CPT Lecture Notes 9: Abstract choice

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ABSTRACT CHOICE THEORY

Distinction between...

Environment of choice: The complete set of observables -could be hypothetical

Model of choice: A strict nonempty subset of the observables defined by an appealing story

Axiom: A strict nonempty subset of the observables satisfying an appealing property

In choice theory we prove results of the form:

$$\text{model} \Leftrightarrow \text{axiom}$$

ENVIRONMENT: CHOICE FUNCTIONS

X is a finite set of alternatives.

A menu is any nonempty subset of X.

$2^X = \{A \subseteq X : A \neq \emptyset\}$ is the full domain of menus.

A choice function is a map $c : 2^X \to X$ satisfying $c(A) \in A$ for every $A \in 2^X$.

Model: Rational choice

A choice function c is rational(izable) if there exists a linear order P (connected, negatively transitive, asymmetric) binary relation on X such that for every menu A,

$$c(A) = \max(A, P).$$

Here by $\max(A, P)$ we mean the necessarily unique alternative in A satisfying $\max(A, P)Pa$ for every $a \in A \setminus \{\max(A, P)\}$.

If c is rational and P is the underlying linear order, then we write $c = c^P$.

Initial observations:

1. For every linear order P, c^P (which maximizes P on every menu A) is a choice function.

2. If $P \neq P'$, then $c^P \neq c^{P'}$. (Why?)

Hence if we observe a rational choice function c^P, we may infer P uniquely. This is called the exercise of revealed preference. But:

1. How can we guarantee that the choice function we observe is rational(izable)?

2. How do we perform the revelation exercise?

A choice function c satisfies α if whenever $x = c(A)$ and $x \in B \subset A$, it follows that $x = c(B)$.

Theorem: A choice function c is rational if and only it c satisfies α.

Proof: ($\Rightarrow$) Suppose c is rational and let P be the underlying linear order. Suppose $x = c(A)$ and $x \in B \subset A$. Then xPa for every $a \in A \backslash \{x\}$. This implies that xPb for every $b \in B \backslash \{x\}$ and therefore $x = c(B)$. Hence c satisfies α.

($\Leftarrow$) Suppose c satisfies α. Define a binary relation P on X as follows.

$$xPy \text{ iff } x \neq y, \text{ and } x = c(\{x, y\}).$$

Note P is asymmetric and connected by definition.

To see that P is negatively transitive, suppose xPy and take any other alternative z. Hence $x = c(\{x, y\})$. Suppose $x \not P z$. Hence $z = c(\{x, z\})$. By α, then, $c(\{x, y, z\})$ is neither x, nor y. It follows that $z = c(\{x, y, z\})$ and by α, $z = c(\{y, z\})$, i.e. zPy as desired. This establishes that P is a linear order.

To finish the proof, take any menu A and let $x = c(A)$. We need to show that $x = \max(A, P)$. Take any $a \in A \backslash \{x\}$. By α, $x = c(\{x, a\})$, i.e. xPa. The proof is complete.

Remarks:

1. Rational versus rationalizable...

2. Kreps' frog legs, Sen's party...

3. We implicitly used the full domain assumption (that c chooses from all members of 2^X) multiple times in the proof. Where? A homework question below will address what happens if we drop the full domain assumption.

4. There are other models of rationality. But those stories can only be told in richer environments.

More axioms:

A choice function c satisfies...

... γ if $x = c(A \cup B)$ whenever $x = c(A) = c(B)$.

... no binary cycles (NBC) if whenever $x = c(\{x, y\})$ and $y = c(\{y, z\})$, then $x = c(\{x, z\})$.

... path independence (PI) if $c(A \cup B) = c(\{c(A), c(B)\})$.

... α_0 if $x = c(\{x, y\})$ whenever $x = c(A)$ and $y \in A$.

... WARP if $c(A) = c(B)$ whenever $c(A) \in B$ and $c(B) \in A$

Homework: Show that the following statements are equivalent. (1) c satisfies α. (2) c satisfies γ and NBC. (3) c satisfies PI. (4) c satisfies α_0. (5) c satisfies WARP.

