_{z^{t+1} \in Z^{t+1}|z^t} \lambda_{2t+1}\left(z^{t+1}\right)(1-\delta) = 0$$

$$\frac{dL}{dc_t(z^t)} = \beta^t \pi(z^t) u'\left[c_t(z^t)\right] - \lambda_{1t}(z^t) = 0$$

$$\Rightarrow \lambda_{1t}(z^t) = \beta^t \pi(z^t) u'\left[c_t(z^t)\right]$$

$$\frac{dL}{di_t(z^t)} = -\lambda_{1t}(z^t) + \lambda_{2t}(z^t) = 0$$

$$\Rightarrow \lambda_{1t}(z^t) = \lambda_{2t}(z^t)$$

$$\frac{dL}{dk_{t+1}(z^t)} = -\lambda_{2t}(z^t)$$
$$+ \sum_{z^{t+1} \in Z^{t+1}|z^t} \lambda_{1t+1}(z^{t+1}) e^{z_{t+1}} f'(k_{t+1}(z^t))$$

$$+ \sum_{z^{t+1} \in Z^{t+1} | z^t} \lambda_{2t+1}(z^{t+1}) (1-\delta) = 0$$

$$\lambda_{1t}(z^t) = \sum_{z^{t+1} \in Z^{t+1} | z^t} \lambda_{1t+1}(z^{t+1}) \left[e^{z_{t+1}} f'(k_{t+1}(z^t)) + (1-\delta) \right]$$

The solution is characterized by:

- Euler equation (stochastic):

$\forall t, \forall z^t$

$$u'\left[c_t\left(z^t\right)\right] = \beta E_t\left\{u'\left[c_{t+1}\left(z^{t+1}\right)\right]\left(e^{z_{t+1}}f'\left[k_{t+1}\left(z^t\right)\right] + (1-\delta)\right)\right\}$$

with: $$E_t x\left(z^{t+1}\right) = \sum_{z^{t+1} \in Z^{t+1} | z^t} \frac{\pi\left(z^{t+1}\right)}{\pi\left(z^t\right)} x\left(z^{t+1}\right)$$

$$\cancel{\beta^t} \pi(\cancel{z^t})\, u'[c_t(z^t)] = \sum_{z^{t+1} \in Z^{t+1} | z^t} \frac{\beta^{\cancel{t}+1} \pi(z^{t+1})\, u'[c_{t+1}(z^{t+1})]}{\pi(z^t)} \times \left[e^{z_{t+1}} f'(k_{t+1}(z^t)) + (1-\delta) \right]$$

- Feasibility condition:

$\forall t, \forall z^t$

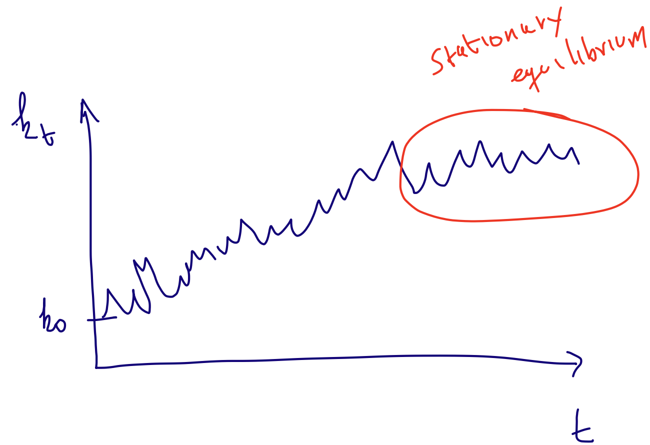
$$c_t\left(z^t\right) = e^{z_t} f\left[k_t\left(z^{t-1}\right)\right] \underbrace{- k_{t+1}\left(z^t\right) + (1-\delta) k_t\left(z^{t-1}\right)}_{i_t(z^t)}$$

