CPT Lecture Notes 10: Expected utility, risk aversion, riskiness

Levent Ülkü

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Expected Utility

Z is a finite set of outcomes (or consequences, or prizes). $L(Z)$ is the set of all lotteries (or probability functions) on Z: $p \in L(Z)$ if and only if $p : Z \rightarrow [0, 1]$ with $\sum_{z \in Z} p(z) = 1$. $p(z)$ is the objective probability of obtaining z. For every z, $[z]$ is the *degenerate lottery* assigning probability one to z.

For n lotteries $p_1, ..., p_n$ and n numbers $\alpha_1, ..., \alpha_n$ with $\alpha_i \geq 0$ and $\sum \alpha_i = 1$, the associated *compound lottery* is the convex combination $\sum_{i=1}^{n} \alpha_i p_i$. The resulting probability that the compound lottery gives prize z is $\sum_{i=1}^{n} \alpha_i p_i(z)$.

Interpretation: the compound lottery plays out in two stages. In stage 1, lottery p_i is realized with probability α_i. If p_i is realized in stage 1, then in stage 2, prize z is realized with probability $p_i(z)$.

We are interested in the following class of preferences over $L(Z)$.

Definition: A binary relation $\succeq$ on $L(Z)$ admits an *expected utility representation* if there exists a utility function $v : Z \to \mathbb{R}$ such that for every $p, q \in L(Z)$

$$p \succeq q \text{ iff } \sum_{z \in Z} p(z)v(z) \geq \sum_{z \in Z} q(z)v(z).$$

Axioms:

A1 (weak order): $\succeq$ is complete and transitive.

A2 (independence): for any $p, q, r \in L(Z)$ and any $\alpha \in (0,1)$

$$p \succeq q \Leftrightarrow \alpha p + (1-\alpha) r \succeq \alpha q + (1-\alpha) r$$

A3 (continuity): if $p \succ q \succ r$, then there exists $\alpha \in (0,1)$ such that $q \sim [\alpha p + (1-\alpha) r]$.

Suppose $\succsim$ is represented by some $V : L(Z) \to \mathbb{R}$.

Then $\succsim$ is a weak order.

If V is continuous, then $\succsim$ is continuous.

If V is linear, then $\succsim$ is linear. (Recall that V is linear if whenever $p_1, \ldots, p_n \in L(Z)$ and $\alpha_1, \ldots, \alpha_n$ are nonnegative numbers such that $\sum \alpha_i p_i \in L(Z)$, then $V(\sum \alpha_i p_i) = \sum \alpha_i V(p_i)$.)

If $V(p) = \sum p(z)v(z)$ for some $v : Z \to \mathbb{R}$, then V is linear and continuous and therefore the preference $\succsim$ represented by V satisfies the three axioms.

The following result says much more.

Theorem (vNM): The following are equivalent:
(1) $\succeq$ admits an expected utility representation.
(2) $\succeq$ satisfies A1, A2 and A3.

Example: Suppose there exists a function $v : Z \to \mathbb{R}$ such that $p \succsim q$ iff $\min\{v(z) : p(z) > 0\} \geq \min\{v(z) : q(z) > 0\}$.

Suppose that $Z = \{z_1, z_2, z_3\}$ and $v(z_1) > v(z_2) > v(z_3)$.

Then $[z_1] \succ [z_2] \succ [z_3]$ but there is no $\alpha \in (0, 1)$ such that $[z_2] \sim \alpha[z_1] + (1 - \alpha)[z_3]$. Hence $\succsim$ is not continuous.

Furthermore $[z_1] \succ [z_2]$ but $\frac{1}{2}[z_1] + \frac{1}{2}[z_2] \sim [z_2]$. Hence $\succsim$ fails independence.

Example: Suppose $p \succsim q$ iff $\max_z p(z) \geq \max_z q(z)$.

Let $V(p) = \max_z p(z)$ for all p. Hence $p \succsim q$ iff $V(p) \geq V(q)$.

Since V is continuous, so is $\succsim$.

However $[z_1] \sim [z_2]$ but $[z_1] \succ \frac{1}{2}[z_1] + \frac{1}{2}[z_2]$. Hence $\succsim$ fails independence.

Example: Suppose that $p \succsim q$ iff $p \geq_L q$, where $\geq_L$ is the lexicographic order on $\mathbb{R}^{|Z|}$.

Then $\succsim$ satisfies independence but fails continuity. (Homework.)

Suppose $\succeq$ satisfies Axiom 2. You should verify that for any p, q and r, and for any $\alpha \in (0, 1)$:

1. $p \succ q \Leftrightarrow \alpha p + (1-\alpha)r \succ \alpha q + (1-\alpha)r$

2. $p \sim q \Leftrightarrow \alpha p + (1-\alpha)r \sim \alpha q + (1-\alpha)r$

3. $p \succeq q \Leftrightarrow p \succeq \alpha p + (1-\alpha)q$

4. $p \succeq q \Leftrightarrow \alpha p + (1-\alpha)q \succeq q$

The following result is useful in the proof of vNM Theorem.

Lemma: Suppose that $\succeq$ satisfies independence. Let $x, y \in Z$ be such that $[x] \succ [y]$ and let $1 \geq \alpha > \beta \geq 0$. Then

$$\alpha[x] + (1-\alpha)[y] \succ \beta[x] + (1-\beta)[y].$$

Proof: If $\alpha = 1$ or $\beta = 0$, then there is nothing to show. Otherwise we have, by independence, $\alpha[x] + (1-\alpha)[y] \succ [y]$. Using independence again,

$$\begin{aligned} \alpha[x] + (1-\alpha)[y] &\succ \frac{\beta}{\alpha}(\alpha[x] + (1-\alpha)[y]) + (1-\frac{\beta}{\alpha})[y] \\ &= \beta[x] + (1-\beta)[y] \end{aligned}$$

and the proof is complete. ■

Proof of VNM Theorem: (2 implies 1)
Suppose $\succeq$ satisfies completeness, transitivity, I and C.
Since Z is finite, there exist $a, b \in Z$ such that $[a] \succeq [z] \succeq [b]$ for all $z \in Z$.
It can be shown that $[a] \succeq p \succeq [b]$ for all $p \in L(Z)$. (Homework!)
Case 1: $[a] \sim [b]$.
It follows that $p \sim q$ for all p, q. Set $v(z) = 0$ for all z so that $U(p) = \sum_{z \in Z} p(z) v(z) = 0$ for all $p \in L(Z)$.

Case 2: $[a] \succ [b]$.
Pick any z.
By C and the Lemma, there is a unique number $v(z)$ such that $v(z)[a] + (1 - v(z))[b] \sim [z]$. (Why?)
In particular, $v(a) = 1$ and $v(b) = 0$.
By I,

$$\begin{aligned} p &= \sum_{z \in Z} p(z)[z] \sim \sum_{z \in Z} p(z)\left(v(z)[a] + (1 - v(z))[b]\right) \\ &= \left[\sum_{z \in Z} p(z)v(z)\right][a] + \left[1 - \sum_{z \in Z} p(z)v(z)\right][b]. \end{aligned}$$

Now by the Lemma $p \succeq q$ if and only if $\sum_z p(z)v(z) \geq \sum_z q(z)v(z)$.
The proof is complete.$\blacksquare$

The vNM utilities are unique up to positive affine transformations.

Fix $\succeq$ on $L(Z)$ which satisfies A1, A2 and A3. Let v be the vNM utility constructed in the proof (using $[z] \sim v(z)[a] + (1 - v(z))[b]$) whose expectation V represents $\succeq$.

Claim 1: If $\alpha > 0$ and $w(z) = \alpha v(z) + \beta$ for all z, then the map $W : L(Z) \to \Re$ given by $W(p) = \sum_{z \in Z} p(z) w(z)$ also represents $\succeq$. (Proof of Claim 1: Homework.)

