Game Theory

Correlated Equilibrium

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Nash equilibrium assumes that players independently randomize their choice of action. In this note, we discuss another solution concept of **correlated equilibrium**, which allows for correlated randomization (Aumann, 1987).

1 Correlated Equilibrium

Example 1. Two players, denoted 1 and 2, decide whether to work (W) or shirk (S) in a joint project. This game, called a teamwork game, is represented as follows:[1]

	W	S
W	$4,4$	$1,5$
S	$5,1$	$0,0$

Table 1: the teamwork game

There are three Nash equilibria: two pure-strategy Nash equilibria (W,S) and (S,W), and one mixed-strategy Nash equilibrium $(\frac{1}{2}\mathrm{W} \oplus \frac{1}{2}\mathrm{S}, \frac{1}{2}\mathrm{W} \oplus \frac{1}{2}\mathrm{S})$, in which each player independently randomizes the two actions with equal probabilities.

We consider the following "correlation device." There are three possible states $\omega \in \{\mathrm{A},\mathrm{B},\mathrm{C}\}$. When state ω is realized, player 1 learns whether "$\omega = \mathrm{A}$" or "$\omega = \mathrm{B}$ or C" but not whether "$\omega = \mathrm{B}$" or "$\omega = \mathrm{C}$". Similarly, player 2 learns whether "$\omega = \mathrm{A}$ or B" or "$\omega = \mathrm{C}$" but not whether "$\omega = \mathrm{A}$" or "$\omega = \mathrm{B}$". This "knowledge" can be schematized as follows:

$$\text{Player 1: } \{\mathrm{A}\}, \{\mathrm{B},\mathrm{C}\},$$
$$\text{Player 2: } \{\mathrm{A},\mathrm{B}\}, \{\mathrm{C}\}.$$

Here, a player learns which set state ω is in but does not learn which element state ω is (if there are multiple states in the set).

Suppose that each ω is realized with probability $\frac{1}{3}$. Then, player 1 will observe $\{\mathrm{A}\}$ with probability $\frac{1}{3}$ and $\{\mathrm{B},\mathrm{C}\}$ with probability $\frac{2}{3}$, while player 2 will observe $\{\mathrm{A},\mathrm{B}\}$ with probability $\frac{2}{3}$ and $\{\mathrm{C}\}$ with probability $\frac{1}{3}$.

Suppose that players have agreed on the following strategy profile (before observing any

[1] This teamwork game is effectively the same as Aumann's (1974) game.

Recap (Partition): A **partition** P of a set $X \neq \varnothing$ is a family of subsets of the set X such that:

- $\varnothing \notin P$.
- $\bigcup_{A \in P} A = X$.
- $A \cap B = \varnothing$ if $A, B \in P$ and $A \neq B$.

information):

$$a_1 = \begin{cases} \text{S} & \text{if player 1 observes } \{\text{A}\} \\ \text{W} & \text{if player 1 observes } \{\text{B}, \text{C}\}, \end{cases} \qquad a_2 = \begin{cases} \text{W} & \text{if player 2 observes } \{\text{A}, \text{B}\} \\ \text{S} & \text{if player 2 observes } \{\text{C}\}. \end{cases}$$

Are they willing to obey the agreement? We examine each player's incentive. Is player 1 (she) willing to obey the agreement, if player 2 (he) obeys it? First, suppose that player 1 observes $\{\text{A}\}$. Then, she believes that player 2 will play $a_2 = \text{W}$ (since he observes $\{A, B\}$. Hence, she has no incentive to deviate. Second, suppose that player 1 observes $\{\text{B}, \text{C}\}$. Then, she believes that player 1 will play $a_2 = \text{W}, \text{S}$ with probabilities $\frac{1}{2}$ (since he observes $\{A, B\}$ and $\{C\}$ with probabilities $\frac{1}{2}$). Hence, player 1's payoff is $\frac{5}{2}$ if she plays $a_1 = \text{W}$ but still is $\frac{5}{2}$ if she plays $a_1 = \text{S}$. She has no (strict) incentive to deviate. That is, she is willing to obey the agreement. Similarly, player 2 is willing to obey the agreement (if player 1 obeys it). This is called a **correlated equilibrium**. The corresponding payoff profile is $\frac{1}{3}(4,4) + \frac{1}{3}(1,5) + \frac{1}{3}(5,1) = (\frac{10}{3}, \frac{10}{3})$, which Pareto-dominates the mixed-strategy Nash equilibrium payoff profile $(\frac{5}{2}, \frac{5}{2})$. $\square$