- Transversality condition:

$$\lim_{t\to\infty} E_0 \beta^t u'\left[c_t\left(z^t\right)\right] k_{t+1}\left(z^t\right) = 0$$

Stationary Equilibrium

- Solvng the equilibrium, we obtain contingent plans for all the variables (quantities, prices)
- Given the stochastic processes for the shocks, these plans define processes for the variables - difficult to characterize
- There is NO stationary equilibrium in a strict sense, the variables move permanently
- We can, however, have an equilibrium in which all variables follow stationary stochastic processes - all their moments (mean, variance, etc.) are constant over time



`Sequential Markets and Arrow-Debreu Markets`

So far we have worked with *sequential markets*:

- In each period a new market is opened, in which goods and production factors of the current period are exchanged
- In the deterministic model there are as many markets as periods
- With uncertainty, there is a market for every history or state in the world
- Therefore, we also have a budget constraint for each period or state of the world

An alternative market structure is the one proposed by *Arrow-Debreu*:

- There is only one market, which opens in the initial period $(t = 0)$
- In this market, promises to deliver goods or productive factors are exchanged in any future period for each state of the world
- There is only one budget constraint
- We interpret the contingent plans as baskets of Arrow-Debreu goods

$p_t\left(z^t\right)$: price in period 0 of one unit of the only good delivered at period t if the history of the shocks is z^t

We normalize $p_0\left(z_0\right) = 1$

An Arrow-Debreu equilibrium is a set of baskets $c_t\left(z^t\right)$, $i_t\left(z^t\right)$, $k_{t+1}\left(z^t\right)$, $Y_t\left(z^t\right)$, $K_t\left(z^t\right)$ and prices $p_t\left(z^t\right)$, $w_t\left(z^t\right)$, $r_t\left(z^t\right)$ such that:

i) Given $k_0 > 0$, z_0, $p_t\left(z^t\right)$, $w_t\left(z^t\right)$, $r_t\left(z^t\right)$ and the process for z, the baskets $c_t\left(z^t\right)$, $i_t\left(z^t\right)$ and $k_{t+1}\left(z^t\right)$ solve:

$$\max \quad \sum_{t=0}^{\infty} \sum_{z^t \in Z^t} \beta^t \pi\left(z^t\right) u\left[c_t\left(z^t\right)\right]$$

$$s.t. \quad \sum_{t=0}^{\infty} \sum_{z^t \in Z^t} p_t\left(z^t\right)\left[c_t\left(z^t\right) + i_t\left(z^t\right)\right] = \sum_{t=0}^{\infty} \sum_{z^t \in Z^t} p_t\left(z^t\right)\left[w_t\left(z^t\right) + r_t\left(z^t\right) k_t\left(z^{t-1}\right)\right]$$

$$k_{t+1}\left(z^t\right) = (1-\delta) k_t\left(z^{t-1}\right) + i_t\left(z^t\right) \qquad \forall z^t, \forall t$$

ii) Given $w_t\left(z^t\right)$ and $r_t\left(z^t\right)$, the baskets $Y_t\left(z^t\right)$ and $K_t\left(z^t\right)$ solve:

$$\max \quad \sum_{t=0}^{\infty} \sum_{z^t \in Z^t} p_t\left(z^t\right)\left[Y_t\left(z^t\right) - w_t\left(z^t\right) - r_t\left(z^t\right) K_t\left(z^t\right)\right]$$

$$s.t. \quad Y_t\left(z^t\right) = e^{z_t} f\left[K_t\left(z^t\right)\right] \qquad \forall z^t, \forall t$$

iii) For each story z^t at period t, markets clear:

$$Y_t\left(z^t\right) = c_t\left(z^t\right) + i_t\left(z^t\right)$$

$$K_t\left(z^t\right) = k_t\left(z^{t-1}\right)$$

Solving the household problem:

