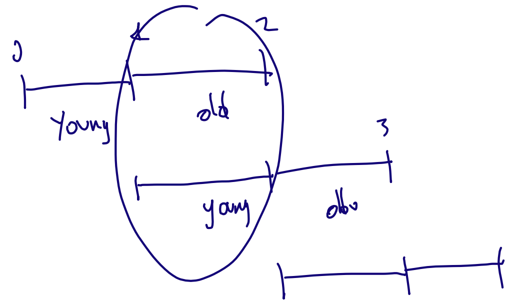
OVERLAPPING GENERATIONS MODELS AND LIFE CYCLE

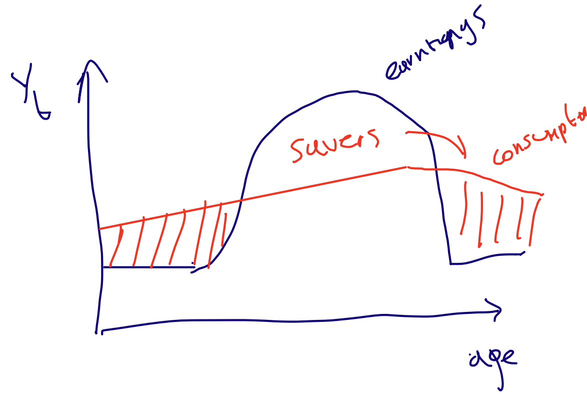
An important dimension in which agents differ is age

- In each moment of time, young and old individuals coexist

- These agents have different income levels and incentives to save

- *Life cycle theory*: given a profile of income throughout their life, individuals smooth consumption by saving and spending at different ages





Basic Model

Individuals live for two periods: young and old (retired)

Exogenous profile of labor productivity for each agent i,

$$\lambda_t^{i,1} = 1 \qquad \lambda_{t+1}^{i,2} = 0$$

retirement

There is no uncertainty in the model

We assume that in each period a continuum of identical young people is born in $[0, 1]$

$i \in [0, 1]$

There is no population growth

All agents are *ex-ante* equal and start with zero assets

Young agent problem

$$\max_{\left\{c_t^1, c_{t+1}^2, a_{t+1}^2\right\}} \underbrace{u\left(c_t^1\right) + \beta u\left(c_{t+1}^2\right)}$$

lifetime utility

$$\begin{aligned} s.t. \qquad c_t^1 + a_{t+1}^2 &= w_t \\ c_{t+1}^2 &= R_{t+1} a_{t+1}^2 \end{aligned}$$

$a_{t+1}^2 \geq 0$

There are no credit restrictions; nor are Ponzi schemes possible in finite horizon

In this model, young workers save to finance their retirement

We impose the transversality condition $a_{t+2}^3 = 0$; note that the old can transform non-depreciated capital into consumer goods

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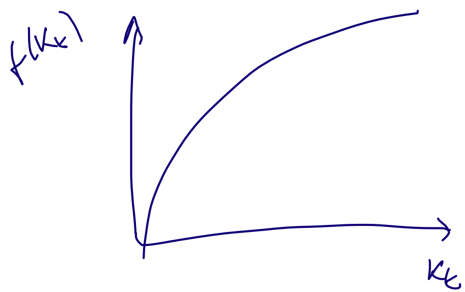


The representative firm combines capital and labor to produce the unique good, according to

$$Y_t = F(K_t, L_t) = F(K_t, 1) = f(K_t)$$

where F has constant returns to scale and the other usual properties

The stock of capital is equal to the assets of the retirees

$$K_t = \int_0^1 a_t^2 di = a_t^2$$

or else

$$K_{t+1} = \int_0^1 a_{t+1}^2 di = a_{t+1}^2$$

tomorrow's capital is equal to the savings of young people

$\{c_t^1\}_{t=0}^{\infty}$

Definition of Equilibrium

A Sequential Competitive Equilibrium for this economy is a set of sequences for individual quantities c_t^1, c_t^2, a_t^2, aggregate quantities Y_t, K_t and prices w_t, R_t such that

i) In each period t, given w_t and R_t, the values c_t^1, c_{t+1}^2, a_{t+1}^2 solves the problem of the young agent:

$$\begin{cases} \max & u\left(c_t^1\right) + \beta u\left(c_{t+1}^2\right) \\ s.t. & c_t^1 + a_{t+1}^2 = w_t \\ & c_{t+1}^2 = R_{t+1} a_{t+1}^2 \end{cases}$$

$$u(c_t^1) + \beta u(c_{t+1}^2) - \lambda_t^1 (c_t^1 + a_{t+1}^2 - w_t) - \lambda_t^2 (c_{t+1}^2 - R_{t+1} a_{t+1}^2)$$

$$\left.\begin{aligned} \frac{\partial}{c_t^1} &= u'(c_t^1) - \lambda_t^1 = 0 \\ \frac{\partial}{\partial c_t^2} &= \beta u'(c_{t+1}^2) - \lambda_t^2 = 0 \\ \frac{\partial}{\partial a_{t+1}^2} &= -\lambda_t^1 + \lambda_{t+1}^2 R_{t+1} = 0 \end{aligned}\right\}$$

$$u'(c_t^1) = \beta u'(c_{t+1}^2) R_{t+1}$$

$$\frac{u'(c_t^1)}{\beta u'(c_{t+1}^2)} = R_{t+1}$$

ii) In the initial period, given a_0^2 and R_0, the value c_0^2 satisfies the condition for the retired agent

$$c_0^2 = R_0 a_0^2$$

iii) In each period t, given w_t and R_t, the values Y_t and K_t solve the firm's problem:

$$\begin{aligned} \max \qquad & Y_t - w_t - [R_t - (1-\delta)] K_t \\ s.t. \qquad & Y_t = f(K_t) \end{aligned}$$

and the benefits are zero

iv) In each period t, markets clear:

$$Y_t = c_t^1 + c_t^2 + K_{t+1} - (1-\delta) K_t$$
$$K_t = a_t^2$$

First Order Conditions

Solving the problem of the young worker, we obtain Euler's equation

$$u'\left(c_t^1\right) = \beta R_{t+1} u'\left(c_{t+1}^2\right)$$

$\forall t$

We can also combine the budget constraints as a single intertemporal constraint in present value:

$$c_t^1 + \frac{c_{t+1}^2}{R_{t+1}} = w_t$$

These two equations implicitly define the consumption functions for young and old agent:

$$c_t^1 = c_t^1\left(w_t, R_{t+1}\right) \qquad c_{t+1}^2 = c_{t+1}^2\left(w_t, R_{t+1}\right)$$

from where we can also get the savings function

$$a_{t+1}^2 = w_t - c_t^1\left(w_t, R_{t+1}\right) = a_{t+1}^2\left(w_t, R_{t+1}\right)$$

$$c^1_t = c^1_t\left(w_t, R_{t+1}\right)$$

$$c^2_{t+1} = c^2_{t+1}\left(w_t, R_{t+1}\right)$$

$$a^2_{t+1} = a^2_{t+1}\left(w_t, R_{t+1}\right)$$

From the problem of the firm, we obtain equilibrium prices

$$R_t = f'(K_t) + (1-\delta) \longrightarrow R_t(K_t)$$

$$w_t = f(K_t) - f'(K_t)\, K_t \longrightarrow w_t(K_t)$$

$$\Longrightarrow \quad a^2_{t+1} = a^2_{t+1}(w_t, R_{t+1})$$

$$= a^2_{t+1}(K_t, K_{t+1})$$

Replacing in the saving function, it implicitly defines

$$a^2_{t+1} = a^2_{t+1}\left(w_t(K_t), R_{t+1}(K_{t+1})\right) \equiv S(K_t, K_{t+1})$$

