CPT Lecture Notes 4: Correspondences

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A correspondence from X to Y (both Euclidean) is a map φ (denoted $\varphi : X \rightrightarrows Y$) which associates with every $x \in X$ a nonempty set $\varphi(x) \subseteq Y$.

Put differently, a correspondence from X to Y is a function from X to $\{A \subseteq Y : A \neq \emptyset\}$.

A correspondence φ is closed-valued if $\varphi(x)$ is a closed subset of Y for every $x \in X$. We similarly define correspondences which are open-valued, compact-valued, etc.

The graph of φ is the set $G_\varphi := \{(x, y) \in X \times Y : y \in \varphi(x)\}$. φ is closed (open) if G_φ is a closed (open) subset of $X \times Y$.

If φ is closed, then it is closed-valued. The reverse implication does not hold. (Let $\varphi : \mathbb{R} \rightrightarrows \mathbb{R}$ be given by $\varphi(x) = [2, 5]$ if $x < 2$ and $\varphi(x) = [3, 4]$ if $x \geq 2$.) Similarly open correspondences are open-valued but not vice versa.

We will need the following two nested notions of inverse.

Let $U \subseteq Y$. The strong and weak inverses of U under φ are defined respectively as follows:

$$\varphi_s^{-1}(U) := \{x \in X : \varphi(x) \subseteq U\}$$
$$\varphi_w^{-1}(U) := \{x \in X : \varphi(x) \cap U \neq \emptyset\}$$

Note:
(1) $\varphi_s^{-1}(U) \subseteq \varphi_w^{-1}(U)$.
(2) If φ is such that $|\varphi(x)| = 1$ for every x, then $\varphi_s^{-1}(U) = \varphi_w^{-1}(U)$.
(3) In general, $\varphi_w^{-1}(U) = (\varphi_s^{-1}(U^c))^c$.

A correspondence $\varphi : X \rightrightarrows Y$ is upper hemi-continuous (UHC) at $x \in X$ if whenever U is an open subset of Y and $x \in \varphi_s^{-1}(U)$, $B(x, \delta) \subseteq \varphi_s^{-1}(U)$ for some $\delta > 0$. φ is UHC if it is UHC at every $x \in X$.

Hence φ is UHC if and only if the strong inverse of any open set in Y under φ is open in X.

A correspondence $\varphi : X \rightrightarrows Y$ is lower hemi-continuous (LHC) at $x \in X$ if whenever U is an open subset of Y and $x \in \varphi_w^{-1}(U)$, $B(x, \delta) \subseteq \varphi_w^{-1}(U)$ for some $\delta > 0$. φ is LHC if it is LHC at every $x \in X$.

Hence φ is LHC if and only if weak inverse of any open set in Y under φ is open in X.

A correspondence is continuous if it is LHC and UHC.

Theorem: Let $f : X \to Y$ be a function and let $\varphi : X \rightrightarrows Y$ be defined as $\varphi(x) = \{f(x)\}$ for every x. Then the following are equivalent:
(1) φ is UHC.
(2) φ is LHC.
(3) f is continuous.

Proof: Homework.

Hence, for instance, if the function $f : X \to Y$ is upper (or lower) **s**emicontinuous, the correspondence $\varphi : X \rightrightarrows Y$ defined by $\varphi(x) = \{f(x)\}$ need not be upper (or lower) **h**emicontinuous.

Observation: Closedness and UHC are not nested.

If $\varphi(0) = \{0\}$ and $\varphi(x) = \{1/x\}$ for $x > 0$, then φ is closed but not UHC. If $\varphi(x) = (0, 1)$ for every x, then φ is UHC but not closed.

The exact relationship between UHC, closedness and closed-valuedness is addressed in the next two important results.

Theorem: If $\varphi : X \rightrightarrows Y$ is UHC and closed-valued, then it is also closed.

Theorem: If $\varphi : X \rightrightarrows Y$ is closed and Y is compact, then φ is UHC.

