Game Theory

Normal-Form Representation of Games

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How can we formalize a strategic environment—i.e., a game? There are multiple ways to represent the game. Among them, **normal-form representation** is the simplest one. It highlights fundamental elements to describe a game.

1 Normal-Form Representation

Example 1 (Prisoners' Dilemma)**.** Two prisoners 1 and 2 are being interrogated. They are in separate rooms and cannot communicate with each other. Prosecutors do not have enough evidence to convict them on the principal charge but have enough evidence to convict them on a lesser charge. Each prisoner chooses either to remain silent or to testify that they have committed the crime. We call the former action C (to cooperate with the other prisoner) and the latter action D (to deviate). Penalty on the prisoners depends on which actions they choose:

- If prisoners 1 and 2 take C and C, they serve 1- and 1-year-long imprisonment.
- If prisoners 1 and 2 take C and D, they serve 3- and 0-year-long imprisonment.
- If prisoners 1 and 2 take D and C, they serve 0- and 3-year-long imprisonment.
- If prisoners 1 and 2 take D and D, they serve 2- and 2-year-long imprisonment.

How would we abstract this story? The minimum element is: **Who does what for what?** The minimum element of the prisoners' dilemma is: Two prisoners 1 and 2 each choose either C or D, to minimize the penalty. $\square$

1.1 Normal-Form Representation

To represent a strategic environment in a normal form, it suffices to list up the minimum elements—i.e., who does what for what?

Definition 1. A **normal-form game**, or the **normal-form representation** of a game, is a tuple $G = \langle I, (A_i, u_i)_i \rangle$ such that:

1. I is the set of players.
   - i is a generic player and $-i$ is all the players but i.
2. A_i is the set of actions, also called pure strategies, available to player i.[1]
   - $a = (a_i)_i$ is an action profile, where $a_i \in A_i$ is player i's action.

[1] We often call the set of actions an action space. Here, "space" is a synonym for "set."

- $A = \prod_i A_i$ is the set of action profiles.

3. $u_i : A \to \mathbb{R}$ is player i's payoff function.

Example 2. The prisoners' dilemma has the normal-form representation G such that:

1. $I = \{1, 2\}$ is the set of players.

2. $A_i = \{C, D\}$ is the set of player i's actions.

3. $u_i : A \to \mathbb{R}$ is player i's payoff function defined by the following table:

	C	D
C	$-1, -1$	$-3, 0$
D	$0, -3$	$-2, -2$

Table 1: prisoners' dilemma

In this table, the "row" player is player 1, while the "column" player is player 2. In each cell, the first number is the row player's payoff and the second number is the column player's payoff. For example, cell "$-3, 0$" means that players 1 and 2 receive payoffs -3 and 0 respectively when they play $(a_1, a_2) = (C, D)$.

This table has all the elements for the normal-form representation. Hence, this **payoff table representation** is equivalent to the normal-form representation. □

Simutaneity of Taking Actions The interpretation of a normal-form game is straightforward. It represents a story in which each player i individually chooses action a_i, after which action profile a is determined. They choose actions *simultaneously*. The "simultaneous" does not necessarily mean that the actions must be taken at the same point in time. In the prisoners' dilemma, the two prisoners need not choose their actions at the same point in time. The key feature is that neither prisoner knows which action the other prisoner has chosen. In summary, for a strategic environment to be represented in a normal-form game, it is important that each player makes a decision *individually*, without being informed of the choices of any other players.

We sometimes refer to a normal-form game as a *simultaneous-move game*, just to emphasize that players move simultaneously.

1.2 Examples

Example 3 (Braess's Paradox)**.** Recall Braess's Paradox in the previous note. There are 10 people, each of whom chooses either route SAG or SBG (where road AB is unavailable). Each person wants to minimize her travel time. Her travel time when she chooses route SAG is $11 + x$, where x is the number of people choosing route SAG (including herself); her travel time when she chooses route SBG is $11 + y$, where y is the number of people choosing route SBG (including herself).

