Game Theory

Rationalizability

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A player is rational if she maximizes her (expected) payoff given her belief about opponents' play.[1] Assume that all players are rational; in addition, assume that (i) they know they are rational, (ii) they know they know are rational, (iii) they know they know they know are rational, and so on.[2] Then, what are all the strategies that they could potentially play based only on this assumption?

1 Correlated Rationalizability

Example 1. Consider the following normal-form game:

	L	R
U	$1,1$	$-2,0$
D	$0,-2$	$0,0$

Table 1: all actions are rationalizable

There are two pure-strategy Nash equilibria: (U, L) and (D, R). However, we can still say that (U, R) is rationalizable in the following sense:

- Player 1 can rationalize playing U if she believes that player 2 will play L.
- Player 2 can rationalize playing R if he believes that player 1 will play D.

In the Nash equilibria, the players have the "correct" beliefs about the opponents' play, but this is not the case in the above logic. For example, player 1 believes that player 2 will play L but actually he will play R; similarly, player 2 believes that player 1 will play D but actually she will play U. □

1.1 Never-Best Responses

Beliefs about Opponents' Strategies In a normal-form game G, let $\mu_{-i} \in \Delta(A_{-i})$ be player i's **belief** about players $-i$' strategies. Given this belief μ_{-i}, if player i plays action $a_i \in A_i$ then her expected payoff is

$$u_i(a_i, \mu_{-i}) = \sum_{a_{-i}} \mu_{-i}(a_{-i}) u_i(a_i, a_{-i}).$$

[1] Rationality is often defined differently in other contexts.
[2] This assumption is called common knowledge of rationality.

Remark 1. Note that $\Delta(\prod_{j\neq i} A_j) \neq \prod_{j\neq i} \Delta(A_j)$. The left-hand side is the set of *correlated* mixed strategies of players $-i$, while the right-hand side is the set of *independent* mixed strategies of players $-i$. □

Remark 2. The beliefs herein are often called "conjectures." We use terminology "beliefs" but the reader should not confuse the "beliefs" here with "beliefs" used in incomplete-information games. □

Never-Best Responses The concept of rationalizability is independently defined by Bernheim (1984) and Pearce (1984).

Definition 1. In a normal-form game G, player i's action $a_i \in A_i$ is a **never-best response** if for her every belief $\mu_{-i} \in \Delta(A_{-i})$, there exists some (mixed) strategy $\sigma_i \in \Sigma_i$ such that

$$u_i(\sigma_i, \mu_{-i}) > u_i(a_i, \mu_{-i}).$$

1.2 Rationalizability

Analogous to the iterated deletion of strictly dominated strategies, we iteratively delete all players' never-best responses. At each round of deletion, we ask "what are actions that could potentially be played by rational players?" Then, each player will conclude that no (rational) player will ever play actions that are a never-best response. Since each player will expect that no (rational) player will play such actions with positive probabilities. Furthermore, it will be common knowledge that players arrive at this conclusion, so that it justifies the deletion of these actions from the game. The iteration proceeds until no further actions can be deleted.

Definition 2. In a normal-form game G, for each $i \in I$ and each $k \in \mathbb{N}$, let $\mathrm{CR}_i^0 = A_i$ and

$$\mathrm{CR}_i^k = \mathrm{CR}_i^{k-1} \setminus \underbrace{\left\{ a_i \in \mathrm{CR}_i^{k-1} : \forall\, \mu_{-i} \in \Delta\left(\mathrm{CR}_{-i}^{k-1}\right) \quad \exists\, \sigma_i \in \Delta\left(\mathrm{CR}_i^{k-1}\right) \quad u_i(\sigma_i, \mu_{-i}) > u_i(a_i, \mu_{-i}) \right\}}_{\text{actions that are never-best responses}}.$$

Let the set CR_i^∞ of player i's **correlated rationalizable actions** be such that

$$\mathrm{CR}_i^\infty = \bigcap_{k=0}^{\infty} \mathrm{CR}_i^k.$$

We often call them rationalizable strategies by omitting "correlated."

From this definition, it follows that

$$a_i \in \mathrm{CR}_i^\infty \quad \Longrightarrow \quad \exists\, \mu_{-i} \in \Delta(\mathrm{CR}_{-i}^\infty) \quad \forall\, \sigma_i \in \Delta(\mathrm{CR}_i^\infty) \quad u_i(a_i, \mu_{-i}) \geq u_i(\sigma_i, \mu_{-i}).$$

In words, any rationalizable action a_i is a best response (in CR_i^∞) to some belief μ_{-i}.

