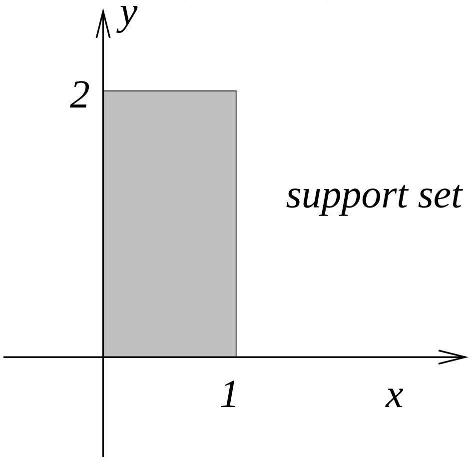
Examples: Joint Densities and Joint Mass Functions

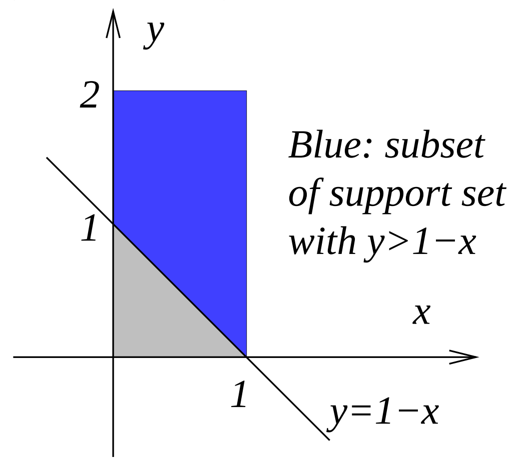
Example 1: X and Y are jointly continuous with joint pdf

$$f(x,y) = \begin{cases} cx^2 + \frac{xy}{3} & \text{if } 0 \le x \le 1,\ 0 \le y \le 2 \\ 0, & \text{otherwise.} \end{cases}$$

(a). Find c. (b). Find $P(X+Y \ge 1)$. (c). Find marginal pdf's of X and of Y. (d). Are X and Y independent (justify!). (e). Find $E(e^X \cos Y)$. (f). Find $cov(X,Y)$.

We start (as always!) by drawing the support set. (See below, left.)





(a). We find c by setting

$$1 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)dydx = \int_0^1\int_0^2 (cx^2 + \frac{xy}{3})dydx = \frac{2c}{3} + \frac{1}{3},$$

so $c = 1$.

(b). Draw a picture of the support set (a 1-by-2 rectangle), and intersect it with the set $\{(x,y) : x+y \ge 1\}$, which is the region above the line $y = 1-x$. See figure above, right. To compute the probability, we double integrate the joint density over this subset of the support set:

$$P(X+Y \ge 1) = \int_0^1\int_{1-x}^2 (x^2 + \frac{xy}{3})dydx = \frac{65}{72}$$

(c). We compute the marginal pdfs:

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)dy = \begin{cases} \int_0^2 (x^2 + \frac{xy}{3})dy = 2x^2 + \frac{2x}{3} & \text{if } 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)dx = \begin{cases} \int_0^1 (x^2 + \frac{xy}{3})dx = \frac{1}{3} + \frac{y}{6} & \text{if } 0 \le y \le 2 \\ 0 & \text{otherwise} \end{cases}$$

(d). NO, X and Y are NOT independent. The support set is a rectangle, so we need to check if it is true that $f(x,y) = f_X(x)f_Y(y)$, for all (x,y). We easily find counterexamples: $f(0.2, 0.3) \neq f_X(0.2)f_Y(0.3)$.

(e).

$$E(e^X \cos Y) = \int_0^1 \int_0^2 (e^x \cos y)(x^2 + \frac{xy}{3})dydx$$

(f).

