MODELS WITH HETEROGENEOUS AGENTS

We can add *heterogeneity* to the neoclassical growth model in various ways:

Different consumers in:

- Initial capital (k_0)
- Labour productivity ($\lambda_t \Leftarrow$ deterministic or random) idiosyncratic shocks
- Preferences (β)

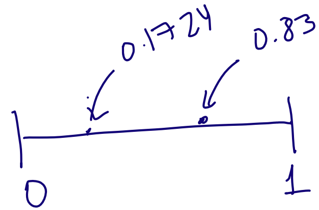
~~Heterogeneous firms in:~~

- ~~Productivity ($A_t \Leftarrow$ deterministic or random)~~
- ~~Specific factors~~

Models with heterogeneous agents can deal with questions about the distribution of income, industrial organization, etc.

They also add new insights to the analysis of classic topics at the aggregate level, such as growth or economic cycles

We are going to focus on models with heterogeneous consumers



A Simple Model with Differences in Assets

We will assume a continuum of agents (consumers / workers) in $[0, 1]$

$i \in [0,1]$

- In each period, agents differ only in their asset holdings $a_t^i = k_t^i + b_t^i$ (composite asset)

difference in initial conditions

$a_0^i \,,\, k_0^i \,,\, b_0^i$

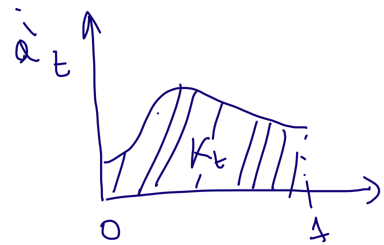
- Individual assets can be positive or negative, but are bounded below $(-B)$ in order to avoid Ponzi schemes

$a_t^i > -B$

- In each period, the distribution of assets a_t^i $[0,1] \to (-B, \infty)$ determines the aggregate stock of capital of the economy:

$$K_t = \int_0^1 a_t^i di$$

$$\int_0^1 b_t^i \, di = 0$$



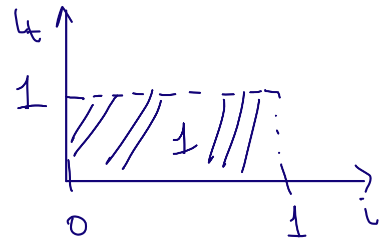
On the production side, a representative firm combines capital and labor to produce the unique good, according to

$$Y_t = F(K_t, L_t)$$

where F features constant returns to scale and the other usual properties

Each individual has a labor endowment that we normalize to one, therefore

$$L_t = \int_0^1 di = 1$$



and we write to simplify

$$Y_t = f(K_t) \equiv F(K_t, 1)$$

Finally, note that by an arbitrage logic the two components of the asset $a_t^i = k_t^i + b_t^i$ must have the same return

$$R_t \equiv \underbrace{r_t + (1-\delta)}_{k_t^i} = \underbrace{1 + \iota_t}_{b_t^i}$$

real interest rate

where

- r_t: rental price of the capital

- ι_t: real interest rate of bonds or financial assets

With that notation, we write the budget constraint of any agent $i \in [0,1]$ in period t as

$$c_t^i + a_{t+1}^i = w_t + R_t a_t^i$$

asset income

labor income

`Definition of Equilibrium`

A Sequential Competitive Equilibrium for this economy is a set of sequences for individual quantities c_t^i, a_t^i, $\forall i \in [0,1]$, aggregate quantities Y_t, K_t and prices w_t, R_t such that:

i) For each agent $i \in [0,1]$, given $a_0^i > -B$, w_t and R_t, the sequences c_t^i, a_{t+1}^i solve the problem:

$$
\begin{aligned}
& \max \quad \sum_{t=0}^{\infty} \beta^t u\left(c_t^i\right) \\
s.t. \quad & c_t^i + a_{t+1}^i = w_t + R_t a_t^i \qquad \forall t \\
& a_{t+1}^i > -B \qquad \forall t
\end{aligned}
$$

ii) In each period t, given w_t and R_t, the values Y_t and K_t solve the firm problem:

$$\max \quad Y_t - w_t - \overbrace{[R_t - (1-\delta)]}^{r_t} K_t$$

$$s.t. \quad Y_t = f(K_t)$$

and benefits are zero

$$Y_t = w_t + [R_t - (1-\delta)] K_t$$

iii) In each period t, markets clear:

$$Y_t = \int_0^1 \left[c_t^i + a_{t+1}^i - (1-\delta) a_t^i\right] di$$

$$K_t = \int_0^1 a_t^i di$$

$$Y_t = C_t + I_t$$

$$= C_t + K_{t+1} - (1-\delta) K_t$$

$$\int_0^1 c_t^i di \qquad \int_0^1 a_{t+1}^i di \qquad \int_0^1 a_t^i di$$

FIRST PROBLEM SET
(QUESTION 1)

