Game Theory

Bayesian Games

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A **complete-information game** is a game whose game structure is common knowledge among all players.[1] In contrast, an **incomplete-information game** is a game whose game structure—for example, the payoff structure—is not common knowledge among all players.

In this note, we study a **Bayesian game**, which is a tractable model of incomplete-information games proposed by Harsanyi (1967, 1968a,b).

1 Example

Example 1 (Akerlof's (1970) market for lemons[2])**.** There are two players, called Seller (S) and Buyer (B). S owns a used car, and B is a potential buyer of that car. S knows a quality θ of the car but B does not. Specifically, we assume that quality θ is uniformly distributed on the interval $\Theta = [0, 1]$, and S learns a realization θ but B does not. That is, there is **information asymmetry** between S and B.

B announces the price $p_{\mathrm{B}} \in [0, 1]$ at which B is willing to pay, and simultaneously S announces the minimum price $p_{\mathrm{S}} \in [0, 1]$, above which S is willing to sell her car. Trade occurs if and only if $p_{\mathrm{B}} \geq p_{\mathrm{S}}$, and the car is traded at B's price p_{B}. Payoffs depend on the quality and the trade price (if trade occurs). The car of quality θ is worth θ to S and worth $k\theta$ to B with a known parameter $k \in (1, 2)$. Hence, if trade occurs at price p, S's payoff is $p - \theta$ and B's payoff is $k\theta - p$.

This setting can be represented in extensive form:

1. Nature draws a car quality θ.
   - θ is called a **state of nature**.
2. S learns a realized state θ but B does not.
   - S's knowledge is represented by an information set $\{\theta\}$, called as S's **type**.
3. S and B simultaneously make their trade decision.

Figure 1 illustrates the game tree (with many edges omitted). On the left edge, we have $p_{\mathrm{B}} \geq p_{\mathrm{S}}$, so that trade occurs at price p_{B}. On the right edge, we have $p_{\mathrm{B}} < p_{\mathrm{S}}$, so that no trade occurs.

[1] That is, they know the game structure, they know they know the game structure, and so on.

[2] A bad-quality used car is called a lemon in American slang.

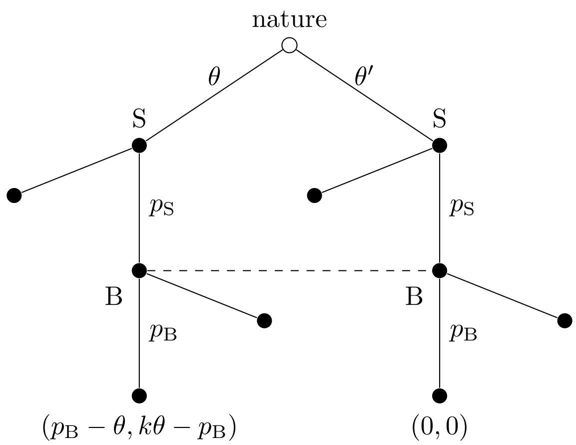

Figure 1: an extensive-form representation (with many edges omitted)

How should this game be played? We provide an informal analysis, leaving a formal analysis for another note. First, we note that S chooses a price $p_{\mathrm{S}} = \theta$ given any quality θ.[3] Second, we consider what price B would offer.

- B does not know quality θ, inferring the expected quality $\mathbb{E}[\theta] = \frac{1}{2}$. Hence, B never chooses any price $p_{\mathrm{B}} > \frac{k}{2}$. There is no trade if the car is of quality $\theta \in (\frac{k}{2}, 1]$.
- B does not know quality θ, inferring the expected quality $\mathbb{E}[\theta \mid \theta \leq \frac{k}{2}] = \frac{k}{4}$. Hence, B never chooses any price $p_{\mathrm{B}} > \frac{k^2}{4}$. There is no trade if the car is of quality $\theta \in (\frac{k^2}{4}, \frac{k}{2}]$.
- B does not know quality θ, inferring the expected quality $\mathbb{E}[\theta \mid \theta \leq \frac{k^2}{4}] = \frac{k^2}{8}$. Hence, B never chooses any price $p_{\mathrm{B}} > \frac{k^3}{8}$. There is no trade if the car is of quality $\theta \in (\frac{k^3}{8}, \frac{k^2}{4}]$.

This argument continues ad infinitum.

Who will be in the market? Only S of quality $\theta = 0$ will be. In this market, only the worst-quality car could be traded. This phenomenon that bad-quality products are selected under information asymmetry is called **adverse selection**. $\square$

2 Bayesian Games

2.1 Bayesian Games as Representation of Incomplete-Information Games

As suggested by Example 1, if we encapsulate all relevant uncertainty in players' types, then we can represent an incomplete-information game as an imperfect-information game of complete information (Harsanyi, 1967, 1968a,b):

[3] Here is the intuition. First, we show that $p_{\mathrm{S}} \geq \theta$. Since S does not sell any car worth θ (to S) at any price lower than that. Second, we show that $p_{\mathrm{S}} \leq \theta$. Suppose, by negation, that S chooses a price $p_{\mathrm{S}} > \theta$. Then, if B chooses a price $p_{\mathrm{B}} \in (\theta, p_{\mathrm{S}})$, S misses the opportunity of profitable trade; besides, if B chooses a price $p_{\mathrm{B}} \in [p_{\mathrm{S}}, 1]$, trade occurs and S gains payoff $p_{\mathrm{B}} - \theta$, independent of price p_{S}.

