RECURSIVE REPRESENTATION AND DYNAMIC PROGRAMMING

Within the Neoclassical Growth Model, we can formulate the social planner problem in a recursive language:

- Given a set of *state variables* today, the social planner chooses decision rules which determine the state of the economy tomorrow

- These decision rules determine the *value* (or lifetime utility) of starting in a given state today

In each period t, we can decompose lifetime utility

$$\underbrace{\sum_{s=0}^{\infty} \beta^s u\left(c_{t+s}\right)}_{v_t} = u\left(c_t\right) + \beta \underbrace{\sum_{s=0}^{\infty} \beta^s u\left(c_{t+1+s}\right)}_{v_{t+1}}$$

The only thing that differentiates the period t from $t+1$ is the stock of capital

If the problem is recursive, we can write

$$v\left(k_t\right) = u\left(c\left(k_t\right)\right) + \beta v\left(k'\left(k_t\right)\right)$$

where $c\left(k_t\right)$ and $k'\left(k_t\right)$ are optimal decision rules for the consumption and tomorrow's capital which depend solely on the state of the economy (today's capital k_t)

`Recursive Social Planner's Problem`

The social planner chooses functions $v(k)$, $c(k)$, $i(k)$, $k'(k)$ which solve the Bellman equation:

$$v(k) = \max_{c,i,k'} \left\{ u(c) + \beta v\left(k'\right) \right\}$$

$$\begin{aligned} s.t. \quad c + i &= f(k) \\ k' &= (1-\delta) k + i \end{aligned}$$

for all $k > 0$

This is a *functional* equation in v

(for simplicity, we are assuming no population growth nor technological change)

The *Principle of Optimality* guarantees that the solution to this problem is equivalent to solving the sequential problem:

$$c_t = c(k_t) \qquad k_{t+1} = k'(k_t)$$

plus

$$v(k_0) = \sum_{t=0}^{\infty} \beta^t u(c_t)$$

Starting from a given k_0 , we can build recursively

$$k_1 = k'(k_0) \qquad k_2 = k'(k_1) = k'\left(k'(k_0)\right) \qquad ...$$

Dynamic Programming Overview

Define the following notation:

- state x: column vector with n components
- state space X : subset of R^n
- control y: column vector with n components
- return function $F : X \times X \to R$
- correspondence $\Omega : X \to X$

We define the value function $v : X \to R$ as the solution to the Bellman equation:

$$v\left(x\right) = \max_{y} \quad \left\{F\left(x, y\right) + \beta v\left(y\right)\right\}$$

$$s.t. \quad y \in \Omega\left(x\right)$$

$\forall x \in X$

Also, we define the decision rule $g : X \to X$:

$$g\left(x\right) = \arg\max_{y} \quad \left\{F\left(x, y\right) + \beta v\left(y\right)\right\}$$

$$s.t. \quad y \in \Omega\left(x\right)$$

$\forall x \in X$, such that: $v\left(x\right) = F\left(x, g\left(x\right)\right) + \beta v\left(g\left(x\right)\right)$

Suppose that:

(i) X is a convex set

(ii) $\Omega(x)$ is compact and nonempty, $\forall x \in X$

(iii) Ω is convex and continuous

(iv) F is bounded and continuous

(v) $\beta < 1$

In many applications, including different versions of the Neoclassical Growth Model, restrictions on technology and preferences ensure that these conditions are met

We now define the operator $T : B(X) \to B(X)$, where $B(X)$ is a set of bounded functions in X, as:

$$T[f(x)] = \max_{y} \quad \{F(x,y) + \beta f(y)\}$$

$$s.t. \quad y \in \Omega(x)$$

T is a functional operator over the metric space $B(X)$, with the norm

$$\|f_1 - f_2\| = \sup_{x \in X} |f_1(x) - f_2(x)|$$

By construction, the value function v that solves the Bellman equation is a fixed point of the operator T ($v = Tv$)

Result 1: (*Contraction mapping*)

If conditions (i)-(v) are met, then the operator T is a *contraction* with module β; in other words

$$\|Tf_1 - Tf_2\| \leq \beta \|f_1 - f_2\| \qquad , \forall f_1, f_2 \in B(X)$$

In simple terms, a contraction is a function (or operator) that shortens distances

