Game Theory

Extensive-Form Representation: Finite-Horizon Case

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We have so far studied the normal-form representation of a game, which assumes that all players move simultaneously. In this note, we study the **extensive-form representation**, which allows players to move sequentially.

1 Extensive-Form Representation

Example 1. Suppose that player 1 plays either action L_1 or R_1 and then player 2 observes player 1's action and plays either action L_2 or R_2. This "sequential move" is illustrated by the "tree" below:

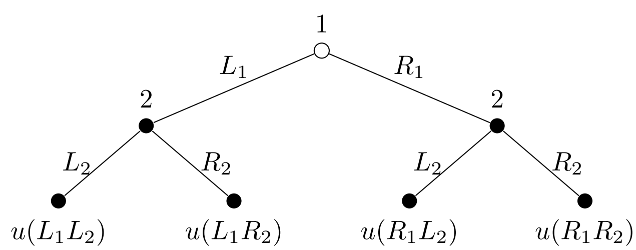

Figure 1: an extensive-form game in which player 2 observes player 1's move

This tree starts at the hollow node, at which player 1 chooses either action L_1 or R_1, which correspond to edge L_1 or R_1 respectively. Player 2, after learning whether he is at the left or right node, chooses either action L_2 or R_2, which correspond to edge L_2 or R_2 respectively. The play then arrives at one of the terminal nodes. Every different play, in which at least one player chooses a different action, leads to a different terminal node. Each terminal node (a_1a_2) is associated with a payoff profile $u(a_1a_2) = (u_1(a_1, a_2), u_2(a_1, a_2))$.

The tree is summarized by the tuple $\Gamma = \langle I, X, P, (u_i)_{i\in I}\rangle$ such that:

1. $I = \{1, 2\}$ is the set of players.
2. $X = \{\varnothing, L_1, R_1, L_1L_2, L_1R_2, R_1L_2, R_1R_2\}$ is the set of nodes.
   - $\varnothing \in X$ is the initial node, depicted by the hollow node.
   - $Z = \{L_1L_2, L_1R_2, R_1L_1, R_1L_2\}$ is the set of all terminal nodes.
3. $P : X \setminus Z \to I$ is the function that assigns to node $x \in X \setminus Z$ a player at node x.
   - $P(\varnothing) = 1$ and $P(L_1) = P(R_1) = 2$.
4. $u_i : Z \to \mathbb{R}$ is the payoff function for player i, for each $i \in I$.

The actions available at each non-terminal node are implicitly defined in this representation. For example, player 2 has two actions L_2, R_2 available at node L_1, each of which is represented by two nodes L_1L_2, L_1R_2, connected to node L_1.

The tree of Figure 1 can be recovered from the tuple $\langle I, X, P, (u_i)_{i\in I}\rangle$. First, we recover the "shape" of the tree from X. Second, we label each non-terminal node x with player $P(x)$. Third, we label each edge with the corresponding action. Finally, we add payoff profiles to all terminal nodes. □

Example 2. We modify Example 1 so that player 2 cannot observe player 1's choice of action. This environment is represented by Figure 2, in which nodes L_1 and R_1 are connected by the dashed line. The dashed line indicates that player 2 does not know which node he is at—that is, which action L_1 or R_1 player 1 has chosen.

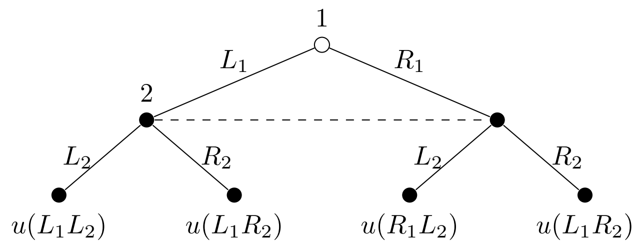

Figure 2: an extensive-form game in which player 2 does not observe player 1's move

This tree is summarized by the tuple $\Gamma = \langle I, X, P, (u_i, H_i)_{i\in I}\rangle$ such that I, X, P, and $(u_i)_{i\in I}$ all remain the same as in Example 1 and that:

5. H_i is the set of player i's information partition.
    - $H_1 = \{\{\varnothing\}\}$ means that player 1 knows that she is at node $\varnothing$.
    - $H_2 = \{\{L_1\}, \{R_1\}\}$ means that player 2 knows that he is at either node L_1 or R_1 but not which.