Proof of the first result:

Suppose that φ is UHC and closed-valued. We will show that G_φ^c is open. Pick $(x, y) \notin G_\varphi$. Then $y \notin \varphi(x)$ and since $\varphi(x)$ is closed, there exists an open ball $B(y, \varepsilon)$ such that $cl(B(y, \varepsilon)) \subset \varphi(x)^c$. Let $U = (cl(B(y, \varepsilon)))^c$ so that U is open and $\varphi(x) \subset U$. By UHC, there must exist some $\delta > 0$ such that $\varphi(z) \subset U$ for every $z \in B(x, \delta)$. Now observe that $B(x, \delta) \times B(y, \varepsilon)$ is an open set containing (x, y). We need to argue that $B(x, \delta) \times B(y, \varepsilon) \subset G_\varphi^c$, i.e., if $(x', y') \in B(x, \delta) \times B(y, \varepsilon)$, then $y' \notin \varphi(x')$. For any $(x', y') \in B(x, \delta) \times B(y, \varepsilon)$, (1) $\varphi(x') \subset U$ and (2) $y' \in cl(B(y, \varepsilon))$, implying $y' \notin (cl(B(y, \varepsilon)))^c = U$. Hence $y' \notin \varphi(x')$, as desired.

Proof of the second result:

Suppose that $\varphi : X \rightrightarrows Y$ is closed and Y is compact, but φ is not UHC. Then there exists x and an open set U such that (1) $\varphi(x) \subset U$, and (2) for every $\delta > 0$, there exists some $z_\delta \in B(x, \delta)$ such that $\varphi(z_\delta) \backslash U \neq \emptyset$. So we can pick a sequence x_n such that $x_n \longrightarrow x$ and $\varphi(x_n) \backslash U \neq \emptyset$ for every n. Let $y_n \in \varphi(x_n) \backslash U$ for every n. Since Y is compact, y_n has a subsequence y_{n_k} with limit y. Since U^c is closed, y must belong to U^c. On the other hand, since $(x_{n_k}, y_{n_k}) \longrightarrow (x, y)$, $(x_{n_k}, y_{n_k}) \in G_\varphi$ and G_φ is closed, it follows that $(x, y) \in G_\varphi$. Hence $y \in \varphi(x) \subseteq U$, which is a contradiction. We conclude that φ is UHC.

We also have the following.

Theorem: If φ is open, then it is LHC.

Proof: Suppose φ is open.

Step 1: We will show that for any $y \in Y$, $\varphi_w^{-1}(\{y\})$ is open. To this end, fix some y and a sequence $\{x_n\}$ in $\varphi_w^{-1}(\{y\})^c$ with limit x. Hence $y \notin \varphi(x_n)$ for all n. Hence $(x_n, y) \in G_\varphi^c$, a closed set. Taking the limit, $(x, y) \in G_\varphi^c$ as well. Thus, $x \in \varphi_w^{-1}(\{y\})^c$, which establishes that $[\varphi_w^{-1}(\{y\})]^c$ is closed. Hence $\varphi_w^{-1}(\{y\})$ is open.

Step 2: Now we will show that for any open set U, $\varphi_w^{-1}(U)$ is open. Now fix x and an open set U such that $x \in \varphi_w^{-1}(U)$. Pick $y \in \varphi(x) \cap U$ so that $x \in \varphi_w^{-1}(\{y\})$. Since $\varphi_w^{-1}(\{y\})$ is open, there exists some $\delta > 0$ such that $B(x, \delta) \subset \varphi_w^{-1}(\{y\})$. Hence if $z \in B(x, \delta)$, then $y \in \varphi(z)$. This, in return, implies that $\varphi(z) \cap U \neq \emptyset$, or, equivalently, that $z \in \varphi_w^{-1}(U)$. The proof is complete.

We will next study sequential characterizations of UHC and LHC. Consider the following two statements.

S1: For every $x \in X$, every sequence $\{x_n\}$ converging to x and every sequence $\{y_n\}$ such that $y_n \in \varphi(x_n)$ for every n, there exists a convergent subsequence of $\{y_n\}$ with limit in $\varphi(x)$.

S2: For every $x \in X$, every sequence $\{x_n\}$ converging to x and every $y \in \varphi(x)$, there exists a sequence $\{y_n\}$ converging to y such that $y_n \in \varphi(x_n)$ for every n.

Homework: What do these statements mean if $\varphi(x) = \{f(x)\}$ for all x and some function f?

