Game Theory

Bayesian Nash Equilibrium

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In this note, we define the equilibrium concept of **Bayesian Nash equilibrium** for Bayesian games (Harsanyi, 1967, 1968a,b).

1 Example

Example 1 (Cournot game with unknown costs)**.** Firms 1 and 2 produce a homogeneous product. Each firm i chooses its quantity $q_i \geq 0$. The price of the product depends on the total quantity $Q = q_1 + q_2$, and the inverse demand function P is such that $P(Q) = 1 - Q$. Assume that firm 1's marginal cost $c_1 = 0$ is commonly known, but firm 2's marginal cost c_2 is not. Specifically, it is either 0 or $\frac{1}{2}$ with probability $\frac{1}{2}$ each. Firm 2 knows the realized marginal cost c_2, but firm 1 does not.

This game is a Bayesian game $\mathcal{G}$ such that:

1. $I = \{1, 2\}$ is the set of players.
2. $\Omega = T_1 \times T_2$ is the set of states of the world.
3. $T_1 = \{0\}$ and $T_2 = \{0, \frac{1}{2}\}$ is the type spaces for players 1 and 2 respectively.
   - Firm 2 is either the low-cost type $c_2 = 0$ or the high-cost type $c_2 = \frac{1}{2}$.
4. $\mathbb{P} \in \Delta(\Omega)$ is the common prior.
   - $\mathbb{P}(0, 0) = \mathbb{P}(0, \frac{1}{2}) = \frac{1}{2}$, where firm 2 has a marginal cost 0 or $\frac{1}{2}$ with probabilities $\frac{1}{2}$.
5. $A_1 = A_2 = \mathbb{R}_+$ are the set of quantities that firm i chooses.
6. $\pi_i : A \times T_i \to \mathbb{R}$ is firm i's payoff function defined as follows:

$$\pi_i(q, t_i) = P(Q)q_i - c_i q_i.$$

Since different types of firm 2 may choose different quantities, we write q_{2L} and q_{2H} for the quantities that the low- and high-cost types choose, respectively.

There are three cases to consider:

- Firm 1 has the profit maximization problem:

$$\max_{q_1} \quad \frac{1}{2}(1 - (q_1 + q_{2L}))q_1 + \frac{1}{2}(1 - (q_1 + q_{2H}))q_1 - 0 \cdot q_1, \tag{1}$$

where firm 1 thinks that firm 2 chooses either q_{2L} or q_{2H} (as the low-cost type or the high-cost type) with probability $\frac{1}{2}$ each.

- The low-cost type of firm 2 has the profit maximization problem:

$$\max_{q_{2L}} \quad (1-(q_1+q_{2L}))q_{2L} - 0 \cdot q_{2L}. \tag{2}$$

- The high-cost type of firm 2 has the profit maximization problem:

$$\max_{q_{2H}} \quad (1-(q_1+q_{2H}))q_{2H} - \frac{1}{2}q_{2H}. \tag{3}$$

If we solve the system of the first-order conditions for problems (1) to (3), then we obtain the triplet $(q_1^*, q_{2L}^*, q_{2H}^*)$, where firm 1 best-responds to the behavior by both types of firm 2 and each type of firm 2 also best-responds to the behavior by firm 1. Hence, we call he triplet a **Bayesian Nash equilibrium**. □

2 Bayesian Nash Equilibrium

Recall that a Bayesian game is similar to a normal-form game with a correlation device. Indeed, a normal-form game with a correlation device can be regarded as a Bayesian game such that a state ω does not affect any players' payoffs but is just a device to manipulate players' behavior. As we shall see, a Bayesian Nash equilibrium for Bayesian games is similar to a correlated equilibrium for normal-form games. For reference, we summarize the basic definitions of correlated equilibrium in the box.

Ex-Ante Definition Here is the definition of Bayesian Nash equilibrium:

Definition 1. In a Bayesian game $\mathcal{G}$ with a finite state space Ω and a common prior $\mathbb{P}$, a strategy profile $\sigma^* = (\sigma_i^*)_i$ is a **Bayesian Nash equilibrium** if for each $i \in I$ and each $\sigma_i : T_i \to A_i$,

$$\sum_{\omega\in\Omega} \mathbb{P}(\omega)u_i(\sigma_i^*(\tau_i(\omega)), \sigma_{-i}^*(\tau_{-i}(\omega)), \omega) \geq \sum_{\omega\in\Omega} \mathbb{P}(\omega)u_i(\sigma_i(\tau_i(\omega)), \sigma_{-i}^*(\tau_{-i}(\omega)), \omega),$$

where $\tau_j(\omega)$ denotes player j's type at state ω for each $j \in I$.

