Game Theory

Weak Perfect Bayesian Equilibrium

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In this note, we study dynamic Bayesian games. First, we see that subgame perfect equilibrium is not necessarily a plausible solution concept. Second, we introduce the concept of **(weak) perfect Bayesian equilibrium**. Finally, we see that this concept has some advantages but is inconsistent with subgame perfection.

1 Example

Example 1. Consider the extensive-form game of Figure 1. It has two Nash equilibria (R_1, R_2) and (L_1, L_2). It has a unique subgame perfect equilibrium (R_1, R_2).

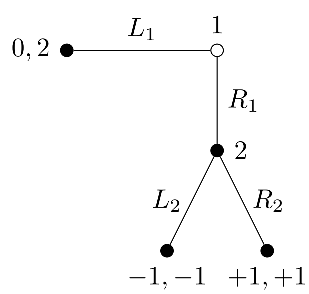

Figure 1: the extensive-form game

Next, consider the variant of Figure 2. It has a redundant action R_1', which is just a copy of action R_1. Player 2 cannot distinguish R_1 and R_1'. Since the variant has no proper subgame, all Nash equilibria are subgame perfect. In particular, (L_1, L_2) is a subgame perfect equilibrium.

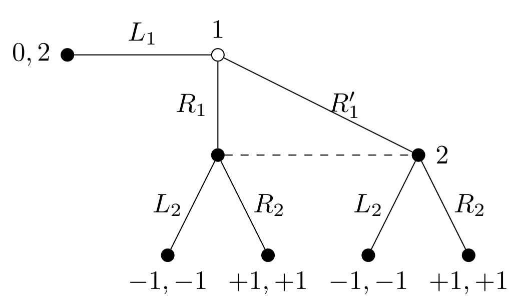

Figure 2: the extensive-form game with “redundant” action R_1'

The two games of Figures 1 and 2 “should” be identical, but as demonstrated above, they have totally different subgame perfect equilibria. $\square$

Box (Finite extensive-form game):

Definition 1. A **finite extensive-form game** is a tuple $\Gamma = \langle I, X, P, (u_i, H_i)_i, \pi \rangle$ such that:

1. I is the set of players.
   - I may or may not include player 0, who is called **nature**.
2. X is the finite set of nodes such that:
   - $\rightarrow \subset X \times X$ is a binary relation that is transitive and asymmetric.
     - It defines the "direction" between nodes.
     - There exists an **initial node** $\varnothing \in X$ such that for each $x \in X$, $\varnothing \rightarrow x$.
   - $z \in X$ is a **terminal node** if there exists no $x \in X$ such that $z \rightarrow x$.
     - Z is the set of all terminal nodes.
   - $\twoheadrightarrow$ is such that $x \twoheadrightarrow y$ if $x \rightarrow y$ but there exists no $x' \in X$ such that $x \rightarrow x' \rightarrow y$.
     - $y \in X \setminus \{\varnothing\}$ has a unique $x \in X$ such that $x \twoheadrightarrow y$.
   - $\varnothing \twoheadrightarrow x_1 \twoheadrightarrow x_2 \twoheadrightarrow \cdots$ is a path (from $\varnothing$).
     - Every path has a last element, and the last element must be a terminal node.
3. $P : X \setminus Z \rightarrow I$ is the function that assigns to node $x \in X \setminus Z$ a player moving at node x.
   - $X_i = \{x \in X : P(x) = i\}$ is the set of player i's nodes.
   - $A(x) = \{y \in X : x \twoheadrightarrow y\}$ is the set of actions for player $P(x)$ at node $x \in X$.
4. $u_i : Z \rightarrow \mathbb{R}$ is player i's payoff function.
5. H_i is player i's **partition** of the set X_i such that:
   - $h_i \in H_i$ is called player i's **information set**.
   - $A(x) = A(x')$ for each $x, x' \in h_i$, which is also denoted by $A(h_i)$.
6. π is a function that assigns to node $x \in X_0$ a distribution $\pi(x) \in \Delta(A(x))$.
   - $0 \in I$ denotes the player called nature, who moves at random.
   - $\pi(x) \in \Delta(A(x))$ is nature's random choice of action $y \in A(x)$ at node $x \in X_0$.

To see how we could fix the issue illustrated in Example 1, we recall the motivation for the notion of subgame perfect equilibrium. In the example, we deem Nash equilibrium (L_1, L_2) implausible because playing L_2 at R_1 is *irrational*. The notion of subgame perfect equilibrium requires that every player be rational at her every *singleton* information set. However, it imposes no restriction on how she behaves at a *non-singleton* information set. Hence, a natural idea to fix the issue is to strengthen the requirement:

> An equilibrium notion should impose some rationality restriction on a player's behavior at her non-singleton information sets.

To define her rational behavior in a non-singleton information set, we define her belief about which node she is at in the information set—i.e., a probability distribution over the nodes in the information set.

2 Weak Perfect Bayesian Equilibrium

Assessment We define a player's belief over the nodes for each information set.