Theorem 1: (a) If φ satisfies S1, then φ is UHC. (b) If φ is UHC and compact-valued, then φ satisfies S1.

Theorem 2: φ satisfies S2 if and only if φ satisfies LHC.

Proof of Theorem 1:

(b) Suppose that φ is UHC and compact-valued. Fix x, $\{x_n\}$ and $\{y_n\}$ such that $x_n \to x$ and $y_n \in \varphi(x_n)$ for every n. For every $m \in \mathbb{N}$, let $U_m = \cup_{z \in \varphi(x)} B(z, 1/m)$. Fix m. Since $\varphi(x) \subset U_m$ and U_m is open, by UHC, there exists $\delta_m > 0$ such that if $x' \in B(x, \delta_m)$, then $\varphi(x') \subset U_m$. Pick a subsequence x_{n_m} such that $x_{n_m} \in B(x, \delta_m)$ for every m, so that $\varphi(x_{n_m}) \subset U_m$. Hence $y_{n_m} \in U_m$ implying that $d(y_{n_m}, \varphi(x)) := \min_{z \in \varphi(x)} d(y_{n_m}, z) < 1/m$. (Recall that d is the Euclidean metric. Hence $d(y_{n_m}, \cdot) : \varphi(x) \to \mathbb{R}$ is a continuous function defined over the compact set $\varphi(x)$, and therefore $\min_{z \in \varphi(x)} d(y_{n_m}, z)$ is well-defined.) Since $\varphi(x)$ is bounded, so is U_1. Since y_{n_m} is a sequence in U_1, it has a convergent subsequence $y_{n_{m_k}}$. Let $y = \lim y_{n_{m_k}}$. Note that $d(y_{n_{m_k}}, \varphi(x)) < 1/m_k \to 0$ as $k \to \infty$. By continuity of $d(\cdot, \varphi(x))$ we have $d(y, \varphi(x)) = 0$ and since $\varphi(x)$ is closed, we have $y \in \varphi(x)$.

(a) Now suppose that φ satisfies S1 but is not UHC. Hence there exist x, U such that U is open, $\varphi(x) \subset U$, but for every n, there exists some $x_n \in B(x, 1/n)$ such that $\varphi(x_n) \backslash U \neq \emptyset$. Note $x_n \to x$. Pick a sequence y_n such that $y_n \in \varphi(x_n) \backslash U$. By S1, there exists a subsequence $\{y_{n_k}\}$ which converges to some $y \in \varphi(x)$. But $\{y_{n_k}\}$ is in the closed set U^c, hence the limit $y \in U^c$ as well. This implies that $\varphi(x) \not\subset U$, a contradiction.

The following examples show that we need the full power of compact-valuedness in part (b) of Theorem 1.

Example 1: Let $\varphi : \mathbb{R} \rightrightarrows \mathbb{R}$ be given by $\varphi(x) = \mathbb{R}$ for all x. Then φ is UHC. (Why?) Take $x = 0$, $x_n = 1/n$ and $y_n = n$ so that $x_n \to x$ and $y_n \in \varphi(x_n)$. Note however that y_n has no convergent subsequence. Hence UHC and closed-valuedness do not imply S1.

Example 2: Let $\varphi : \mathbb{R} \rightrightarrows \mathbb{R}$ be given by $\varphi(x) = (0, 1]$ for all x. Then φ is UHC. Take $x = 0$, $x_n = 1/n$ and $y_n = 1/n$ so that $x_n \to x$ and $y_n \in \varphi(x_n)$. Note however that all subsequences of y_n converge to $0 = \lim y_n$ and $0 \notin \varphi(0)$. Hence UHC and bounded-valuedness do not imply S1.

Corollary to Theorem 1: If $\varphi : X \rightrightarrows Y$ is UHC and compact-valued and $K \subseteq X$ is compact, then $\varphi(K) := \cup_{x \in K} \varphi(x)$ is compact.

(Recall: A subset S of a metric space is compact if and only if every sequence in S has a subsequence that converges to a point in S.)