$$L = \sum_{t=0}^{\infty} \sum_{z^t \in Z^t} \left\{ \beta^t \pi\left(z^t\right) u\left[c_t\left(z^t\right)\right] \right.$$
$$-\lambda_1 p_t\left(z^t\right)\left[c_t\left(z^t\right)+i_t\left(z^t\right)-w_t\left(z^t\right)-r_t\left(z^t\right)k_t\left(z^{t-1}\right)\right]$$
$$\left. -\lambda_{2t}\left(z^t\right)\left[k_{t+1}\left(z^t\right)-(1-\delta)k_t\left(z^{t-1}\right)-i_t\left(z^t\right)\right]\right\}$$

with first order conditions:

$$\frac{\partial L}{\partial c_t\left(z^t\right)} = \beta^t \pi\left(z^t\right) u'\left[c_t\left(z^t\right)\right] - \underbrace{\lambda_1 p_t\left(z^t\right)}_{\lambda_{1t}(z^t)} = 0$$

$$\frac{\partial L}{\partial i_t\left(z^t\right)} = -\lambda_1 p_t\left(z^t\right) + \lambda_{2t}\left(z^t\right) = 0$$

$$\frac{\partial L}{\partial k_{t+1}\left(z^t\right)} = \sum_{z^{t+1}\in Z^{t+1}|z^t} \overbrace{\lambda_1 p_{t+1}\left(z^{t+1}\right)}^{\lambda_{1t+1}(z^{t+1})} r_{t+1}\left(z^{t+1}\right)$$
$$-\cancel{\lambda_{2t}\left(z^t\right)}+\sum_{z^{t+1}\in Z^{t+1}|z^t} \lambda_{2t+1}\left(z^{t+1}\right)(1-\delta)$$

$$\lambda_1 p_t(z^t) = \beta^t \pi(z^t) u'[c_t(z^t)]$$

$$\lambda_1 p_{t+1}(z^{t+1}) = \beta^{t+1}\pi(z^{t+1}) u'[c_{t+1}(z^{t+1})]$$

$$\lambda_1 p_t(z^t) = \sum_{z^{t+1}\in z^{t+1}|z^t} \lambda_1 p_{t+1}(z^{t+1})\left\{r_{t+1}(z^{t+1}) + (1-\delta)\right\}$$

$$\boxed{\beta^t \pi(z^t) u'[c_t(z^t)]} = \sum_{z^{t+1}\in Z^{t+1}|z^t} \beta^{t+1}\pi(z^{t+1}) u'[c_{t+1}(z^{t+1})]$$
$$\{ \ldots (z^{t+1}) \ldots (1-\delta)\}$$

$$\frac{p_{t+1}(z^{t+1})}{p_t(z^t)} = \beta \frac{\pi(z^{t+1})}{\pi(z^t)} \frac{u'[c_{t+1}(z^{t+1})]}{u'[c_t(z^t)]}$$

$$\frac{p_1(z^1)}{p_0(z_0)} = \beta\, \pi(z^1) \frac{u'[c_1(z^1)]}{u'[c_0(z^0)]} \rightarrow \frac{p_2(z^2)}{p_0(z_0)} = \beta^2 \pi(z^2) \frac{u'[c_2(z^2)]}{u'[c_0(z_0)]}$$

The Arrow-Debreu equilibrium is characterized by:

- Stochastic Euler Equation:

$$u'\left[c_t\left(z^t\right)\right] = \beta E_t\left\{u'\left[c_{t+1}\left(z^{t+1}\right)\right]\left(r_{t+1}\left(z^{t+1}\right) + (1-\delta)\right)\right\}$$

- Feasibility Condition:

$$c_t\left(z^t\right) = e^{z_t} f\left[k_t\left(z^{t-1}\right)\right] - k_{t+1}\left(z^t\right) + (1-\delta)\, k_t\left(z^{t-1}\right)$$

- Arrow-Debreu Prices:

$$\frac{p_t\left(z^t\right)}{p_{t+1}\left(z^{t+1}\right)} = \frac{\pi\left(z^t\right) u'\left[c_t\left(z^t\right)\right]}{\beta\pi\left(z^{t+1}\right) u'\left[c_{t+1}\left(z^{t+1}\right)\right]}$$