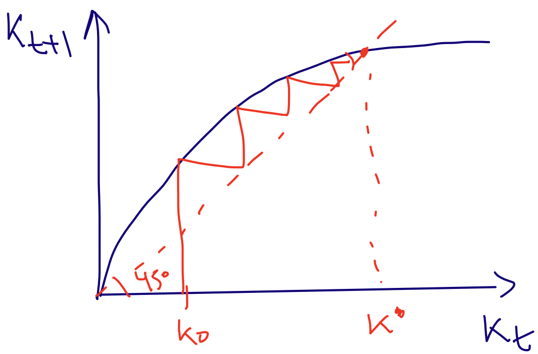
In equilibrium, we obtain

$$K_{t+1} = S(K_t, K_{t+1})$$

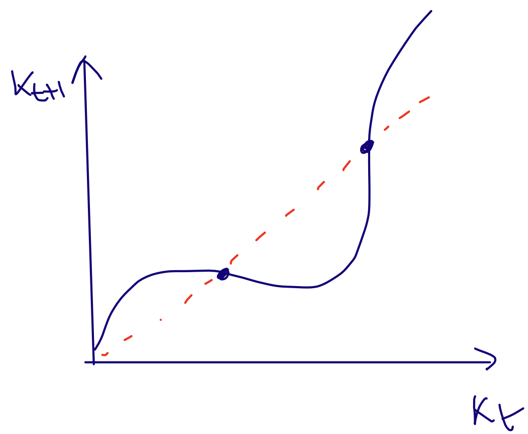
$$K_{t+1} = S(K_t, K_{t+1})$$

a first order difference equation in K_t

This equation characterizes the dynamics of aggregate capital in the transition; depending on the utility and production functions, it can be very complicated



"nice" case



Stationary Equilibrium and Dynamic Efficiency

We will focus now on a stationary equilibrium, in which aggregate quantities and prices are constant

$K_t = K^*$ $\Rightarrow$ w^*, R^* constant

This also implies that the consumption levels for each age are stationary

$$c_t^1 = c_{t+1}^1 = c^1 \qquad \text{, etc.}$$

but does NOT imply that consumption is independent of age

$$c^1 \neq c^2 \qquad \text{, in general}$$

In a stationary equilibrium individual agents do not have a constant behavior since their decisions depend on age

... even though at the aggregate level the economy is stationary

In a stationary equilibrium, the Euler equation

$$\frac{u'\left(c^1\right)}{\beta u'\left(c^2\right)} = R\left(K^*\right)$$

and the intertemporal budget constraint

$$c^1 + \frac{c^2}{R\left(K^*\right)} = w\left(K^*\right)$$

hold, where $w\left(K^*\right)$ and $R\left(K^*\right)$ are equilibrium prices

We can derive in a similar way the consumption of each type of agent and the savings of the economy as a functions of aggregate capital

$$a^2 = S\left(K^*, K^*\right)$$

The value of aggregate capital in a stationary equilibrium solves the equation $S\left(K^*, K^*\right) = K^*$

$K_{t+1} = S(K_t, K_{t+1})$

$K^* = S(K^*, K^*)$

$\beta R(K^*) \overset{?}{\gtrless} 1$

X

$f'(K^*) + (1-\delta)$

$1+i$

$R(K^*) \leq 1$

dynamic inefficiency

(non-positive real interest rate)

Proposition: *All stationary equilibria in which* $R(K^*) > 1$ *are efficient* ✓

lifetime utility generation born at date t

Let $(c^1,\ c^2$ y $K^*)$ be an stationary equilibrium. We can verify that it satisfies the first order conditions of the problem of a social planner

$$\max \sum_{t=0}^{\infty} \gamma^t \left[u\left(c_t^1\right) + \beta u\left(c_{t+1}^2\right)\right]$$

$$(\lambda_t) \quad s.t. \quad c_t^1 + c_t^2 = f\left(K_t\right) + (1-\delta) K_t - K_{t+1} \quad \text{feasibility}$$

$\gamma \in (0,1)$

which discounts the weight of future generations at the rate γ

The solution to this problem is the Pareto optimal

born at 0 $\longrightarrow$ weight 1

born at 1 $\longrightarrow$ weight γ

born at 2 $\longrightarrow$ weight γ^2

F.O.C.

