MODELS WITH IDIOSYNCRATIC SHOCKS AND INCOMPLETE MARKETS

These models were developed in the seminal articles of:

• Bewley (*Journal of Economic Theory*, 1977)

• Aiyagari (*Quarterly Journal of Economics*, 1994, y *Journal of Political Economy*, 1995)

• Huggett (*Journal of Economic Dynamics and Control*, 1993, y *Journal of Monetary Economics*, 1996)

and have won an important place in the literature on saving and income distribution

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Basic Model
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a^0 same

We assume a continuum of agents in $[0, 1]$

- In each period, agents differ in their assets a_t^i and in the realization of an idiosyncratic productivity shock λ_t^i

$w_t \lambda^i_t$

- Productivity shocks follow an $AR(1)$ process with mean normalized to 1

$$\lambda_t^i = (1-\rho) + \rho\lambda_{t-1}^i + \varepsilon_t$$

mean = 1

where $\varepsilon_t \sim N\left(0, \sigma^2\right)$

- We approximate this process with a Markov chain with lower and upper limits $\lambda_{\text{min}}, \lambda_{\text{max}} > 0$ and transition matrix Π

$\lambda^i_t \in \{\lambda_{min}, \lambda_{max}\}$

$\Pi = \begin{bmatrix} \cdot & \cdot \\ \cdot & \cdot \end{bmatrix}$

$\hookrightarrow$ Tauchen's process

- Agents are equal *ex-ante* (they have the same initial assets a_0^i and realization $\lambda_0^i = 1$), but they will diverge over time according to their own history of shocks

$$\lambda^{i,t} \equiv \left(\lambda_0^i, \lambda_1^i, ..., \lambda_t^i\right)$$

Therefore, all agents solve the same problem in period 0 $\longrightarrow$ Contingent plans are the same

$$c_t^i(\lambda^{i,t}) = c_t(\lambda^t)$$

- Using the law of large numbers, there is uncertainty at the individual level, but not in the aggregate

$$L_t \equiv \int_0^1 \lambda_t^i di = 1$$

- We also assume that markets are incomplete: agents can only self-insure against a bad productivity draw by accumulating a safe asset. In addition, we impose a credit limit to individual's borrowing.

Each agent solves the problem:

budget constraint

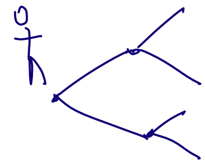
$$\max \quad E_0 \sum_{t=0}^{\infty} \beta^t u(c_t)$$

$$\begin{aligned} s.t. \quad c_t + a_{t+1} &= w_t \lambda_t + R_t a_t \\ a_{t+1} &\geq -\phi \\ a_0, \lambda_0 &= 1 \text{given} \end{aligned}$$

borrowing limit

where λ_t is the realization of the labor productivity shock and ϕ it is a credit limit (I will talk about it later) more strict than necessary to avoid Ponzi schemes

Definition of Equilibrium



A Sequential Competitive Equilibrium for this economy is a set of contingent plans for individual quantities $c_t\left(\lambda^t\right)$, $a_{t+1}\left(\lambda^t\right)$ and sequences for aggregated quantities Y_t, K_t and prices w_t, R_t such that:

i) Given $a_0 \geq -\phi$, $\lambda_0 = 1$, w_t, R_t and the stochastic process for λ, the plans $c_t\left(\lambda^t\right)$, $a_{t+1}\left(\lambda^t\right)$ solve the problem:

$$\max \quad \sum_{t=0}^{\infty} \sum_{\lambda^t \in \Lambda^t} \beta^t \pi\left(\lambda^t\right) u\left(c_t\left(\lambda^t\right)\right)$$

$$\begin{aligned} s.t. \quad c_t\left(\lambda^t\right) + a_{t+1}\left(\lambda^t\right) &= w_t \lambda_t + R_t a_t\left(\lambda^{t-1}\right) \\ a_{t+1}\left(\lambda^t\right) &\geq -\phi \qquad \forall \lambda^t, \forall t \end{aligned}$$

ii) In each period t, given w_t and R_t, the values Y_t and K_t solves the firm problem:

$$\max \quad Y_t - w_t - [R_t - (1-\delta)] K_t$$

$$s.t. \quad Y_t = f(K_t)$$

$$R_t = f'(K_t) + (1-\delta)$$

$$w_t = f(K_t) - f'(K_t) K_t$$

and profits are zero

iii) In each period t, markets clear:

$$Y_t = \sum_{\lambda^t} \pi\left(\lambda^t\right)\left[c_t\left(\lambda^t\right) + a_{t+1}\left(\lambda^t\right) - (1-\delta)\, a_t\left(\lambda^{t-1}\right)\right]$$

$$K_t = \sum_{\lambda^t} \pi\left(\lambda^t\right) a_t\left(\lambda^{t-1}\right)$$