1.1 Correlated Equilibrium

Correlated Device

Definition 1. For a finite normal-form game G, a **correlation device** is a triple (Ω, π, H) such that:

1. Ω is a finite set of states.
2. $\pi \in \Delta(\Omega)$ is a full-support prior.[2]
3. $H = (H_i)_i$ is the profile of player i's partition H_i of Ω.
   - $H_i(\omega)$ is the (unique) element $h_i \in H_i$ such that $\omega \in h_i$.
   - H_i is called player i's **information partition** and $h_i \in H_i$ his **information set**.

Example 2. The correlation device of Example 1 is such that:

1. $\Omega = \{\text{A}, \text{B}, \text{C}\}$.

[2] That is, $\pi(\omega) > 0$ for each $\omega \in \Omega$. This full-support assumption is without loss of generality.

2. $\pi(\omega) = \frac{1}{3}$ for each $\omega \in \Omega$.

3. $H_1 = \{\{\mathrm{A}\}, \{\mathrm{B}, \mathrm{C}\}\}$ and $H_2 = \{\{\mathrm{A}, \mathrm{B}\}, \{\mathrm{C}\}\}$.

For example, if $\omega = \mathrm{A}$, $H_1(\omega) = \{\mathrm{A}\}$ and $H_2(\omega) = \{\mathrm{A}, \mathrm{B}\}$. □

Correlated Strategies In a finite normal-form game G without a correlation device, players do not have any information about the game (other than the setting itself), but now since they may learn some information, their choice of action should be conditioned on the information they learn.

Definition 2. For a finite normal-form game G, let (Ω, π, H) be a correlation device. Player i's **(correlation) strategy** is a function $f_i : \Omega \to A_i$ such that for each $\omega, \omega' \in \Omega$,[3]

$$H_i(\omega) = H_i(\omega') \quad \Longrightarrow \quad f_i(\omega) = f_i(\omega').$$

Let $F_i(\Omega, H_i)$ be the set of all player i's strategies. A **(correlation) strategy profile** $f = (f_i)_i$ is the profile consisting of each player's strategy $f_i \in F_i(\Omega, H_i)$. Let $F(\Omega, H)$ be the set of all the strategy profiles.

Remark 1. Player i's strategy f_i does not allow for randomization except through observations of an information set h_i. This is without loss of generality, if we enlarge the state space Ω.

Using Example 1, we illustrate this idea. We attempt to let player 1 randomize W and S with equal probabilities when observing $\{\mathrm{B}, \mathrm{C}\}$. Enlarge the state space Ω to $\Omega' = \{\mathrm{A}_H, \mathrm{A}_T, \mathrm{B}_H, \mathrm{B}_T, \mathrm{C}_H, \mathrm{C}_T\}$. When a state $\omega' \in \Omega'$ is realized, player 1 observes the same information as in the original setting (i.e., $\{\mathrm{A}\}$ or $\{\mathrm{B}, \mathrm{C}\}$), plus the result of a coin toss (heads or tails). Player 1's information partition is as follows:

$$\text{Player 1: } \{\mathrm{A}_H\}, \{\mathrm{A}_T\}, \{\mathrm{B}_H, \mathrm{C}_H\}, \{\mathrm{B}_T, \mathrm{C}_T\},$$

where subscripts H and T denote heads and tails of the coin toss respectively. Let f_1 be her strategy such that $f_1(\mathrm{A}_H) = f_1(\mathrm{A}_T) = \mathrm{S}$, $f_1(\mathrm{B}_H) = f_1(\mathrm{C}_H) = \mathrm{W}$, and $f_1(\mathrm{B}_T) = f_1(\mathrm{C}_T) = \mathrm{S}$. If each $\omega' \in \Omega'$ is realized with equal probability $\frac{1}{6}$ then f_1 randomizes W and S with equal probabilities when observing $\{\mathrm{B}, \mathrm{C}\}$ in the original setting. □