Claim 2: If W is the expectation of $w : Z \to \mathbb{R}$, i.e., $W(p) = \sum_{z \in Z} p(z)w(z)$ for all q, and if W represents $\succeq$, then there exist numbers $\alpha > 0$ and β such that $w(z) = \alpha v(z) + \beta$ for all z.

To see this first suppose that $[a] \sim [b]$. Then $v(z) = 0$ for all z. Since the expectation of w represents $\succeq$, for some constant k, $w(z) = k$ for all z. Hence $w(z) = 3v(z) + k$ for all z. Now suppose $[a] \succ [b]$. Let $\alpha = w(a) - w(b)$ and $\beta = w(b)$. Note $\alpha > 0$. For any z, $[z] \sim v(z)[a] + (1 - v(z))[b]$, so

$$\begin{aligned} w(z) &= v(z)w(a) + (1 - v(z))w(b) \\ &= v(z)(\alpha + \beta) + (1 - v(z))\beta \\ &= \alpha v(z) + \beta, \end{aligned}$$

as desired.

What does this mean? Let $Z = \{a, b, c\}$ and consider two utility functions:

	a	b	c
v	0	1	2
w	0	1	3

Note that both utilities represent the same preference over Z. But they do not induce the same preferences over $L(Z)$ through their expectation. Taking expectations of v, we get $[b] \sim \frac{1}{2}[a] + \frac{1}{2}[c]$ and taking expectations of w, we get $[b] \prec \frac{1}{2}[a] + \frac{1}{2}[c]$.

Allais Paradox: First compare:

$$L_1 = \frac{1}{4}[3000] + \frac{3}{4}[0] \text{ and } L_2 = \frac{1}{5}[4000] + \frac{4}{5}[0]$$

Now compare:

$$L_3 = 1[3000] \text{ and } L_4 = \frac{4}{5}[4000] + \frac{1}{5}[0]$$

Note $L_1 = \frac{1}{4}L_3 + \frac{3}{4}[0]$ and $L_2 = \frac{1}{4}L_4 + \frac{3}{4}[0]$ and hence the independence axiom implies that $L_3 \succ L_4 \Rightarrow L_1 \succ L_2$. However, there is experimental evidence for the preference pattern $L_2 \succ L_1$ and $L_3 \succ L_4$.

Risk Aversion

Let $Z = \mathbb{R}$. A prize $z \in Z$ is interpreted as "earning z dollars." Let $\tilde{Z}$ be the set of all lotteries on Z with finite support. In other words,

$$p \in \tilde{Z} \Leftrightarrow \left\{ \begin{array}{ll} p : Z \longrightarrow [0, 1] & \text{and} \\ \exists Y \subset Z \text{ such that} & \left\{ \begin{array}{l} Y \text{ is finite} \\ \sum_{z \in Y} p(z) = 1 \end{array} \right\} \end{array} \right\}$$

Recall that $[z]$ is the lottery that gives the reward z with probability one.

Definition: A preference $\succeq$ on $\tilde{Z}$ exhibits *risk aversion* if

$$[tx + (1 - t)y] \succeq t[x] + (1 - t][y].$$

for every $x, y \in Z$ and every $t \in [0, 1]$.

Remark:

If v is a vNM utility for $\succeq$, then $\succeq$ exhibits risk aversion if and only if v is a concave function.

Reason: Fix x, y and t.

If v is concave then $v(tx + (1-t)y) \geq tv(x) + (1-t)v(y)$, implying that $[tx + (1-t)y] \succeq t[x] + (1-t][y]$.

If $[tx + (1-t)y] \succeq t[x] + (1-t][y]$, then $v(tx + (1-t)y) \geq tv(x) + (1-t)v(y)$ implying that v is concave.

Note: Risk loving behavior can be defined analogously and is associated with convex v.

Remark:

Jensen's inequality: If v is concave, then for every $p \in \tilde{Z}$

$$v\left(\sum_{x \in \mathrm{supp}(p)} p(x)x\right) \geq \sum_{x \in \mathrm{supp}(p)} p(x)v(x).$$

In other words the utility of the "average prize" is at least as large as the expected utility of the lottery.

More generally, if F is the distribution of a random variable taking values on $\mathbb{R}$ and v is concave, then

$$v(\int_{-\infty}^{+\infty} xdF) \geq \int_{-\infty}^{+\infty} v(x)dF$$

Example: Insurance

Assume that your v is increasing and concave.
You have an asset which is worth

$$\begin{array}{ll} w-d & \text{with probability } p \\ w & \text{with probability } 1-p \end{array}$$

What is the largest insurance premium π for which you would be willing to exchange your asset with $[w-\pi]$?

Call such π, π^*. We can solve for π^* through

$$v(w-\pi^*) = pv(w-d) + (1-p)v(w)$$

Claim: $\pi^* \geq pd$. (Interpretation?)
Reason: We have

$$\begin{aligned} v(w-\pi^*) &= pv(w-d) + (1-p)v(w) \\ &\leq v(p(w-d) + (1-p)w) \\ &= v(w-pd) \end{aligned}$$

where the inequality is by concavity. Since v is increasing, we must have $w - \pi^* \leq w - pd$ or $pd \leq \pi^*$. (Draw a picture!) Let $w = 16, d = 7, p = \frac{1}{2}$ and $v(x) = \sqrt{x}$ and compute π^*. Verify that $\pi^* \geq \frac{7}{2} = pd$.

Example: Gambling

Suppose that your v is increasing and convex and you are faced with the following decision problem.You have to decide between $[w]$ and a lottery that gives $w - c + R$ with probability p and $w - c$ with probability $1 - p$. (Interpretation: c is the cost of gambling, R is the reward if you win, and p is the probability of winning.)
What is the largest c for which you would choose to gamble? Denote it by c^*. We find c^* by

$$v(w) = pv(w - c^* + R) + (1 - p)v(w - c^*)$$

Claim: $c^* \geq pR$. (Interpretation?)

Reason: We have

$$\begin{aligned} v(w) &= pv(w - c^* + R) + (1-p)v(w - c^*) \\ &\geq v(w - c^* + pR) \end{aligned}$$

implying that $w \geq w - c^* + pR$ or $c^* \geq pR$. (Picture!)

Measuring Risk Aversion

Aim: To quantify risk aversion for comparative static purposes

Running assumptions: All vNM utilities are smooth (twice differentiable) and they have strictly positive first derivatives.

Definition: The (Arrow-Pratt) *measure of absolute risk aversion* for v at wealth level x is

$$\frac{-v''(x)}{v'(x)}$$

Definition: The Arrow-Pratt *measure of relative risk aversion* for v at wealth level x is

$$\frac{-xv''(x)}{v'(x)}$$

Note:

$$\frac{-v''(x)}{v'(x)} \cong \frac{-\left[\dfrac{v'(x+\Delta x)-v'(x)}{v'(x)}\right]}{\Delta x} = -\frac{\%\Delta \text{ in } MU}{\Delta \text{ in wealth}}$$

$$\frac{-xv''(x)}{v'(x)} \cong \frac{-\left[\dfrac{v'(x+\Delta x)-v'(x)}{v'(x)}\right]}{\dfrac{\Delta x}{x}} = -\frac{\%\Delta \text{ in } MU}{\%\Delta \text{ in wealth}}$$

Constant absolute risk aversion (CARA): for every x, $\frac{-v''(x)}{v'(x)} = \alpha$. Which utilities exhibit CARA?

Integrating, we get, for every x,

$$\begin{aligned}
\int_0^x \frac{v''(t)}{v'(t)} dt &= -\int_0^x \alpha dt \\
&\Rightarrow \ln v'(x) - \ln v'(0) = -\alpha(x-0) \\
&\Rightarrow \ln \frac{v'(x)}{v'(0)} = -\alpha x \\
&\Rightarrow v'(x) = v'(0) e^{-\alpha x} \\
&\Rightarrow \int_0^x v'(t) dt = v'(0) \int_0^x e^{-\alpha t} dt. \\
&\Rightarrow v(x) = \frac{v'(0)}{-\alpha}(e^{-\alpha x} - 1) + v(0)
\end{aligned}$$

Thus any affine transformation of $v(x) = -e^{-\alpha x}$ would exhibit CARA.