This game is the same as the normal-form game, in which players 1 and 2 simultaneously choose actions, since neither player observes her opponent's action. This example illustrates how we can represent a normal-form game in an extensive form. □

Example 3. Consider the teamwork-game example for correlated equilibrium. There are two players denoted 1 and 2, and there is a state $\omega \in \Omega = \{\mathrm{A}, \mathrm{B}, \mathrm{C}\}$. Player 1's information partition is $\{\{\mathrm{A}\}, \{\mathrm{B}, \mathrm{C}\}\}$, and player 2's information partition is $\{\{\mathrm{A}, \mathrm{B}\}, \{\mathrm{C}\}\}$. After learning their information, they simultaneously choose either action W or S.

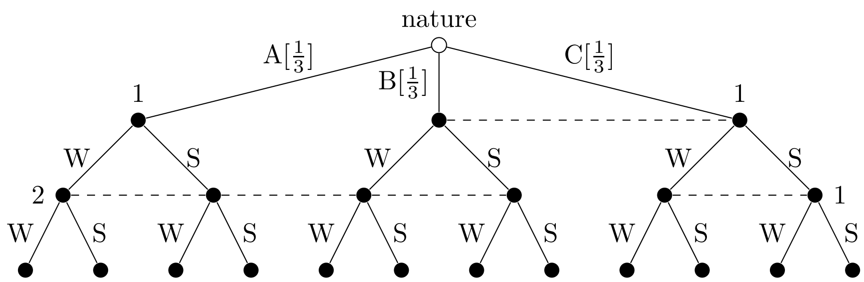

Figure 3: an extensive-form representation of the teamwork game

This game is represented by Figure 3, where we omit the labels of some edges and payoffs. The random choice of state ω is interpreted as the (virtual) player called **nature** randomly choosing an edge from the set $\{\mathrm{A}, \mathrm{B}, \mathrm{C}\}$. The numbers in the square brackets denote the probabilities of nature's choice of action.

The tree is summarized by the tuple $\langle I, X, P, (u_i, H_i)_{i \in I}, \pi \rangle$ such that:

1. $I = \{1, 2\} \cup \{0\}$ is the set of players, where player 0 denotes nature.
2. X is the set of nodes.
   - $\varnothing \in X$ is the initial node, depicted by the hollow node.
   - Z is the set of all terminal nodes.
   - For example, node AWS denotes the (second left) terminal node that is reached when players 0, 1, and 2 choose actions A, W, and S respectively.
3. $P : X \setminus Z \to I$ is the function that maps node $x \in X \setminus Z$ to a player at node x.
   - $P(\varnothing) = 0$.
   - $P(\mathrm{A}) = P(\mathrm{B}) = P(\mathrm{C}) = 1$.
   - $P(\mathrm{AW}) = P(\mathrm{AS}) = P(\mathrm{BW}) = P(\mathrm{BS}) = P(\mathrm{CW}) = P(\mathrm{CS}) = 2$.
4. $u_i : Z \to \mathbb{R}$ is the payoff function for player i for each $i \in I \setminus \{0\}$.
5. H_i is the set of player i's information partition for each $i \in I \setminus \{0\}$.
   - $H_1 = \{\{\mathrm{A}\}, \{\mathrm{B}, \mathrm{C}\}\}$.
   - $H_2 = \{\{\mathrm{AW}, \mathrm{AS}, \mathrm{BW}, \mathrm{BS}\}, \{\mathrm{CW}, \mathrm{CS}\}\}$.
6. π is a function that specifies nature's random moves at nature's nodes.
   - $\{x \in X : P(x) = 0\} = \{\varnothing\}$ is the (singleton) set of nature's nodes.
   - $\pi(\varnothing) \in \Delta(\Omega)$ assigns probability $\frac{1}{3}$ to each $\omega \in \Omega$. □

1.1 Extensive-Form Representation

As discussed above, extensive-form representation incorporates sequential moves. It is a tree endowed with additional structures that formalize the rules of the game: the order of play, the actions and information available to players, outcomes, and players' preferences over the outcomes, etc.