Correlated Equilibrium

Definition 3. In a finite normal-form game G, a **correlated equilibrium** is a pair $(\Omega, \pi, H; f^*)$ of a correlation device (Ω, π, H) and a strategy profile $f^* \in F(\Omega, H)$ such that for each $i \in I$

[3] That is, her strategy f_i must be constant over each $h_i \in H_i$. In terms of measure theory, it must be measurable with respect to the sigma-algebra generated by the partition H_i.

and each $f_i \in F_i(\Omega, H_i)$,

$$\sum_{\omega \in \Omega} \pi(\omega) u_i\big(f_i^*(\omega), f_{-i}^*(\omega)\big) \geq \sum_{\omega \in \Omega} \pi(\omega) u_i\big(f_i(\omega), f_{-i}^*(\omega)\big). \tag{1}$$

Coffee Break ☕. If you have never seen this kind of setting (e.g., information partitions), you might be being puzzled. No worries! What is modeled here is actually simple. I would recommend you to compare the formal definitions with the example, again and again; you would be get used to the modeling soon. □

Ex-Ante versus Interim Perspectives In Definition 3, we take the **ex-ante** perspective—that is, we evaluate player i's expected payoff using the prior probability π, which she has before she learns her information $H_i(\omega)$. In Example 1, however, we take the **interim** perspective—that is, we evaluate player i's expected payoff using the posterior probability $\pi(\cdot \mid H_i(\omega))$, which she forms after she learns her information (but does not know players $-i$'s information). It turns out that these two formulations are equivalent.

Theorem 1. *For a finite normal-form game G, let (Ω, π, H) be a correlation device. Let $f^* \in F(\Omega, H)$ be a strategy profile. Then, the following two are equivalent:*

(i) For each $f_i \in F_i(\Omega, H_i)$,

$$\sum_{\omega \in \Omega} \pi(\omega) u_i\big(f_i^*(\omega), f_{-i}^*(\omega)\big) \geq \sum_{\omega \in \Omega} \pi(\omega) u_i\big(f_i(\omega), f_{-i}^*(\omega)\big). \tag{2}$$

(ii) For each $h_i \in H_i$ and each $a_i \in A_i$,

$$\sum_{\omega \in h_i} \pi(\omega \mid h_i) u_i\big(f_i^*(\omega), f_{-i}^*(\omega)\big) \geq \sum_{\omega \in h_i} \pi(\omega \mid h_i) u_i\big(a_i, f_{-i}^*(\omega)\big), \tag{3}$$

where $\pi(\omega \mid h_i)$ is player i's posterior based on the information set h_i:

$$\pi(\omega \mid h_i) = \begin{cases} \frac{\pi(\omega)}{\pi(h_i)} & \text{if } \omega \in h_i \\ 0 & \text{if } \omega \notin h_i. \end{cases}$$

Proof. See Appendix A. ■

1.2 Revelation-Principle-Like Result

A correlated device specifies which information each player observes with what probability. It does not specify the resulting action distribution itself. Each player i's strategy f_i converts information into action. Combining them, we will have the resulting action distribution.

Definition 4. In a finite normal-form game G, we say that a distribution $\mu \in \Delta(A)$ is a **correlated equilibrium action distribution** if there exists some correlated equilibrium, denoted $(\Omega, \pi, H; f)$, such that for each $a \in A$,

$$\mu(a) = \sum_{\omega : f(\omega) = a} \pi(\omega).$$

Example 3. In Example 1, we assume that each state $\omega \in \{\mathrm{A}, \mathrm{B}, \mathrm{C}\}$ is drawn with equal probability $\frac{1}{3}$. We then find correlated equilibrium strategies f_1, f_2 such that $f_1(\{\mathrm{A}\}) = f_2(\{\mathrm{C}\}) = \mathrm{S}$ and $f_1(\{\mathrm{B}, \mathrm{C}\}) = f_2(\{\mathrm{A}, \mathrm{B}\}) = \mathrm{W}$. Hence, the resulting action distribution $\mu \in \Delta(A)$ is such that $\mu(\mathrm{W}, \mathrm{W}) = \mu(\mathrm{W}, \mathrm{S}) = \mu(\mathrm{S}, \mathrm{W}) = \frac{1}{3}$ and $\mu(\mathrm{S}, \mathrm{S}) = 0$. □

Direct Correlated Device It sounds a daunting task to find all correlated equilibrium action distributions, because a state space Ω can be taken arbitrarily. However, we note that when we define a correlated equilibrium, the end-result that players would care about is only the induced action distribution. This is because players do not care about a state ω itself, aside from its indirect effect on the induced distribution over action profiles.