Constant relative risk aversion (CRRA): for every x, $\frac{-xv''(x)}{v'(x)} = \alpha$. Which utilities exhibit CRRA?

Assume $\alpha \neq 1$ and rewrite as $v''(x) = -\alpha \frac{v'(x)}{x}$. Let $u(x) = v'(x)$ and rewrite as $\frac{du}{dx} = -\alpha \frac{u}{x}$. This differential equation is "seperable" (between u and x.)

The usual trick: $\frac{du}{u} = -\alpha \frac{dx}{x}$ and integrate to get: $\ln u = -\alpha \ln x + c_1$. Do some algebra to get $u = c_2 [e^{\ln x}]^{-\alpha} = c_2 x^{-\alpha}$ (where $c_2 > 0$). Now solving for v:

$$v(x) = \int_{-\infty}^{x} u(t)dt = \int_{-\infty}^{x} c_2 t^{-\alpha} dt = c_3 + c_4 x^{1-\alpha},$$

where $c_4 > 0$. We conclude that any affine transformation of $v(x) = x^{-\alpha+1}$ exhibits CRRA.

Application: Portfolio Choice

You have w dollars to split between a riskless asset r whose return is $1 + r$ and am risky asset $\tilde{R}$ whose return is a random variable $1 + \tilde{R}$. Suppose that you have a vNM utility v with $v' > 0$ and $v'' < 0$. (You are risk averse.)

Let x be the amount you invest in the risky asset. (You invest $w - x$) in the riskless asset.

Your wealth is now a random variable
$\tilde{w}(x) = x(1 + \tilde{R}) + (w - x)(1 + r)$.

In this simple portfolio choice problem, you will choose x to solve

$$\max_{0 \le x \le w} E[v(\tilde{w}(x))]$$

Question: What happens to the optimal x when w goes up?

Let $f(x) = E[v(\tilde{w}(x))]$.
Assumption: The data allows the interchange of the order of limit operations, i.e.,

$$\frac{d}{dx}E[f(x)] = E[\frac{d}{dx}f(x)].$$

Since $x(1+\tilde{R}) + (w-x)(1+r) = x(\tilde{R}-r) + w(1+r)$, we can compute

$$\begin{aligned} f'(x) &= E[v'(\tilde{w}(x)) \cdot (\tilde{R}-r)] \\ f''(x) &= E[v''(\tilde{w}(x)) \cdot (\tilde{R}-r)^2] < 0 \end{aligned}$$

indicating that f is concave.

We will analyze two cases.
Case 1: $E(\tilde{R}) \leq r$.
In this case, we have

$$\begin{aligned} f'(0) &= E[v'(w(1+r)) \cdot (\tilde{R} - r)] \\ &= v'(w(1+r))(E(\tilde{R}) - r) \\ &\leq 0 \end{aligned}$$

and f achieves max on $[0, w]$ at $x^* = 0$. Picture should help. (We should have expected this!)

Case 2: $E(\tilde{R}) > r$.
In this case, $f'(0) > 0$ and there are two possibilities. Either f achieves its max at $x^* = w$ or at some $x^* \in (0, w)$. Picture should help. We are interested in the second case where the maximum is interior so that calculus can be used. This is also the economically more interesting case of diversification. At an interior optimum x^*, $f'(x^*) = 0$, in other words,

$$E[v'(x^*(\tilde{R} - r) + w(1 + r)) \cdot (\tilde{R} - r)] = 0$$

giving us an implicit function of the form $F(w, x^*) = 0$. Using the implicit function rule, we find

$$\frac{dx^*}{dw} = -\frac{E[v''(x^*(\tilde{R} - r) + w(1 + r)) \cdot (\tilde{R} - r) \cdot (1 + r)]}{E[v''(x^*(\tilde{R} - r) + w(1 + r)) \cdot (\tilde{R} - r)^2]}$$

and since the denominator is negative, the sign of the comparative static of interest depends on the sign of the numerator $E[v''(x^*(\tilde{R} - r) + w(1 + r)) \cdot (\tilde{R} - r) \cdot (1 + r)]$.

Let $\rho(x)$ denote the Arrow-Pratt measure of absoulte risk aversion, i.e., $\rho(x) = \dfrac{-v''(x)}{v'(x)}$ for every x.

We will show that

$$\rho'(x) \geq 0 \Rightarrow \frac{dx^*}{dw} \leq 0.$$

Suppose that $\rho'(x) \geq 0$ and let R be a realization of $\tilde{R}$.
If $R > r$, then $x^*(R-r) + (1+r)w > (1+r)w$, giving

$$\rho((1+r)w) \leq \rho(x^*(R-r) + (1+r)w)$$

and

$$\rho((1+r)w)(R-r) \leq \rho(x^*(R-r) + (1+r)w)(R-r)$$

If $R < r$, then $x^*(R-r) + (1+r)w < (1+r)w$ giving

$$\rho((1+r)w) \geq \rho(x^*(R-r) + (1+r)w)$$

and

$$\rho((1+r)w)(R-r) \leq \rho(x^*(R-r) + (1+r)w)(R-r)$$

Thus for any realization of $\tilde{R}$, the same conclusion holds. (This is trivial to show if $R = r$.)

Next, we observe that for every realization $R > r$

$$\begin{aligned} -\frac{v''(x^*(R-r)+(1+r)w)\cdot(R-r)}{v'(x^*(R-r)+(1+r)w)\cdot(R-r)} &= \rho(x^*(R-r)+(1+r)w) \\ &\geq \rho((1+r)w) \end{aligned}$$

implying

$$-v''(x^*(R-r)+(1+r)w)\cdot(R-r) \geq \rho((1+r)w)\cdot v'(x^*(R-r)+(1+r)$$

The same inequality follows if $R < r$. (Why?) Taking expectations over $\tilde{R}$

$$
\begin{aligned}
-&E[v''(x^*(\tilde{R}-r)+(1+r)w)\cdot(\tilde{R}-r)] \\
&\geq \rho((1+r)w)\cdot \underbrace{E[v'(x^*(\tilde{R}-r)+(1+r)w)\cdot(\tilde{R}-r)]}_{=f'(x^*)=0}
\end{aligned}
$$

implying what we wanted to show:

$$E[v''(x^*(\tilde{R}-r)+(1+r)w)\cdot(\tilde{R}-r)] \leq 0$$

Summary:

$$\frac{d}{dw}\left[-\frac{v''(w)}{v'(w)}\right] \geq 0 \Rightarrow \frac{dx^*(w)}{dw} \leq 0$$

More risk averse than

Recall...

The (Arrow-Pratt) *measure of absolute risk aversion* for v at wealth level x is:

$$\rho(x) := \frac{-v''(x)}{v'(x)}$$

Story: You have w dollars to split between a riskless asset r whose return is $1+r$ and a risky asset $\tilde{R}$ whose return is a random variable $1+\tilde{R}$. Suppose that you have a vNM utility v with $v' > 0$ and $v'' < 0$. Let x be the amount you invest in the risky asset. If $x^*(w)$ is an interior solution to the problem $\max_{0 \le x \le w} E[v(x(1+\tilde{R}) + (w-x)(1+r))]$, then

$$\rho'(x) \ge 0 \Rightarrow \frac{dx^*}{dw} \le 0.$$

Fix a strictly increasing vNM utility v.

Definition: For any random asset X and any wealth level w, the *risk premium* of X is the number $\pi(w, X)$ satisfying

$$v(w - \pi(w, X)) = E[v(w + X)]$$

Question: What does it mean for v_2 to be more risk averse than v_1?