$$cov(X,Y) = E(XY) - E(X)E(Y)$$

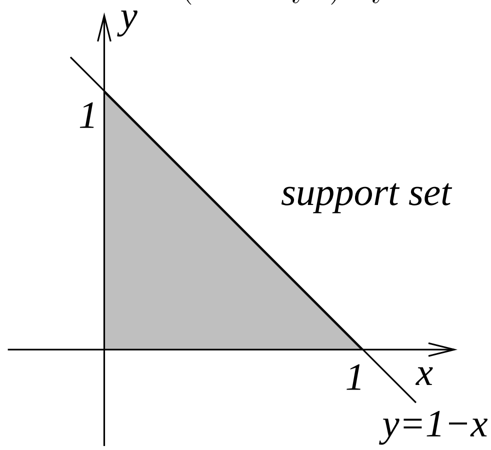
$$= \int_0^1 \int_0^2 xy(x^2 + \frac{xy}{3})dydx - \left[\int_0^1 \int_0^2 x(x^2 + \frac{xy}{3})dydx\right]\left[\int_0^1 \int_0^2 y(x^2 + \frac{xy}{3})dydx\right]$$

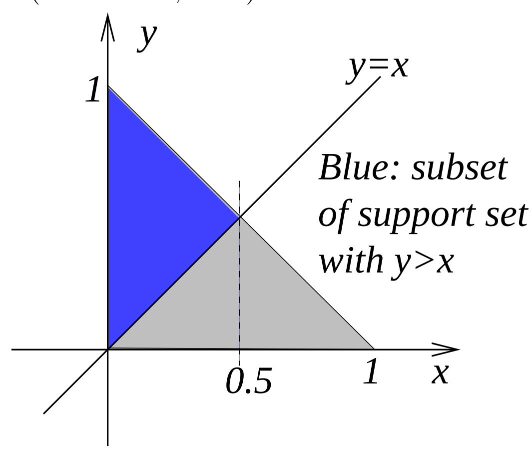
Example 2: X and Y are jointly continuous with joint pdf

$$f(x,y) = \begin{cases} cxy & \text{if } 0 \leq x,\ 0 \leq y,\ x+y \leq 1 \\ 0, & \text{otherwise.} \end{cases}$$

(a). Find c. (b). Find $P(Y > X)$. (c). Find marginal pdf's of X and of Y. (d). Are X and Y independent (justify!).

We start (as always!) by drawing the support set. (See below, left.)





(a). We find c by setting

$$1 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y)dydx = \int_0^1 \int_0^{1-x} cxydydx = \frac{c}{24},$$

so $c = 24$.

(b). Draw a picture of the support set (a triangle), and intersect it with the set $\{(x,y) : y \geq x\}$, which is the region above the line $y = x$; this yields a triangle whose leftmost x-value is 0 and whose rightmost x-value is $1/2$ (which is only seen by drawing the figure!). See figure above, right. To compute the probability, we double integrate the joint density over this subset of the support set:

$$P(Y \geq X) = \int_0^{1/2} \int_x^{1-x} 24xydydx$$

(c). We compute the marginal pdfs:

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)dy = \begin{cases} \int_0^{1-x} 24xydy = 12x(1-x)^2 & \text{if } 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)dx = \begin{cases} \int_0^{1-y} 24xydx = 12y(1-y)^2 & \text{if } 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

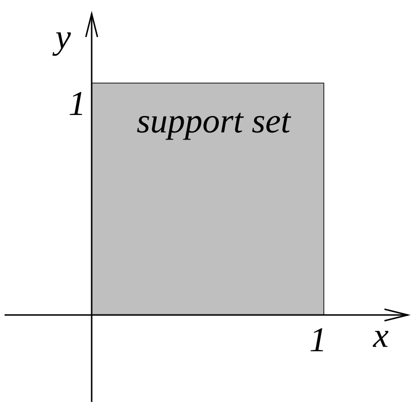
(d). NO, X and Y are NOT independent. The support set is not a rectangle or generalized rectangle, so we know we can find points (x,y) where it fails to be true that $f(x,y) = f_X(x)f_Y(y)$. In particular, $f(0.7, 0.7) = 0 \ne f_X(0.7)f_Y(0.7) > 0$.