Recap (Never-Best Responses for Mixed Strategies): From the straightforward generalization of Definition 1, we reach the following definition:

Definition 3. In a normal-form game G, player i's (mixed) strategy $\sigma_i \in \Sigma_i$ is a **never-best response** if for her every belief $\mu_{-i} \in \Delta(A_{-i})$, there exists some (mixed) strategy $\sigma_i' \in \Sigma_i$ such that

$$u_i(\sigma_i', \mu_{-i}) > u_i(\sigma_i, \mu_{-i}).$$

1.3 Rationalizability for Mixed Strategies*

So far we have defined the concept of (correlated) rationalizability for actions, or pure strategies, but we can extend it for mixed strategies. Indeed, we can extend Definition 1 for mixed strategies just by replacing action a_i with a mixed strategy. This observation is summarized in the recap box.

Subtleties emerge when we define the set of rationalizable mixed strategies. One may jump to the (wrong) conclusion that it is merely the set $\Delta(\text{CR}_i^\infty)$ of all distributions over CR_i^∞, but this is not true in general. That is, mixed strategies in CR_i^∞ are not always rationalizable.

Example 2. Consider the following normal-form game:

	L	R
U	$1, 0$	$-2, 0$
M	$-2, 0$	$1, 0$
D	$0, 0$	$0, 0$

Table 2: a mixed strategy in $\Delta(\text{CR}_i^\infty)$ is a never-best response

For each player, all actions are rationalizable: $\text{CR}_i^\infty = A_i$. For example, D is rationalizable, because it is optimal for player 1's belief that player 2 plays L and R with equal probabilities. However, player 1's mixed strategy $\frac{1}{2}U \oplus \frac{1}{2}M \in \Delta(\text{CR}_i^\infty)$ is a never-best response and thus is not rationalizable. □

2 Correlated Rationalizability versus Iterated Strict Dominance

Correlated rationalizability and iterated strict dominance ask complementary questions to each other. While iterated strict dominance deletes all strategies that a player will never play under common knowledge of rationality, correlated rationalizability identifies all strategies that a player could potentially play under common knowledge of rationality. Then, how are these two concepts related to each other?

Never-Best Response ⇔ Strict Dominance Because rationalizability is based on the notion of never-best responses and iterated strict dominance on the notion of strict dominance, it is

Recap (Never-Best Responses versus Strict Dominated Strategies): The notion of never-best responses is similar to the notion of strictly dominated strategies, which we review below:

Remark 3. In a normal-form game G, player i's action $a_i \in A_i$ is **strictly dominated** by her (mixed) strategy $\sigma_i \in \Sigma_i$ if for player $-i$'s every (correlated) strategy profile $\mu_{-i} \in \Delta(A_{-i})$,

$$u_i(\sigma_i, \mu_{-i}) > u_i(a_i, \mu_{-i}).$$

Recall that strict dominance requires the above inequality for each action profile $a_{-i} \in A_{-i}$. From this observation, we have the above inequality for a (correlated) strategy profile $\mu_{-i} \in \Delta(A_{-i})$ as well. □

In light of Remark 3, we can compare the two notions as follows:

Action a_i is a **never-best response**:	$\forall\, \mu_{-i} \in \Delta(A_{-i})$	$\exists\, \sigma_i \in \Delta(A_i)$	$u_i(\sigma_i, \mu_{-i}) > u_i(a_i, \mu_{-i})$.
Action a_i is **strictly dominated**:	$\exists\, \sigma_i \in \Delta(A_i)$	$\forall\, \mu_{-i} \in \Delta(A_{-i})$	$u_i(\sigma_i, \mu_{-i}) > u_i(a_i, \mu_{-i})$.

That is, the only difference between the two notions is the order of quantifiers $\forall$ and $\exists$.

natural to examine the relationship between the two notions.

Theorem 1. *In a finite normal-form game G, player i's action $a_i \in A_i$ is a never-best response if and only if it is strictly dominated.*

Proof. We show the "only if" part, as the "if" part is immediate. We show the contrapositive: If a_i is not strictly dominated then it is not a never-best response. Suppose that a_i is not strictly dominated. That is, there exists no $\sigma_i \in \Sigma_i$ such that for each $a_{-i} \in A_{-i}$, $u_i(\sigma_i, a_{-i}) > u_i(a_i, a_{-i})$. There are, in total, $n = \prod_{j \neq i} |A_j|$ possible action profiles for players $-i$, which we enumerate by $a^1_{-i}, a^2_{-i}, \ldots, a^n_{-i}$. To use the Separating Hyperplane Theorem, we define a set Y by

$$Y = \left\{ \begin{pmatrix} u_i(\sigma_i, a^1_{-i}) - u_i(a_i, a^1_{-i}) \\ \vdots \\ u_i(\sigma_i, a^n_{-i}) - u_i(a_i, a^n_{-i}) \end{pmatrix} : \sigma_i \in \Delta(A_i) \right\} \subset \mathbb{R}^n.$$