Definition 2. In a finite extensive-form game Γ with perfect recall, player i's **(behavioral)**

strategy β_i is a function $\beta_i : H_i \to \Delta(A_i)$ such that for each $h_i \in H_i$, $\beta_i(h_i) \in \Delta(A(h_i))$. Call $\beta = (\beta_i)_i$ a (behavioral) strategy profile.

Definition 3. In a finite extensive-form game Γ with perfect recall, a **belief system** μ consists of probability distributions $\mu(\cdot \mid h)$, one for each information set h. Call $\mu(\cdot \mid h)$ the **belief at information set** h.

Definition 4. In a finite extensive-form game Γ with perfect recall, an **assessment** is the pair (β, μ) of a strategy profile $\beta = (\beta_i)_i$ and a belief system μ.

Expected Payoff Suppose that a finite extensive-form game Γ is played according to a given assessment (β, μ). Then, player i's expected payoff at a given node x is $\mathbb{E}_\beta[u_i(z) \mid x]$, where $\mathbb{E}_\beta$ is the expectation with respect to the outcome distribution generated by β. Accordingly, the payoff that player i is expected to receive at an information set h is

$$\mathbb{E}_{\beta,\mu}[u_i(z) \mid h] = \sum_{x \in h} \mu(x \mid h)\mathbb{E}_\beta[u_i(z) \mid x].$$

Sequential Rationality We define the rationality concept—called sequential rationality—that is suitable in extensive-form games. The sequential rationality requires that given player $-i$'s strategy profile β_{-i} and the belief system μ, player i has no profitable deviation.

Definition 5. In a finite extensive-form game Γ with perfect recall, an assessment (β, μ) is **sequentially rational** if for each $i \in N$, each β'_i, and each $h_i \in H_i$, it holds that

$$\mathbb{E}_{\beta_i,\beta_{-i},\mu}[u_i(z) \mid h_i] \geq \mathbb{E}_{\beta'_i,\beta_{-i},\mu}[u_i(z) \mid h_i].$$

Consistency A player updates her belief according to Bayes' rule, when she observes an event. However, Bayes's rule applies only when the observed event has non-zero probability. In other words, Bayes's rule by itself does not specify a belief when the event has zero probability.

Example 2. In Figure 2, if player 1 chooses L_1, R_1, R'_1 with probabilities $\frac{1}{2}, \frac{1}{3}, \frac{1}{6}$ respectively, what belief should player 2 have? As usual, we assume that player 2 forms his belief according to Bayes' rule. His belief should assign to the left node probability $\frac{1/3}{1/3+1/6} = \frac{2}{3}$ and to the right node probability $\frac{1/6}{1/3+1/6} = \frac{1}{3}$.

What if player 1 chooses L_1, R_1, R'_1 with probabilities $1, 0, 0$ respectively? Bayes' rule no longer applies to player 2's information set. In other words, Bayes' rule does not exclude any belief on that information set. □

Therefore, we require that players update their beliefs according to Bayes' rule *whenever possible*. Let us introduce a useful terminology:

Definition 6. In an extensive-form game Γ with perfect recall, an information set is **on-path** if it is reached with strictly positive probability when a given strategy profile β is played. Otherwise, it is called **off-path**.

In this terminology, we use Bayes' rule to beliefs over on-path information sets.

Definition 7. In a finite extensive-form game Γ with perfect recall, an assessment (β, μ) is **on-path consistent** if $\mu(\cdot \mid h)$ is a Bayes-updated belief for every on-path information set h.[1]

Weak Perfect Bayesian Equilibrium Weak Perfect Bayesian equilibrium is an assessment that is sequentially rational and on-path consistent.

Definition 8 (Mas-Colell et al. 1995)**.** In a finite extensive-form game Γ with perfect recall, an assessment (β, μ) is a **weak perfect Bayesian equilibrium** if the following conditions are satisfied:

1. (β, μ) is sequentially rational.
2. (β, μ) is on-path consistent.

Example 3. In Figure 2, every Nash equilibrium is subgame perfect. In particular, we have a "strange" subgame perfect equilibrium in which player 1 plays L_1 and player 2 plays L_2.

However, there is no perfect Bayesian equilibrium in which player 1 plays L_1. Player 2 has a belief at his information set h, which consists of two nodes R_1 and R_1'. We write $\mu(R_1 \mid h)$ and $\mu(R_1' \mid h)$ for his belief. Regardless of his belief, he always accommodates as a sequentially rational response. If player 2 chooses R_2, player 1 would deviate from L_1 to R_1.