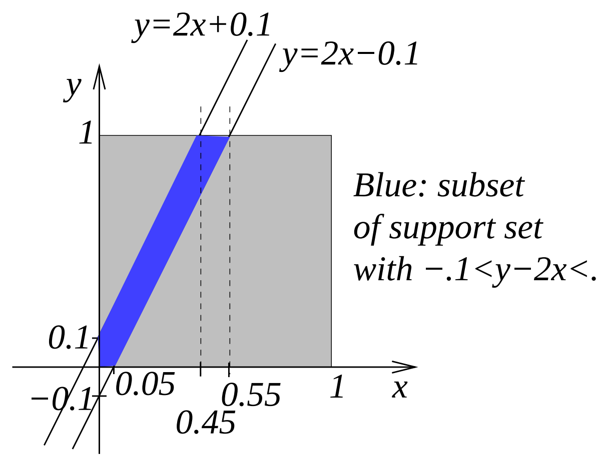
Example 3: X and Y are jointly continuous with joint pdf

$$f(x,y) = \begin{cases} cxy & \text{if } 0 \le x \le 1,\ 0 \le y \le 1 \\ 0, & \text{otherwise.} \end{cases}$$

(a). Find c. (b). Find $P(|Y-2X| \le 0.1)$. (c). Find marginal pdf's of X and of Y. (d). Are X and Y independent (justify!).

We start (as always!) by drawing the support set, which is just a unit square in this case. (See below, left.)





(a). We find c by setting

$$1 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)dydx = \int_0^1\int_0^1 cxydydx = \frac{c}{4},$$

so $c = 4$.

(b). Draw a picture of the support set (unit square), and intersect it with the set $\{(x,y) : |y-2x| \le 0.1\} = \{(x,y) : -0.1 \le y-2x \le 0.1\} = \{(x,y) : 2x-0.1 \le y \le 2x+0.1\}$, which is the region above the line $y = 2x-0.1$ and below the line $y = 2x+0.1$. See figure above, left. (You will not be able to figure out the limits of integration without it!) To compute the probability, we double integrate the joint density over this subset of the support set:

$$P(|Y-2X| \le 0.1) = \int_0^{0.1}\int_0^{(y+0.1)/2} 4xydxdy + \int_{0.1}^{1}\int_{(y-0.1)/2}^{(y+0.1)/2} 4xydxdy$$

(c). We compute the marginal pdfs:

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)dy = \begin{cases} \int_0^1 4xydy = 2x & \text{if } 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)dx = \begin{cases} \int_0^1 4xydx = 2y & \text{if } 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

(d). YES, X and Y are independent, since

$$f_X(x)f_Y(y) = \begin{cases} 2x \cdot 2y = 4xy & \text{if } 0 \le x \le 1 \text{ and } 0 \le y \le 1 \\ 0 otherwise \end{cases}$$

is exactly the same as $f(x,y)$, the joint density, for all x and y.