Example 4. Consider the game of Example 1. Suppose that players have access to a "direct" correlation device $\mu \in \Delta(A)$: Action profile $a = (a_1, a_2)$ is drawn according to μ, and players 1 and 2 are "recommend" to play a_1 and a_2 respectively (but neither knows the recommendation to the opponent). If player 1 is recommended to play a_1 then she knows that either (a_1, W) or (a_1, S) is drawn; by Bayes's Theorem, she believes that the probability that player 2 is recommended to play a_2 is $\frac{\mu(a_1, a_2)}{\mu(a_1, \mathrm{W}) + \mu(a_1, \mathrm{S})}$. The same applies to player 2.

	W	S
W	$\mu(\mathrm{W}, \mathrm{W})$	$\mu(\mathrm{W}, \mathrm{S})$
S	$\mu(\mathrm{S}, \mathrm{W})$	$\mu(\mathrm{S}, \mathrm{S})$

Table 2: a recommendation distribution

Suppose that $\mu(\mathrm{W}, \mathrm{W}) = \mu(\mathrm{W}, \mathrm{S}) = \mu(\mathrm{S}, \mathrm{W}) = \frac{1}{3}$ and $\mu(\mathrm{S}, \mathrm{S}) = 0$. Then, both players are willing to obey the recommendations (why?), and thus it should constitute a "correlated equilibrium," which yields a payoff profile $\frac{1}{3}(4, 4) + \frac{1}{3}(1, 5) + \frac{1}{3}(5, 1) = (\frac{10}{3}, \frac{10}{3})$. □

In a direct correlation device, as we shall see below, each player will receive a recommendation of action that she should play, rather than an arbitrary information set.

Definition 5. In a finite normal-form game G, a **direct correlation device** is a correlation device $(A, \mu, (A_i)_i)$ such that:

1. A is the set of action profiles.

$$
\begin{tikzcd}
& H_i(\omega) \arrow[dr, "f_i"] & \\
\omega \arrow[ur, "H_i"] \arrow[rr, dashed] & & a_i
\end{tikzcd}
$$

Figure 1: the revelation principle

2. $\mu \in \Delta(A)$ is a distribution.[4]

3. $A_i = \{\{a_i\} \times A_{-i}\}_{a_i \in A_i}$ by abuse of notation.

For each $a \in A$, let $A_i(a)$ denote the (unique) element $\{a_i\} \times A_{-i}$, denoted a_i by abuse of notation.

Direct Correlated Equilibrium

Definition 6. In a finite normal-form game G with a direct correlation device $(A, \mu, (A_i)_i)$, we say that $\mu \in \Delta(A)$ is a **direct correlated equilibrium** if for each $i \in I$ and each $a_i, a_i' \in A_i$,

$$\sum_{a_{-i}} \mu(a_{-i} \mid a_i) u_i(a_i, a_{-i}) \geq \sum_{a_{-i}} \mu(a_{-i} \mid a_i) u_i(a_i', a_{-i}). \tag{4}$$

where $\mu(a_{-i} \mid a_i)$ is player i's posterior when she is recommended to play action a_i:

$$\mu(a_{-i} \mid a_i) = \begin{cases} \frac{\mu(a_i, a_{-i})}{\sum_{a'_{-i}} \mu(a_i, a'_{-i})} & \text{if } \sum_{a'_{-i}} \mu(a_i, a'_{-i}) > 0 \\ 0 & \text{if } \sum_{a'_{-i}} \mu(a_i, a'_{-i}) = 0. \end{cases}$$

The left-hand side of (4) is player i's payoff when she "obeys" the recommendation to play action a_i, while the right-hand side is her payoff when she "deviates" to playing action a_i'. Hence, the correlated equilibrium is such that every player is willing to obey the recommendation. Such a property is called the obedience or incentive compatibility in other contexts.