Theorem:
Let v_1 and v_2 be twice differentiable vNM utility functions with $v_1', v_2' > 0$. The following three statements are equivalent.
(1) for all x, $\rho_2(x) \geq \rho_1(x)$
(2) there exists a function f such that $f' > 0$, $f'' \leq 0$ and $v_2 = f \circ v_1$
(3) for every w and X, $\pi_2(w, X) \geq \pi_1(w, X)$

Proof: We will show $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 1$.
$1 \Rightarrow 2$: Suppose that for all x, $\rho_2(x) \geq \rho_1(x)$. For every y in the range of v_1, let $f(y) = v_2(v_1^{-1}(y))$. Then

$$\begin{aligned} f'(y) &= v_2'(v_1^{-1}(y)) \frac{d}{dy}[v_1^{-1}(y)] \\ &= v_2'(v_1^{-1}(y)) \frac{1}{v_1'(v_1^{-1}(y))} > 0 \end{aligned}$$

Now we need to show that $f''(y) \leq 0$. Let $\phi(y) = \frac{v_2'(y)}{v_1'(y)}$ and note that

$$\begin{aligned}
\phi'(y) &= \frac{v_2''(y)v_1'(y) - v_2'(y)v_1''(y)}{[v_1'(y)]^2} \\
&= \underbrace{\frac{v_2'(y)}{v_1'(y)}}_{>0}\underbrace{\left[\frac{v_2''(y)}{v_2'(y)} - \frac{v_1''(y)}{v_1'(y)}\right]}_{\leq 0} \\
&\leq 0.
\end{aligned}$$

Finally note that $f'(y) = \phi(v_1^{-1}(y))$ and therefore

$$\begin{aligned}
f''(y) &= \phi'(v_1^{-1}(y))\frac{d}{dy}[v_1^{-1}(y)] \\
&\leq 0.
\end{aligned}$$

$2 \Rightarrow 3$: Suppose that there is a smooth increasing and concave f such that $v_2 = f \circ v_1$. By Jensen's inequality $E(f(Y)) \leq f(E(Y))$ for every random variable Y. To see this note that for every realization y of Y,

$$f(y) \leq f(E(Y)) + f'(E(Y))[y - E(Y)].$$

Taking expectations yields

$$E(f(Y)) \leq f(E(Y)) + 0.$$

Fix a random variable X. We have

$$\begin{aligned} v_2(w - \pi_1(w, X)) &= f[v_1(w - \pi_1(w, X))] \\ &= f[E(v_1(w + X))] \\ &\geq E[f(v_1(w + X))] \\ &= E[v_2(w + X)] \\ &= v_2(w - \pi_2(w, X)) \end{aligned}$$

implying that $w - \pi_1(w, X) \geq w - \pi_2(w, X)$, or $\pi_2(w, X) \geq \pi_1(w, X)$.

$3 \Rightarrow 1$: Suppose that for every w and X, $\pi_2(w, X) \geq \pi_1(w, X)$. Choose a random variable Y with $E(Y) = 0$ and $var(Y) = \sigma^2$. Let $X = tY$ so that $E(X) = 0$ and $var(X) = t^2\sigma^2$. Fix w and define

$$\lambda_i(t) := \pi_i(w, tY) \text{ for } i = 1, 2$$

so that $\lambda_2(t) \geq \lambda_1(t)$.
First note that $\lambda_i(0) = 0$ since
$v_i(w - \lambda_i(0)) = E[v_i(w + 0)] = v_i(w)$ and $v_i' > 0$.

Next compute $\lambda_i'(0)$. Note that $v_i(w - \lambda_i(t)) = E[v_i(w + tY)]$ for every t, so

$$v_i'(w - \lambda_i(t))(-\lambda_i'(t)) = E[v_i'(w + tY)Y].$$

At $t = 0$,

$$\begin{aligned} v_i'(w)(-\lambda_i'(0)) &= v_i'(w)E(Y) \\ &= 0 \end{aligned}$$

and since $v_i'(w) > 0$, we get $\lambda_i'(0) = 0$.

Next we will compute $\lambda_i''(0)$. We have

$$v_i''(w-\lambda_i(t))[\lambda_i'(t)]^2 + v_i'(w-\lambda_i(t))(-\lambda_i''(t)) = E[v_i''(w+tY)Y^2]$$

and at $t=0$,

$$0 + v_i'(w)(-\lambda_i''(0)) = v_i''(w)\sigma^2$$

so $\lambda_i''(0) = \rho_i(w)\sigma^2$.

Now choose $t > 0$ and note that by the MVT, there is some $c_i \in (0, t)$ with

$$\begin{aligned}
\lambda_i(t) &= \underbrace{\lambda_i(0)}_{=0} + t\underbrace{\lambda_i'(0)}_{=0} + \frac{t^2}{2}\lambda_i''(c_i) \\
&= \frac{t^2}{2}\lambda_i''(c_i).
\end{aligned}$$

Since $\lambda_2(t) \geq \lambda_1(t)$, we also have $\lambda_2''(c_2) \geq \lambda_1''(c_1)$. By smoothness of λ_i'', if take $t \to 0$, (so that $c_1, c_2 \to 0$ as well) we get $\lambda_2''(0) \geq \lambda_1''(0)$. This gives us $\sigma^2\rho_2(w) \geq \sigma^2\rho_1(w)$ and consequently $\rho_2(w) \geq \rho_1(w)$.

This completes the proof of the theorem. ∎

Example: Insurance continued

Recall

$$\begin{aligned} v_i(w - \pi_i^*) &= p v_i(w-d) + (1-p) v_i(w) \\ &= E[v_i(w+X)] \end{aligned}$$

where

$$\begin{aligned} X &= -d \text{ with probability } p \\ X &= 0 \text{ with probability } 1-p \end{aligned}$$

The number π_i^* is the maximum i is willing to pay for insurance. We would expect that if 2 is more risk averse than 1, then $\pi_2^* \geq \pi_1^*$. Let us verify this.

We have:

$$\begin{aligned}
v_1(w-\pi_1^*) &= pv_1(w-d)+(1-p)v_1(w) \\
v_2(w-\pi_2^*) &= pv_2(w-d)+(1-p)v_2(w) \\
&= pf[v_1(w-d)]+(1-p)f[v_1(w)] \;\; (f'>0,\ f''\leq 0) \\
&\leq f[pv_1(w-d)+(1-p)v_1(w)] \\
&= f[v_1(w-\pi_1^*)] \\
&= v_2(w-\pi_1^*)
\end{aligned}$$

and since $v_2'>0$, $\pi_1^*\leq\pi_2^*$.

Riskiness

We move on to the related but different question of what it means for a lottery to be riskier than another lottery. We will identify the lotteries with their distribution functions. We will restrict attention to lotteries with prizes in $[0,1]$.

A lottery is identified by a distribution F on $[0,1]$ of a random variable X with the usual interpretation $F(s) = \Pr\{X \leq s\}$.

A distribution is necessarily continuous from the right, nondecreasing and satisfies $F(1) = 1$. We will also assume that $F(0) = 0$.

First order stochastic dominance A distribution F FOSD a distribution G, denoted $F \succeq_1 G$, if

$$\int_0^1 v(x)dF \geq \int_0^1 v(x)dG \text{ whenever } v \text{ is nondecreasing.}$$

In other words $F \succeq_1 G$ if every individual with an increasing vNM utility prefer the uncertainty defined by F over the uncertainty defined by G.

Remark: If $F \succeq_1 G$, then $\mu_F \geq \mu_G$ where

$$\mu_F = \int_0^1 xdF \text{ and } \mu_G = \int_0^1 xdG$$

are the associated expectations. (Why?) The converse implication is false. (Why?)