Definition 1. A **finite-horizon extensive-form game**, or the **finite-horizon extensive-form representation** of a game, is a tuple $\Gamma = \langle I, X, P, (u_i, H_i)_i, \pi \rangle$ such that:

1. I is the set of players.
   - I may or may not include player 0, who is called **nature**.
2. X is the set of nodes such that:
   - $\rightarrow \subset X \times X$ is a binary relation that is transitive and asymmetric.[1,2]
     - It defines the "direction" between nodes.
     - There exists an **initial node** $\varnothing \in X$ such that for each $x \in X$, $\varnothing \rightarrow x$.
   - $z \in X$ is a **terminal node** if there exists no $x \in X$ such that $z \rightarrow x$.
     - Z is the set of all terminal nodes.
   - $\twoheadrightarrow$ is such that $x \twoheadrightarrow y$ if $x \rightarrow y$ but there exists no $x' \in X$ such that $x \rightarrow x' \rightarrow y$.
     - $y \in X \setminus \{\varnothing\}$ has a unique $x \in X$ such that $x \twoheadrightarrow y$.
   - $\varnothing \twoheadrightarrow x_1 \twoheadrightarrow x_2 \twoheadrightarrow \cdots$ is a path (from $\varnothing$).
     - Every path has a last element, and the last element is a terminal node.
3. $P : X \setminus Z \rightarrow I$ is the function that assigns to node $x \in X \setminus Z$ a player moving at node x.
   - $X_i = \{x \in X : P(x) = i\}$ is the set of player i's nodes.
   - $A(x) = \{y \in X : x \twoheadrightarrow y\}$ is the set of actions for player $P(x)$.
4. $u_i : Z \rightarrow \mathbb{R}$ is player i's payoff function.
5. H_i is player i's **partition** of X_i such that:
   - $h_i \in H_i$ is called player i's **information set**.
   - $A(x) = A(x')$ for each $x, x' \in h_i$, which is also denoted by $A(h_i)$.
6. π is a function that assigns to node $x \in X_0$ a distribution $\pi(x) \in \Delta(A(x))$.
   - $0 \in I$ denotes the player called nature, who moves at random.
   - $\pi(x) \in \Delta(A(x))$ is nature's random choice of action $y \in A(x)$ at node $x \in X_0$.

Perfect/Imperfect Information In Example 1, each player knows the past play. Such an extensive-form game is of perfect information. In Example 2, player 2 does not know player 1's past play. In Example 3, neither player knows the past play by nature. Such extensive-form games are of imperfect information.

[1] Transitivity: for each $x, y, z \in X$, if $x \rightarrow y$ and $y \rightarrow z$ then $x \rightarrow z$.
[2] Asymmetry: for each $x, y \in X$, if $x \rightarrow y$ then $y \not\rightarrow x$.

Definition 2. A finite-horizon extensive-form game Γ is of **perfect information** if each $h_i \in H_i$ is a singleton for each $i \in I$, and is of **imperfect information** otherwise.

Remark 1. If a finite-horizon extensive-form game Γ is of perfect information, we omit players' information partitions H_i. Similarly, if it has no (random) move by nature, we omit nature's play π. □

1.2 Perfect Recall*

Assume that all players never forget what they have ever learned in the past. This assumption is called **perfect recall**. To better understand the assumption, we see an example that does not satisfy the assumption.

Example 4. Consider the extensive-form game represented below:

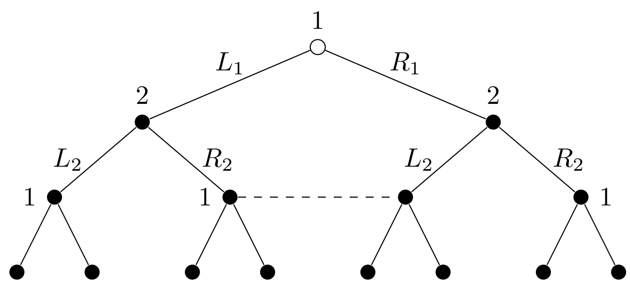

Figure 4: an extensive-form game of imperfect recall

Player 1 has four information sets: $\{\varnothing\}$, $\{L_1L_2\}$, $\{L_1R_2, R_1L_2\}$, and $\{R_1R_2\}$. Note that when reaching the information set $\{L_1R_2, R_1L_2\}$, player 1 "forgets" which action L_1 or R_1 she had taken at the information set $\{\varnothing\}$. □

To formalize the notion of perfect recall, we introduce the notion of experience.[3]

Definition 3. In a finite-horizon extensive-form game Γ, player i's **experience** $\texttt{Exp}(x_i)$ at a node x_i is the sequence consisting of the information sets that player i encounters all the way to the node x_i and all the actions that she has taken, in the order that these events occur.