Example 4: X and Y are independent continuous random variables, each with pdf

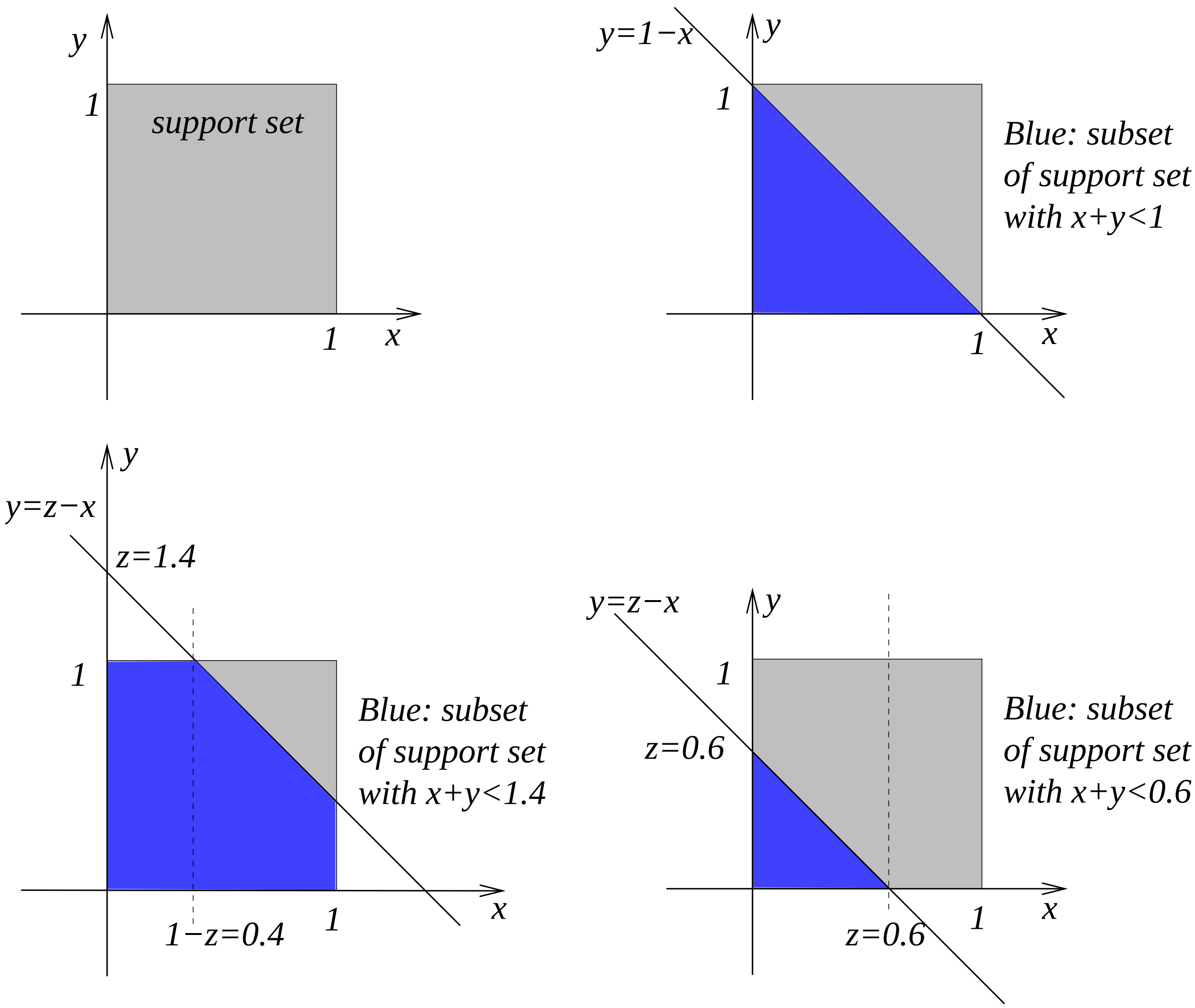
$$g(w) = \begin{cases} 2w & \text{if } 0 \le w \le 1 \\ 0, & \text{otherwise.} \end{cases}$$

(a). Find $P(X+Y \le 1)$. (b). Find the cdf and pdf of $Z = X+Y$.

Since X and Y are independent, we know that

$$f(x,y) = f_X(x)f_Y(y) = \begin{cases} 2x \cdot 2y & \text{if } 0 \le x \le 1 \text{ and } 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

We start (as always!) by drawing the support set, which is a unit square in this case. (See below, left.)



Case: 1<z<2 Case: 0<z<1

(a). Draw a picture of the support set (unit square), and intersect it with the set $\{(x, y) : x+y \leq 1\}$, which is the region below the line $y = 1-x$. See figure above, right. To compute the probability, we double integrate the joint density over this subset of the support set:

$$P(X+Y \leq 1) = \int_0^1 \int_0^{1-x} 4xydydx = \frac{1}{6}$$

(b). Refer to the figure (lower left and lower right). To compute the cdf of $Z = X + Y$, we use the definition of cdf, evaluating each case by double integrating the joint density over the subset of the support set corresponding to $\{(x, y) : x + y \leq z\}$, for different cases depending on the value of z:

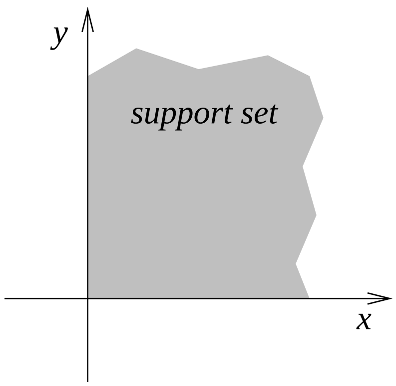
$$F_Z(z) = P(Z \leq z) = P(X+Y \leq z) = P(Y \leq -X+z)$$
$$= \begin{cases} 0 & \text{if } z \leq 0 \\ \int_0^z \int_0^{z-x} 4xydydx & \text{if } 0 \leq z \leq 1 \\ \int_0^{1-z} \int_0^1 4xydydx + \int_{1-z}^1 \int_0^{z-x} 4xydydx & \text{if } 1 \leq z \leq 2 \\ 1 & \text{if } z \geq 2 \end{cases}$$