Theorem:

$$F \succeq_1 G \Leftrightarrow F(t) \leq G(t) \text{ for every } t \in [0,1].$$

Proof:

$\Rightarrow$: Suppose that $F \succeq_1 G$. Pick $t \in [0,1]$. Let

$$v(x) = \begin{cases} 1 & \text{if } x \geq t \\ 0 & \text{if } x < t \end{cases}$$

and note that v is nondecreasing. So

$$\begin{aligned} 1 - F(t) &= \int_t^1 v(x)dF = \int_0^t v(x)dF + \int_t^1 v(x)dF \\ &= \int_0^1 v(x)dF \\ &\geq \int_0^1 v(x)dG \\ &= 1 - G(t) \text{ (similarly)} \end{aligned}$$

implying that $F(t) \leq G(t)$.

$\Leftarrow$: Suppose that v is nondecreasing. Also assume, to avoid heavier math, that v is differentiable. If $F(t) \leq G(t)$ for every t, then

$$\begin{aligned}
& \int_0^1 v(x)dF - \int_0^1 v(x)dG \\
= \; & \int_0^1 v(x)d(F-G) \\
= \; & \underbrace{[v(t)(F(t)-G(t))]_{t=0}^{t=1}}_{=0} - \int_0^1 \underbrace{[F(t)-G(t)]}_{\leq 0}\underbrace{v'(t)}_{\geq 0}dt \\
\geq \; & 0
\end{aligned}$$

and the proof is complete. ■

Second order stochastic dominance F SOSD G, denoted $F \succeq_2 G$, if

$$\int_0^1 v(x)dF \geq \int_0^1 v(x)dG \text{ whenever } v \text{ is nondecreasing and concave.}$$

Interpretation?

Theorem:

$$F \succeq_2 G \Leftrightarrow \int_0^t F(x)dx \leq \int_0^t G(x)dx \text{ for every } t \in [0,1].$$

Picture should help!

Proof:
$\Rightarrow$: Suppose that $F \succeq_2 G$, i.e., $\int_0^1 v(x)dF \geq \int_0^1 v(x)dG$ whenever v is nondecreasing and concave.
Choose $t \in [0, 1]$ and define

$$v(x) = \begin{cases} x & \text{if } 0 \leq x \leq t \\ t & \text{if } t \leq x \leq 1 \end{cases}$$

Integration yields:

$$\begin{aligned}\int_0^1 v(x)dF(x) &= \int_0^t xdF(x) + \int_t^1 tdF(x)\\ &= [xF(x)]_{x=0}^{x=t} - \int_0^t F(x)dx + t[F(1) - F(t)]\\ &= tF(t) - \int_0^t F(x)dx + t - tF(t)\\ &= t - \int_0^t F(x)dx\end{aligned}$$

and similarly we have

$$\int_0^1 v(x)dG(x) = t - \int_0^t G(x)dx$$

We therefore must also have

$$
\begin{aligned}
t - \int_0^t F(x)dx &= \int_0^1 v(x)dF(x) \\
&\geq \int_0^1 v(x)dG(x) \ (v' \geq 0,\ v'' \leq 0 \text{ and } F \succeq_2 G) \\
&= t - \int_0^t G(x)dx
\end{aligned}
$$

Equivalently:

$$\int_0^t F(x)dx \leq \int_0^t G(x)dx.$$

$\Leftarrow$: In the other direction, let v be a nondecreasing and concave. Assume v is twice differentiable. Then

$$
\begin{aligned}
& \int_0^1 v(x)dF(x) - \int_0^1 v(x)dG(x) \\
= \; & \int_0^1 v(x)d[F(x) - G(x)] \\
= \; & \underbrace{[v(x)(F(x) - G(x))]_{x=0}^{x=1}}_{=0} - \int_0^1 v'(x)[F(x) - G(x)]dx \\
= \; & -\left(\left[v'(x) \int_0^x (F(t) - G(t))dt \right]_{x=0}^{x=1} - \int_0^1 v''(x) \left[\int_0^x (F(t) - G(t))d \right. \right. \\
= \; & \underbrace{-\underbrace{v'(1)}_{\geq 0} \underbrace{\int_0^1 (F(t) - G(t))dt}_{\leq 0}}_{\geq 0} + \underbrace{\int_0^1 \underbrace{\underbrace{v''(x)}_{\leq 0} \underbrace{\left[\int_0^x (F(t) - G(t))dt \right]}_{\leq 0}}_{\geq 0} dx}_{\geq 0} \\
\geq \; & 0. \qquad \blacksquare
\end{aligned}
$$

Thus, we have FOSD⇒SOSD:

$$
\begin{aligned}
F(t) &\leq G(t) \text{ for every } t \in [0,1]. \\
&\Rightarrow \\
\int_0^t F(x)dx &\leq \int_0^t G(x)dx \text{ for every } t \in [0,1].
\end{aligned}
$$

Example:

Let F be the uniform distribution and $G(x) = 0$ if $x < \frac{1}{2}$ and $G(x) = 1$ otherwise. Note that the lottery identified by G places probability one on the outcome $\frac{1}{2}$. Then $G \succeq_2 F$ but $G \not\succeq_1 F$. To see that $G \succeq_2 F$, we can use the characterization result, calculate areas under F and G and compare... Or we can use the definition directly: $\int_0^1 v dG = v(\frac{1}{2})$. If v is increasing and concave, then

$$\begin{aligned}
\int_0^1 v(x)dF(x) &= \int_0^1 v(x)dx \\
&\leq v\left(\int_0^1 x dx\right) \\
&= v(\frac{1}{2}) \\
&= \int_0^1 v(x)dG(x).
\end{aligned}$$

To see that $G \not\succeq_1 F$ we can also use the characterization result, or the definition directly. For example, let $v(x) = x^2$. Then

$$\begin{aligned} \int_0^1 v(x) dF(x) &= \frac{1}{3} \\ \int_0^1 v(x) dG(x) &= \frac{1}{4}. \qquad \spadesuit \end{aligned}$$

Next we will study a result due to Rothschild and Stiglitz on the notion of "being riskier."

Lemma: The following are equivalent: (i) $\mu_F = \mu_G$, (ii) $\int_0^1 F(t)dt = \int_0^1 G(t)dt$.

Proof:
The result follows from the observation that

$$\mu_F = \int_0^1 x dF(x) = [xF(x)]_{x=0}^{x=1} - \int_0^1 F(x)dx = 1 - \int_0^1 F(x)dx. \qquad \blacksquare$$

Theorem: (Rothschild and Stiglitz) The following two statetments are equivalent:
(1) (a) $\mu_F = \mu_G$ and (b) $\int_0^1 v(x)dF(x) \geq \int_0^1 v(x)dG(x)$ whenever v is concave.
(2) (c) $\int_0^1 F(t)dt = \int_0^1 G(t)dt$ and (d) $\int_0^t F(x)dx \leq \int_0^t G(x)dx$ for every $t \in [0,1]$.

Note: If (a,b) (or equivalently (a,d) or (c,d) or (c,b)) holds, then we say that F dominates G in the RS sense.

Proof:

$\Leftarrow$: Suppose that (c) and (d) hold. We know by Lemma, that (c)$\Rightarrow$(a). To see that (c,d)$\Rightarrow$(b), recall that

$$\begin{aligned}
& \int_0^1 v(x)dF(x) - \int_0^1 v(x)dG(x) \\
= \; & \underbrace{-\underbrace{v'(1)}_{?}\underbrace{\int_0^1 (F(t)-G(t))dt}_{=0}}_{=0} + \underbrace{\int_0^1 \underbrace{\underbrace{v''(x)}_{\leq 0}\underbrace{\left[\int_0^x (F(t)-G(t))dt\right]}_{\leq 0}}_{\geq 0} dx}_{\geq 0} \\
\geq \; & 0.
\end{aligned}$$

$\Rightarrow$: Suppose that (a) and (b) hold. We know by Lemma, that (a)$\Rightarrow$(c). But we can use the argument in the proof of the result which characterizes SOSD verbatim, to see that (b)$\Rightarrow$(c) and the proof is complete. ■

Remark:

If $\mu_F = \mu_G$ and if $\int_0^1 v(x)dF(x) \geq \int_0^1 v(x)dG(x)$ whenever v is concave, then we can conclude that $var(F) \leq var(G)$.