$\gamma^t u'(c_t^1) - \lambda_t = 0 \qquad \frac{\partial}{\partial c_t^1} = 0$

$\gamma^{t-1} u'(c_t^2)\beta - \lambda_t = 0 \qquad \frac{\partial}{\partial c_t^2} = 0$

$\lambda_{t+1} - \lambda_t\left[f'(K_{t+1}) + (1-\delta)\right] = 0$

$$\frac{u'(c_t^1)}{\gamma u'(c_{t+1}^1)} = \underbrace{f'(K_{t+1}) + (1-\delta)}_{R}$$

$u'(c_t^1) = \gamma \beta^{-1} u'(c_t^2)$

K_{t+1}

The first order conditions of this problem are

$$\frac{u'\left(c_t^1\right)}{\beta u'\left(c_{t+1}^2\right)} = f'\left(K_{t+1}\right) + (1-\delta)$$

and

$$\frac{u'\left(c_t^1\right)}{\beta u'\left(c_t^2\right)} = \frac{1}{\gamma}$$

also

$$c_t^1 + c_t^2 = f\left(K_t\right) + (1-\delta) K_t - K_{t+1}$$

We can easily show that the stationary equilibrium satisfies these conditions with the discount rate $\gamma = R\left(K^*\right)^{-1}$

Note that we need $R\left(K^*\right) > 1$ such that $\gamma < 1$ and the problem of the social planner is well defined ■

Stationary equilibrium

$$\frac{u'(c^1)}{\beta u'(c^2)} = R(K^*)$$

$$\frac{u'(c^1)}{\beta u'(c^2)} = \frac{1}{\gamma}$$

$$\Rightarrow R(K^*) = 1/\gamma > 1$$

On the contrary, if $R(K^*) < 1$ the equilibrium *is dynamically inefficient*

- $R(K^*) < 1$ implies that $f'(K^*) < \delta$, then there is over-accumulation of capital above its efficient limit

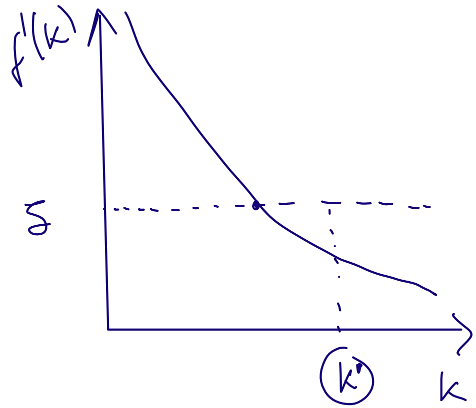
- Despite having a negative return, young agents save even more because they need to have an income for retirement

- In this context, public intervention through a social security system can improve welfare

$R(K^*) < 1$

$f'(K^*) + (1-\delta) < 1$

$f'(K^*) < \delta$



$u'(c) = 1/c$

An Illustrative Example

Simple case with log utility and Cobb-Douglas production function

$$u(c_t) = \log(c_t) \qquad f(K_t) = K_t^{\alpha}$$

From the problem of the young agent we obtain the Euler equation

$$c_{t+1}^2 = \beta R_{t+1} c_t^1$$

$$c_t^1 + \frac{c_{t+1}^2}{R_{t+1}} = w_t$$

from where

$$c_t^1 = \frac{w_t}{1+\beta} \qquad c_t^2 = \frac{\beta R_t w_{t-1}}{1+\beta}$$

$$c_t^1 = \frac{1}{1+\beta} w_t$$

and

$$a_{t+1}^2 = \frac{\beta w_t}{1+\beta}$$

Note that savings does not depend on R_{t+1}; the income and substitution effects of the interest rate cancel out

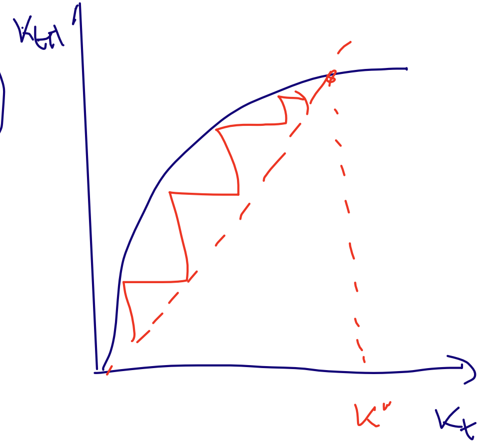
$$a_{t+1}^2 = w_t - c_t^1 = \frac{\beta}{1+\beta} w_t$$