With the notion of experience, we mean by perfect recall that no player ever forgets her experience.

Definition 4. A finite-horizon extensive-form game Γ is of **perfect recall** if for each $i \in I$, $\texttt{Exp}(x_i) = \texttt{Exp}(x_i')$ for any nodes x, x' belonging to the same information set h_i of player i. It is of **imperfect recall** if it is not of perfect recall.

Example 5. In Figure 4, player 1 has the following experiences:

[3] This is taken from Osborne & Rubinstein (1994, p.203).

Recap (Remark on Perfect Recall): We often define the notion of perfect recall as follows:

Definition 5. A finite-horizon extensive-form game Γ is of **perfect recall** if the following is true:

- For each $i \in I$ and each $h_i, h_i' \in H_i$, if there exist some $x \in h_i$, $y \in h_i'$, and $a \in A(x)$ such that $(x, a) \to y$, then for each $x' \in h_i$ there exists some $y' \in h_i'$ such that $(x', a) \to y'$.

It is of **imperfect recall** if it is not of perfect recall.

We can read this definition as follows: If player i's nodes x' and x'' are indistinguishable to player i, then for all past nodes and actions that she recalls at x', there must be an indistinguishable node and action that she recalls at x''. If such a node or action did not exist, it would imply that player i must have forgotten.

Example 6. In Example 4, let $x = x' = \varnothing$ and $y = (L_1, R_2)$, $y' = (R_1, L_2)$ in the notation of Definition 5. The only action a such that $(x, a) \to y$ is $a = L_1$. However, $(x', a) \not\to y'$. Hence, this game does not satisfy the assumption of perfect recall. □

1. The experience at her node $x_1^1 = \varnothing$ is $\texttt{Exp}(x_1^1) = (\{\varnothing\})$.
2. The experience at her node $x_1^2 = L_1L_2$ is $\texttt{Exp}(x_1^2) = (\{\varnothing\}, L_1, \{L_1L_2\})$.
3. The experience at her node $x_1^3 = L_1R_2$ is $\texttt{Exp}(x_1^3) = (\{\varnothing\}, L_1, \{L_1R_2, R_1L_2\})$.
4. The experience at her node $x_1^4 = R_1L_2$ is $\texttt{Exp}(x_1^4) = (\{\varnothing\}, R_1, \{L_1R_2, R_1L_2\})$.
5. The experience at her node $x_1^5 = R_1R_2$ is $\texttt{Exp}(x_1^5) = (\{\varnothing\}, R_1, \{R_1R_2\})$.

Player 1's nodes x_1^3 and x_1^4 are in the same information set $\{L_1R_2, R_1L_2\}$, but her experiences are different. That is, at that information set, she has forgotten which action she had taken before. This game is of imperfect recall. □

Remark 2. Is the assumption of perfect recall too strong? Not necessarily. If a player does not have perfect recall, then we can think of that player as a team or an organization consisting of multiple agents (with the same payoffs) in which there may be miscommunication between the agents, rather than a single forgetful agent. □

2 Strategies

2.1 Pure Strategies

In a finite-horizon extensive-form game, a pure strategy σ_i for player i is a *complete contingent plan*, which specifies which action she plays at her *every* information set.

Example 7. In Example 1, player 2 has two information sets $\{L_1\}$ and $\{R_1\}$. His pure strategy s_2 specifies which action L_2 or R_2 he plays at each of $\{L_1\}$ and $\{R_1\}$. Hence, if player 2 chooses L_2 at $\{L_1\}$ and R_2 at $\{R_1\}$ then this strategy is described by a function s_2 such that $s_2(\{L_1\}) = L_2$ and $s_2(\{R_1\}) = R_2$.