To see this let $\mu_F = \mu_G = \mu$. Then, since $var(F) = \int (x-\mu)^2 dF$ and $var(G) = \int (x-\mu)^2 dG$, we have

$$-var(F) = \int \underbrace{[-(x-\mu)^2]}_{\text{concave!!!}} dF \geq \int \underbrace{[-(x-\mu)^2]}_{\text{concave!!!}} dG = -var(G)$$

Some more theory...

Definition. G is a *mean preserving spread* of F if there exist numbers $0 \leq a \leq b \leq c \leq 1$ such that

$$\begin{aligned} F(x) &\leq G(x) \text{ if } a \leq x \leq b \\ F(x) &\geq G(x) \text{ if } b \leq x \leq c \\ F(x) &= G(x) \text{ otherwise} \\ \int_a^b (G(x) - F(x))dx &= \int_b^c (F(x) - G(x))dx \end{aligned}$$

Picture should help. Note that if G is a mean preserving spread of F then

$$\begin{aligned} \int_0^t F(x)dx &\leq \int_0^t G(x)dx \text{ for every } t \in [0,1] \text{ and} \\ \int_0^1 F(x)dx &= \int_0^1 G(x)dx \end{aligned}$$

in other words $\mu_F = \mu_G$.

Example:
Let $F(x) = x$ and G be given by $G(0) = 0$, $G(x) = \frac{1}{2}$ for every $x \in (0, 1)$ and $G(1) = 1$. Note that if Y is the rv defined by G, then $Y = 0$ with probability $\frac{1}{2}$ and $Y = 1$ with probability $\frac{1}{2}$. Note that G is a mean preserving spread of F. Also note that F dominates G is the RS sense since $\mu_F = \mu_G$ and $.\int_0^t F(x)dx \leq \int_0^t G(x)dx$. Indeed, (b) also follows since if v is concave, then

$$\int_0^1 v(x)dF(x) = \int_0^1 v(x)dx \geq \frac{v(0) + v(1)}{2} = \int_0^1 v(x)dG(x).$$

Example:
Suppose that w is income, v is a concave vNM utility X is a risky asset and s is the fraction of your income you invest in X. Then your random income tomorrow is swX.
Optimization problem: Suppose that a decision-maker solves

$$\max_{0 \le s \le 1} v((1-s)w) + E[v(swX)].$$

We are assuming separability of utility in the random and certain components.
Natural question: What happens to optimal s if X becomes riskier?

Suppose that $F \succeq_1 G$. Let s_F solve

$$\max_{0 \le s \le 1} v((1-s)w) + \underbrace{\int_0^1 v(swx)\,dF(x)}_{=: E_F[v(swX)]}$$

and s_G solve

$$\max_{0 \le s \le 1} v((1-s)w) + \underbrace{\int_0^1 v(swx)\,dG(x)}_{=: E_G[v(swX)]}.$$

Question: When can we conclude that $s_G \le s_F$?

Suppose that s_F anf s_G are interior optima, i.e., they belong to $(0, 1)$. Let

$$h_F(s) = v((1-s)w) + E_F[v(swX)]$$

so that

$$\begin{aligned} h'_F(s) &= -wv'((1-s)w) + wE_F[v'(swX)X] \\ h''_F(s) &= w^2 \underbrace{v''((1-s)w)}_{\leq 0} + w^2 \underbrace{E_F[v''(swX)X^2]}_{\leq 0} \end{aligned}$$

implying that h_F is concave and therefore $h'_F(s_F) = 0$. Similarly defining $h_G(s)$, we get $h'_G(s_G) = 0$ as well.

Now suppose that $g(x) := v'(swx)x$ is increasing for all s and w. Then $F \succeq_1 G$ implies

$$\begin{aligned} h'_F(s_G) &= -wv'((1-s_G)w) + wE_F[v'(s_G wX)X] \\ &\geq -wv'((1-s_G)w) + wE_G[v'(s_G wX)X] \\ &= h'_G(s_G) \\ &= 0 \\ &= h'_F(s_F). \end{aligned}$$

Since h_F is concave and $h'_F(s_G) \geq 0 = h'_F(s_G)$, we conclude that $s_G \leq s_F$.

Exercise 1: Consider the portfolio choice problem

$$\max_{0 \leq x \leq w} E[u(xY + (w - x)r)]$$

where the investor's vNM utility function u satisfies $u' > 0$ and $u'' < 0$. Y is a random variable with $E(Y) > r$. Suppose that $0 < x^*(r) < w$ solves the problem for every r and that x^* is differentiable. Suppose that the investor's Arrow-Pratt measure of absolute risk aversion is a strictly increasing function of wealth, i.e. that $\frac{d}{dx}\left[-\frac{u''(x)}{u'(x)}\right] > 0$. Show that $\frac{dx^*}{dr} < 0$, i.e. that the optimal investment in the risky asset is a decreasing function of r.

Solution to Exercise 1:
Letting $x^* = x^*(r)$ and given y, define

$$\begin{aligned} W &= x^* y + (w - x^*(r))r \\ &= x^*(y - r) + wr. \end{aligned}$$

If $y - r > 0$, then $W > rw$ implies that

$$-\frac{u''(W)}{u'(W)} \geq -\frac{u''(rw)}{u'(rw)}$$

so that

$$u''(W)(y - r) \leq \frac{u''(rw)}{u'(rw)} u'(W)(y - r)$$

Similarly, if $y - r < 0$, then $W < rw$ and therefore

$$-\frac{u''(W)}{u'(W)} \leq -\frac{u''(rw)}{u'(rw)}$$

so that

$$u''(W)(y - r) \leq \frac{u''(rw)}{u'(rw)} u'(W)(y - r).$$

Consequently,

$$
\begin{aligned}
E[u''(x^*Y+(w-x^*)r)(Y-r)] \\
&\le \frac{u''(rw)}{u'(rw)}E[u'(x^*Y+(w-x^*)r)(Y-r)] \\
&= 0
\end{aligned}
$$

Since

$$E[u'(x^*(Y-r)+rw)(Y-r)]=0$$

it follows that

$$
\begin{aligned}
E[u''(x^*(Y-r)+rw)(Y-r)(w-x^*+\frac{dx^*(r)}{dr}(Y-r))] \\
- E[u'(x^*(Y-r)+rw)] = 0.
\end{aligned}
$$

Solving for $\frac{dx^*(r)}{dr}$ we obtain

$$\frac{dx^*(r)}{dr} = \frac{E[u'(x^*(Y-r)+rw)] - (w-x^*)E[u''(x^*(Y-r)+rw)(Y-r)]}{E[u''(x^*(Y-r)+rw)(Y-r)^2]} < 0.$$

Exercise 2: Consider an individual whose vNM utility v satisfies $v' > 0$. Suppose that $w \geq 0$ and $p \in \mathbb{R}^2_+$ satisfies $p_1 + p_2 = 1$. Let

$$A(w, p) := \{(x_1, x_2) \in \mathbb{R}^2 | p_1 v(w + x_1) + p_2 v(w + x_2) \geq v(w)\}$$

denote the *acceptable set* when the individual's initial wealth is w and the probability of state i is p_i. The acceptable set has an obvious interpretation: if $(x_1, x_2) \in A(w, p)$ then the simple lottery that yields $w + x_i$ with probability p_i results in an expected utility that is not less than the utility of the agent's initial wealth. Now assume that $p_2 > 0$. The implicit function theorem implies that there exists $\varepsilon > 0$ and a function $g : B(0, \varepsilon) \to \mathbb{R}$ that satisfies the equation

$$p_1 v(w + t) + p_2 v(w + g(t)) = v(w) \text{ for all } t \in B(0, \varepsilon).$$