Notice:

- The contingent plans are the same for all agents
- The consumption and assets of each individual depend on their own history $\lambda^{i,t}$
- Using again the law of large numbers, the probability $\pi\left(\lambda^t\right)$ represents the fraction of agents with history λ^t

full-insurance

`Complete Markets and Arrow-Debreu`

Before characterizing the sequential competitive equilibrium with incomplete markets, we can define the *efficient allocation* as the aggregate consumption and capital sequences that solve the problem of the social planner

$$
\begin{aligned}
\max \quad & \sum_{t=0}^{\infty} \beta^t u\left(C_t\right) \\
s.t. \quad C_t + K_{t+1} - (1-\delta) K_t \; &= \; f\left(K_t\right) \qquad \forall t \\
& \quad K_0 \text{ given}
\end{aligned}
$$

feasibility

Since individuals are equal ex-ante, the planner distributes consumption equally $(c_t = C_t)$

This solution insures perfectly agents

With incomplete markets and credit restrictions the efficient solution can not be reached

However, we can decentralize the efficient solution with *complete markets*

Sequentially, it would only change the budget constraint of households

$$c_t\left(\lambda^t\right) + \sum_{\lambda^{t+1}|\lambda^t} a_{t+1}\left(\lambda^{t+1}\middle|\,\lambda^t\right) = w_t\lambda_t + R_t a_t\left(\lambda^t\middle|\,\lambda^{t-1}\right)$$

where $a_{t+1}\left(\lambda^{t+1}\middle|\,\lambda^t\right)$ it is a contingent asset that pays R_t in λ^{t+1} and zero otherwise

Another way to decentralize the efficient solution is through *Arrow-Debreu* markets

The budget constraint would be given by

$$\sum_{t=0}^{\infty} \sum_{\lambda^t \in \Lambda^t} p_t\left(\lambda^t\right) c_t\left(\lambda^t\right) = \sum_{t=0}^{\infty} \sum_{\lambda^t \in \Lambda^t} p_t\left(\lambda^t\right) w_t \lambda_t$$

where $p_t\left(\lambda^t\right)$ is the price of one unit of the only good delivered in the period t if the history of individual shocks is fulfilled λ^t

In this structure, there is a single market that opens in period zero and where these contingent goods are exchanged

Recursive Formulation

Returning to sequential competitive equilibrium with incomplete markets, we continue our analysis using recursive language

- The individual state variables are a and λ
- In each period, $\mu(a^*, \lambda^*)$ denotes the fraction of agents with assets $a \leq a^*$ and productivity $\lambda \leq \lambda^*$

$$\mu : S \equiv [-\phi, \infty) \times [\lambda_{\text{min}}, \lambda_{\text{max}}] \rightarrow [0, 1]$$

$$\lim_{a \rightarrow \infty} \mu_t(a, \lambda_{\text{max}}) = 1$$

- The distribution or measure μ is the aggregate state variable

$$\int_S a d\mu(a, \lambda) = K \qquad \int_S \lambda d\mu(a, \lambda) = L = 1$$

A Recursive Competitive Equilibrium is a set of functions $v(a,\lambda,\mu)$, $c(a,\lambda,\mu)$, $a'(a,\lambda,\mu)$, prices $w(\mu)$ and $R(\mu)$, capital demand $K(\mu)$ and law of motion $\Gamma(\mu)$ such that:

i) For each triple (a,λ,μ), given functions w, r and Γ, the value function $v(a,\lambda,\mu)$ solves the Bellman equation:

$$v(a,\lambda,\mu) = \max_{c,a'} \left\{ u(c) + \beta E_\lambda v\left(a',\lambda',\mu'\right)\right\}$$

$$\begin{aligned} s.t. \quad c + a' &= w(\mu)\lambda + R(\mu)a \\ a' &\geq -\phi \\ \lambda' &\sim \Pi(\lambda) \\ \mu' &= \Gamma(\mu) \end{aligned}$$

and $c(a,\lambda,\mu)$, $a'(a,\lambda,\mu)$ are optimal decision rules for this problem

ii) For each distribution μ, prices satisfies conditions:

$$R(\mu) = f'(K(\mu)) + (1-\delta)$$

$$w(\mu) = f(K(\mu)) - f'(K(\mu))\,K(\mu)$$

iii) For each distribution μ, markets clear:

$$f(K(\mu)) = \int_S \left[c(a,\lambda,\mu) + a'(a,\lambda,\mu) - (1-\delta)\,a\right] d\mu(a,\lambda)$$

$$K(\mu) = \int_S a\,d\mu(a,\lambda)$$

$$1 = \int_S \lambda\,d\mu(a,\lambda)$$

iv) For each distribution μ, the law of motion Γ is consistent with individual decisions