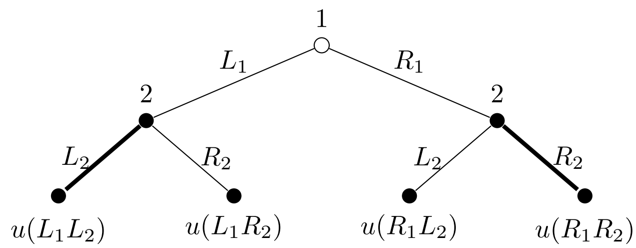

Figure 5: player 2's pure strategy in Figure 1

In Example 2, player 2 has only one information set $\{L_1, R_1\}$. His pure strategy s_2 specifies which action L_2 or R_2 she plays at $\{L_1, R_1\}$. Hence, if player 2 chooses L_2 at $\{L_1, R_1\}$ then this strategy is described by a function s_2 such that $s_2(\{L_1, R_1\}) = L_2$.

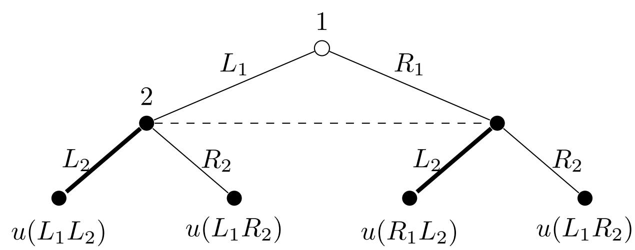

Figure 6: player 2's pure strategy in Figure 2

Player 2 is not allowed to choose different actions at nodes L_1 and R_1 (included in the same information set $\{L_1, R_1\}$). $\square$

We will now define pure strategies in extensive-form games by generalizing this observation.

Definition 6. In a finite-horizon extensive-form game Γ, a **pure strategy** for player i is a function $s_i : H_i \to \bigcup_{h_i} A_i(h_i)$ such that $s_i(h_i) \in A(h_i)$ for each $h_i \in H_i$. Let S_i be the set of all pure strategies for player i.

2.2 Mixed Strategies and Behavioral Strategies

There are two ways to model player i's random choice of action in an extensive-form game. First, we introduce the concept of mixed strategies. In an extensive-form game, as in a normal-form game, we define a mixed strategy for player i as a randomization over the set of her pure strategies.

Definition 7. In a finite-horizon extensive-form game Γ, a **mixed strategy** for player i is a probability distribution σ_i defined over the set S_i of her pure strategies. Let Σ_i be the set of all mixed strategies for player i.

Example 8. Recall that player 2's pure strategy s_2 is a function $s_2 : \{\{L_1\}, \{R_1\}\} \to \{L_2, R_2\}$ in Example 1. Hence, a mixed strategy for him is a probability distribution over the set of these pure-strategies. $\square$

Second, we introduce the concept of behavioral strategies. This strategy allows each player to make a random choice of action at her every information set.

Definition 8. In a finite-horizon extensive-form game Γ, a **behavioral strategy** for player i is a function $\beta_i : H_i \to \Delta(A_i)$ such that for each $h_i \in H_i$, $\beta_i(h_i) \in \Delta(A(h_i))$. Let $\beta_i(a_i \mid h_i)$ denote the probability that player i chooses action $a_i \in A_i(h_i)$ at an information set $h_i \in H_i$.

Example 9. Recall that player 2's pure strategy s_2 is a function $s_2 : \{\{L_1\}, \{R_1\}\} \to \{L_2, R_2\}$ in Example 1. A behavioral strategy for him is a function $\beta_2 : \{\{L_1\}, \{R_1\}\} \to \Delta(\{L_2, R_2\})$, where $\beta_2(\{L_1\}), \beta_2(\{R_1\}) \in \Delta(\{L_2, R_2\})$ are probability distributions over player 2's actions at his information sets $\{L_1\}$ and $\{R_1\}$ respectively. $\square$

Remark 3. A mixed strategy for a player makes a random choice of a pure strategy (i.e., a complete contingent plan) once and for all at the beginning of the game, while a behavioral strategy for the player makes a random choice of action at her every information set. $\square$

2.3 Kuhn's Theorem

We have introduced two kinds of random strategies—i.e., mixed strategies and behavioral strategies. Are they equivalent? If so, when? To answer these questions, we need to formalize the "equivalence" between two strategies.

In a finite-horizon extensive-form game Γ, since players' payoffs are defined over terminal nodes, they care about which terminal nodes will be reached with what probabilities, if they play a strategy profile $\sigma = (\sigma_i)_{i \in I}$ of either mixed or behavioral strategies.