Revelation-Principle-Like Result The following result—akin to the revelation principle in mechanism design—claims that it suffices to consider Definition 6, not Definition 3, to find correlated equilibrium action distributions. The idea is illustrated in Figure 1. The solid arrows indicate how players choose their actions under an (indirect) correlation device. The dashed arrow is a "composite" of the two solid arrows. The revelation-principle-like result claims that these two are, in effect, the same.

[4] We no longer assume the full-support. That is, there may exist some $a \in A$ such that $\mu(a) = 0$.

Theorem 2. *In a finite normal-form game G, $\mu \in \Delta(A)$ is a correlated equilibrium action distribution if and only if μ is a direct correlated equilibrium.*

Proof. See Appendix A. ∎

Remark 2. Correlated equilibrium is often defined in the fashion of Definition 6, without mentioning the direct correlation device. □

1.3 How to Compute Correlated Equilibrium

We illustrate how to compute a correlated equilibrium.

Example 5. Using Example 1, we compute a correlated equilibrium. By Theorem 2, we can focus on a direct correlated equilibrium. For example, we find a (symmetric) correlated equilibrium that maximizes the sum of two players' payoffs, where $\mu(\mathrm{W}, \mathrm{S}) = \mu(\mathrm{S}, \mathrm{W})$ by symmetry.

	W	S
W	$\mu(\mathrm{W}, \mathrm{W})$	$\mu(\mathrm{W}, \mathrm{S})$
S	$\mu(\mathrm{S}, \mathrm{W})$	$\mu(\mathrm{S}, \mathrm{S})$

Table 3: a recommendation distribution

Constraint (4) for a correlated equilibrium is reduced to the following conditions:[5]

$$\begin{cases} \mu(\mathrm{W}, \mathrm{S}) \geq \mu(\mathrm{W}, \mathrm{W}), \\ \mu(\mathrm{S}, \mathrm{W}) \geq \mu(\mathrm{S}, \mathrm{S}). \end{cases}$$

Hence, the desired correlated equilibrium is characterized by the following problem:

$$\max_{\mu} \quad 8\mu(\mathrm{W}, \mathrm{W}) + 6\mu(\mathrm{W}, \mathrm{S}) + 6\mu(\mathrm{S}, \mathrm{W})$$

$$\text{subject to} \quad \begin{cases} \mu(\mathrm{W}, \mathrm{S}) = \mu(\mathrm{S}, \mathrm{W}), \\ \mu(\mathrm{W}, \mathrm{S}) \geq \mu(\mathrm{W}, \mathrm{W}), \\ \mu(\mathrm{S}, \mathrm{W}) \geq \mu(\mathrm{S}, \mathrm{S}). \end{cases}$$

There is a unique solution $\mu(\mathrm{W}, \mathrm{W}) = \mu(\mathrm{W}, \mathrm{S}) = \mu(\mathrm{S}, \mathrm{W}) = \frac{1}{3}$ and $\mu(\mathrm{S}, \mathrm{S}) = 0$, which is the desired correlated equilibrium. It is associated with the payoff vector $(\frac{10}{3}, \frac{10}{3})$. □

Remark 3. As illustrated above, the computation of correlated equilibrium is reduced to a linear programming problem. This property is in contrast to Nash equilibrium, because Nash equilibrium is, in general, not a solution to a linear programming problem. Hence, correlated equilibrium is "easy" to solve numerically, but Nash equilibrium is "difficult." □

[5] We avoid the subtleties arising from zero-probability events.

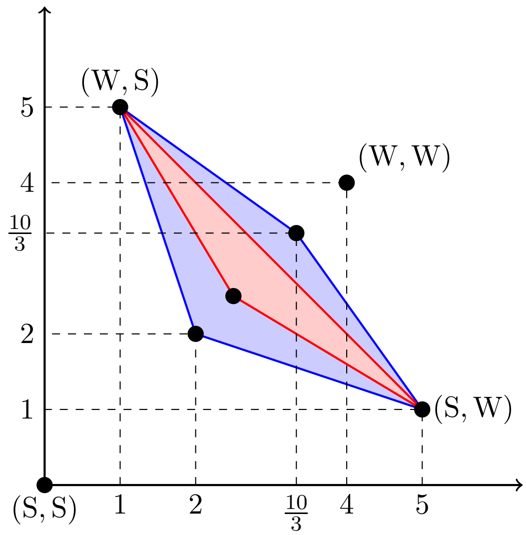

Figure 2: equilibrium payoffs for Example 1 (red: Nash equilibria; blue: correlated equilibria)

1.4 Convexity of Correlated Equilibria

Correlated equilibrium has desirable properties. We study the example and then generalize it.