First Order Conditions

Solving the consumer's problem for each agent i, we obtain Euler equation

$$\frac{u'\left(c_t^i\right)}{\beta u'\left(c_{t+1}^i\right)} = R_{t+1}$$

and the budget constraint

$$c_t^i + a_{t+1}^i = w_t + R_t a_t^i$$

Using the *transversality* condition

$$\lim_{t\to\infty} \rho_t a_{t+1}^i = 0$$

with $\rho_t = \prod_{j=0}^{t} \frac{1}{R_j}$, we can write the restriction in present value

$$\underbrace{R_0 \sum_{t=0}^{\infty} \rho_t c_t^i}_{\text{PV consumption}} = \underbrace{R_0 \left[a_0^i + \sum_{t=0}^{\infty} \rho_t w_t \right]}_{\text{PV labor income}} \equiv W_0^i \leftarrow \text{lifetime wealth}$$

From the firm's problem, we obtain the equilibrium prices

$$\begin{cases} R_t = f'(K_t) + (1-\delta) \\ w_t = f(K_t) - f'(K_t) K_t \end{cases}$$

$$\frac{u'(C_t)}{\beta u'(C_{t+1})} \text{ ?}$$

Combining, we have the system

$$\begin{cases} \dfrac{u'\left(c_t^i\right)}{\beta u'\left(c_{t+1}^i\right)} = f'(K_{t+1}) + (1-\delta) & \forall i \in [0,1] \\ \displaystyle\int_0^1 c_t^i di = f(K_t) + (1-\delta) K_t - K_{t+1} \end{cases}$$

which, together with the initial distribution of capital and the transversality condition, characterizes the equilibrium trajectories for individual consumption and aggregate capital

Stationary Equilibrium

In a stationary equilibrium, aggregate quantities (K_t, C_t) and prices (w_t, R_t) are constant in time

What does this imply for individual quantities (c_t^i, a_t^i)?

- It is enough that

$$C_t = \int_0^1 c_t^i di \qquad K_t = \int_0^1 a_t^i di$$

  be constant, for which the distribution of assets must be stationary

- In the deterministic model, the individual quantities are also constant

From the first order conditions, we can easily see that in a stationary equilibrium

$$R^* = f'(K^*) + (1-\delta) = \frac{1}{\beta} \quad \Rightarrow \quad K^*$$

The interest rate is given by the inverse of the discount factor

In a steady state, preferences and technology determine aggregate capital, regardless of the distribution of assets

Homotheticity and Aggregation

Homothetic preferences: the ratio of the consumption of two goods depends only on the relative price between the two, not on the level of income

Examples:

- Homothetic: $u(c) = \log(c) \Rightarrow \frac{c^i_{t+1}}{c^i_t} = \beta R_{t+1}$

$$\frac{u'(c_t)}{u'(c_{t+1})} = \frac{c_{t+1}}{c_t}$$

- Non-homotetic: $u(c) = \log(c - c_{\text{min}}) \Rightarrow \frac{c^i_{t+1} - c_{\text{min}}}{c^i_t - c_{\text{min}}} = \beta R_{t+1}$, then

$$\frac{c^i_{t+1}}{c^i_t} = \beta R_{t+1} + \underbrace{(1 - \beta R_{t+1})}_{<0} \frac{c_{\text{min}}}{c^i_t}$$

$$\frac{u'(c_t)}{u'(c_{t+1})} = \frac{c_{t+1} - c_{min}}{c_t - c_{min}} = \beta R_{t+1}$$

$$c_{t+1} - c_{min} = \beta R_{t+1}(c_t - c_{min})$$

$$c_{t+1} = \beta R_{t+1} c_t + (1 - \beta R_{t+1}) c_{min}$$

$$a_t^{RICH} > a_t^{POOR}$$

$$c_t^{RICH} > c_t^{POOR}$$

$$c_{t+1}/c_t^{\,RICH} > c_{t+1}/c_t^{\,POOR}$$

$$c^i_{t+1} = \beta R_{t+1} c^i_t \Rightarrow \int^1 c^i_{t+1}\, di = \beta R_{t+1} \int^1 c^i_t\, di$$