Definition 9. In a finite-horizon extensive-form game Γ, for any (mixed or behavioral) strategy profile σ, the **outcome** $O(\sigma) \in \Delta(Z)$ is the probability distribution over terminal nodes that results when players play σ.

Definition 10. In a finite-horizon extensive-form game Γ, a mixed strategy σ_i and a behavioral strategy β_i for player i are **outcome-equivalent** if regardless of player $-i$'s every pure strategy profile s_{-i}, both σ_i and β_i induce the same outcome: $O(\sigma_i, s_{-i}) = O(\beta_i, s_{-i})$.

We will show that mixed strategies and behavioral strategies are outcome-equivalent in an extensive-form game of perfect recall.

Example 10. In Example 1, we consider player 2's mixed strategy σ_2 that assigns probabilities $(\frac{1}{10}, \frac{2}{10}, \frac{3}{10}, \frac{4}{10})$ to pure strategies $(L_2L_2, L_2R_2, R_2L_2, R_2R_2)$. If player 1 plays L_1 then we have outcome $(\frac{1}{3}, \frac{2}{3}, 0, 0)$, while if player 1 plays R_1 then we have outcome $(0, 0, \frac{3}{7}, \frac{4}{7})$, where we denote by $(p_{LL}, p_{LR}, p_{RL}, p_{RR})$ an outcome that assigns these probabilities to terminal nodes L_1L_2, L_1R_2, R_1L_2, and R_1R_2 respectively.

We find player 2's behavioral strategy β_2 that is outcome-equivalent to σ_2. Let β_2 be such that he mixes actions L_2, R_2 with probabilities $\frac{1}{3}, \frac{2}{3}$ at information set L_1 and actions L_2, R_2 with probabilities $\frac{3}{7}, \frac{4}{7}$ at information set R_1. Then, β_2 is outcome-equivalent to σ_2. □

Kuhn's Theorem Kuhn (1953) shows that for any mixed strategy, there is an outcome-equivalent behavioral strategy and vice versa, in a finite extensive-form game of perfect recall, where "finite" means that there are finitely many nodes.

Definition 11. A (finite-horizon) extensive-form game Γ is **finite** if the set X of nodes is finite.

Theorem 1 (Kuhn 1953)**.** *In a finite extensive-form game Γ of perfect recall, the following two hold:*

1. *For any mixed strategy σ_i, there is an outcome-equivalent behavioral strategy β_i.*
2. *For any behavioral strategy β_i, there is an outcome-equivalent mixed strategy σ_i.*

Proof. See Osborne & Rubinstein (1994, p.214-5). ■

The assumption of perfect recall is necessary for Kuhn's Theorem, as illustrated below.

Example 11. Consider the extensive-form game represented below:

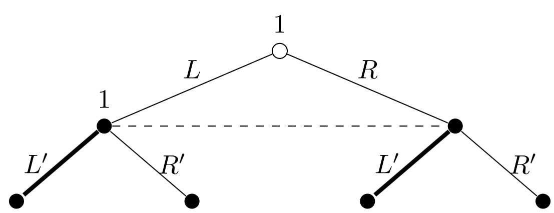

Figure 7: the assumption of perfect recall is necessary for Kuhn's Theorem

This one-player game is of imperfect recall. The player "forgets" which action L or R she had taken when arriving at the information set $\{L, R\}$.[4]

Suppose that she plays a mixed strategy that assigns probability $\frac{1}{2}$ to pure-strategies LL' and RR' each. It induces an outcome $(\frac{1}{2}, 0, 0, \frac{1}{2})$, where we denote by $(p_{LL'}, p_{LR'}, p_{RL'}, p_{RR'})$ an outcome that assigns these probabilities to terminal nodes LL', LR', RL', and RR' respectively. However, there is no behavioral strategy that induces the same outcome. □

[4]The two nodes L and R are in the same information set, but the experiences are different: $\texttt{Exp}(L) = (\varnothing, L)$ and $\texttt{Exp}(R) = (\varnothing, R)$.

References

Kuhn, H. W. (1953). Extensive games and the problem of information. In H. W. Kuhn & A. W. Tucker (Eds.), *Contribution to the theory of games* (p. 193-216). Princeton University Press.

Osborne, M. J., & Rubinstein, A. (1994). *A course in game theory*. MIT Press.