Example 6. In Example 5, we have found a (symmetric) correlated equilibrium that maximizes the sum of two players' payoffs, with payoffs $(\frac{10}{3}, \frac{10}{3})$. Similarly, we can find a (symmetric) correlated equilibrium that minimizes the sum of their payoffs, with payoffs $(2, 2)$.

Is it possible that the two players randomize the two equilibria with equal probabilities? Yes, it is. Suppose that they have a fair coin and publicly observe the result of the coin toss before they play either correlated equilibrium. They play the "good" correlated equilibrium when the coin lands heads and the "bad" correlated equilibrium when the coin lands tails.

Along this line of analysis, we find that any convex combination of two correlated equilibria is also a correlated equilibrium. That is, the set of correlated equilibria is convex. It is immediate, from this result, that the set of equilibrium payoffs is also convex, as illustrated in blue in Figure 2. □

It is straightforward to generalize this observation.

Theorem 3. *In a finite normal-form game G, the set of correlated equilibrium action distributions is convex.*

Proof. Take any two correlated equilibrium action distributions μ, μ', which we identify with direct correlated equilibria, still denoted μ, μ'. Consider a biased coin that lands heads with probability λ and tails with probability $1 - \lambda$. All players *publicly* observe the result of the coin toss. Assume that they play μ if the coin lands heads and μ' if the coin lands tails. Then, the mixture $\lambda\mu + (1 - \lambda)\mu'$ is still a correlated equilibrium. ■

1.5 Correlated Equilibrium versus Nash Equilibrium

We compare correlated equilibrium with Nash equilibrium.

Example 7. We compare correlated equilibria with Nash equilibria in the teamwork game of Example 1. Recall that there are three Nash equilibria: (W, S), (S, W), and $(\frac{1}{2}\mathrm{W} \oplus \frac{1}{2}\mathrm{S}, \frac{1}{2}\mathrm{W} \oplus \frac{1}{2}\mathrm{S})$ with respective payoff profiles $(1, 5)$, $(5, 1)$, and $(\frac{5}{2}, \frac{5}{2})$. We draw the convex hull of the Nash equilibrium payoff set in red in Figure 2. Note that the correlated equilibrium payoff set contains the convex hull of the Nash equilibrium payoff set. □

Theorem 4. *In a finite normal-form game G, a mixed strategy profile $\sigma^* \in \prod_i \Delta(A_i)$ is a Nash equilibrium if and only if it is a direct correlated equilibrium with a distribution $\mu(a) = \prod_i \sigma^*(a_i)$.*

Proof. Since the "if" part is obvious, we prove the "only if" part. Take any action a_i such that $\mu_i(a_i) \equiv \sum_{a_{-i}} \mu(a_i, a_{-i}) > 0$. For each $a_i' \in A_i$, it must be that

$$\sum_{a_{-i}} \mu(a_{-i} \mid a_i) u_i(a_i', a_{-i}) = \sum_{a_{-i}} \prod_{j \neq i} \sigma_j^*(a_j) u_i(a_i', a_{-i}) = u_i(a_i', \sigma_{-i}^*).$$

Hence,

$$\sum_{a_{-i}} \mu(a_{-i} \mid a_i) u_i(a_i, a_{-i}) = u_i(a_i, \sigma_{-i}^*) \geq u_i(a_i', \sigma_{-i}^*) = \sum_{a_{-i}} \mu(a_{-i} \mid a_i) u_i(a_i', a_{-i}),$$

as desired. ■

Corollary 1. *In a finite normal-form game G, the convex hull of Nash equilibrium action distributions is a correlated equilibrium action distribution.*

Proof. Immediate from Theorems 3 and 4. ■

2 Correlated Equilibrium as Bayesian Rationality*

In the Bayesian view of decision-making under uncertainty, a player first forms a belief about a (relevant) state of interest and then optimizes her action. In the context of a game, the (relevant) state for each player is about how the other players in the game will play. If a player were given this information, she would know the best course of action for himself.