Stationary Equilibrium (at the aggregate level $\nRightarrow$ individual stationarity)

A stationary equilibrium is a equilibrium in which the aggregate quantities C_t, K_t and prices w_t, R_t are constant

In recursive language, a stationary equilibrium is an invariant distribution μ^* such that $\mu^* = \Gamma(\mu^*)$

We are only going to study the stationary equilibrium of the model, which involves solving and characterizing the Bellman equation:

$$v(a,\lambda) = \max_{c,a'} \left\{ u(c) + \beta E_\lambda v\left(a',\lambda'\right)\right\}$$

$$\begin{aligned} s.t. \quad c + a' &= w^*\lambda + R^* a \\ a' &\geq -\phi \\ \lambda' &\sim \Pi(\lambda) \end{aligned}$$

Analyzing the transition to this stationary equilibrium is much more difficult

`Parenthesis: About the Credit Limit`

From the budget constraint, we know that in a stationary equilibrium $c_t \geq 0$ only if

$$a_t \geq \frac{1}{R^*}\left(a_{t+1} - w^* \lambda_t\right)$$

Iterating forward and using the transversality condition, $c_{t+j} \geq 0$, $\forall j > 0$, only if

$$a_t \geq -\frac{1}{R^*} \sum_{j=0}^{\infty} \left(\frac{1}{R^*}\right)^j w^* \lambda_{t+j}$$

that is, if the value of today's debt is not greater than the present value of the future labor income stream

To ensure that agents do not have a negative consumption in the future even in the worst possible case, we must impose the *natural credit limit*:

$c_t \geq 0$

$$\boxed{a_t \geq -\frac{1}{R^*}\sum_{j=0}^{\infty}\left(\frac{1}{R^*}\right)^j w^*\lambda_{\text{min}}} = -\frac{w^*\lambda_{\text{min}}}{R^*-1}$$

Any credit limit lower than the natural limit is 'ad-hoc', in the sense that it does not come from the non-negativity condition of consumption

The results that we will see next correspond to an arbitrary limit

$$\phi < \frac{w^*\lambda_{\text{min}}}{R^*-1}$$

End of parentheses

The Interest Rate with Incomplete Markets

Returning to the Bellman equation

$$v(a,\lambda) = \max_{c,a'} \left\{ u(c) + \beta E_\lambda v\left(a',\lambda'\right)\right\}$$

$$\begin{aligned} s.t. \quad c + a' &= w^*\lambda + R^* a \\ a' &\geq -\phi \end{aligned}$$

the first order condition with respect to a' requires:

$$-u'(c) + \beta E_\lambda v_a\left(a',\lambda'\right) \leq 0 \qquad \left(= \text{ if } a' > -\phi\right)$$

Using the envelope condition of Benveniste-Scheinkman,

$$v_a(a,\lambda) = R^* u'(c)$$

we get Euler's equation:

$$u'(c) \geq \beta R^* E_\lambda u'\left(c'\right) \qquad \left(= \text{ si } a' > -\phi\right)$$

(borrowing constraint not binding)

Euler
$=$ $\quad a' > -\phi$
$<$ $\quad a' = -\phi$

borrowing constraint binding

first-best (full insurance) $\Rightarrow R^*_{plan} = 1/\beta$

Result: *In any stationary equilibrium for this economy,* $R^* < \frac{1}{\beta}$

$R^* < R^*_{plan}$

$K^* > K^*_{plan}$

(there is over-accumulation of capital)

$\downarrow$

precautionary savings motive

We are going to test this result for i.i.d. shocks (λ' independent of λ). The steps are:

1. Define total resources as $z \equiv w^*\lambda + R^*a + \phi$ an rewrite the problem:

$v(a,\lambda) = \max\{u(c) + \beta E v(a', \lambda')\}$

s.t. $c + a' = w^*\lambda + R^*a \Rightarrow$

$a' \geq -\phi$

$$v(z) = \max_{c,a'} \quad \left\{u(c) + \beta E v\left(z'\right)\right\}$$

$$\begin{aligned} s.t. \quad c + a' &= z - \phi \\ a' &\geq -\phi \\ z' &= w^*\lambda' + R^*a' + \phi \end{aligned}$$

As income from capital and labor are perfect substitutes, agents only care about the sum