Proof: Suppose that φ is UHC and compact-valued and $K \subseteq X$ is compact. By the previous theorem, φ satisfies S1. Pick a sequence y_n in $\varphi(K)$. We will show that y_n has a subsequence with limit in $\varphi(K)$. By definition of $\varphi(K)$, there exists a sequence x_n in K such that $y_n \in \varphi(x_n)$ for every n. Since K is compact, x_n has a subsequence x_{n_m} with limit $x \in K$. Let y_{n_m} be the associated subsequence of y_n. Now we can use the sequential characterization of UHC to conclude that y_{n_m} has a subsequence $y_{n_{m_k}}$ which converges to some $y \in \varphi(x)$. Noting that $y_{n_{m_k}}$ is a subsequence of y_n as well and that its limit $y \in \varphi(K)$ finishes the proof.

Proof of Theorem 2: In one direction, suppose that φ is LHC. Pick x, y and $\{x_n\}$ such that $x_n \to x$ and $y \in \varphi(x)$. Note that for every $m \in \mathbb{N}$, $B(y, 1/m)$ is open and $\varphi(x) \cap B(y, 1/m) \neq \emptyset$. By LHC, for every m, there exists $\delta_m > 0$ such that for every $x' \in B(x, \delta_m)$, $\varphi(x') \cap B(y, 1/m) \neq \emptyset$. Since x_n converges to x, for every m, there exists an integer $n(m)$ such that $x_n \in B(x, \delta_m)$ for every $n \geq n(m)$. Now let $n_1 = n(1)$ and $n_m = \max\{n_{m-1}, n(m)\} + 1$ for every $m > 1$. Hence n_m strictly increases in m. This means that $\{x_{n_m}\}$ is a subsequence of $\{x_n\}$ with the property that if $n \geq n_m$, then $\varphi(x_n) \cap B(y, 1/m) \neq \emptyset$. (Why? Note that $n_m \geq n(m)$. Hence $x_n \in B(x, \delta_m)$.) Now we **can** choose a sequence y_n as follows:

$$\begin{aligned} y_n &\in \varphi(x_n) \text{ if } n < n_1 \\ y_n &\in \varphi(x_n) \cap B(y, 1/m) \text{ if } n_m \leq n < n_{m+1} \text{ for every } m. \end{aligned}$$

Note that $y_n \in \varphi(x_n)$ for every n and $y_n \to y$ as desired.

In the other direction, suppose that φ satisfies S2 but is not LHC. Hence there exist x and U such that U is open and $\varphi(x) \cap U \neq \emptyset$, yet there is a sequence $x_n \to x$ such that $\varphi(x_n) \cap U = \emptyset$ for every n. Pick $y \in \varphi(x) \cap U$. There must, then, exist a sequence $y_n \to y$ with $y_n \in \varphi(x_n)$ for every n. Since U is open there exists some n' such that $y_n \in U$ for every $n > n'$. Then $y_n \in \varphi(x_n) \cap U$ for every $n > n'$, a contradiction.

Application: The Maximum Theorem

Let T and X be Euclidean sets. Let $\varphi : T \rightrightarrows X$ and $f : X \times T \to \mathbb{R}$. For every $t \in T$, consider the problem

$$\max_{x \in \varphi(t)} f(x,t).$$

Define

$$\mu(t) = \arg\max_{x \in \varphi(t)} f(x,t) = \{x \in \varphi(t) : \forall x' \in \varphi(t)\ f(x,t) \geq f(x',t)\}.$$

Suppose that $\mu(t) \neq \emptyset$ for every t. (This follows, for instance, if f is continuous and φ is compact-valued by WT.) We can define a function $g : T \to \mathbb{R}$ as follows:

$$g(t) = f(x,t) \text{ for some } x \in \mu(t).$$

Theorem: (The Maximum Theorem) If φ is compact-valued, UHC and LHC and if f is continuous, then (1) $\mu : T \rightrightarrows X$ is compact-valued and UHC, and (2) $g : T \to \mathbb{R}$ is continuous.

Proof: Suppose that φ is compact-valued, UHC and LHC and f is continuous.

Step 1: By Weierstrass Theorem, $\mu(t) \neq \emptyset$ for all t. Hence $\mu : T \rightrightarrows X$ and $g : T \to \mathbb{R}$.