Interim Definition In Definition 1, player i's payoff maximization is in term of the **ex-ante** perspective—that is, before all players learn their types. In Example 1, however, we take the **interim** perspective—that is, after firm 2 learns its type c_2. As in the case of correlated equilibrium, it turns out that these two formulations are equivalent.

Here is another definition of Bayesian Nash equilibrium:

Note (Correlated Equilibrium):

Definition 3. For a normal-form game G, a **correlation device** is a triple $(\Omega, \mathbb{P}, H)$ such that:

1. Ω is a finite set of states.
2. $\mathbb{P} \in \Delta(\Omega)$ is a prior.
3. $H = (H_i)_i$ is the profile of player i's information partition H_i of Ω.
   - $H_i(\omega)$ is the (unique) element $h_i \in H_i$ such that $\omega \in h_i$.

Definition 4. Let a normal-form game G be endowed with a correlation device $(\Omega, \mathbb{P}, H)$. Player i's **(correlation) strategy** is a function $f_i : H_i \to A_i$. Let $F_i(\Omega, H_i)$ denote the set of all player i's strategies.[a]

Definition 5. Let a normal-form game G be endowed with a correlation device $(\Omega, \mathbb{P}, H)$. We have two equivalent definitions of **correlated equilibrium**:

(i) A strategy profile f^* is a correlated equilibrium if for each $f_i \in F_i(\Omega, H_i)$,

$$\sum_{\omega \in \Omega} \mathbb{P}(\omega) u_i\big(f_i^*(\omega), f_{-i}^*(\omega)\big) \geq \sum_{\omega \in \Omega} \mathbb{P}(\omega) u_i\big(f_i(\omega), f_{-i}^*(\omega)\big).$$

(ii) A strategy profile f^* is a correlated equilibrium if for each $h_i \in H_i$ and each $a_i \in A_i$,

$$\sum_{\omega \in h_i} \mathbb{P}(\omega \mid h_i) u_i\big(f_i^*(\omega), f_{-i}^*(\omega)\big) \geq \sum_{\omega \in h_i} \mathbb{P}(\omega \mid h_i) u_i\big(a_i, f_{-i}^*(\omega)\big),$$

where $\mathbb{P}(\omega \mid h_i)$ is player i's posterior conditional on her information set h_i.

[a] This definition is equivalent to the definition that appears in the note on correlated equilibrium.

Definition 2. In a Bayesian game $\mathcal{G}$ with a finite state space Ω and a common prior $\mathbb{P}$, a strategy profile $\sigma^* = (\sigma_i^*)_i$ is a **Bayesian Nash equilibrium** if for each $i \in I$, each $t_i \in T_i$, and each $a_i \in A_i$,

$$\sum_{\omega \in \Omega} \mathbb{P}(\omega \mid t_i) u_i\big(\sigma_i^*(t_i), \sigma_{-i}^*(\tau_{-i}(\omega)), \omega\big) \geq \sum_{\omega \in \Omega} \mathbb{P}(\omega \mid t_i) u_i\big(a_i, \sigma_{-i}^*(\tau_{-i}(\omega)), \omega\big),$$

where $\mathbb{P}(\cdot \mid t_i)$ denotes player i's posterior conditional on her type t_i, and $\tau_j(\omega)$ denotes player j's type at state ω for each $j \in I$.

Remark 1. In Definition 2, we regard two distinct types $t_i \neq t_i'$ of player i as if they were different players. In Example 1, we treat the low-cost type and high-cost type of firm 2 as if they were different firms and were solving their profit maximization problems (2) and (3) independently.

This view helps us to see that a Bayesian Nash equilibrium is, indeed, a Nash equilibrium of the normal-form game with the set of players consisting of all (i, t_i), or type t_i of player i. □

Ex-Ante versus Interim Perspectives As mentioned above, the two definitions, ex-ante and interim, are equivalent.