Homework: Let us remove the full-domain assumption. Let $\mathcal{D} \subseteq 2^X$. A choice function is a map $c : \mathcal{D} \to X$ satisfying $c(A) \in A$ for every $A \in \mathcal{D}$. A choice function c is rational(izable) if there exists a linear order P such that $c(A) = \max(A, P)$ for every $A \in \mathcal{D}$.

(i) Consider the following variation of α. A choice function c satisfies α' if $x = c(B)$ whenever $x = c(A)$, $x \in B \subset A$ and $B \in \mathcal{D}$. Prove or find a counter example: c is rational if and only if it satisfies α'.

(ii) For any c, define a binary relation P^c as follows: xP^cy iff there exists $A \in \mathcal{D}$ such that $x = c(A)$ and $y \in A \backslash \{x\}$. Prove or find a counter example: c is rational if and only if P^c is acyclic.

More homework: Consider the following condition.

For any n, for any $x_1, ..., x_n$, if $x_i = c(\{x_i, x_{i+1}\})$ for $i = 1, ..., n-1$, then $x_1 = c(\{x_1, x_n\})$.

It must be clear that this condition implies NBC. (Just take $n = 3$.) Show that NBC implies this condition.

Before we move on to other models, beware: not every story is a model, regardless of how appealing it may look.

Story: The DM has a utility function $u : X \to \mathbb{R}$ which she maximizes in every menu as follows... $c(A) \in \arg\max_{x \in A} u(x)$.

Observation: For every choice function c, there is some $u : X \to \mathbb{R}$ such that $c(A) \in \arg\max_{x \in A} u(x)$ for every A. (Why?)

Hence the story can not be refuted. It does not correspond to a strict subset of the set of observables. There is no axiom which it necessarily satisfies. It has no empirical content.

Homework: For every A, there exists a linear order P_A such that $c(A) = \max(A, P_A)$. Does this story have empirical content?

… and why do we need other models? Here are two very related possible reasons.

1. Empirical evidence suggests that people make choices which do not satisfy α.

2. There are interesting features of decision making which do not appear in the story of rationality.

Model: Rational shortlisting

A choice function c is a rational shortlisting method (RSM) if there exists a pair (P_1, P_2) of asymmetric binary relations on X such that for every $A \in 2^X$

$$\{c(A)\} = \max(\max(A, P_1), P_2).$$

Here, by $\max(B, P_i)$ we mean $\{x \in B : yP_ix \text{ for no } y \in B\}$.

The asymmetric binary relations P_1 and P_2 are called rationales.

Question: Is it common to approach choice problems with more than one rationale in mind?

The DM goes through 2 sequential rounds of elimination of alternatives. In the first round, he only retains the elements that are maximal according to P_1 This is the shortlist. In the second round he retains only the element (singular!) of the shortlist that is maximal according to P_2.

Crucially the rationales and the order in which they are applied do not depend on the choice set. P_1 shortlists and P_2 chooses from the shortlist in every menu.

If P_1 is a linear order, the resulting RSM is rationalizable.

P_2 may or may not be complete but it needs to be decisive on the shortlist, i.e., select from it a single element.

Homework: Let $X = \{x, y, z\}$, xP_1y, yP_2z and zP_2x. Construct the RSM.

Homework: Argue that the following choice function is not a RSM.

xy	yz	xz	xyz
x	y	x	y

Initial observations:

1. If the pair (P_1, P_2) of rationales gives rise to the RSM c, we write $c = c^{(P_1,P_2)}$. Note however that not every pair (P_1, P_2) gives rise to a RSM.

2. We may have $(P_1, P_2) \neq (P_1', P_2')$, yet $c^{(P_1,P_2)} = c^{(P_1',P_2')}$.

Hence even if we knew that c is a RSM, there is no hope of identifying the underlying rationales uniquely. This problem is sometimes referred to as underidentification. The two questions we wish to resolve with our axiomatization of RSMs are:

1. How can we guarantee that the choice function we observe is a RSM?

2. Can we perform the revelation exercise for RSMs at least partially?

First consider the following condition.

WARP: If $x = c(S)$, $y \in S \backslash \{x\}$ and $x \in T$, then $y \neq c(T)$.

Homework: Show that WARP is equivalent to α.

Here is a weakening of WARP.

W-WARP: Let $x \neq y$. If $\{x, y\} \subset B \subset A$ and $x = c(\{x, y\}) = c(A)$, then $y \neq c(B)$.