$C_{t+1} = \beta R_{t+1} C_t$

In general, with homothetic preferences, the marginal rate of substitution between present consumption and future consumption depends only on the growth rate of consumption

$$\frac{u'\left(c_t^i\right)}{\beta u'\left(c_{t+1}^i\right)} = \Psi\left(\frac{c_{t+1}^i}{c_t^i}\right)$$

then, by Euler's equation,

$$\frac{c_{t+1}^i}{c_t^i} = \frac{c_{t+1}^j}{c_t^j} = \frac{C_{t+1}}{C_t}$$

Therefore, the Euler equation at the aggregate level is also fulfilled

$$\Psi\left(\frac{C_{t+1}}{C_t}\right) = \frac{u'\left(C_t\right)}{\beta u'\left(C_{t+1}\right)} = R_{t+1}$$

That is, with homothetic preferences the aggregate quantities satisfy the system

$$\frac{u'(C_t)}{\beta u'(C_{t+1})} = f'(K_{t+1}) + (1-\delta)$$

$$C_t = f(K_t) + (1-\delta) K_t - K_{t+1}$$

and their equilibrium trajectories are independent of the distribution of assets

In other words, with homothetic preferences we can represent the aggregate dynamics of the economy as if it came from the problem of maximization of a single *representative agent*

Even in this case, the model has predictions about the evolution of the distribution of wealth over time

Evolution of the Distribution of Wealth

How do we measure the disparity in the distribution of wealth?

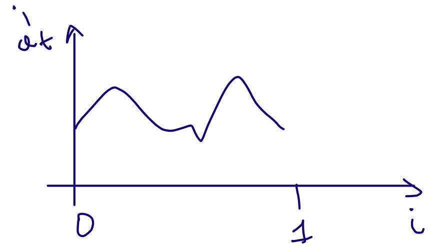
- Using the distribution of assets a_t^i

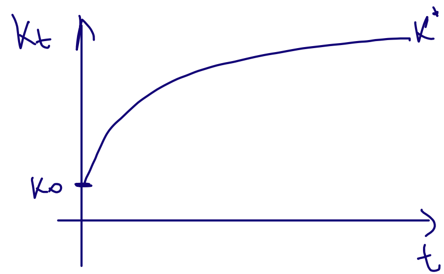
- Using the distribution of permanent wealth

$$W_t^i \equiv R_t \left[a_t^i + \sum_{j=t}^{\infty} \left(\prod_{s=t}^{j} \frac{1}{R_s} \right) w_j \right]$$

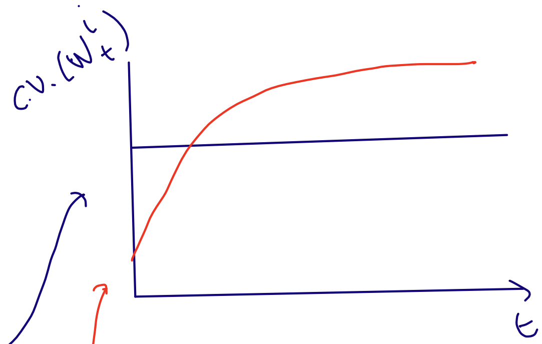
measured as today's assets plus the present value of the future labor income stream

In both cases, we will measure in each period the dispersion in the distribution of wealth as the *coefficient of variation* (the standard deviation on the mean) of wealth among all the agents





$$C.V.(x^i) = \frac{S.D.(x^i)}{\text{mean}(x^i)}$$



The question is, starting from an initial level of capital $K_0 < K^*$, what happens to the disparity in the distribution of wealth as the economy converges to the steady state?