For example, the real function $h : [0,1] \to [0,1]$ is a contraction if

$$|h(x) - h(y)| \leq \kappa |x - y|$$

that is, if the slope of the function is less than a constant $\kappa < 1$ called contraction module

As we can see clearly in this example, a contraction has a single fixed point $x^* = h(x^*)$, which can be reached iteratively from any $x^0 \in [0,1]$ and calculating $x^{n+1} = h(x^n)$

The *Banach Theorem* generalizes this result to complete metric spaces, such as the one we are analyzing

Corollary *(Existence and uniqueness of the value function):*

If conditions (i)-(v) are met, the value function v exists and is unique. Additionally, starting from *any* $v^0 \in B(X)$, the sequence v^n constructed as:

$$\begin{aligned} v^1 &= Tv^0 \\ v^2 &= Tv^1 = T^2v^0 \\ &\ldots\ldots\ldots\ldots \end{aligned}$$

converges to the function v, the only fixed point of the operator T

This corollary is the basis of the numerical iteration method of the value function

The previous corollary implies that if the operator T preserves a certain property (for example, continuity), the value function will also have this property. With that argument, we can demonstrate:

Corollary 2: (*Properties of the value function*)

If conditions (i)-(v) are met, plus (vi) F is strictly concave, then the value function v is continuous, bounded and strictly concave

For instance, in the case of concavity, we only need to show that f weakly concave implies that Tf is strictly concave

Let $x_1, x_2 \in X$, $0 < \alpha < 1$, and $\hat{x} = \alpha x_1 + (1-\alpha) x_2$; also let $y_1 = g(x_1)$ and $y_2 = g(x_2)$ attain the maximum for x_1, x_2, and define $\hat{y} = \alpha y_1 + (1-\alpha) y_2$

Then, by strict concavity of F and weak concavity of f,

$$\begin{aligned} F(\hat{x}, \hat{y}) + \beta f(\hat{y}) &> \alpha [F(x_1, y_1) + \beta f(y_1)] \\ &\quad + (1-\alpha)[F(x_2, y_2) + \beta f(y_2)] \\ &= \alpha T f(x_1) + (1-\alpha) T f(x_2) \end{aligned}$$

But, by definition,

$$Tf(\hat{x}) \geq F(\hat{x}, \hat{y}) + \beta f(\hat{y})$$

so the desired result follows

$$Tf(\hat{x}) > \alpha T f(x_1) + (1-\alpha) T f(x_2)$$

What about the properties of the optimal policy rule $g(x)$?

Result 2: (*Properties of the optimal policy rule*)

> If conditions (i)-(vi) are met, the optimal decision rule g exists and is unique; plus, $g(x)$ is continuous in x

The existence of $g : X \to X$ follows form *Weierstrass Theorem*, once it has been proven that the value function v is continuous and bounded

... the uniqueness of g comes from the strict concavity of v

... the continuity of g is an application of the *Maximum Theorem*

Finally, we want to know in which cases the value function is differentiable, which would allow us working with the first order conditions

Result 3: (*Differentiability of the value function*)

> If conditions (i)-(v) are met and (vii) F is continuously differentiable, then, for each $x^0 \in int(X)$ with $g\left(x^0\right) \in int\left(\Omega\left(x^0\right)\right)$, the value function v is continuously differentiable in x^0 and its derivatives can be found according to:
>
> $$\frac{\partial v}{\partial x_i}\left(x^0\right) = \frac{\partial F}{\partial x_i}\left(x^0, g\left(x^0\right)\right)$$

Benveniste and Scheinkman propose this set of conditions

`First Order Conditions`

Coming back to the recursive social planner's problem, we have:

$$v(k) = \max_{k'} \quad \left\{u\left[f(k) + (1-\delta)k - k'\right] + \beta v\left(k'\right)\right\}$$

$$s.t. \quad k' \in [0, f(k) + (1-\delta)k]$$

a particular case of the problem described earlier with

$$x = k \qquad y = k' \qquad X = [0, k_{\max}]$$

$$F(x,y) = u[f(x) + (1-\delta)x - y] \qquad \Omega(x) = [0, f(x) + (1-\delta)x]$$

where $k_{\max}$ is the maximum sustainable level of capital that satisfies $f(k_{\max}) = \delta k_{\max}$