Homework: Show that WARP implies W-WARP, but not vice versa. (Note: WARP and W-WARP both have the same "then part" but the "if part" of W-WARP is stronger than the "if part" of WARP.)

Theorem: A choice function c is a RSM if and only if it satisfies γ and W-WARP.

Proof: ($\Rightarrow$) Homework.

($\Leftarrow$) Suppose that c satisfies γ and W-WARP. For $x \neq y$, say
xP_1y if there is no $S \subseteq X$ such that $y = c(S)$ and $x \in S$,
xP_2y if $x = c(\{x, y\})$.
The relations P_1 and P_2 are asymmetric. (Why?) Now fix x and S such that $x = c(S)$. We need to show that

$$\{x\} = \max(\max(S, P_1), P_2).$$

Step 1: $x \in \max(S, P_1)$. This follows since $x = c(S)$, and therefore there is no $y \in S \backslash \{x\}$ such that yP_1x. Hence x must be shortlisted.

Step 2: $x \in \max(\max(S, P_1), P_2)$. Suppose $z \in S$ and zP_2x. Towards a contradiction, suppose z survives to the second round, i.e. for every $y \in S\backslash\{z\}$, $y\not P_1 z$. Hence for every $y \in S\backslash\{z\}$, there is some $A_{yz} \subseteq X$ such that $z = c(A_{yz})$ and $y \in A_{yz}$. Then, by γ, $z = c(\cup_{y\in S\backslash\{z\}}A_{yz})$. Note $S \subseteq \cup_{y\in S\backslash\{z\}}A_{yz}$. If $S = \cup_{y\in S\backslash\{z\}}A_{yz}$, we get a contradiction to $x = c(S)$. If $S \subset \cup_{y\in S\backslash\{z\}}A_{yz}$, since $c(\{x, z\}) = z$, we get a contradiction to W-WARP.

Step 3: To finish the proof, we need to show that x is the **only** alternative in S which survives both rounds of elimination. Suppose some $y \neq x$ also also belongs to $\max(\max(S, P_1), P_2)$. Case 1: $c(\{x, y\}) = y$. In this case yP_2x by definition of P_2. But we showed that for any $z \in S$, if zP_2x then z is not shortlisted, a contradiction. Case 2: $c(\{x, y\}) = x$. In this case xP_2y. But since $x \in \max(S, P_1)$, x eliminates y in the shortlist and $y \notin \max(\max(S, P_1), P_2)$, a contradiction. This finishes the proof.

Remark: Note: $xP_1y \Rightarrow xP_2y$. Is this a problem? In particular, does this mean that we can eliminate the first rationale? Definitely not... Consider the choice function c on $\{x, y, z\}$ given by $c(\{x, y\}) = x$, $c(\{y, z\}) = y$, $c(\{x, z\}) = z$ and $c(\{x, y, z\}) = x$. The construction in the proof gives: xP_1y and yP_1z for the first rationale and xP_2y, yP_2z and zP_2x for the second rationale. (Recall that all we impose on the rationales is asymmetry. In particular we do not require transitivity or completeness.) Now it is easy to check that $\{c(A)\} = \max(\max(A, P_1), P_2)$ for every $A \subseteq \{x, y, z\}$. This verifies that c is an RSM. Note that whenever two alternatives are ordered by the first rationale, they are ordered in exactly the same way by the second rationale. Consider the alternate choice procedure obtained by eliminating P_1 and maximizing only P_2 in a single stage. Such a procedure would not be able to choose out of $\{x, y, z\}$. Hence the existence of a first stage is essential! As a further remark, note that the choice function in this example is also a RSM with different rationales: $yP_1'z$, $xP_2'y$ and $zP_2'x$.

Corollary: If a RSM is not rationalizable, it fails NBC.

Proof: Suppose a RSM c fails α. Since α is equivalent to γ and NBC, c has to fail either γ or NBC (or both). By Theorem 1, c satisfies γ, hence it must fail NBC. ■

Hence all violations of "rationality" by an RSM is of a specific type involving pairwise cycles of choice.

Homework: Show that W-WARP and γ are logically independent.

More homework:

1. Show that W-WARP is equivalent to the following condition: If $x \neq y$, $x, y \in A'' \subset A' \subset A$ and $x = c(A'') = c(A)$, then $y \neq c(A')$.