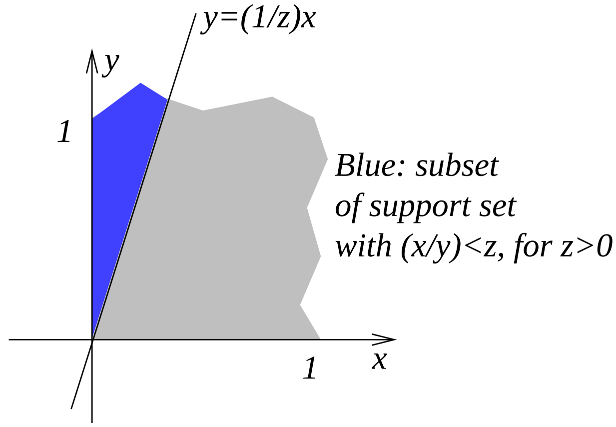
Example 5: X and Y are jointly continuous with joint pdf

$$f(x,y) = \begin{cases} e^{-(x+y)} & \text{if } 0 \le x,\ 0 \le y \\ 0, & \text{otherwise.} \end{cases}$$

Let $Z = X/Y$. Find the pdf of Z.

The first thing we do is draw a picture of the support set (which in this case is the first quadrant); see below, left.





To find the density, $f_Z(z)$, we start, as always, by finding the cdf, $F_Z(z) = P(Z \le z)$, and then differentiating: $f_Z(z) = F'_Z(z)$. Thus, using the definition, and a picture of the support set, we start by handling the cases,

$$F_Z(z) = P(Z \le z) = P(X/Y \le z)$$
$$= \begin{cases} 0 & \text{if } z < 0 \\ P(Y \ge (1/z)X) & \text{if } z > 0, \end{cases}$$

where we have used the fact that X and Y are both nonnegative (with probability 1), so multiplying both sides of the inequality by Y does not flip the inequality; note, however, that when we divide both sides by z, to obtain $Y \ge (1/z)X$, we were making the assumption that $z > 0$ (otherwise the inequality would flip).

Now, we consider the picture of the support set, together with the halfplane specified by $y \ge (1/z)x$; see the figure above, right. We double integrate the joint density over the portion of the support set where $y \ge (1/z)x$, obtaining

$$F_Z(z) = P(Z \le z) = P(X/Y \le z)$$
$$= \begin{cases} 0 & \text{if } z < 0 \\ P(Y \ge (1/z)X) & \text{if } z > 0, \end{cases}$$
$$= \begin{cases} 0 & \text{if } z < 0 \\ \int_0^\infty \int_{(1/z)x}^\infty e^{-(x+y)} dy dx = \frac{z}{z+1} & \text{if } z > 0. \end{cases}$$

Then, to get the pdf, we take the derivative:

$$f_Z(z) = \begin{cases} 0 & \text{if } z < 0 \\ \frac{(z+1)\cdot 1 - z \cdot 1}{(z+1)^2} = \frac{1}{(z+1)^2} & \text{if } z > 0 \end{cases}$$

Example 6: X and Y are independent, each with an exponential(λ) distribution. Find the density of $Z = X + Y$ and of $W = Y - X^2$.

Since X and Y are independent, we know that $f(x,y) = f_X(x)f_Y(y)$, giving us

$$f(x,y) = \begin{cases} \lambda e^{-\lambda x}\lambda e^{-\lambda y} & \text{if } x, y \geq 0 \\ 0 & \text{otherwise} \end{cases}.$$

The first thing we do is draw a picture of the support set: the first quadrant.

(a). To find the density, $f_Z(z)$, we start, as always, by finding the cdf, $F_Z(z) = P(Z \leq z)$, and then differentiating: $f_Z(z) = F_Z'(z)$. Thus, using the definition, and a picture of the support set together with the halfplane $y \leq -x + z$, we get

$$F_Z(z) = P(Z \leq z) = P(X + Y \leq z) = P(Y \leq -X + z)$$

$$= \begin{cases} 0 & \text{if } z < 0 \\ \int_0^z \int_0^{z-x} \lambda e^{-\lambda x}\lambda e^{-\lambda y} dy dx = 1 - e^{-\lambda z} - \lambda z e^{-\lambda z} & \text{if } z \geq 0 \end{cases}$$

This gives the pdf,

$$f_Z(z) = \begin{cases} 0 & \text{if } z < 0 \\ \lambda^2 z e^{-\lambda z} & \text{if } z \geq 0 \end{cases},$$

which is the pdf of a Gamma(2,λ). Thus, Z is Gamma(2,λ) random variable.