Chatterjee (1994) shows that the disparity in the distribution of permanent wealth $C.V.\left(W_t^i\right)$

... remains constant with homothetic preferences

... *increases* throughout the transition with non-homothetic preferences of the type $u(c) = v(c - c_{\text{min}})$, with v homothetic and $c_{\text{min}} > 0$

Intuition: with $c_{\text{min}} = 0$, the growth rate of optimal consumption is the same for poor and rich agents; with $c_{\text{min}} > 0$, the growth rate of optimal consumption is lower for poor agents, who therefore save less

In the example with $u(c) = \log(c - c_{\text{min}})$

$$\frac{c_{t+1}^i}{c_t^i} = \beta R_{t+1} + (1 - \beta R_{t+1}) \frac{c_{\text{min}}}{c_t^i}$$

In addition, we can show that in this same example consumption is a function of permanent wealth

$$c_t^i = (1 - \beta) W_t^i + c_{\text{min}} D_t$$

therefore the disparity in consumption follows the disparity in permanent wealth

What happens to the disparity in the distribution of assets $C.V.\left(a_t^i\right)$?

Intuition is more complicated; not only the growth rate of consumption matters, but also the change in relative prices $(R_t,\ w_t)$ throughout the transition, which affect the proportion of permanent wealth that agents hold in financial assets or in work

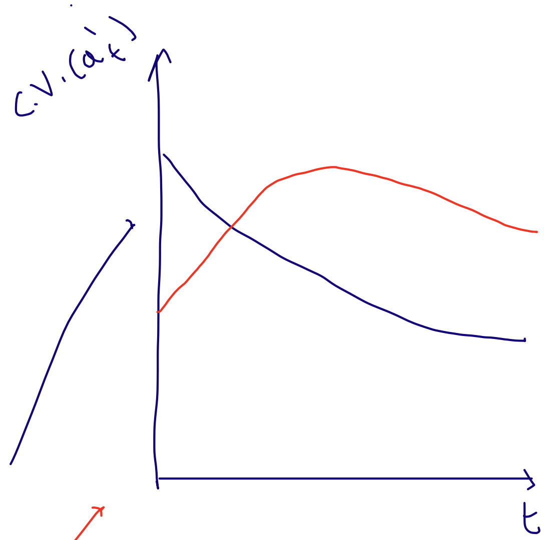
Obiols-Homs and Urrutia (2004) prove for the case $u(c) = \log(c - c_{\text{min}})$ and $f(K) = AK^{\alpha}$ that dispersion in the distribution of assets

... *diminishes* monotonically, if $c_{\text{min}} \leq 0$ or if $c_{\text{min}} > 0$ and K_0 is close enough to your steady state level K^*

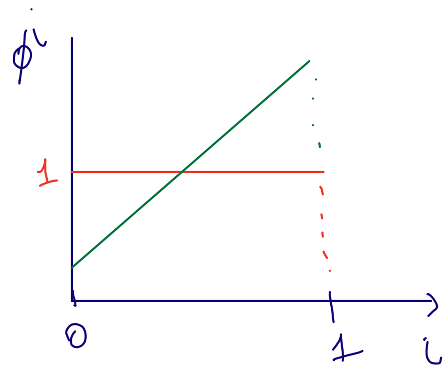
... otherwise, it may exhibit a non-monotonic dynamic that recalls the *Kuznets curve*

$c_{\min} > 0$

$c_{\min} > 0$ and K_0 well below K^*



Social Planner's Problem and Pareto Efficiency



How do we define the problem of the social planner with heterogeneous agents?

Consider a planner that assigns a weight ϕ^i to agent i:

$$\phi : [0,1] \to R \qquad\qquad \int_0^1 \phi^i di = 1$$

Given K_0, the planner chooses sequences for individual consumption c_t^i and aggregate capital K_t that solves:

$$\max \quad \sum_{t=0}^{\infty} \beta^t \int_0^1 \phi^i u\left(c_t^i\right) di$$

$$s.t. \quad \int_0^1 c_t^i di + K_{t+1} - (1-\delta) K_t = f(K_t) \qquad \forall t$$

$$L = \sum_{t=0}^{\infty} \left[\beta^t \int_0^1 \phi^i u(c_t^i) di - \lambda_t \left(\int_0^1 c_t^i di + K_{t+1} - (1-\delta) K_t - f(K_t) \right) \right]$$