2. Consider the following condition called weak path independence (WPI): If $x = c(A)$ and $y = c(A \backslash \{x\})$, then $x = c(\{x, y\})$. Show that γ implies WPI but not vice versa. (ii) Show that W-WARP and WPI together are not sufficient for a choice function to be a RSM.

3. Consider the following condition: If $x \neq y$, $\{x, y\} \subset B \subset A$ and $x = c(\{x, y\}) = c(A)$, then $x = c(B)$. Show that this condition strengthens W-WARP. Show that RSMs need not satisfy this condition.

4. Characterize the following model: There exist (1) a weak order P_1 and (2) a linear order P_2 such that $\{c(A)\} = \max(\max(A, P_1), P_2)$.

Model: Choice with limited attention

A filter is a map $\Gamma : 2^X \rightarrow 2^X$ satisfying $\Gamma(A) \subseteq A$ for every menu A. The interpretation is that faced with the menu A, the DM only considers the subset $\Gamma(A)$ for choice.

Homework: Suppose that there exist (1) a linear order P and (2) a filter Γ such that for every menu A, $c(A) = \max(\Gamma(A), P)$. Does this story constitute a model?

A filter Γ is an attention filter if it satisfies the following condition:

$$x \notin \Gamma(A) \Rightarrow \Gamma(A) = \Gamma(A \backslash x).$$

The interpretation is that, the DM only pays attention to members of $\Gamma(A)$ and her attention span is not affected if an alternative which she has not paid attention to is removed from the menu.

A choice function c is a choice with limited attention (CLA) whenever there is a linear order P and an attention filter Γ such that for all nonempty $A \subseteq X$,

$$c(A) = \max(\Gamma(A), P).$$

Clearly, the trivial filter given by $\Gamma(A) = A$ for all A is an attention filter (Why?) and yields the rational choice benchmark.

Example: Consider the following choice function:

xy	yz	xz	xyz
x	y	z	x

Is this behavior consistent with the CLA model? Note the violation of α, as $x = c(xyz)$ is not chosen in xz, where it belongs. This implies, if a CLA produced this behavior, that $y \in \Gamma(xyz)$ and xPy. Now suppose $\Gamma(xyz) = xy$, $\Gamma(xz) = z$, $\Gamma(yz) = y$ and $\Gamma(xy) = xy$. Note Γ satisfies the attention order property. If the underlying preference is given by $xPyPz$, the consequent CLA produces the behavior in the example.

This example indicates that the primitives of a CLA (like those of a RSM) cannot be uniquely identified.

Example: Let $X = \{x, y, z\}$, and consider the following choice function:

xy	yz	xz	xyz
x	y	x	z

Is this behavior consistent with the CLA model? Note there are two violations of α, as $z = c(xyz)$ is not chosen in xz and yz. Hence there is more to learn from. If this behavior is produced by a CLA, it must be the case that x and y were paid attention to in xyz, as their removal led to choice reversals. Hence, $\Gamma(xyz) = xyz$, zPy and zPx. Now suppose $\Gamma(xy) = \Gamma(xz) = x$ and $\Gamma(yz) = y$. Note Γ satisfies the attention order property. If the underlying preference is given by $zPyPx$, the consequent CLA produces the behavior in the example.

Note that with more α violations, more can be identified.

Homework: If $X = \{x, y, z\}$, then all 24 choice functions are CLAs.

Homework: Construct a choice function which is not a CLA.

Recall...

WARP: For every A and B if $c(A) \in B$ and $c(B) \in A$, then $c(A) = c(B)$.

Here is another way of phrasing WARP.

WARP-rephrased: For every A, there exists $x \in A$ such that

$$\left.\begin{array}{c} x \in B \\ c(B) \in A \end{array}\right\} \Rightarrow x = c(B).$$

Here is a weakening of the rephrased version of WARP:

WARP-LA: For every A, there exists $x \in A$ such that

$$\left.\begin{array}{c} c(B) \in A \\ c(B) \neq c(B \backslash x) \end{array}\right\} \Rightarrow x = c(B).$$

The axiom says that every set A should have a special element, which, if present in a set B whose choice lies in A, and "if paid attention to in B", is the choice in B.