If it is common knowledge that every player is rational at every state, then what does play correspond to? Aumann (1987) shows that the action distribution induced by such play is given by some correlated equilibrium.

Let G be a normal-form game. Let Ω be a finite set of states of the world with typical element ω. The state ω of the world encodes all possible information about the relevant uncertainty, which, in this case, is how players play. Thus if the players knew a state ω, they would know exactly what will happen. Assume that an information partition H_i is given to each player i.

Common Prior Assumption Suppose that player i has her own prior $\pi^i \in \Delta(\Omega)$. To form her (posterior) belief, she does Bayesian inference based on the information that she observes.

Definition 7. Let G be a normal-form game. Let Ω be the set of states of the world. Suppose that player i has a prior $\pi^i \in \Delta(\Omega)$. There is a **common prior** $\pi \in \Delta(\Omega)$ if $\pi = \pi^i$ for each $i \in I$.

The **Common Prior Assumption** (CPA) is the assumption that admits the existence of a common prior.[6] The CPA does **not** assume that all players must have the same information. Player i's subjective belief about a state ω is her posterior (after Bayesian updating) based on the information that she has. Therefore, different players may have different subjective beliefs if they have different information. The CPA rather says that differences in subjective beliefs arise **only** from differences in information, rather than a priori differences in beliefs.

Remark 4. The CPA has been pervasive in economic models, but it is known that the CPA has very strong implications on economic modeling. We will revisit the CPA when we study games of incomplete information. □

Correlated Equilibrium as Bayesian Rationality

Definition 8. Let G be a normal-form game. Let Ω be the finite set of states of the world with a common prior π, and let H_i be player i's information partition for each $i \in I$. For each $i \in I$, let $f_i : \Omega \to A_i$ be a function such that for each $\omega, \omega' \in \Omega$,

$$H_i(\omega) = H_i(\omega') \quad \Longrightarrow \quad f_i(\omega) = f_i(\omega'),$$

and let $f(\omega) = (f_i(\omega))_i$. Player i is **Bayesian rational** at $\omega \in \Omega$ if for each $a_i \in A_i$,

$$\sum_{\omega' \in H_i(\omega)} \pi(\omega' \mid H_i(\omega)) u_i(f_i(\omega'), f_{-i}(\omega')) \geq \sum_{\omega' \in H_i(\omega)} \pi(\omega' \mid H_i(\omega)) u_i(a_i, f_{-i}(\omega')).$$

Theorem 5. *Let G be a normal-form game. Let Ω be the finite set of states of the world with a common prior π, and let H_i be player i's information partition for each $i \in I$. Suppose that (it is common knowledge that) every player is Bayesian rational at each $\omega \in \Omega$. Let $\mu \in \Delta(A)$ be the action distribution such that for each $a \in A$,*

$$\mu(a) = \sum_{\omega : f(\omega) = a} \pi(\omega).$$

Then, μ is a correlated equilibrium action distribution.

[6] The CPA is sometimes called the Harsanyi doctrine, named after Harsanyi (1967, 1968a,b).

Proof. Interpret (Ω, π, H) as a correlation device. Since all players are Bayesian rational at each ω, f is a correlated equilibrium with respect to (Ω, π, H). Hence, μ is a correlated equilibrium action distribution. ■

The proof is simple, but the implication is far-reaching. One conclusion that we draw from Theorem 5 is that if an action distribution μ is not a correlated equilibrium action distribution but is the action distribution induced by such an uncertainty model, then either the players must have started out with uncommon priors or there must be at least one state ω at which at least one player is not acting as a Bayesian rational player (that is, the common knowledge of Bayes rationality is violated).

A Proofs

A.1 Theorem 1

Proof of (i) ⇒ (ii) Suppose that (i) holds. For each $f_i \in F_i(\Omega, H_i)$,

$$\sum_{\omega \in h_i} \pi(\omega \mid h_i) u_i(f_i^*(\omega), f_{-i}^*(\omega)) \geq \sum_{\omega \in h_i} \pi(\omega \mid h_i) u_i(f_i(\omega), f_{-i}^*(\omega)),$$

Taking the summation across all $h_i \in H_i$, we have

$$\sum_{h_i \in H_i} \pi(h_i) \sum_{\omega \in h_i} \pi(\omega \mid h_i) u_i(f_i^*(\omega), f_{-i}^*(\omega)) \geq \sum_{h_i \in H_i} \pi(h_i) \sum_{\omega \in h_i} \pi(\omega \mid h_i) u_i(f_i(\omega), f_{-i}^*(\omega)),$$

which implies (ii).