$$\frac{\lambda_t}{\lambda_{t+1}} = \frac{u'(c_t^i)}{\beta u'(c_{t+1}^i)} \qquad \forall i \in [0,1]$$

$$\frac{dL}{dc_t^j} = \beta^t \phi^j u'(c_t^j) - \lambda_t = 0$$

fix $j=0$ $\qquad \lambda_t = \beta^t \phi^0 u'(c_t^0) \qquad \frac{u'(c_t^i)}{u'(c_t^0)} = \frac{\phi^0}{\phi^i}$

comparing i and 0 $\lambda_t = \beta^t \phi^i u'(c^i_t)$

$$\frac{dL}{dK_{t+1}} = -\lambda_t + \lambda_{t+1}\left[f'(K_{t+1}) + (1-\delta)\right]$$

These sequences will be efficient from Pareto's point of view

Using the first order conditions:

$$\phi^i u'\left(c_t^i\right) = \phi^0 u'\left(c_t^0\right) \rightarrow \phi^i > \phi^0 \rightarrow u'(c^i_t) < u'(c^0_t)$$

$$c^i_t > c^0_t$$

$$\frac{u'\left(c_t^i\right)}{\beta u'\left(c_{t+1}^i\right)} = f'\left(K_{t+1}\right) + (1-\delta)$$

EULER equation

$$\int_0^1 c_t^i di = f\left(K_t\right) + (1-\delta) K_t - K_{t+1} \rightarrow \text{feasibility}$$

we can prove the Welfare Theorems for this economy for a given a_0^i $\Longrightarrow$ $\tilde{c}_0^i$

$$\frac{u'(c_0^0)}{u'(c_0^i)} = \frac{\tilde{\phi}^i}{\tilde{\phi}^0}$$

WT1: *Let* $(\tilde{c}_t^i,\ \tilde{K}_t)$ *be the sequences obtained as part of a competitive equilibrium. There are weights* $\tilde{\phi}$ *for which* $(\tilde{c}_t^i,\ \tilde{K}_t)$ *solves the social planner's problem*

To prove this result, we simply build the weights

$$\frac{\tilde{\phi}^i}{\tilde{\phi}^0} = \frac{u'\left(\tilde{c}_0^0\right)}{u'\left(\tilde{c}_0^i\right)}$$

and verify the first order conditions of the social planner problem for a given ϕ^i

WT2: *Let* $(\hat{c}_t^i, \hat{K}_t)$ *be the sequences obtained solving the ~~social planner's~~ problem. There exist price sequences* $(\hat{R}_t, \hat{w}_t)$ *and individual assets* $\hat{a}_t^i$ *that support a competitive equilibrium*

→ initial distribution $\hat{a}_0^i$

The proof involves constructing prices:

$$\hat{R}_t = f'\left(\hat{K}_t\right) + (1-\delta) \qquad \hat{w}_t = f\left(\hat{K}_t\right) - \cancel{f'\left(\hat{K}_t\right)\hat{K}_t}$$

and, for each agent i, a sequence for assets

$$\hat{a}_{t+1}^i = \hat{w}_t + \hat{R}_t^i \hat{a}_t^i - \hat{c}_t^i$$

with

$$\hat{a}_0^i = \sum_{t=0}^{\infty} \left(\prod_{j=0}^{t} \frac{1}{\hat{R}_j} \right) \left[\hat{c}_t^i - \hat{w}_t \right]$$

Finally, we verify that the first order conditions for a competitive equilibrium are satisfied

Competitive Recursive Equilibrium

How do we describe the equilibrium of a model with heterogeneous agents in the recursive language of dynamic programming?

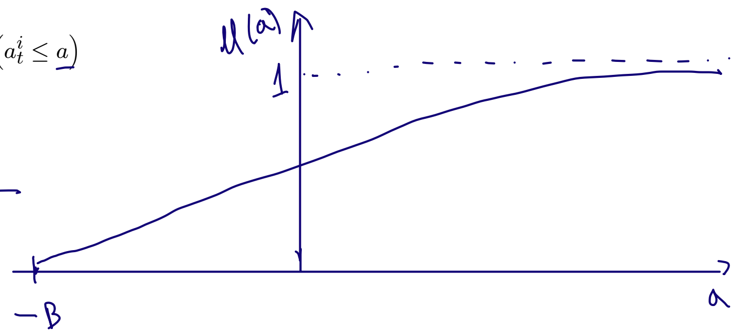
We need first to define the state variables