$$S(w_t, R_{t+1}) = \left(\frac{\beta}{1+\beta}\right) w_t$$

$$R_t = \alpha K_t^{\alpha-1} + (1-\delta)$$

$$w_t = K_t^{\alpha} - \alpha K_t^{\alpha-1} K_t = (1-\alpha) K_t^{\alpha}$$

$$K_{t+1} = \left(\frac{\beta}{1+\beta}\right)(1-\alpha)\, K_t^{\alpha}$$



From the problem of the firm

$$w_t = (1-\alpha)\, K_t^{\alpha} \qquad R_t = \alpha K_t^{\alpha-1} + (1-\delta)$$

Combining, we obtain the savings function

$$S(K_t) = \frac{\beta(1-\alpha)\, K_t^{\alpha}}{1+\beta}$$

and the first order difference equation

$$K_{t+1} = \frac{\beta(1-\alpha)}{1+\beta} K_t^{\alpha}$$

that characterizes the dynamics of capital

stationary equilibrium

$$K^* = \frac{\beta(1-\alpha)}{1+\beta} K^{*\alpha}$$

$$K^* = \left(\frac{\beta(1-\alpha)}{1+\beta}\right)^{1/(1-\alpha)}$$

In this particular case, there is a single stationary equilibrium with positive capital

$$K^{*}=\left[\frac{\beta\left(1-\alpha\right)}{1+\beta}\right]^{\frac{1}{1-\alpha}}$$

in which

$\alpha (K^{*})^{\alpha-1} + (1-\delta)$

$$R\left(K^{*}\right)=\left(\frac{\alpha}{1-\alpha}\right)\left(\frac{1+\beta}{\beta}\right)+\left(1-\delta\right)$$

and this equilibrium will be dynamically inefficient if $\left(\frac{\alpha}{1-\alpha}\right)\left(\frac{1+\beta}{\beta}\right)<\delta$ $R(K^{*}) < 1$

We can also show in this case that starting from any $K_0 > 0$ the aggregate capital converges monotonically to its value of stationary equilibrium (*global stability*)

With other utility and / or production functions:

- There can be multiple stationary equilibria
- Some stationary equilibria can be dynamically efficient, others can not
- Some stationary balances may be stable, others may not
- The transition to one of the stable stationary equilibria can be non-monotonic (oscillatory, even chaotic)

Parenthesis: Altruism in Dynamic Models

Consider now altruistic individuals who are concerned not only about their utility, but about future generations (children, etc.).

In particular,

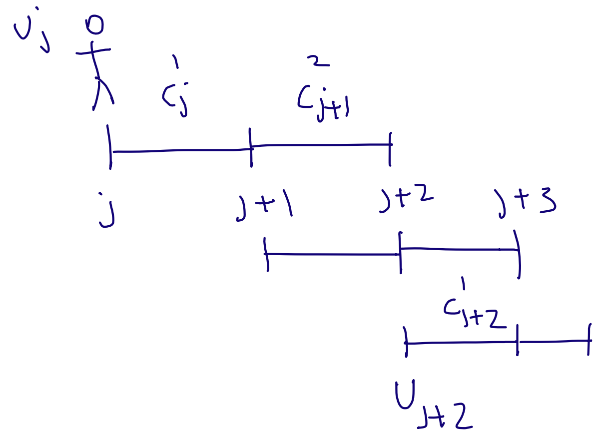
$$U_j = u\left(c_j^1\right) + \beta\left[u\left(c_{j+1}^2\right) + \underline{\psi U_{j+2}}\right]$$

where U_j is the utility of a young worker born in the period j and $\psi \in (0, \beta]$ measures the degree of *altruism*; then

$$U_j = \sum_{s=0}^{\infty} (\beta\psi)^s \left[u\left(c_{j+2s}^1\right) + \beta u\left(c_{j+2s+1}^2\right)\right]$$

where s indexes the current and future generations (*dynasties*)

This version of the model looks like an infinitely lived agent model!