Proof of (ii) ⇒ (i) Suppose that (2) holds. Suppose, for a contradiction, that there exist some $\hat{h}_i \in H_i$ and some $\hat{a}_i \in A_i$,

$$\sum_{\omega \in \hat{h}_i} \pi(\omega \mid \hat{h}_i) u_i(\hat{a}_i, f_{-i}^*(\omega)) > \sum_{\omega \in \hat{h}_i} \pi(\omega \mid \hat{h}_i) u_i(f_i^*(\omega), f_{-i}^*(\omega)). \tag{5}$$

Let $\hat{f}_i \in F_i(\Omega, H_i)$ be player i's strategy such that

$$\hat{f}_i(\omega) = \begin{cases} f_i^*(\omega) & \text{if } \omega \notin \hat{h}_i \\ \hat{a}_i & \text{if } \omega \in \hat{h}_i. \end{cases}$$

That is, player i "follows" strategy f_i if $h_i \neq \hat{h}_i$ but "deviates" to action $\hat{a}_i$ if $\hat{h}_i$. Then, for each $h_i \neq \hat{h}_i$,

$$\sum_{\omega \in h_i} \pi(\omega \mid h_i) u_i(\hat{f}_i(\omega), f_{-i}(\omega)) = \sum_{\omega \in h_i} \pi(\omega \mid h_i) u_i(f_i(\omega), f_{-i}(\omega)). \tag{6}$$

Summing (5) and (6) with weights $\pi(h_i) = \sum_{\omega \in h_i} \pi(\omega)$ for all $h_i \in H_i$, we have

$$\sum_{h_i \in H_i} \pi(h_i) \sum_{\omega \in h_i} \pi(\omega \mid h_i) u_i(\hat{f}_i(\omega), f^*_{-i}(\omega)) > \sum_{h_i \in H_i} \pi(h_i) \sum_{\omega \in h_i} \pi(\omega \mid h_i) u_i(f^*_i(\omega), f^*_{-i}(\omega)).$$

This is equivalent to

$$\sum_{\omega \in \Omega} \pi(\omega) u_i(\hat{f}_i(\omega), f^*_{-i}(\omega)) > \sum_{\omega \in \Omega} \pi(\omega) u_i(f^*_i(\omega), f^*_{-i}(\omega)),$$

but it contradicts (2).

A.2 Theorem 2

We show the "only if" part, as the "if" part is immediate. Since $\mu \in \Delta(A)$ is a correlated equilibrium action distribution, there is a correlated equilibrium $(\Omega, \pi, H; f)$ such that for each $a \in A$,

$$\mu(a) = \sum_{\omega : f(\omega) = a} \pi(\omega). \tag{7}$$

For each $a_i, a'_i \in A_i$, it must be that

$$\sum_{a_{-i}} \mu(a_i, a_{-i}) u_i(a'_i, a_{-i}) = \sum_{h_i \in f_i^{-1}(a_i)} \pi(h_i) \sum_{\omega \in \Omega} \pi(\omega \mid h_i) u_i(a'_i, f_{-i}(\omega)),$$

where $f_i^{-1}(a_i) = \{h_i \in H_i : f_i(\omega) = a_i \text{ for each } \omega \in h_i\}$. From Definition 3, it follows that for each $h_i \in H_i$ and each $a'_i \in A_i$,

$$\sum_{\omega \in \Omega} \pi(\omega \mid h_i) u_i(f_i(\omega), f_{-i}(\omega)) \geq \sum_{\omega \in \Omega} \pi(\omega \mid h_i) u_i(a'_i, f_{-i}(\omega)).$$

From (7), it follows that

$$\sum_{a_{-i}} \mu(a_i, a_{-i}) u_i(a_i, a_{-i}) \geq \sum_{a_{-i}} \mu(a_i, a_{-i}) u_i(a'_i, a_{-i}).$$

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