Using the price normalization $p_0(z_0) = 1$

$$p_t\left(z^t\right) = \beta^t \frac{u'\left[c_t\left(z^t\right)\right]}{u'\left[c_0\left(z_0\right)\right]} \pi\left(z^t\right)$$

- Factor prices:

$$w_t\left(z^t\right) = e^{z_t} f\left[K_t\left(z^t\right)\right] - e^{z_t} f'\left[K_t\left(z^t\right)\right] K_t\left(z^t\right)$$

$$r_t\left(z^t\right) = e^{z_t} f'\left[K_t\left(z^t\right)\right]$$

- Transversality condition:

$$\lim_{t\to\infty} \sum_{z^t\in Z^t} p_t\left(z^t\right) k_{t+1}\left(z^t\right) = 0$$

$$\{ r_{t+1}(z^{t+1}) + (1-\delta)$$

Complete Markets

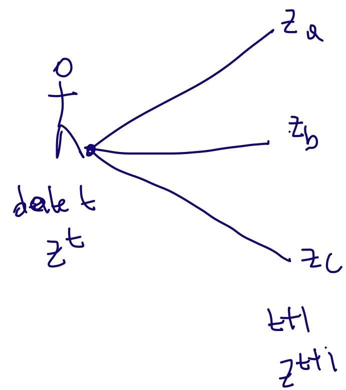
A structure of sequential and complete markets requieres the existence of assets for each posible state of the world

- $b_{t+1}\left(z^{t+1}\right)$: contingent bond purchased in period t, with return:

  z^t $$\begin{cases} 1, \text{ if } z^{t+1} \text{ ocurs} \\ 0, \text{ in other case} \end{cases}$$

- $q_t\left(z^{t+1}\right)$: price of the contingent bond in period t

Given that the contingent bonds are traded between all consumers, the net supply is equal zero



$z^{t+1} = \{z^t, z_a\}$

$z^{t+1} = \{z^t, z_b\}$

$z^{t+1} = \{z^t, z_c\}$

$b_{t+1}(\{z^t, z_a\})$ pays $\begin{cases} 1 & \text{if } z_{t+1} = z_a \\ 0 & \text{otherwise} \end{cases}$

$\downarrow$

$q_t(\{z^t, z_a\})$

A competitive equilibrium with complete markets is a set of contingent plans $c_t\left(z^t\right)$, $i_t\left(z^t\right)$, $b_{t+1}\left(z^{t+1}\right)$, $k_{t+1}\left(z^t\right)$, $Y_t\left(z^t\right)$, $K_t\left(z^t\right)$ and prices $q_t\left(z^{t+1}\right)$, $w_t\left(z^t\right)$, $r_t\left(z^t\right)$ such that

i) Given $k_0 > 0$, z_0, $b_0 = 0$, $q_t\left(z^{t+1}\right)$, $w_t\left(z^t\right)$, $r_t\left(z^t\right)$ and the process for z, the plans $c_t\left(z^t\right)$, $i_t\left(z^t\right)$, $b_{t+1}\left(z^{t+1}\right)$ and $k_{t+1}\left(z^t\right)$ solve:

$$\max \quad \sum_{t=0}^{\infty} \sum_{z^t \in Z^t} \beta^t \pi\left(z^t\right) u\left[c_t\left(z^t\right)\right]$$

$$s.t. \quad c_t\left(z^t\right)+i_t\left(z^t\right)+\sum_{z^{t+1}\in Z^{t+1}|z^t} q_t\left(z^{t+1}\right) b_{t+1}\left(z^{t+1}\right)$$

$$= w_t\left(z^t\right)+r_t\left(z^t\right) k_t\left(z^{t-1}\right)+b_t\left(z^t\right) \qquad \forall z^t, \forall t$$