$$U_j = \left[u(c_j^1) + \beta u(c_{j+1}^2)\right] + \beta\psi\left[u(c_{j+2}^1) + \beta u(c_{j+3}^2)\right]$$

$$+ \beta^2\psi^2\left[u(c_{j+4}^1) + \beta u(c_{j+5}^2)\right] + \cdots \beta^3\psi^3$$

The problem of a young worker in period zero would be

$$\max \sum_{s=0}^{\infty} (\beta\psi)^s \left[u\left(c_{2s}^1\right) + \beta u\left(c_{2s+1}^2\right)\right]$$

(to future generation)
bequests

Initial assets
(previous generation)

$$\begin{aligned} s.t. \qquad c_{2s}^1 + a_{2s+1}^2 &= w_{2s} + R_{2s}a_{2s}^1 \\ c_{2s+1}^2 + a_{2s+2}^1 &= R_{2s+1}a_{2s+1}^2 \qquad \forall t \geq 0 \\ & a_0^1 \quad \text{given} \end{aligned}$$

where a_{2s+2}^1 are *bequests* left to the next generation (perhaps negative), then

$$K_t = a_t^1 + a_t^2$$

The production side of the model is the same

F.O.C.

$$\frac{u'(c_t^1)}{\beta u'(c_{t+1}^2)} = R_{t+1} \quad \checkmark$$

$$\frac{u'(c_{t+1}^2)}{\psi u'(c_{t+2}^1)} = R_{t+2} \quad \text{(new)}$$

The Euler equations obtained from the dynastic model are

$$\frac{u'\left(c_t^1\right)}{\beta u'\left(c_{t+1}^2\right)} = R_{t+1} \qquad \frac{u'\left(c_{t+1}^2\right)}{\psi u'\left(c_{t+2}^1\right)} = R_{t+2}$$

(reindexing $t = 2s$) and therefore

$$u'\left(c_t^1\right) = \beta\psi R_{t+1} R_{t+2} u'\left(c_{t+2}^1\right)$$

$$u'(c_t^1) = \beta R_{t+1} u'(c_{t+1}^2)$$

$$= \beta R_{t+1} \left[\psi R_{t+2} u'(c_{t+2}^1) \right]$$

$$= (\beta\psi) R_{t+1} R_{t+2} u'(c_{t+2}^1)$$

In an stationary equilibrium, $c_t^1 = c_{t+2}^1 = c^1$ implies $R(K^*) = (\beta\psi)^{-\frac{1}{2}}$ > 1

$$u'(\cancel{c^1}) = (\beta\psi) R(K^*)^2 u'(\cancel{c^1})$$

In the dynastic model there is a single interest rate consistent with a stationary equilibrium; its value depends only on the discount factor and the degree of altruism

It is easy to show again that this stationary equilibrium is efficient

$\psi \uparrow \Rightarrow R^* \downarrow$

$\Rightarrow K^* \uparrow$

Notice finally that if the degree of altruism increases, the interest rate in the stationary equilibrium decreases

- Greater altruism incentivizes young agents to accumulate more capital, not only to consume when they are old but to provide bequests to future generations

- With perfect altruism ($\psi = \beta$), the interest rate is the inverse of the discount factor $R(K^*) = \beta^{-1}$ as in the model with infinite horizon

End of parenthesis

Recursive Competitive Equilibrium

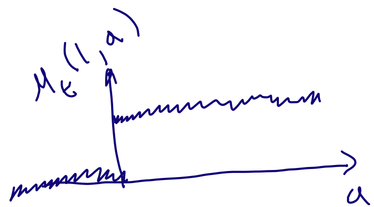
The state variables are

- Individual state: age $e = \{1, 2\}$ and assets $a \in (-B, \infty)$
- Aggregate state: distribution or measure $\mu_t\,(e, a)$ of agents over ages and assets

  This distribution satisfies

$$\lim_{a \to -B} \mu_t\,(2, a) = 0 \qquad \lim_{a \to \infty} \mu_t\,(2, a) = 1$$

$$\mu_t\,(1, a) = \begin{cases} 0 & , \forall a < 0 \\ 1 & , \forall a \geq 0 \end{cases}$$