Theorem: A choice function is a CLA if and only if it satisfies WARP-LA.

Proof: ($\Rightarrow$) Suppose c is a CLA defined by linear order P and attention filter Γ. We need to show it satisfies WARP-LA. Take any A and let $x = \max(A, P)$. Take B such that $c(B) \in A$ and $c(B) \neq c(B \backslash x)$. Since $c(B) \neq c(B \backslash x)$, it follows that $x \in \Gamma(B)$ and $c(B)Px$. On ther other hand $c(B) \in A$, giving $xPc(B)$. Since P is a linear order, it must be that $x = c(B)$, as desired. This establishes WARP-LA.

($\Leftarrow$) Now suppose c satisfies WARP-LA. We need to show that c is a CLA. Define a binary relation P_0 as follows:

$$xP_0y \text{ iff for some } A,\ c(A) = x \neq c(A \backslash y).$$

We will show that P_0 is *acyclic*, i.e., there do not exist distinct alternatives $x_1, ..., x_n$ such that $x_1 P_0 x_2, ..., x_{n-1} P_0 x_n$, and $x_n P_0 x_1$.

Suppose P_0 contains such a cycle: there exist distinct alternatives $x_1, ..., x_n$ such that $x_1 P_0 x_2, ..., x_{n-1} P_0 x_n$ and $x_n P_0 x_1$. This implies that there exist menus $B_1, ..., B_n$ such that

$$\begin{array}{rcl} x_1 & = & c(B_1) \neq c(B_1 \backslash x_2) \\ x_2 & = & c(B_2) \neq c(B_2 \backslash x_3) \\ & & ... \\ x_n & = & c(B_n) \neq c(B_n \backslash x_1). \end{array}$$

Then WARP-LA fails in the set $A = \{x_1, ..., x_n\}$: Setting $x_0 = x_n$ and $B_0 = B_n$, for every $x_i \in A$, there exists a set B_{i-1} such that $x_i \in B_{i-1}$, $x_{i-1} = c(B_{i-1}) \in A$, $c(B_{i-1} \backslash x_i) \neq c(B_i)$ but $x_i \neq c(B_{i-1})$. This is a contradiction which establishes P_0 is acyclic.

Now we will use the following fact without proving it: For every acyclic binary relation, there is a linear order which contains it.

Hence there exists a linear order P which contains P_0, in the sense that whenever xP_0y, xPy as well. Now, using P, define a map Γ as follows: for every A

$$\Gamma(A) = \{c(A)\} \cup \{a \in A : c(A)Pa\}.$$

Clearly $\Gamma(A)$ is a nonempty subset of A. We need to show two things to finish the proof: (1) that Γ satisfies the attention filter property, (2) that $c(A) = \max(\Gamma(A), P)$. The second statement follows by construction of Γ. To see (1), fix A and suppose $x \notin \Gamma(A)$. There is nothing to show if $x \notin A$. Suppose $x \in A$. Then $x \neq c(A)$ and $xPc(A)$. Then $c(A)\not{P}_0x$, in other words, for every set T such that $c(A) = c(T)$, $c(A) = c(T\backslash x)$ as well. This implies that $c(A) = c(A\backslash x)$. Hence, by definition of Γ, $\Gamma(A) = \Gamma(A\backslash x)$ and Γ is an attention filter. This completes the proof.

More homework:

1. Consider the following story. There exist (1) a linear order P, and (2) a filter Γ satisfying [if $x, y \in \Gamma(A)$, $A \subset B$, $y \in \Gamma(B)$, then $x \in \Gamma(B)$ as well] such that $c(S) = \max(\Gamma(S), P)$ for every menu S. Interpret the story, in particular the condition it imposes on Γ. Does the story have empirical content? If yes, formulate an axiom which it satisfies.

2. Consider the following story. There exist (1) a linear order P, and (2) a filter Γ satisfying [if $x \in A \subset B$, and $x \in \Gamma(B)$, then $x \in \Gamma(A)$ as well] such that $c(S) = \max(\Gamma(S), P)$ for every menu S. Interpret the story, in particular the condition it imposes on Γ. Does the story have empirical content? If yes, formulate an axiom which it necessarily satisfies.