Solving this problem, the first order conditions for an internal solution are:

$$\frac{\partial}{\partial k'}\left(k, k'\right) = -u'\left[f\left(k\right) + \left(1-\delta\right)k - k'\right] + \beta v'\left(k'\right) = 0$$

but, using Benveniste-Scheinkman:

$$v'\left(k'\right) = u'\left[f\left(k'\right) + \left(1-\delta\right)k' - k''\right]\left(f'\left(k'\right) + \left(1-\delta\right)\right)$$

Replacing and simplifying, we get Euler's equation:

$$\frac{u'\left[f\left(k\right) + \left(1-\delta\right)k - k'\right]}{\beta u'\left[f\left(k'\right) + \left(1-\delta\right)k' - k''\right]} = f'\left(k'\right) + \left(1-\delta\right)$$

Notice the similarity with the Euler equation obtained from the sequential problem

Stationary Equilibrium

In recursive language, a steady equilibrium is a value for k^* such that:

$$k^* = k'\left(k^*\right)$$

where k' is the optimal decision rule of the social planner

Using the Euler equation:

$$k^* = \left(f'\right)^{-1}\left(\frac{1}{\beta} - \left(1-\delta\right)\right)$$

The Inada conditions over f guarantee the existence of a unique steady state k^*

Moreover, we can prove that the stationary equilibrium is stable: starting from any $k_0 > 0$, the economy converges in the long run to the only steady state k^*

The steps of the proof are the following:

1. Show that the value function of the social planner$v(k)$ is concave (using the operator T)

2. Using the concavity of $v(k)$ and the first order conditions, show that the decision rule $k'(k)$ is increasing

3. The monotonicity of $k'(k)$ implies that the optimal sequence $\{k_0, k_1, ...\}$ is also monotone

4. Show that the optimal sequence $\{k_0, k_1, ...\}$ is bounded

5. Conclude, using the *Monotone Convergence Theorem*, that $\{k_0, k_1, ...\}$ converges to k^*

`Recursive Competitive Equilibrium`

To move from the problem of the social planner to the recursive equilibrium, we have to distinguish the *individual* state variable k from the *aggregate* state variable K

• The prices depend of the aggregate capital, not of the individual (*perfect competition*)

• Consumers choose the law of motion for individual capital $k'(k, K)$

... taking as given the law of the motion for aggregate capital $K' = \Gamma(K)$

In equilibrium, both laws of motion must be consistent

A Recursive Competitive Equilibrium is a set of functions $v(k, K)$, $c(k, K)$, $i(k, K)$, $k'(k, K)$, prices $w(K)$ and $r(K)$ and aggregate law of motion $\Gamma(K)$ such that:

i) For each pair (k, K), given the functions w, r and Γ, the value function $v(k, K)$ solves the Bellman equation:

$$v(k, K) = \max_{c,i,k'} \left\{ u(c) + \beta v\left(k', K'\right) \right\}$$

$$\begin{aligned} s.t. \quad c + i &= w(K) + r(K)k \\ k' &= (1-\delta)k + i \\ K' &= \Gamma(K) \end{aligned}$$

and $c(k, K)$, $i(k, K)$, $k'(k, K)$ are the optimal decision rules for this problem

ii) For each K, prices satisfy the marginal conditions (from the firm maximization problem):

$$r(K) = f'(K)$$

$$w(K) = f(K) - f'(K)K$$

iii) For each K, markets clear:

$$f(K) = c(K,K) + i(K,K)$$

iv) For each K, the aggregate law of motion is consistent with individual decisions:

$$\Gamma(K) = k'(K,K)$$

Once the recursive equilibrium is solved, starting from a given $k_0 > 0$, we can construct the sequences for the stock of capital:

$$\begin{aligned} k_1 &= k'(k_0, k_0) \\ k_2 &= k'(k_1, k_1) = k'\left(k'(k_0, k_0), k'(k_0, k_0)\right) \\ &\ldots\ldots\ldots\ldots\ldots \end{aligned}$$

and for the other variables:

$$\begin{aligned} c_t &= c(k_t, k_t) \qquad i_t = i(k_t, k_t) \\ w_t &= w(k_t) \qquad\quad r_t = r(k_t) \end{aligned}$$

The *Principle of Optimality* guarantees that those sequences are the same as one would get by solving the sequential competitive equilibrium