A Recursive Competitive Equilibrium is a set of functions $v(e, a, \mu)$, $c(e, a, \mu)$, $a'(e, a, \mu)$, prices $w(\mu)$ and $R(\mu)$, capital demand $K(\mu)$ and law of motion $\Gamma(\mu)$ such that:

R

i) For each pair (a, μ), given functions w and Γ, the value function $v(1, a, \mu)$ solves the Bellman equation for the young agent:

$$v(1, a, \mu) = \max_{c, a'} \left\{ u(c) + \beta v\left(2, a', \mu'\right) \right\}$$

$$\begin{aligned} s.t. \quad c + a' &= w(\mu) + R(\mu)\, a \\ \mu' &= \Gamma(\mu) \end{aligned}$$

and $c(1, a, \mu)$, $a'(1, a, \mu)$ are optimal decision rules for this problem

ii) For each pair (a, μ), given the function R, the value function $v(2, a, \mu)$ of the retired agent satisfies:

$$v(2, a, \mu) = u(R(\mu) a)$$

and we define $c(2, a, \mu) = R(\mu) a, \; a'(2, a, \mu) = 0$

iii) For each μ, prices satisfy the marginal conditions of the representative firm:

$$R(\mu) = f'(K(\mu)) + (1 - \delta)$$

$$w(\mu) = f(K(\mu)) - f'(K(\mu)) K(\mu)$$

iv) For each μ, markets clear:

$$f\left(K\left(\mu\right)\right)=\sum_{e=1}^{2}\int_{-B}^{\infty}\left[c\left(e,a,\mu\right)+a'\left(e,a,\mu\right)-\left(1-\delta\right)a\right]d\mu\left(e,a\right)$$

$$K\left(\mu\right)=\sum_{e=1}^{2}\int_{-B}^{\infty}ad\mu\left(e,a\right)$$

v) For each μ, the law of motion Γ is consistent with the optimal decisions of the agents

Solving Numerically a Stationary Equilibrium

In a stationary equilibrium, the aggregate state of the economy reduces to total capital K^*

We need to discretize the space of possible values of individual assets

$$a \in \{a_1, a_2, ..., a_N\}$$

The idea is to iterate on K^* until achieving convergence

For each value of K^*, we solve first the problem of the old agent and we go back to the younger agents (*backwards induction*)

Iterative algorithm:

1. Propose a value for K^* and calculate de corresponding prices $R(K^*)$ and $w(K^*)$

old

2. Given the prices, calculate the value function of the retired agent $v(2, a_i) = u(R(K^*)\, a_i)$ for each point $a_i \in \{a_1, a_2, ..., a_N\}$

3. Given the prices and $v(2, a)$, calculate the value function of the young agent $v(1, 0)$ solving the problem

$$v(1,0) = \max_{a_j \in \{a_1, a_2, ..., a_N\}} \left\{ u\left(\underbrace{w(K^*) - a_j}_{c}\right) + \beta v\left(2, a_j\right) \right\}$$

and store $a'(1) = a_j$

4. Calculate the aggregate capital corresponding to the decision rule found in step 3

$$\widehat{K} = a'(1)$$

5. Compare K^* and $\widehat{K}$

   - If they are equal (subject to a margin of tolerance), we are done

   - If they are different, go back to step 1 with a new K^*

$$K^*_{new} = \frac{K^*_{initial} + \widehat{K}}{2}$$

Solving numerically overlapping generations models is simpler than the corresponding models with infinite life horizon

- The reason is that to find the value function of each agent you do not have to find a fixed point in the Bellman equation

The algorithm can be easily adapted to more periods of life

It can also be modified to calculate the *transition* to stationary equilibrium

- Idea: Assume that the stationary equilibrium is reached in T periods and iterate over a vector $\{K_0, K_1, ..., K_T\}$

Some Implications of the Model and Extensions

- The age structure and life cycle income profile are key to explain the differences in savings rates between poor and rich (Huggett and Ventura, JME 2000)

- Demographic changes affect the sustainability of the tax and social security system (Auerbach and Kotlikoff, Dynamic Fiscal Policy 1987)

  Generational accounting

- Technological change and its impact on agents' education decisions are consistent with the increase in income inequality in the United States (Heckman, Lochner and Taber, RED 1998)