Since Y is a non-empty convex set and $Y \cap \mathbb{R}^n_{++} = \varnothing$, it follows from the Separating Hyperplane Theorem that there exist some $c \in \mathbb{R}$ and some $v \in \mathbb{R}^n \setminus \{0\}$ such that for each $x \in \mathbb{R}^n_{++}$ and each $y \in Y$,

$$v \cdot x > c \geq v \cdot y.$$

It is immediate that $v \in \mathbb{R}^n_+ \setminus \{0\}$.[3] Letting $\bar{v} = \sum_l v_l > 0$, we define a vector $\mu_{-i} = v/\bar{v}$. Since all elements of μ_{-i} are non-negative and the sum of them is one, it follows that $\mu_{-i} \in \Delta(A_{-i})$.

[3] To see $v \in \mathbb{R}^n_+$, suppose that $v = (v_1, v_2, \ldots, v_n) \notin \mathbb{R}^n_+$. Then, there is some $l \in \{1, 2, \ldots, n\}$ such that $v_l < 0$. Then, $v \cdot x$ is arbitrarily small and thus less than c if we take $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n_{++}$ such that x_l is large enough and x_k is close to zero, but this contradicts inequality $v \cdot x > c$. It must be that $v \in \mathbb{R}^n_+$. Since $v \neq 0$ by the Separating Hyperplane Theorem, it follows that $v \in \mathbb{R}^n_+ \setminus \{0\}$.

Then, $\mu_{-i} \cdot x \geq c/\bar{v} \geq \mu_{-i} \cdot y$. Since $\mu_{-i} \cdot x > 0$, it follows that $c \leq 0$. Hence, $0 \geq \mu_{-i} \cdot y$ for each $y \in Y$. That is, for each σ_i,

$$\sum_{l=1}^{n} u_i\left(\sigma_i, a_{-i}^l\right)\mu_{-i}\left(a_{-i}^l\right) \leq \sum_{l=1}^{n} u_i\left(a_i, a_{-i}^l\right)\mu_{-i}\left(a_{-i}^l\right),$$

or equivalently $u_i(\sigma_i, \mu_{-i}) \leq u_i(a_i, \mu_{-i})$. That is, a_i is not a never-best response. ■

Remark 4. Pearce's (1984) proof is based on Nash's Existence Theorem and the Minimax Theorem. The Minimax Theorem itself can be proven by (a corollary of) the Separating Hyperplane Theorem. □

Remark 5. From Theorem 1, it follows that a strategy that is weakly dominated but not strictly dominated is a best response to some belief. Indeed, a weakly dominated strategy may be played in a Nash equilibrium. □

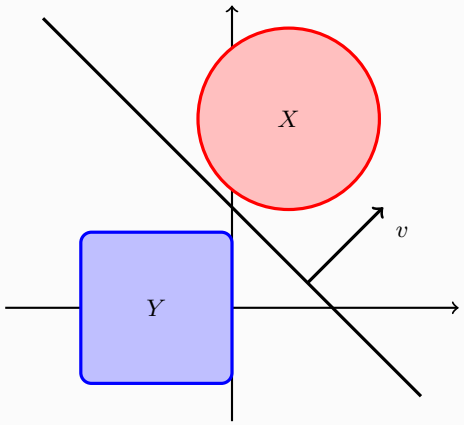
Recap (Separating Hyperplane Theorem):

Theorem 2. *Let $X, Y \subset \mathbb{R}^n$ be two disjoint non-empty convex subsets for $n \in \mathbb{N}$. Then, there exist some $c \in \mathbb{R}$ and some $v \in \mathbb{R}^n \setminus \{0\}$ such that for each $x \in X$ and each $y \in Y$,*

$$v \cdot x > c \geq v \cdot y.$$

That is, the hyperplane $\{z \in \mathbb{R}^n : v \cdot z = c\}$, with the normal vector v, separates X and Y.

This theorem is illustrated below. It is intuitive, in the two-dimensional case, that there exists a line that separates the two disjoint non-empty convex sets $X, Y \subset \mathbb{R}^2$.