Assuming that g is twice differentiable in the open ball $B(0, \varepsilon)$, show that there exists a positive constant K such that

$$g''(0) = K\left[-\frac{v''(w)}{v'(w)}\right].$$

Solution to Exercise 2:
From the definition of g, it follows that

$$v(w + g(0)) = v(w).$$

Therefore, $g(0) = 0$ since v is strictly increasing. Differentiating twice, we obtain

$$0 = p_1 v'(w + t) + p_2 v'(w + g(t))g'(t)$$

and

$$0 = p_1 v''(w + t) + p_2 v''(w + g(t))(g'(t))^2 + p_2 v'(w + g(t))g''(t).$$

Therefore $g(0) = 0$ implies that

$$0 = p_1 v'(w) + p_2 v'(w)g'(0)$$

and

$$0 = p_1 v''(w) + p_2 v''(w)(g'(0))^2 + p_2 v'(w)g''(0).$$

We conclude from the first equation that

$$g'(0) = -\frac{p_1}{p_2}$$

so that

$$0 = p_1 v''(w) + p_2 v''(w)(-\frac{p_1}{p_2})^2 + p_2 v'(w) g''(0).$$

Solving for $g''(0)$, we obtain

$$\begin{aligned} g''(0) &= -\frac{\left[p_1 + p_2(-\frac{p_1}{p_2})^2\right] v''(w)}{p_2 v'(w)} \\ &= -\frac{p_1}{p_2}\left[1 + \frac{p_1}{p_2}\right]\frac{v''(w)}{v'(w)} \\ &= -\frac{p_1}{p_2^2}\frac{v''(w)}{v'(w)}. \end{aligned}$$

Exercise 3: Consider an investor with initial wealth $w > 0$ and differentiable vNM utility function u satisfying $u'(x) > 0$ and $u''(x) < 0$ for all $x \in \mathbb{R}$. Let $\alpha \in [0,1]$ denote the fraction of the investor's initial wealth that he invests in a risky asset with random return Y. $(1-\alpha)w$ denotes the amount of the investor's initial wealth that he invests in a riskless asset with return $r > 0$ where $E(Y) > r$. Consequently, the investor's final wealth is the r.v. $w[\alpha Y + (1-\alpha)r]$. The investor's final wealth is subject to taxation. If $t \in (0,1)$ denotes the tax rate, then the investor's portfolio choice problem is given by

$$\max_{0 \leq \alpha \leq 1} E\left[u((1-t)w[\alpha Y + (1-\alpha)r])\right].$$

Let $t \mapsto \hat{\alpha}(t)$ denote the solution to this problem and suppose that $0 < \hat{\alpha}(t) < 1$ for each t. Show that $\hat{\alpha}'(t) \geq 0$ if u exhibits strictly increasing Arrow-Pratt relative risk aversion.

Solution to Exercise 3:
The function $t \mapsto \hat{\alpha}(t)$ satisfies the condition: for each $t \in (0,1)$

$$(1-t)wE\left[u'([(1-t)w[\hat{\alpha}(Y-r)+r])(Y-r)\right] = 0.$$

Eliminating $(1-t)w$ and differentiating wrt t yields

$$E\left[(Y-r)\left[-w[r+\hat{\alpha}(Y-r)]+(1-t)w\hat{\alpha}'(Y-r))\right] u''([(1-t)w[\hat{\alpha}(Y\right.$$
$$= 0$$

Defining $W(y) = (1-t)w[r+\hat{\alpha}(y-r)]$, we obtain

$$\begin{aligned}\hat{\alpha}'(t) &= \frac{E\left[(Y-r)w[r+\hat{\alpha}(Y-r)]u''(W(Y))\right]}{(1-t)wE\left[(Y-r)^2u''(W(Y))\right]}\\ &= \frac{E\left[W(Y)u''(W(X))(Y-r)\right]}{(1-t)^2wE\left[(Y-r)^2u''(W(Y))\right]}\end{aligned}$$

so it suffices to show that $E\left[W(Y)u''(W(Y))(Y-r)\right] \leq 0$.

To show this we use the usual argument:

$$
\begin{aligned}
y &> r \Rightarrow W(y) > w(1-t)r \\
&\Rightarrow -\frac{W(y)u''(W(y))}{u'(W(y))} > -\frac{[w(1-t)r]\,u''(w(1-t)r)}{u'(w(1-t)r)} \\
&\Rightarrow -\frac{W(y)u''(W(y))(y-r)}{u'(W(y))} > -\frac{[w(1-t)r]\,u''(w(1-t)r)(y-r)}{u'(w(1-t)r)}.
\end{aligned}
$$

and

$$
\begin{aligned}
y &< r \Rightarrow W(y) < w(1-t)r \\
&\Rightarrow -\frac{W(y)u''(W(y))}{u'(W(y))} < -\frac{[w(1-t)r]\,u''(w(1-t)r)}{u'(w(1-t)r)} \\
&\Rightarrow -\frac{W(y)u''(W(y))(y-r)}{u'(W(y))} > -\frac{[w(1-t)r]\,u''(w(1-t)r)(y-r)}{u'(w(1-t)r)}.
\end{aligned}
$$

Consequently,

$$W(y)u''(W(y))(y-r) \leq \left[\frac{[w(1-t)r]\,u''(w(1-t)r)}{u'(w(1-t)r)}\right] u'(W(y))(y-r)$$

for all realizations y of Y. This implies that

$$E\left[W(Y)u''(W(Y))(Y-r)\right] \leq \left[\frac{[w(1-t)r]\,u''(w(1-t)r)}{u'(w(1-t)r)}\right] E[u'(W(Y)$$

Since

$$E[u'(W(Y))(Y-r)] = 0,$$

we conclude that $E\left[W(Y)u''(W(Y))(Y-r)\right] \leq 0$.

Exercise 4: Suppose that an individual's preferences with respect to risk are represented by the Von Neumann-Morgenstern utility function u exhibiting constant absolute risk aversion $\alpha > 0$. If F is a distribution function, let $E_F(u)$ denote the expectation of u with respect to F. The *certainty equivalent* for F is a real number c_F satisfying the equation

$$u(c_F) = E_F(u)$$

Prove the following statement or construct a counterexample:

$$\frac{c_F + c_G}{2} = c_{\frac{F+G}{2}}$$

for all distribution functions F and G.

Solution to Exercise 4:
Suppose that $u(x) = -e^{-\alpha x}$. Consider the lotteries F and G satisfying

$$\begin{aligned} F(x) &= 1 \text{ if } x \geq 0 \\ &= 0 \text{ if } x < 0 \end{aligned}$$

$$\begin{aligned} G(x) &= 1 \text{ if } x \geq 1 \\ &= 0 \text{ if } x < 1 \end{aligned}$$

Then F is the distribution function of a rv X where $X \equiv 0$ and G is the distribition function of a rv Y where $Y \equiv 1$. Therefore, then

$$-e^{-\alpha c_F} = u(c_F) = E_F(u) = E(X) = u(0) = -1 \Rightarrow c_F = 0$$

and, similarly, $c_G = 1$.