(b). To find the density, $f_W(w)$, we start, as always, by finding the cdf, $F_W(w) = P(W \leq w)$, and then differentiating: $f_W(w) = F_W'(w)$. Thus, using the definition, and a picture of the support set together with the region specified by $y \leq x^2 + w$, we get

$$F_W(w) = P(W \leq w) = P(Y - X^2 \leq w) = P(Y \leq X^2 + w)$$

$$= \begin{cases} \int_0^\infty \int_0^{x^2+w} \lambda e^{-\lambda x}\lambda e^{-\lambda y} dy dx & \text{if } w > 0 \\ \int_{\sqrt{-w}}^\infty \int_0^{x^2+w} \lambda e^{-\lambda x}\lambda e^{-\lambda y} dy dx & \text{if } w < 0 \end{cases}$$

Then, we differentiate to get $f_W(w) = F_W'(w)$. (Go ahead and evaluate the integral, then take the derivative.)

Example 7: X and Y are jointly continuous with (X,Y) uniformly distributed over the union of the two squares $\{(x,y) : 0 \leq x \leq 1, 1 \leq y \leq 1\}$ and $\{(x,y) : 0 \leq x \leq 1, 3 \leq y \leq 4\}$.

(a). Find $E(Y)$. (b). Find the marginal densities of X and Y. (c). Are X and Y independent? (d). Find the pdf of $Z = X + Y$.

Solution to be provided. (possibly in class)

Example 8: X and Y have joint density

$$f(x,y) = \begin{cases} x + y & \textit{if } 0 \leq x, y \leq 1 \\ 0, & \textit{otherwise.} \end{cases}$$

Find the joint cdf, $F_{X,Y}(x,y)$, for all x and y. Compute the covariance and correlation of X and Y.

Solution to be provided. (possibly in class)

Example 9: Suppose that X and Y have joint mass function as shown in the table below. (Here, X takes on possible values in the set $\{-2, 2\}$, Y takes on values in the set $\{-2, 0, 2, 3.1\}$.)

	-2	0	2	3.1
-2	.04	.08	.12	.16
2	.06	.12	.18	.24

(a). (6 points) Compute $P(|X + Y^2| < 1)$. (b). (6 points) Find the marginal mass function of Y and plot it. (be very explicit!) (c). (6 points) Compute $var(X^2 - Y)$ and $cov(X, Y)$. (d). (2 points) Are X and Y independent? (Why or why not?)

Solution to be provided. (possibly in class)

Example 10: Two fair dice are rolled. Let X be the larger of the two values shown on the dice, and let Y be the absolute value of the difference of the two values shown. Give the joint pmf of X and Y. Compute $cov(X, Y)$, $E(X)$, $E(Y^X)$, $P(X > 2Y)$.

The sample space is the set $S = \{(1, 1), (1, 2), \ldots, (6, 6)\}$; there are 36 equally likely outcomes.

Note that $X \in \{1, 2, \ldots, 6\}$ and $Y \in \{0, 1, \ldots, 5\}$.

$p(1, 0) = P(X = 1, Y = 0) = P(\{(1, 1)\}) = 1/36$, where "(1,1)" is the outcome in which the first die is a "1" and the second die is also a "1" (so that the larger die is "1" and the difference of the two values is "0"). Similarly, $p(i, 0) = P(X = i, Y = 0) = P(\{(i, i)\}) = 1/36$, for $i = 1, 2, \ldots, 6$. in which the first die is a "i" and the second die is also a "i"

Now, $p(1, 1) = p(1, 2) = \cdots = p(1, 6) = 0$, since, if the larger of the two dice shows "1", the difference cannot be 1 or more.

Now, $p(2, 1) = P(X = 2, Y = 1) = P(\{(2, 1), (1, 2)\}) = 2/36$. Similarly, $p(3, 1) = p(3, 2) = p(4, 1) = p(4, 2) = p(4, 3) = p(5, 1) = \cdots = p(5, 4) = p(6, 1) = \cdots = p(6, 5) = 2/36$, since each corresponding event is a subset of two outcomes from S.