> A Bayesian game can be represented as a complete- but imperfect-information game such that each player i learns her type t_i, which nature draws according to given priors.

To formulate Bayesian games, we need no new machinery. We can reuse the framework of (extensive-form) games of imperfect information.

Remark 1. The extensive-form game of Figure 1 is reminiscent of an extensive-form game with a correlation device. However, the former is more general than the latter because the former allows the payoffs of S and B to depend on a state θ but the latter does not.[4] $\square$

Bayesian Games Next, we formalize a Bayesian game in normal-form-like representation:

Definition 1. A **Bayesian game** is a tuple $\mathcal{G} = \langle I, \Omega, \Theta, (T_i, \mathbb{P}_i, A_i, u_i)_i \rangle$ such that:

1. I is the set of players.
2. $\Omega \subset \Theta \times T$ is the set of **states of the world**, with a typical element ω.
3. Θ is the set of **states of nature**, with a typical element θ.
4. T_i is the set of player i's **types**, with a typical element t_i.
    - $T = \prod_i T_i$ is the set of type profiles, with a type profile $t = (t_i)_i$.
5. $\mathbb{P}_i \in \Delta(\Omega)$ is player i's prior over the states of the world.
    - $\mathbb{P}_i(\cdot \mid t_i) \in \Delta(\Omega)$ is the posterior of player i of type t_i.
6. A_i is player i's action space.
    - $A = \prod_i A_i$ is the set of action profiles.
7. $u_i : A \times \Omega \to \mathbb{R}$ is the payoff function for player i.

Remark 2. In some applications, we omit a state space Θ or player i's type space T_i. In Example 1, we can identify a state θ and S's type t_{S}, and we can omit B's type t_{B}. $\square$

Strategies In a Bayesian game $\mathcal{G}$, player i's choice of action may depend on her type t_i.

Definition 2. In a Bayesian game $\mathcal{G}$, a **pure strategy** for player i is a function $s_i : T_i \to A_i$. In particular, a **mixed strategy** for player i is a function $\sigma_i : T_i \to \Delta(A_i)$.[5]

Remark 3. Player i's choice of action may depend on her type t_i. This corresponds to the setting that in an extensive-form game, her choice of action may depend on her information set. $\square$

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[4] Historically, Bayesian games were proposed earlier than correlation equilibrium for normal-form games.

[5] The functions s_i and σ_i must be measurable (with respect to given sigma-algebras), but we disregard such measure-theoretic subtleties.

Example 2. Example 1 is formalized as a Bayesian game $\mathcal{G}$ such that:

1. $I = \{\mathrm{S}, \mathrm{B}\}$ is the set of players.
2. $\Theta = [0, 1]$ is the set of states of nature.
3. $\mathbb{P}_{\mathrm{S}} \in \Delta(\Theta)$ and $\mathbb{P}_{\mathrm{B}} \in \Delta(\Theta)$ are S and B's priors respectively.
   - Both S and B share the same prior: $\mathbb{P}_{\mathrm{S}} = \mathbb{P}_{\mathrm{B}}$.
4. S learns a state θ as her type θ, and B learns nothing.
5. $A_{\mathrm{S}} = [0, 1]$ and $A_{\mathrm{B}} = [0, 1]$ are the set of prices that S and B choose respectively.
   - p_{S} is the minimum price above which S is willing to sell her car
   - p_{B} is the price at which B is willing to pay.
6. u_{S} and u_{B} are payoff functions for S and B respectively.

Then, S's pure strategy is a function $\sigma_{\mathrm{S}} : \Theta \to [0, 1]$, where $\sigma_{\mathrm{S}}(\theta)$ denotes S's announced price when she learns state θ, while B's pure strategy is a number $p_{\mathrm{B}} \in [0, 1]$. □

2.2 Common Prior Assumption

Definition 1 allows players to possess different priors. However, many economic models, including Example 1, assume that all players have the same prior. This assumption is called the common prior assumption defined below:

Definition 3. In a Bayesian game $\mathcal{G}$, a **common prior** is a prior $\mathbb{P} \in \Delta(\Omega)$ such that $\mathbb{P} = \mathbb{P}_i$ for each $i \in I$.[6]

The common prior assumption means that differences in beliefs are **only** due to differences in information (encapsulated by players' types), rather than differences in prior beliefs. If players learn precisely the same information then they must have precisely the same beliefs. As discussed in the remark below, the common prior assumption is, in fact, a strong assumption, but it will be maintained in most applications.

Remark 4. The common prior assumption may seem reasonable, but it is a strong assumption. There are important economic phenomena that cannot be captured by any common-prior Bayesian games. For example, Aumann (1976) shows that players with a common prior cannot agree to disagree. That is, if at some state of the world, it is commonly known that player 1 assigns probability p_1 to an event E and player 2 assigns probability p_2 to the same event E, then it must be that $p_1 = p_2$. Milgrom & Stokey (1982) show the No-Trade Theorem: Rational players with a common prior cannot believe that they both can gain from a trade or a bet between them. □

[6] The common-prior assumption is sometimes called the Harsanyi doctrine.

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