Theorem 1. *In a Bayesian game $\mathcal{G}$ with a finite state space Ω and a common prior $\mathbb{P}$, Definitions 1 and 2 are equivalent to each other.*

Proof. Analogous to the corresponding result of correlated equilibrium. ■

Remark 2. In Definitions 1 and 2, we assume that a state space Ω is finite, but this assumption is dispensable. For a general state space Ω, we may use integrals instead of summation. □

Remark 3. In Definitions 1 and 2, we assume that there is a common prior $\mathbb{P}$, but this assumption is not necessary. We may replace the common prior $\mathbb{P}$ with player i's prior $\mathbb{P}_i$ in the defining inequalities. □

3 More Examples

3.1 Adverse Selection

Example 2. Recall that the lemons model is represented as a Bayesian game $\mathcal{G}$ such that:

1. $I = \{\mathrm{S}, \mathrm{B}\}$ is the set of players, where S and B denote a seller and a buyer.
2. $\Theta = [0, 1]$ is the set of states of nature.
3. $\mathbb{P} \in \Delta(\Theta)$ is the common prior that is uniform.
4. S learns a state θ as her type θ, and B learns nothing.
5. $A_\mathrm{S} = [0, 1]$ and $A_\mathrm{B} = [0, 1]$ are the set of prices that S and B choose respectively.
   - p_S is the minimum price above which S is willing to sell her car
   - p_B is the price at which B is willing to pay.
6. u_S and u_B are payoff functions for S and B respectively:

$$u_\mathrm{S}(p_\mathrm{S}, p_\mathrm{B}, \theta) = \begin{cases} p_\mathrm{B} - \theta & \text{if } p_\mathrm{B} \geq p_\mathrm{S} \\ 0 & \text{otherwise,} \end{cases} \qquad u_\mathrm{B}(p_\mathrm{S}, p_\mathrm{B}, \theta) = \begin{cases} k\theta - p_\mathrm{B} & \text{if } p_\mathrm{B} \geq p_\mathrm{S} \\ 0 & \text{otherwise,} \end{cases}$$

   with a given parameter $k \in (1, 2)$.

In the previous note, we have informally seen that in a Bayesian Nash equilibrium (in pure strategies), the object that could be traded is of quality $\theta = 0$.

Note (Leibniz Rule): Let $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a continuous function such that its partial derivative $\frac{\partial f}{\partial x}$ is continuous. Let $a, b : \mathbb{R} \to \mathbb{R}$ be continuously differentiable functions. Then, Leibniz rule holds:

$$\frac{d}{dx}\int_{a(x)}^{b(x)} f(x,t)dt = \int_{a(x)}^{b(x)} \frac{\partial f}{\partial x}(x,t)dt + f(x,b(x))b'(x) - f(x,a(x))a'(x).$$

We show this result. Recall that S of any type θ chooses a price $p_{\mathrm{S}} = \theta$ (i.e., S plays a strategy $\sigma_{\mathrm{S}} : \Theta \to [0,1]$ such that $\sigma_{\mathrm{S}}(\theta) = \theta$), as noted in the previous note.[1] Then, what price does B offer? Given S's strategy $p_{\mathrm{S}} = \theta$, B maximizes his expected payoff:

$$\max_{p_{\mathrm{B}}} \quad \int_0^{p_{\mathrm{B}}} (k\theta - p_{\mathrm{B}})d\theta,$$

where we have the integral interval $[0, p_{\mathrm{B}}]$ because under S's strategy $p_{\mathrm{S}} = \theta$, trade occurs if and only if $\theta \leq p_{\mathrm{B}}$. By Leibniz rule, an optimal price p_{B}^* satisfies the first-order condition:

$$\int_0^{p_{\mathrm{B}}^*} (-1)d\theta + (kp_{\mathrm{B}}^* - p_{\mathrm{B}}^*) = 0.$$

That is, $(k-2)p_{\mathrm{B}}^* = 0$. Since $k < 2$, it follows that $p_{\mathrm{B}}^* = 0$. Hence, the Bayesian Nash equilibrium is such that B offers price $p_{\mathrm{B}} = 0$ and S of type θ offers price $p_{\mathrm{S}} = \theta$. $\square$

3.2 First-Price Sealed-Bid Auction

Example 3. There is a single object for sale, and there are two potential buyers, called bidders $i = 1, 2$. Each bidder i gains a private value t_i from the object if she gets it. Assume that each t_i is a realization of a random variable $\boldsymbol{t}_i$ that is independently and identically distributed on an interval $[0,1]$ according to an increasing distribution function F with a full-support continuous density f. Each bidder i observes her own value t_i (but not the other bidder $-i$'s value t_{-i}).