Next, note that $\frac{F+G}{2}$ is the distribution function of a rv taking the value 0 with prob $1/2$ and the value 1 wirth prob $1/2$. Therefore,

$$E_{\frac{F+G}{2}}(u) = [-\exp(-\alpha)] \cdot \frac{1}{2} + [-\exp(0)] \cdot \frac{1}{2} = \frac{-e^{-\alpha} - 1}{2}.$$

From the definition of certainty equivalent, it follows that $c_{\frac{F+G}{2}}$ is defined by the equation

$$-\exp[-\alpha c_{\frac{F+G}{2}}] = \frac{-e^{-\alpha} - 1}{2}$$

so that

$$c_{\frac{F+G}{2}} = \frac{-1}{\alpha} \ln \left[\frac{e^{-\alpha} + 1}{2} \right] \neq \frac{1}{2}.$$

Exercise 5: Consider the following two stage stochastic consumer choice problem. In stage 1, the price vector q^0 is known with certainty while in stage 2, the price vector is a random variable taking one of finitely many values $q^1, ..., q^m$ with respective probabilities $p_1, ..., p_m$. In stage 1, the consumer chooses bundle $x \in \mathbb{R}^n_+$. At the beginning of stage 2, the price vector is revealed to the consumer who then chooses $z^i \in \mathbb{R}^n_+$ if the realized price vector is q^i. The intertemporal budget constraints are given by $p \cdot x + q^i \cdot z^i \leq y^i$ for each $i \geq 1$ where $y^1, .., y^m$ are the incomes in each state. Let $v(p)$ denote the optimal value for the problem

$$\max u(x) + \sum_{i=1}^{m} u(z^i) p_i \text{ s.t } \left\{ \begin{array}{l} q^0 \cdot x + q^i \cdot z^i \leq y^i \text{ for each } i \\ x \in \mathbb{R}^n_+, z^1 \in \mathbb{R}^n_+, .., z^m \in \mathbb{R}^n_+ \end{array} \right.$$

Define $\Delta = \{p \in \mathbb{R}^m_+ | p_1 + \cdots + p_m = 1\}$ and let $v : \Delta \to \mathbb{R}$ denote the optimal value function. Prove that v is convex on Δ .

Solution to Exercise 5:
Choose $p, p' \in \Delta, t \in [0,1]$ and define $\overline{p} = tp + (1-t)p'$. Furthermore, let $(x,z),(x',z')$ and $(\overline{x},\overline{z})$ solve the respective two stage problems for p, p' and $\overline{p}$. Clearly, $(\overline{x},\overline{z})$ is feasible for the problems defined by p and p'. Therefore,

$$u(\overline{x}) + \sum_{i=1}^{m} u(\overline{z}^i)p_i \leq u(x) + \sum_{i=1}^{m} u(z^i)p_i$$

and

$$u(\overline{x}) + \sum_{i=1}^{m} u(\overline{z}^i)p_i' \leq u(x') + \sum_{i=1}^{m} u(z'^i)p_i'$$

from which it follows that

$$
\begin{aligned}
v(\overline{p}) &= u(\overline{x}) + \sum_{i=1}^{m} u(\overline{z}^{i})\overline{p}_i \\
&= t[u(\overline{x}) + \sum_{i=1}^{m} u(\overline{z}^{i})p_i] + (1-t)[u(\overline{x}) + \sum_{i=1}^{m} u(\overline{z}^{i})p'_i] \\
&\leq t[u(x) + \sum_{i=1}^{m} u(z^{i})p_i] + (1-t)[u(x') + \sum_{i=1}^{m} u(z'^{i})p'_i] \\
&\leq tv(p) + (1-t)v(p').
\end{aligned}
$$

Exercise 6: Suppose that X_1 is uniformly distributed on the interval $[0, \theta_1]$ and that X_2 is uniformly distributed on the interval $[0, \theta_2]$.
(i) If $\theta_1 > \theta_2 > 0$, show that X_1 FOSDs X_2.
(ii) Suppose that u is increasing, $\theta \mapsto \varphi(\theta) = \frac{1}{\theta} \int_0^\theta u(x)dx$ is defined and differentiable on $\mathbb{R}_{++}$ and that differentiation and integration commute. Show that φ is an increasing function of θ.

Solution to Exercise 6:
(i) If $\theta_1 > \theta_2$, then

$$\begin{aligned} F_{X_1}(x) &= 0, \text{ if } x < 0 \\ &= \frac{x}{\theta_1}, \text{ if } 0 \le x < \theta_1 \\ &= 1, \text{ if } \theta_1 \le x \end{aligned}$$

and

$$\begin{aligned} F_{X_2}(x) &= 0, \text{ if } x < 0 \\ &= \frac{x}{\theta_2}, \text{ if } 0 \le x < \theta_2 \\ &= 1, \text{ if } \theta_2 \le x < \theta_1 \\ &= 1, \text{ if } \theta_1 \le x. \end{aligned}$$

so $F_{X_1}(x) - F_{X_2}(x) \le 0$ for all x.

(ii) Define

$$\varphi(\theta) = \frac{1}{\theta}\int_0^\theta u(x)dx$$

so that

$$\varphi'(\theta) = \frac{-1}{\theta^2}\int_0^\theta u(x)dx + \frac{1}{\theta}u(\theta)$$

If u is increasing, it follows that $u(x) \leq u(\theta)$ for all $x \in [0, \theta]$ implying that

$$\begin{aligned}\varphi'(\theta) &= \frac{-1}{\theta^2}\int_0^\theta u(x)dx + \frac{1}{\theta}u(\theta) \\ &\geq \frac{-1}{\theta^2}\int_0^\theta u(\theta)dx + \frac{1}{\theta}u(\theta) \\ &= 0.\end{aligned}$$

Exercise 7: Consider the problem $\max_{0 \leq x \leq w} u(w-x) + E[u(xY)]$.
(i) Suppose that $w = 1$ and $u(x) = \sqrt{x}$. Suppose that Y_1 and Y_2 are nonnegative valued rvs for which $E(\sqrt{Y_i})$ exist and Y_1 FOSDs Y_2. For each i, compute the solution $\overline{x}_i$ to the problem $\max_{0 \leq x_i \leq 1} u(1-x_i) + E[u(x_i Y_i)]$ and show that $\overline{x}_1 \geq \overline{x}_2$. Explain this result by computing the A-P measure of relative risk aversion for this agent.
(ii) Suppose that $w = 1$ and $u(x) = -\frac{1}{x}$. Suppose that Y_1 and Y_2 are positive valued rvs for which $E(\frac{1}{Y_i})$ exist and Y_1 FOSDs Y_2. For each i, compute the solution $\overline{x}_i$ to the problem $\max_{0 \leq x_i \leq 1} u(1-x_i) + E[u(x_i Y_i)]$ and show that $\overline{x}_1 \leq \overline{x}_2$. Explain this result by computing the A-P measure of relative risk aversion for this agent.

Solution to Exercise 7:
(i) The problem becomes

$$\text{maximize } \sqrt{1-x} + \sqrt{x}E[\sqrt{Y_i}] \text{ subject to } 0 \leq x \leq 1$$

with solution

$$\bar{x}_i = \frac{\left(E[\sqrt{Y_i}]\ \right)^2}{1 + \left(E[\sqrt{Y_i}]\ \right)^2}.$$

Since $y \mapsto \sqrt{y}$ is an increasing function, it follows that $E[\sqrt{Y_1}] \geq E[\sqrt{Y_2}]$ since $Y_1 \succsim_1 Y_2$. Therefore $\bar{x}_1 \geq \bar{x}_2$. This result is consistent with the general result proved in lecture where it was shown that, for increasing, concave u, the effect of a FOSD shift in the return is determined by the sign of

$$1 - \left[-\frac{xu''(x)}{u'(x)}\right]$$

In this example,

$$-\frac{xu''(x)}{u'(x)} \equiv \frac{1}{2} < 1$$

and a FOSD shift results in an increase in second period investment.

(ii) The problem becomes

$$\text{maximize } -\frac{1}{1-x} - \frac{1}{x}E(\frac{1}{Y_i}) \text{ subject to } 0 \leq x \leq 1$$

with solution

$$\overline{x}_i = \frac{\sqrt{E(\frac{1}{Y_i})}}{1+\sqrt{E(\frac{1}{Y_i})}}.$$

Since $y \mapsto \frac{1}{y}$ is a decreasing function, it follows that $E(\frac{1}{Y_1}) \leq E(\frac{1}{Y_2})$ since $Y_1 \succsim_1 Y_2$. Therefore $\overline{x}_1 \leq \overline{x}_2$. This result is consistent with the general result proved in lecture where it was shown that, for increasing, concave u, the effect of a FOSD shift in the return is determined by by the sign of

$$1 - \left[-\frac{xu''(x)}{u'(x)}\right]$$

In this example,

$$-\frac{xu''(x)}{u'(x)} \equiv 2 > 1$$

and a FOSD shift results in an decrease in second period investment.