$$k_{t+1}\left(z^t\right) = (1-\delta) k_t\left(z^{t-1}\right) + i_t\left(z^t\right) \qquad \forall z^t, \forall t$$

$$b_{t+1}\left(z^{t+1}\right) > -B \qquad \forall z^{t+1} \in Z^{t+1}, \forall z^t, \forall t$$

no-Ponzi

ii) For each story z^t at each period t, given $w_t\left(z^t\right)$ and $r_t\left(z^t\right)$, the values $Y_t\left(z^t\right)$ and $K_t\left(z^t\right)$ solve:

$$\max \quad Y_t\left(z^t\right) - w_t\left(z^t\right) - r_t\left(z^t\right) K_t\left(z^t\right)$$

$$s.t. \quad Y_t\left(z^t\right) = e^{z_t} f\left[K_t\left(z^t\right)\right]$$

iii) For each story z^t at each period t, markets clear:

$$Y_t\left(z^t\right) = c_t\left(z^t\right) + i_t\left(z^t\right)$$

$$K_t\left(z^t\right) = k_t\left(z^{t-1}\right)$$

$$b_{t+1}\left(z^{t+1}\right) = 0 \qquad \forall z^{t+1} \in Z^{t+1}\big|z^t$$

The equilibrium is characterized by:

- Stochastic Euler Equation:

$$u'\left[c_t\left(z^t\right)\right] = \beta E_t\left\{u'\left[c_{t+1}\left(z^{t+1}\right)\right]\left(r_{t+1}\left(z^{t+1}\right) + (1-\delta)\right)\right\}$$

- Feasibility Condition:

$$c_t\left(z^t\right) = e^{z_t} f\left[k_t\left(z^{t-1}\right)\right] - k_{t+1}\left(z^t\right) + (1-\delta)\, k_t\left(z^{t-1}\right)$$

- Bond Prices:

$$q_t\left(z^{t+1}\right) = \frac{\beta\pi\left(z^{t+1}\right)u'\left[c_{t+1}\left(z^{t+1}\right)\right]}{\pi\left(z^t\right)u'\left[c_t\left(z^t\right)\right]} \qquad \forall z^{t+1} \in$$

- Factor Prices:

$$w_t\left(z^t\right) = e^{z_t} f\left[K_t\left(z^t\right)\right] - e^{z_t} f'\left[K_t\left(z^t\right)\right] K_t\left(z^t\right)$$

$$r_t\left(z^t\right) = e^{z_t} f'\left[K_t\left(z^t\right)\right]$$

- Transversality Condition:

$$\lim_{t\to\infty} \sum_{z^t \in Z^t} \left(\prod_{j=1}^{t} q_{j-1}\left(z^j\right)\right) k_{t+1}\left(z^t\right) = 0$$

$$\lim_{t\to\infty} \sum_{z^t \in Z^t} \left(\prod_{j=1}^{t} q_{j-1}\left(z^j\right)\right) b_t\left(z^t\right) = 0$$

- With complete markets, the Arrow-Debreu and sequential markets equilibria are equivalent:

  i) the contingent plans $c_t\left(z^t\right)$, $i_t\left(z^t\right)$, $k_{t+1}\left(z^t\right)$, $Y_t\left(z^t\right)$, $K_t\left(z^t\right)$ are the same

  ii) the factor prices $w_t\left(z^t\right)$, $r_t\left(z^t\right)$ are the same

  iii) the bond prices satisfies:

$$q_t\left(z^{t+1}\right)=\frac{p_{t+1}\left(z^{t+1}\right)}{p_t\left(z^t\right)} \qquad \forall z^{t+1}\in Z^{t+1}\Big|z^t$$

- Given that we have a unique (representative) agent, the net supply of each financial asset is zero

  In equilibrium, the representative consumer is restricted by market clearing conditions

  Therefore, we could implement the Arrow-Debreu equilibrium with only one asset (capital)