This setting is formalized as follows:

1. $I = \{1, 2\}$ is the set of bidders.
2. $T_i = [0,1]$ is the type spaces for bidder i.
3. $\mathbb{P} \in \Delta(T)$ is the prior.
   - The prior on the event $\{t = (t_1, t_2) : t_1 \leq \boldsymbol{t}_1, t_2 \leq \boldsymbol{t}_2\}$ is $F(t_1)F(t_2)$.
4. $A_i = [0,1]$ for bidder i.

Bidders' payoff structure depends on an auction rule. In this example, we consider the **first-price (sealed-bid) auction**, where the highest bidder gets the object and pays the highest

[1] More precisely, this strategy is weakly dominant for S.

bid (which is her own bid). Under this rule, we define bidder i's payoff function $u_i^{\mathrm{I}} : A \times T_i \to \mathbb{R}$ as follows: For each $b \in [0,1]^2$ and each $t_i \in [0,1]$,

$$u_i^{\mathrm{I}}(b, t_i) = \begin{cases} t_i - b_i & \text{if } b_i > b_j \\ \frac{1}{2}(t_i - b_i) & \text{if } b_i = b_j \\ 0 & \text{if } b_i < b_j. \end{cases}$$

Next, we define bidder i's (pure-strategy) strategies. In this example, we focus on bidder i's strategies $\beta_i : T_i \to [0,1]$, which assign to her type t_i her bid $\beta_i(t_i)$, that are strictly increasing and continuously differentiable.[2]

In what follows, we characterize a symmetric (pure-strategy) Bayesian Nash equilibrium, in which bidders 1 and 2 play the same strategy $\beta^{\mathrm{I}} \equiv \beta_1 = \beta_2$.

Claim 1. *In the first-price sealed-bid auction, a symmetric Bayesian Nash equilibrium strategy β^{I} is such that*

$$\beta^{\mathrm{I}}(t) = \frac{1}{F(t)} \int_0^t s f(s) ds. \tag{4}$$

Proof. By symmetry, we focus on bidder 1. Suppose that bidder 2 plays a strategy β (which is strictly increasing and continuously differentiable). Then, type-t_1 bidder 1's payoff from submitting a bid b_1 is

$$\underbrace{\mathbb{P}(b_1 > \beta(\boldsymbol{t}_2))}_{= F(\beta^{-1}(b_1))} \times (t_1 - b_1) + \underbrace{\mathbb{P}(b_1 < \beta(\boldsymbol{t}_2))}_{= 1 - F(\beta^{-1}(b_1))} \times 0 = F(\beta^{-1}(b_1))(t_1 - b_1).$$

Hence, we have bidder 1's problem:

$$\max_{b_1} \quad F(\beta^{-1}(b_1))(t_1 - b_1).$$

An optimal bid b_1^* at type t_1 must satisfy the first-order condition:

$$f(\beta^{-1}(b_1^*)) \underbrace{\left(\frac{d}{db_1}\beta^{-1}(b_1^*)\right)}_{= \frac{1}{\beta'(\beta^{-1}(b_1^*))}} (t_1 - b_1^*) - F(\beta^{-1}(b_1^*)) = 0.$$

At a symmetric equilibrium, bidder 1's optimal strategy must coincide with the strategy β. That is, the optimal bid b_1^* must be such that $b_1^* = \beta^{-1}(t_1)$. Substituting it into the first-order

[2] The strictly increasingness means that bidder i submits a higher bid when her evaluation t_i is higher.

condition, we have

$$\underbrace{f(t_1)\beta(t_1)+F(t_1)\beta'(t_1)}_{=\frac{d}{dt_1}(F(t_1)\beta(t_1))}=t_1 f(t_1).$$

By solving this differential equation, we find the unique solution:

$$\beta^{\mathrm{I}}(t)=\frac{1}{F(t)}\int_0^t sf(s)ds,$$

which is strictly increasing and continuous.

However, we are not yet done, because the above equation is derived from a *necessary* condition that a symmetric equilibrium strategy must satisfy.

Still we have to show that bidder 1 is willing to play β^{I} when player 2 plays β^{I}. Since it is not optimal for bidder 1 to bid any $b>1$, we assume that she submits a bid $b\le 1$.