This situation has the normal-form representation G such that:

1. $I = \{1, 2, \dots, 10\}$ is the set of peoples.
2. $A_i = \{1, 0\}$ is person i's action space, where action 1 is route SAG and action 0 is route SBG.
3. $u_i : A \to \mathbb{R}$ is person i's payoff function defined by

$$u_i(a) = \begin{cases} -11 - \sum_{j \in I} a_j & \text{if } a_i = 1 \\ -11 - \sum_{j \in I}(1 - a_j) & \text{if } a_i = 0, \end{cases}$$

where $\sum_{j \in I} a_j$ is the number of people choosing route SAG and $\sum_{j \in I}(1 - a_j)$ is the number of people choosing route SBG. $\square$

Example 4 (Cournot Game)**.** Firms 1 and 2 produce a homogeneous product. Each firm i chooses its quantity $q_i \geq 0$. The price of the product depends on the total quantity $Q = q_1 + q_2$, and the inverse demand function P is such that $P(Q) = \max\{1 - Q, 0\}$. Each firm i's production cost is zero regardless of q_i, and it maximizes its profit $P(Q)q_i$.

The Cournot game is the normal-form game G such that:

1. $I = \{1, 2\}$ is the set of firms.
2. $A_i = \mathbb{R}_+$ is firm i's action space.
3. $\pi_i : A \to \mathbb{R}$ is firm i's payoff function defined by $\pi_i(q_i, q_{-i}) = P(Q)q_i$. $\square$

Example 5 (Bertrand Game)**.** Firms 1 and 2 produce a homogeneous product. Each firm i chooses its price $p_i \geq 0$. Assume that there is a unit mass $Q = 1$ of consumers. They buy the products from the firm with the lowest price, or if the prices are equal, the consumers are equally split between the two firms. Each firm i's production cost is zero regardless of the demanded quantity, and it wants to maximizes its profit.

The Bertrand game is the normal-form game G such that:

1. $I = \{1, 2\}$ is the set of firms.
2. $A_i = \mathbb{R}_+$ is firm i's action space.
3. $\pi_i : A \to \mathbb{R}$ is firm i's payoff function defined by

$$\pi_i(p_i, p_{-i}) = \begin{cases} p_i & \text{if } p_i < p_{-i} \\ \frac{p_i}{2} & \text{if } p_i = p_{-i} \\ 0 & \text{if } p_i > p_{-i}. \end{cases}$$

Note that the payoff function π_i is discontinuous. □

Coffee Break ☕. Which is the "right" model of oligopoly competition, Cournot or Bertrand? The reader may be wondering which one to use. It depends on the situation. Neither is "right." Neither is "wrong." Some real-world situations are closer to Cournot competition, while others are closer to Bertrand competition.

1. An example that is close to Bertrand competition is the competition between two neighboring gas stations. They sell a homogeneous product (gasoline), and each gas station i offers a price of p_i (per liter).

2. An example that is close to Cournot competition is the competition in a local fish market. Fishing boats (as players) go fishing to catch fish (i.e., chooses their production) and land them in a market near the local port. Once landed, the fish are sold in the market at the market price.

3. How about electronics such as TVs? This is a complicated case, but since producers cannot choose the retail price and retailers compete with each other to set the price, we can think of competition (among producers) as similar to Cournot competition, where each producer ships a certain quantity of product and then the market sets the price. □

2 "Solution" to Games

Now that we have set up a game, we "solve" it. But what does "solving a game" mean? It means to predict how the game will be played. The prediction is referred to as a "solution" that describes which strategies players will adopt and thus the result of the game. The rule to make a prediction is called a "solution concept" or an "equilibrium concept."

To this end, we describe what players may play in a given game. There are (at least) three approaches to predicting play:

1. **Self-enforcement:** Suppose that players have reached a non-binding agreement about how they play a game, where "non-binding" means that no player is compelled (by any external forces) to obey the agreement. Are all the players willing to obey the agreement? If so, we can expect that they will play the game as agreed; that is, the agreement is self-enforcing. In contrast, if there is at least one player who wants to deviate from the agreement, we cannot expect that they will play the game as agreed. This approach leads to the following solution concepts, for example:
   - **Nash equilibrium**
   - **Correlated equilibrium**

2. **Introspection:** Before choosing strategies, players reason how they should play a game. This approach examines how they reason their way to playing the game. Suppose that

they never play “bad” strategies. Then, the fact that player 1 never plays any bad strategies may change player 2’s way of playing the game, and vice versa. This approach leads to the following solution concepts, for example:

- **Iterated strict dominance**
- **Rationalizability**

3. **Learning:** As players play a given game many times, their play may converge to a particular play through a process by which they learn how they should play and how others play. This approach leads to the following solution concept, for example:

   - **Self-confirming equilibrium**

While these approaches are quite different, we will see that the predictions that they entail are closely related. In terms of predictions of play, these approaches (at least for normal-form games) are not very different, but the interpretations of how agents arrive to play according to the predictions is quite different.