- Individual state: $a \in (-B, \infty)$

μ

- Aggregate state: distribution or measure $\mu_t(a)$ of agents with assets less than a

$$\mu_t : (-B, \infty) \to [0, 1] \qquad \mu_t(a) = \#\left(a_t^i \leq a\right)$$



$$K = \int_{-B}^{\infty} a \, d\mu(a)$$

$$K' = \Gamma(K)$$

$$= \Gamma(\mu)$$

The distribution $\mu_t(a)$ satisfies

$$\lim_{a \to -B} \mu_t(a) = 0 \qquad \lim_{a \to \infty} \mu_t(a) = 1$$

The agents must know the distribution μ today and its *law of motion*

$$\mu' = \Gamma(\mu)$$

to predict the prices of today and tomorrow

The law of motion Γ it is a complicated object, the result of the decisions of all agents in equilibrium

A Recursive Competitive Equilibrium is a set of functions $v(a,\mu)$, $c(a,\mu)$, $a'(a,\mu)$, prices $w(\mu)$ and $R(\mu)$, capital demand $K(\mu)$ and law of motion $\Gamma(\mu)$ such that:

i) For each pair (a,μ), given functions w, R and Γ, the value function $v(a,\mu)$ solves the Bellman equation:

$$v(a,\mu) = \max_{c,a'} \left\{ u(c) + \beta v\left(a',\mu'\right) \right\}$$

$$\begin{aligned} s.t. \quad c + a' &= w(\mu) + R(\mu)\, a \\ a' &> -B \\ \mu' &= \Gamma(\mu) \end{aligned}$$

and $c(a,\mu)$, $a'(a,\mu)$ are optimal decision rules

ii) For each μ, prices satisfy the marginal conditions for the representative firm:

$$R(\mu) = f'(K(\mu)) + (1-\delta)$$

$$w(\mu) = f(K(\mu)) - f'(K(\mu))\,K(\mu)$$

iii) For each μ, markets clear:

$$f(K(\mu)) = \int_{-B}^{\infty} \left[c(a,\mu) + a'(a,\mu) - (1-\delta)\,a\right] d\mu(a)$$

$$K(\mu) = \int_{-B}^{\infty} a\,d\mu(a) \longrightarrow \int_{-B}^{\infty} a \, \frac{d\mu(a)}{da} \, da$$

iv) For each μ, the law of motion Γ is consistent with the optimal decisions of the agents (*rational expectations*)

Once the recursive equilibrium has been solved, starting from μ_0 we can generate a sequence of distributions μ_t iteratively:

$$\{\mu_t\}_{t=0}^{\infty} \longrightarrow \mu^*$$

$$
\begin{aligned}
\mu_1 &= \Gamma(\mu_0) \\
\mu_2 &= \Gamma(\mu_1) = \Gamma(\Gamma(\mu_0)) \\
&\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}
$$

A recursive stationary equilibrium is an *invariant distribution* μ^* such that $\mu^* = \Gamma(\mu^*)$

We can also find the sequence for the aggregate capital stock

$$
\begin{aligned}
K_0 &= K(\mu_0) \\
K_1 &= K(\mu_1) \\
&\ldots\ldots\ldots
\end{aligned}
$$

and equilibrium prices

Finally, we can follow any agent $i \in [0,1]$, with a_0^i given, over time:

$$\begin{aligned} a_1^i &= a'\left(a_0^i, \mu_0\right) \\ a_2^i &= a'\left(a_1^i, \mu_1\right) = a'\left(a'\left(a_0^i, \mu_0\right), \Gamma\left(\mu_0\right)\right) \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}$$

$$\{a_t^i\}_{t=0}^{\infty}$$

$$\begin{aligned} c_1^i &= c\left(a_0^i, \mu_0\right) \\ c_2^i &= c\left(a_1^i, \mu_1\right) \\ &\ldots\ldots\ldots\ldots \end{aligned}$$

$$\{c_t^i\}_{t=0}^{\infty}$$

The resulting sequences are equivalent to those obtained by solving the sequential competitive equilibrium