- Suppose that bidder 1 of type t_1 submits the bid $\beta^{\mathrm{I}}(t_1)$. Then, her expected payoff is

$$\underbrace{\mathbb{P}\Big(\beta^{\mathrm{I}}(t_1)>\beta^{\mathrm{I}}(\boldsymbol{t}_2)\Big)}_{=F(t_1)}\Big(t_1-\beta^{\mathrm{I}}(t_1)\Big)=F(t_1)t_1-\int_0^{t_1} sf(s)ds. \tag{5}$$

- Suppose that bidder 1 of type t_1 deviates to any bid $b_1\neq\beta^{\mathrm{I}}(t_1)$. Let τ_1 denote bidder 1's type who would submit this bid b_1 under the strategy β^{I}. That is, $b_1=\beta^{\mathrm{I}}(\tau_1)$. Then, her expected payoff is

$$\underbrace{\mathbb{P}\Big(b_1>\beta^{\mathrm{I}}(\boldsymbol{t}_2)\Big)}_{=F(\tau_1)}\Big(t_1-\beta^{\mathrm{I}}(\tau_1)\Big)=F(\tau_1)t_1-\int_0^{\tau_1} sf(s)ds. \tag{6}$$

Bidder 1 has no profitable deviation if and only if

$$(5)-(6)=(F(t_1)-F(\tau_1))t_1-\int_{\tau_1}^{t_1} sf(s)ds\ge 0.$$

By integration by parts, (5) − (6) is equal to

$$(F(t_1)-F(\tau_1))t_1-\int_{\tau_1}^{t_1} sf(s)ds=-(t_1-\tau_1)F(\tau_1)+\int_{\tau_1}^{t_1} F(s)ds.$$

This is strictly positive if $t_1>\tau_1$ and is zero if $t_1<\tau_1$. ■

3.3 Second-Price Sealed-Bid Auction

Example 4. We examine the same auction model as Example 3, but here we consider the **second-price (sealed-bid) auction**, where the highest bidder gets the object and pays the second highest bid. Under this rule, we define bidder i's payoff function $u_i^{\mathrm{II}} : A \times T_i \to \mathbb{R}$ as follows: For each $b \in [0,1]^2$ and each $t_i \in [0,1]$,

$$u_i^{\mathrm{II}}(b, t_i) = \begin{cases} t_i - b_{-i} & \text{if } b_i > b_{-i} \\ \frac{t_i - b_i}{2} & \text{if } b_i = b_{-i} \\ 0 & \text{if } b_i < b_{-i}. \end{cases}$$

Claim 2. *In the second-price sealed-bid auction, the following "truthful" strategy is weakly dominant for each $i \in I$:*

$$\beta^{\mathrm{II}}(t_i) = t_i.$$

Hence, a symmetric Bayesian Nash equilibrium consists of this strategy β^{II}.

Proof. By symmetry, we focus on bidder 1. It suffices to compare her truthful bid $b_i = t_i$ with any other bid $b_i \neq t_i$ and to show that the truthful bid $b_i = t_i$ is weakly better. There are two cases to consider:

- We compare the truthful bid t_i with any bid $b_i < t_i$.

	payoff from bid b_1	payoff from bid t_1
$b_2 \leq b_1 < t_1$	$t_1 - b_1$	$t_1 - b_1$
$b_1 < b_2 < t_1$	0	$t_1 - b_2 > 0$
$b_1 < t_1 \leq b_2$	0	0

  As implied by this table, the truthful bid t_i weakly dominates any bid $b_i < t_i$.

- We compare the truthful bid t_i with any bid $b_i > t_i$.

	payoff from bid t_1	payoff from bid b_1
$b_2 \leq t_1 < b_1$	$t_1 - b_2$	$t_1 - b_2$
$t_1 < b_2 < b_1$	0	$t_1 - b_2 < 0$
$t_1 < b_1 \leq b_2$	0	0

  As implied by this table, the truthful bid t_i weakly dominates any bid $b_i > t_i$. ■

Remark 4. Once we have derived the (symmetric) Bayesian Nash equilibrium strategies for both first- and second-price sealed-bid auction, we can compare the expected revenue (for the auctioneer). It turns out that each bidder's expected payment is the same across the two auction rules; therefore, the expected revenue is also the same. This result is a special case of the **Revenue Equivalence Theorem**, which shows that under a more general setting, any “standard” auction rules (which satisfy mild conditions) yields the same expected revenue.

□

References

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