There is a perfect Bayesian equilibrium in which player 1 plays R_1 or R_1' and player 2 plays R_2. Since, as seen above, it is sequentially rational that player 1 plays R_1 or R_1' and player 2 plays R_2 (for any belief $\mu(\cdot \mid h)$), it suffices to construct an on-path consistent belief system. This is not difficult. Since player 1's strategy β_1 is such that for any $b \in [0,1]$,

$$\beta_1(L_1) = 0, \quad \beta_1(R_1) = b, \quad \beta_1(R_1') = 1 - b,$$

we have the belief $\mu(\cdot \mid h)$ such that $\mu(R_1 \mid h) = b$ and $\mu(R_1' \mid h) = 1 - b$. □

3 More Examples

3.1 Unconvincing Prediction by Weak Perfect Bayesian Equilibrium

Example 4. Consider the game of Figure 3. Since each player has two actions and one information set, we represent player i's strategy by probability $s_i \in [0,1]$ of choosing action L_i. There is one non-singleton information set $\{L_1L_2, L_1R_2\}$ for player 3, denoted h_3. We denote

[1] This terminology of "on-path" consistency is not standard but is used for a pedagogical reason.

the belief $\mu(\cdot \mid h_3)$ by probability $\nu \in [0,1]$ on the left node. Hence, let (s_1, s_2, s_3, ν) denote an assessment.

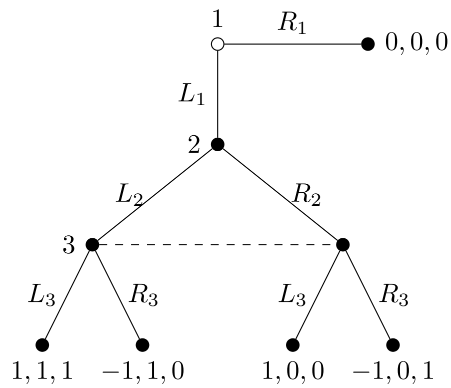

Figure 3: perfect Bayesian equilibrium

Player 2 has a unique sequentially rational strategy $s_2 = 1$ (regardless of player 3's strategy). This is simply because he gains payoff 1 from L_2 but payoff 0 from R_2, regardless of player 3's strategy.

An assessment $(s_1, s_2, s_3, \nu) = (1,1,1,1)$ is a weak perfect Bayesian equilibrium. Player 1's strategy $s_1 = 1$ is sequentially rational because given players -1's strategies, player 1 receives payoff 1 from L_1 but payoff 0 from R_1. Player 2's sequential rationality is discussed above. Player 3's strategy $s_3 = 1$ is sequential rational because given the belief (putting probability 1 on the left node), player 3 receives payoff 1 from L_3 but payoff 0 from R_3. His belief $\mu(\cdot \mid h_3)$ is on-path consistent because his information set h_3 is on-path and follows from Bayes' rule.

An assessment $(s_1, s_2, s_3, \nu) = (0,1,0,0)$ is a weak perfect Bayesian equilibrium. Player 1's strategy $s_1 = 1$ is sequentially rational because given players -1's strategies, player 1 receives payoff -1 from L_1 but payoff 0 from R_1. Player 2's sequential rationality is discussed above. Player 3's strategy $s_3 = 1$ is sequentially rational because given the belief (putting probability 1 on the right node), player 3 receives payoff 0 from L_3 but payoff 1 from R_3. His belief $\mu(\cdot \mid h_3)$ is on-path consistent because his information set h_3 is off-path, which allows for any belief.

However, the equilibrium $(s_1, s_2, s_3, \nu) = (0,1,0,0)$ is "strange." Since player 2 plays L_2, it follows that whenever player 3 had his turn, he should believe that he is at the left node. How can we eliminate this equilibrium? The key idea is to put some restriction on an off-path belief. The resulting concept is called sequential equilibrium. $\square$

3.2 Weak Perfect Bayesian Equilibrium versus Subgame Perfect Equilibrium

Weak perfect Bayesian equilibrium is neither a stronger nor weaker equilibrium concept than subgame perfect equilibrium.

Proposition 1. *In a finite extensive-form game, it holds that:*

1. *Not every weak perfect Bayesian equilibrium is a subgame perfect equilibrium.*
2. *Not every subgame perfect equilibrium is a weak perfect Bayesian equilibrium.*

Proof. To see the first claim, note that in Example 4, a unique subgame perfect equilibrium is such that each player i plays L_i but a weak perfect Bayesian equilibrium admits another strategy profile. To see the second claim, recall Examples 1 and 3. ■

Proposition 1 is responsible for the modifier "weak." The "weakness" results from that fact that Definition 8 does not impose any restriction on the beliefs on off-path information sets. Hence, the refinement of weak perfect Bayesian equilibrium requires to somehow restrict the beliefs on off-path information sets. Notable refinement notions include (non-weak) perfect Bayesian equilibrium (Remark 1) and sequential equilibrium, which we will discuss in another note.

Remark 1. We mention the notion of perfect Bayesian equilibrium (without the modifier "weak"). Gibbons (1992) refers, as (non-weak) perfect Bayesian equilibrium, to the weak perfect Bayesian equilibrium whose beliefs on off-path information sets are determined by Bayes' rule and the equilibrium strategies *whenever possible*. Gibbons (1992) does not give a formal statement that avoids the vague instruction, "whenever possible." Fudenberg & Tirole (1991) gives a formal definition of the (non-weak) perfect Bayesian equilibrium. □

References

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