Homework: A partition of X is a collection $(S_1, ..., S_n)$ such that (1) $n \in \{1, ..., |X|\}$, (2) $S_i \neq \emptyset$ for every i, (3) $S_i \cap S_j = \emptyset$ for any distinct i and j, and (4) $\bigcup_{i=1}^{n} S_i = X$. (Hence a partition defines an equivalence class on X.) For any partition $(S_1, ..., S_n)$ of X and any $x \in X$, let S^x be the (necessarily unique) set in the partition which contains x.

Suppose that for some rationalizable choice function d, integer n, partition $(S_1, ..., S_n)$ of X and linear order P,

$$c(A) = \max(S^{d(A)} \cap A, P).$$

(0) Interpret c.
(1) How does c behave if $S^x = \{x\}$ for every x? What if $S^x = X$ for every x?
(2) Is c rational?
(3) Formulate an axiom which c satisfies.
(4) How are the primitives of this model revealed by choice behavior?

ENVIRONMENT: CHOICE CORRESPONDENCES

X is a finite set, menus are nonempty subsets of X, 2^X is the set of menus.

A choice correspondence is a map $c : 2^X \to 2^X$ satisfying $c(A) \subseteq A$ for every $A \in 2^X$.

Note: $c(A) \neq \emptyset$. $x \in c(A)$ iff x is "choosable" in A.

Model: Rational choice

A choice correspondence c is rational(izable) if there exists a weak order P (negatively transitive, asymmetric) binary relation on X such that for every menu A,

$$c(A) = \max(A, P).$$

Here by $\max(A, P)$ we mean $\{x \in A : yPx \text{ for no } y \in A\}$

If c is rational and P is the underlying weak order, then we write $c = c^P$.

Initial observations:

1. For every weak order P, c^P (which maximizes P on every menu A) is a choice correspondence.

2. If $P \neq P'$, then $c^P \neq c^{P'}$. (Why?)

Hence if we observe a rational choice correspondence c^P, we may infer P uniquely. This is called the exercise of revealed preference. But:

1. How can we guarantee that the choice correspondence we observe is rational(izable)?

2. How do we perform the revelation exercise?

Here are some axioms.

α: if $x \in c(A)$ and $x \in B \subset A$, then $x \in c(B)$.

γ: if $x \in c(A) \cap c(B)$, then $x \in c(A \cup B)$.

NBC: if $x \in c(\{x, y\})$ and $y \in c(\{y, z\})$, then $x \in c(\{x, z\})$.

IIA: if $x \notin c(A)$, then $c(A) = c(A \backslash \{x\})$.

PI: $c(A \cup B) = c(c(A) \cup c(B))$. (HW: PI iff IIA and α.)

β: if $x, y \in c(B)$, $B \subset A$ and $y \in c(A)$, then $x \in c(A)$.

WARP: if $x, y \in A \cap B$, $x \in c(A)$ and $y \in c(B)$, then $x \in c(B)$.

Arrow's Axiom: if $B \subset A$ and $B \cap c(A) \neq \emptyset$, then $c(B) = c(A) \cap B$.

Weak Arrow: For every A, there exists $x \in A$ such that if $B \subset A$ and $B \cap c(A) \neq \emptyset$, then $c(B) = c(A) \cap B$.

Anchor Axiom: For every A, there exists $x \in A$ such that if $x \in S \subset T \subseteq A$, then $x \in c(S) \subseteq c(T)$.

Dominant Alternative Axiom: For every A, there exists $x \in A$ such that if $y \in c(B) \cap A$ and $x \in B$, then $x \in c(B)$.

Dominant Anchor Axiom: For every A, there exists $x \in A$ such that (i) if $x \in S \subset T \subseteq A$, then $x \in c(S) \subseteq c(T)$, (ii) if $y \in c(B) \cap A$ and $x \in B$, then $x \in c(B)$.

Homework: Do the Anchor Axiom and the Dominant Alternative Axiom together imply the Dominant Anchor Axiom?

Theorem: The following are equivalent.

(1) c is rational.

(2) c satisfies α, γ and NBC.

(3) c satisfies α and β.

(4) c satisfies WARP.

(5) c satisfies ARROW.

Proof: Homework.

Homework: Prove the following statement. There exists an acyclic binary relation P such that $c(A) = \{x \in A : yPx \text{ for no } y \in A\}$ if and only if c satisfies α and γ.