$\Rightarrow$ optimal decision rules

$c(z)$

$a'(z)$

$T_f(z) = \max_{a'} \left\{ u(z - \phi - a') + \beta E f(w^* \lambda' + R^* a' + \phi) \right\}$

s.t. $a' \in [-\phi, \infty)$

$v = Tv$ $\Leftarrow$

$\Rightarrow v'$ strictly decreasing

2. Show that $v(z)$ is strictly concave (use T Bellman operator properties)

3. Applying again Benveniste-Scheinkman, we know that:

$$v'(z) = u'(c(z))$$

$\rightarrow$ optimal decision rule for c

4. Assets will have an upper limit if there is a $\bar{z}$ such that:

$$\bar{z} = z'(\bar{z}) = w^* \lambda_{\text{max}} + R^* a'(\bar{z}) + \phi$$

that is, even with the best productivity draw the agents do not increase their total resources

today $\bar{z}$ λ_{max} $\Rightarrow$ tomorrow $\bar{z}$

$u'(c(z)) \geq \beta R^* E u'(c(z'))$

$v'(z) \geq \beta R^* E v'(z')$

5. Write Euler's equation (using Benveniste-Scheinkman 'backwards') as:

$$v'(z) \geq \beta R^* E v'\left(z'\right) = \beta R^* E v'\left(w^* \lambda' + R^* a'(z) + \phi\right)$$

$\forall z$

then, in particular:

$z = \bar{z}$

$$v'(\bar{z}) \geq \beta R^* E v'\left(w^* \lambda' + R^* a'(\bar{z}) + \phi\right)$$

6. Given that v' is strictly decreasing (see step 2):

$$Ev'\left(w^* \lambda' + R^* a'(\bar{z}) + \phi\right) = \sum_{\lambda} \pi(\lambda) v'\left(w^* \lambda + R^* a'(\bar{z}) + \phi\right)$$

$$> v'\left(w^* \lambda_{\text{max}} + R^* a'(\bar{z}) + \phi\right) = v'(\bar{z})$$

$v'(\bar{z}) > \beta R^* v'(\bar{z})$

$\Rightarrow \beta R^* < 1$

$R^* < 1/\beta$

7. Combining the results obtained in steps 5 and 6:

$$v'(\bar{z}) > \beta R^* v'(\bar{z})$$

then $R^* \geq \frac{1}{\beta}$ implies a contradiction $(v'(\bar{z}) > v'(\bar{z}))$

8. Conclude that $R^* \geq \frac{1}{\beta}$ implies that there is no upper limit $\bar{z}$, so the individual assets grow without limit

   Then, in any stationary equilibrium we should have $R^* < \frac{1}{\beta}$ ■

R^*_{eq}

Precautionary Savings

Coming back to the social planner's problem, representing the solution with complete markets,

$$\max \quad \sum_{t=0}^{\infty} \beta^t u(C_t)$$

$$\begin{aligned} s.t. \quad C_t + K_{t+1} - (1-\delta) K_t &= f(K_t) \qquad \forall t \\ & K_0 \text{ given} \end{aligned}$$

We obtain the first order conditions

$$\begin{aligned} \frac{u'(C_t)}{\beta u'(C_{t+1})} &= f'(K_{t+1}) + (1-\delta) \\ C_t &= f(K_t) + (1-\delta) K_t - K_{t+1} \end{aligned}$$

Then, in a stationary solution

$$R^* = f'(K^*) + (1 - \delta) = \frac{1}{\beta} \quad \Rightarrow R'plan$$

$$C^* = f(K^*) - \delta K^*$$

the interest rate is equal to the inverse of the discount factor

Therefore, comparing the efficient solution of the problem of the social planner with the competitive equilibrium with incomplete markets and credit constraints

$$R^*_{eq} < R^*_{plan} = \frac{1}{\beta}$$

where

$$K^*_{eq} > K^*_{plan}$$

In the model with idiosyncratic shocks, incomplete markets and credit constraints, agents save more than the efficient amount

The reason is that they need a buffer of savings to absorb the impact of bad realizations of productivity shocks

When a negative shock occurs, agents may be limited in their ability to borrow; to avoid reducing consumption, they must use their own savings

$\Rightarrow$ Agents then save for *precautionary reasons*

We can measure the size of precautionary saving as K^*_{eq}/K^*_{plan}

HANK

Other Implications of the Model

- Relatively small differences in income can generate large differences in the distribution of wealth (Huggett, JME 96)

- A tax on capital can be efficient, since it reduces the incentives for over-accumulation of wealth (Aiyagari, JPE 95)

- The risk-free interest rate may be as low as in the data still with reasonable values for risk aversion (Huggett, JEDC 93)