All other values of $p(i, j)$ are 0. Check that the sum of all values $p(i, j)$ is 1, as it must be!

Thus, in summary,

$$p(i, j) = \begin{cases} 0 & \text{if } j \geq i, \text{ and } i \in \{1, 2, \ldots, 6\}, j \in \{0, 1, \ldots, 5\} \\ \frac{1}{36} & \text{if } (i, j) \in \{(1, 0), (2, 0), \ldots, (6, 0)\} \\ \frac{2}{36} & \text{otherwise} \end{cases}$$

(It is convenient to arrange all these numbers in a table.)

We can also compute

$$E(X) = \sum_{i=1}^{6} \sum_{j=0}^{5} i \cdot p(i, j) = \frac{161}{36},$$

$$cov(X, Y) = E(XY) - [E(X)][E(Y)] = \sum_{i=1}^{6} \sum_{j=0}^{5} ij \cdot p(i, j) - \left[\sum_{i=1}^{6} \sum_{j=0}^{5} i \cdot p(i, j)\right] \left[\sum_{i=1}^{6} \sum_{j=0}^{5} j \cdot p(i, j)\right]$$

Example 11: Alice and Bob plan to meet at a cafe to do AMS311 homework together. Alice arrives at the cafe at a random time (uniform) between noon and 1:00pm today; Bob independently arrives at a random time (uniform) between noon and 2:00pm today. (a). What is the expected amount of time that somebody waits for the other? (b). What is the probability that Bob has to wait for Alice?

Let X be the number of hours past noon that Alice arrives at the cafe. Let Y be the number of hours past noon that Bob arrives at the cafe. Then, we know that X is Uniform(0,1), and Y is Uniform(0,2). Since, by the stated assumption, X and Y are independent, we know that the joint density is given by

$$f(x,y) = \begin{cases} 1/2 & \text{if } 0 \le x \le 1,\ 0 \le y \le 2 \\ 0 & \text{otherwise} \end{cases}$$

We begin (as always) by plotting the support set: it is simply a rectangle of width 1 and height 2.

(a). Let $W = \max\{X,Y\} - \min\{X,Y\}$; then, W is the amount of time (in hours) that somebody has to wait. We want to compute $E(W)$.

Now, W is a function of X and Y. So we just use the law of the unconscious statistician:

$$E(W) = E(\max\{X,Y\} - \min\{X,Y\}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[\max\{x,y\} - \min\{x,y\}]f(x,y)dydx$$

$$= \int_0^1\int_0^2[\max\{x,y\} - \min\{x,y\}](1/2)dydx$$

Now, in order to write the function $[\max\{x,y\}-\min\{x,y\}]$ explicitly, we break into two cases: If $x < y$, then $\max\{x,y\} - \min\{x,y\} = y - x$; if $x > y$, then $\max\{x,y\} - \min\{x,y\} = x - y$. Thus, we integrate to get

$$E(W) = E(\max\{X,Y\}-\min\{X,Y\}) = \int_0^1\int_0^x(x-y)(1/2)dydx + \int_0^1\int_x^2(y-x)(1/2)dydx = \frac{1}{12} + \frac{3}{4} = \frac{5}{6}.$$

Thus, the expected time waiting is 5/6 hours (or 50 minutes).

(Note that it is wrong to reason like this: Alice expects to arrive at 12:30; Bob expects to arrive at 1:00; thus, we expect that Bob will wait 30 minutes for Alice.)

(b). We want to compute the probability that Bob has to wait for Alice, which is $P(Y < X)$, which we do by integrating the joint density, $f(x,y)$, over the region where $y < x$. Draw a picture! (Show the support set (a rectangle), and the line $y = x$.)

$$P(Y < X) = \int_0^1\int_0^x(1/2)dydx = \frac{1}{4}$$

Example 12: Suppose X and Y are independent and that X is exponential with mean 0.5 and Y has density

$$f_Y(y) = \begin{cases} 3e^{-3y} & \textit{if } y > 0 \\ 0 & \textit{otherwise} \end{cases}$$

Find the density of the random variable $W = \min\{X,Y\}$ and the random variable $Z = X + Y$.