Remark 6. When should we try to use the Separating Hyperplane Theorem? When we want to show the existence of some non-negative vector (e.g., probabilities or prices), the Separating Hyperplane Theorem may come into play. In the proof of the Second Welfare Theorem, for example, we use the Separating Hyperplane Theorem to prove the existence of a (positive) price vector that supports a Pareto-efficient allocation as a quasi-Walrasian equilibrium. In the proof of Theorem 1, we use the Separating Hyperplane Theorem to show the existence of a belief, which is a non-negative vector with all elements summing up to 1. □

Correlated Rationalizability ⇔ Iterated Strict Dominance Recall that ND_i^∞ denotes the set of player i's actions that survive iterated deletion of strictly dominated actions.

Corollary 1. *In a finite normal-form game G, $\mathrm{ND}_i^\infty = \mathrm{CR}_i^\infty$ for each $i \in I$,*

Proof. By definition, $\mathrm{ND}_i^0 = \mathrm{CR}_i^0 = A_i$. By Theorem 1, $\mathrm{ND}_i^k = \mathrm{CR}_i^k$ for each $k \in \mathbb{N}$. Hence, $\mathrm{ND}_i^\infty = \mathrm{CR}_i^\infty$. ■

3 Correlated Rationalizability versus Correlated Equilibrium

Next, we compare correlated rationalizability with correlated equilibrium.

Theorem 3. *In a finite normal-form game G, let $\mu \in \Delta(A)$ be a direct correlated equilibrium. Then, for each $i \in I$ and each $a_i \in A_i$, if $\mu(\{a_i\} \times A_{-i}) > 0$ then $a_i \in \mathrm{CR}_i^\infty$.*

Proof. It suffices to show that for each $k \in \mathbb{N} \cup \{0\}$, if a correlated equilibrium μ assigns non-zero probability on player j playing action a_j then $a_j \in \mathrm{CR}_j^k$. We prove this claim by induction. First, the claim is obvious for $k = 0$. Second, we suppose that the claim is true for $k = K$ and then show that it is true for $k = K + 1$. For each $a'_{-i} \in A_{-i}$,

$$\mu_{-i}(a'_{-i}) = \frac{\mu(a_i, a'_{-i})}{\sum_{a''_{-i} \in A_{-i}} \mu(a_i, a''_{-i})}.$$

By the induction hypothesis, $\mu_{-i} \in \Delta(\mathrm{CR}_i^K)$. By the definition of correlated equilibrium, action a_i is a best response to μ_{-i}. That is, $a_i \in \mathrm{CR}_i^{K+1}$. Hence, $a_i \in \bigcap_{k=0}^\infty \mathrm{CR}_i^k = \mathrm{CR}_i^\infty$. ■

4 Independent Rationalizability*

In the concept of correlated rationalizability, player i's belief allows for correlation between players $-i$' strategies. Indeed, her belief μ_{-i} is defined on $\Delta(\mathrm{CR}_{-i}^{k-1}) = \Delta(\prod_{j \neq i} \mathrm{CR}_j^{k-1})$. In a relevant concept of independent rationalizability, her belief no longer allows for correlation between the opponents' strategies.

Definition 4. In a normal-form game G, let $\mathrm{IR}_i^0 = A_i$ and for each $k \in \mathbb{N}$,

$$\mathrm{IR}_i^k = \mathrm{IR}_i^{k-1} \setminus \Big\{ a_i \in \mathrm{IR}_i^{k-1} : \forall\, \mu_{-i} \in \prod_{j \neq i} \Big(\Delta\Big(\mathrm{IR}_j^{k-1} \Big) \Big) \quad \exists\, \sigma'_i \in \Delta\Big(\mathrm{IR}_i^{k-1} \Big) \quad u_i(\sigma'_i, \mu_{-i}) > u_i(a_i, \mu_{-i}) \Big\}.$$

We define player i's set of **independent rationalizable pure strategies** by

$$\mathrm{IR}_i^\infty = \bigcap_{k=0}^{\infty} \mathrm{IR}_i^k.$$

Remark 7. It is immediate that $\mathrm{IR}_i^\infty \subset \mathrm{CR}_i^\infty$ in general. If there are two players, it is obvious that $\mathrm{IR}_i^\infty = \mathrm{CR}_i^\infty$. In contrast, if there are more than two players, it is often the case that $\mathrm{IR}_i^\infty \neq \mathrm{CR}_i^\infty$. For such a game, see Osborne & Rubinstein (1994, Figure 58.1 together with the discussion). □

References

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