Step 2: We will show that μ is compact-valued. Pick any t. Since $\mu(t) \subseteq \varphi(t)$ for all t and φ is compact-valued, it suffices to show that $\mu(t)$ is closed. Take a sequence $\{x_n\}$ in $\mu(t)$ with limit x. We need to show that $x \in \mu(t)$. Note that $\{x_n\}$ is also in $\varphi(t)$ and, since $\varphi(t)$ is closed, $x \in \varphi(t)$. If $x' \in \varphi(t)$, then $f(x_n, t) \geq f(x', t)$ for every n. Since f is continuous, $f(x, t) \geq f(x', t)$ as well and therefore $x \in \mu(t)$.

Step 3: We will show that μ is UHC. It suffices to show that μ satisfies the sequential property S1. Take $t \in T$, a sequence $\{t_n\}$ with $\lim t_n = t$ and a sequence $\{x_n\}$ such that $x_n \in \mu(t_n)$ for every n. We need to find a convergent subsequence of $\{x_n\}$ with limit in $\mu(t)$. Note that $x_n \in \varphi(t_n)$ for every n. Since φ is UHC and compact-valued, it satisfies S1, i.e., $\{x_n\}$ contains a subsequence $\{x_{n_m}\}$ with limit $x \in \varphi(t)$. We will show that in fact such $x \in \mu(t)$ to finish the argument.

To that end take any $z \in \varphi(t)$. Since φ is LHC, there exists a sequence $\{z_n\}$ such that $z_n \in \varphi(t_n)$ for all n and $\lim z_n = z$. Hence $f(x_{n_m}, t_{n_m}) \geq f(z_{n_m}, t_{n_m})$ for all m. Using the continuity of f and taking limits, we get $f(x, t) \geq f(z, t)$. This establishes that $x \in \mu(t)$, as desired.

Step 4: We will show that g is upper and lower semi-continuous. Let α be a number.

(a) For LSC, we need to show that $\{s \in T : g(s) \leq \alpha\}$ is closed. Take any convergent sequence $\{t_n\}$ such that $g(t_n) \leq \alpha$ for all n, with limit t. We will argue that $g(t) \leq \alpha$ as well. Choose $z \in \varphi(t)$. Since φ is LHC, there is a sequence $\{z_n\}$ which converges to z such that $z_n \in \varphi(t_n)$ for all n. Hence $f(z_n, t_n) \leq g(t_n) \leq \alpha$. Taking limits and using the continuity of f, $f(z, t) \leq \alpha$ as well. Since z was arbitrary, $g(t) \leq \alpha$ as desired.

(b) For USC, we need to show that $\{s \in T : g(s) \geq \alpha\}$ is closed. Take any convergent sequence $\{t_n\}$ such that $g(t_n) \geq \alpha$ for all n, with limit t. We will show that $g(t) \geq \alpha$ as well. Let $\{x_n\}$ be a sequence such that $x_n \in \mu(t_n)$ for all n. (Recall, $\mu(t_n)$ is nonempty.) As we showed in Step 3, μ satisfies the sequential property S1. Hence $\{x_n\}$ has a subsequence $\{x_{n_m}\}$ which converges to some $x \in \mu(t)$. Note, $f(x_{n_m}, t_{n_m}) = g(t_{n_m}) \geq \alpha$ for every m. Taking limit $f(x, t) \geq \alpha$ as well. The proof is complete.

Example:

For every $p \in \mathbb{R}^n_{++}$ and $w \in \mathbb{R}_{++}$, let $B(p, w) = \{x \in \mathbb{R}^n_+ : p \cdot x \leq w\}$. Hence $B : \mathbb{R}^{n+1}_{++} \rightrightarrows \mathbb{R}^n_+$ is UHC, LHC and compact-valued. (Why?) Let $u : \mathbb{R}^n_+ \to \mathbb{R}$ be continuous. Consider the problem

$$\max_{x \in B(p,w)} u(x)$$

for every (p, w). Let $d(p, w) = \arg\max_{x \in B(p,w)} u(x)$. Then $d : \mathbb{R}^{n+1}_{++} \rightrightarrows \mathbb{R}^n_+$ is compact-valued and UHC. Let $v(p, w) = \max_{x \in B(p,w)} u(x)$. Then $v : \mathbb{R}^{n+1}_{++} \to \mathbb{R}$ is continuous. (d is the demand correspondence, and v is the indirect utility function.)