Homework: Prove the following statement. There exists a partial order P such that $c(A) = \{x \in A : yPx \text{ for no } y \in A\}$ if and only if c satisfies α, γ and IIA.

Homework: Prove the following statement. There exists an interval order P such that $c(A) = \{x \in A : yPx \text{ for no } y \in A\}$ if and only if c satisfies α, γ, IIA and the Dominant Alternative Axiom. (Also equivalent to α, γ, IIA and the Anchor Axiom. Also equivalent to α and the Anchor Axiom.)

Homework: Prove the following statement. There exists a semiorder P such that $c(A) = \{x \in A : yPx \text{ for no } y \in A\}$ if and only if c satisfies α, γ, IIA and the Dominant Anchor Axiom. (Also equivalent to α and the Dominant Anchor Axiom.)

Some bddly rational choice correspondences

Monotone semiorder maximization equivalent to the Dominant Anchor Axiom.

Monotone interval order maximization is equivalent to the Anchor Axiom.

Similarity-based mistakes type 1 is equivalent to Weak Arrow.

Similarity-based mistakes type 2 is equivalent to the Anchor Axiom.

For some n and linear orders $P_1, ..., P_n$, $c(A) = \bigcup_{i=1}^{n} \max(A, P_i)$ iff PI.

ENVIRONMENT: STOCHASTIC CHOICE

X is a finite set of alternatives, menus are nonempty subsets of X, 2^X is the set of menus.

A stochastic choice function is a map $p : X \times 2^X \rightarrow [0, 1]$ satisfying
(1) $p(x, A) = 0$ for every A, every $x \notin A$, and
(2) $\sum_{x \in A} p(x, A) = 1$ for every A.

Interpretation: time series versus cross section

Model: Random preferences

A stochastic choice function p is a random preference model (RPM) if there exists a probability μ over linear orders on X such that

$$p(x, A) = \sum_{P : x = \max(A, P)} \mu(P)$$

for every A, every $x \in A$.

The model is sometimes referred to as random utility model (RUM).

p satisfies regularity if $p(x, A) \leq p(x, B)$ whenever $x \in B \subset A$.

Homework: (i) every RPM satisfies regularity, (ii) there are regular stochastic choice functions which are not RPMs.

Failures of regularity are often observed: attraction effect, compromise effect.

Here is a condition on p.

For every A, every $x \in A$,

$$\sum_{B \supseteq A} (-1)^{|B \setminus A|} p(x, B) \geq 0.$$

The condition is called Block-Marschak polynomial condition.

Theorem: A stochastic choice function is a RPM if and only if it satisfies the Block-Marschak polynomial condition.

Proof: Skip.

Model: Luce

p is a Luce model if there exists some $w : X \to (0,1)$ such that

$$p(x, A) = \frac{w(x)}{\sum_{y \in A} w(y)}$$

for every A, every $x \in A$.

Theorem: Every Luce model is a RPM.

Proof: Skip.

Homework: Find a RPM which is not a Luce model.

Here are two conditions on p.

Luce IIA: For every A, every $x, y \in A$,

$$\frac{p(x, \{x, y\})}{p(y, \{x, y\})} = \frac{p(x, A)}{p(y, A)}.$$

Positivity: For every A, every $x \in A$, $p(x, A) > 0$.

Theorem: p is a Luce model if and only if p satisfies Positivity and Luce IIA.

Proof: Homework.

Homework: Suppose that for some linear order P and a probability σ on 2^X, the choice frequencies are as follows:

$$
\begin{aligned}
p(x, A) &= \sum_{S: x=\max(A\cap S, P)} \sigma(S) + \sum_{S: A\cap S=\emptyset} \sigma(S) \text{ if } x = \max(A, P) \\
p(x, A) &= \sum_{S: x=\max(A\cap S, P)} \sigma(S) \text{ if } x \neq \max(A, P)
\end{aligned}
$$

for every A, every $x \in A$.

(0) Interpret p.
(1) Is p a stochastic choice function?
(2) Does p satisfy regularity?
(3) Is p a RPM?
(4) Formulate an axiom which p satisfies.
(5) How are the primitives P and σ of this model revealed by choice behavior?