Game Theory

Collusion

Tetsuya Hoshino

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In this note, we study **collusion** in repeated games with perfect monitoring. For example, we think of the following two possibilities:

1. In the Cournot competition, firms produce more than the monopoly level in the short-term, but in the long-term, they may collude to produce less.

2. In the Bertrand competition, firms set lower prices than the monopoly level in the short-term, but in the long-term, they may collude to set higher prices.

In what follows, we study collusion, using the model that is often used in applied economics.[1]

1 Bertrand Competition with Differentiated Products

Firms 1 and 2 are exclusive producers of differentiated products 1 and 2, respectively. Each firm i sets a price $p_i \geq 0$ for product i. Assume that the demand for product i under prices (p_i, p_{-i}) is given in the following logit form:

$$q_i(p_i, p_{-i}) = \frac{\exp\left\{\frac{a_i - p_i}{\mu}\right\}}{1 + \exp\left\{\frac{a_i - p_i}{\mu}\right\} + \exp\left\{\frac{a_{-i} - p_{-i}}{\mu}\right\}},$$

where:

- a_i is the parameter of vertical differentiation.
  - a_i can be interpreted as the product i's quality.
  - $a_i \uparrow$ implies $q_i(p_i, p_{-i}) \uparrow$.
- μ is the parameter of horizontal differentiation.
  - $\mu \to 0+$ makes the two products perfect substitutes.

Remark 1. Unlike the homogeneous product case, even if firm i sets a lower price $p_i < p_{-i}$, it does not take the entire demand in the market. Consumers may also choose to buy neither product (which is modeled by the "1" term in the denominator). Hence, $q_1(p) + q_2(p) < 1$ at a price profile $p = (p_1, p_2)$. $\square$

This game is summarized as a normal-form game $G = \langle I, (A_i, u_i)_i \rangle$ such that:

[1] You can easily find repeated Cournot/Bertrand games in many references.

1. $I = \{1, 2\}$ is the set of firms.

2. $A_i = \mathbb{R}_+$ is firm i's action space.

3. $\pi_i : A \to \mathbb{R}$ is firm i's payoff function defined by $\pi_i(p_i, p_{-i}) = (p_i - c_i)q_i(p_i, p_{-i})$.
   - $c_i \geq 0$ is firm i's constant marginal cost.

Simplification In the rest of this note, we will make the following assumption.

Assumption 1. *Assume that the firms' technologies are symmetric in the following sense:*

1. *The two products have the same quality:* $a \equiv a_1 = a_2$.

2. *The two products have the same marginal cost:* $c \equiv c_1 = c_2$.

1.1 Stage-Game Nash Equilibrium

Lemma 1. *Under Assumption 1, a symmetric Nash equilibrium* (p^N, p^N) *is characterized by the following equation:*

$$\mu = (p^N - c)\left[1 - \frac{\exp\left\{\frac{a-p^N}{\mu}\right\}}{1 + 2\exp\left\{\frac{a-p^N}{\mu}\right\}}\right].$$

For example, $p^N \approx 1.473$ *with each firm's profit* $\pi^N \equiv \pi_i(p^N, p^N) \approx 0.223$ *at the parameters* $a = 2$, $c = 1$, *and* $\mu = 0.25$.

Proof. Immediate from the first-order conditions for each firm i's profit maximization problem:

$$\max_{p_i} \quad (p_i - c)q_i(p_i, p^N_{-i}),$$

together with the symmetry condition $p_i = p_{-i}$. ■

1.2 Stage-Game Monopoly Profit

Next, we imagine what prices firms 1 and 2 would choose if they were merged into a monopoly firm. Then, the "monopolist" would choose prices to maximize the total profit.

Lemma 2. *Under Assumption 1, a symmetric "monopoly" price profile* (p_M, p_M) *that maximizes the total profit, is characterized by the following equation:*

$$\mu = (p^M - c)\left[1 - \frac{2\exp\left\{\frac{a-p^M}{\mu}\right\}}{1 + 2\exp\left\{\frac{a-p^M}{\mu}\right\}}\right].$$

For example, $p^M \approx 1.925$ *with each firm's profit* $\pi^M \equiv \pi_i(p^M, p^M) \approx 0.337$ *at the parameters* $a = 2$, $c = 1$, *and* $\mu = 0.25$.

Note (Repeated Games with Perfect Monitoring):

Definition 1. Fix a (finite or infinite) normal-form game $G = \langle I, (A_i, u_i)_i \rangle$ and a discount factor $\delta \in (0,1)$. An **(infinitely) repeated game $G^\infty(\delta)$ with perfect monitoring** is an infinite-horizon extensive-form game such that:

1. Every period t, each player i simultaneously chooses action $a_i^t \in A_i$.
2. Every period t, each player i observes the history of past play $h^t = (a^1, a^2, \dots, a^{t-1})$.
   - $h^1 = \varnothing$ denotes that each player i has no information at period 1.
   - $h^\infty = (a^1, a^2, \dots)$ denotes an infinite history of play.
3. Each player i's payoff at history h^∞ is

$$\sum_{t=1}^{\infty} \delta^{t-1} u_i(a^t). \tag{1}$$

The stage game of the repeated game $G^\infty(\delta)$ refers to the normal-form game G.

Proof. Immediate from the first-order conditions for the total profit maximization problem:

$$\max_{p_1, p_2} \quad (p_1 - c)q_1(p_1, p_2) + (p_2 - c)q_2(p_1, p_2),$$

together with the symmetry condition $p_1 = p_2$. ■

Remark 2. By this lemma, it is impossible to earn more than the (total) monopoly profit $2\pi^M$ in a single period. Fir any p_{-i}, it follows that $\max_{p_i}\{\pi_i(p_i, p_{-i})\} \leq 2\pi^M$. □

2 Repeated Bertrand Competition with Differentiated Products

Firms 1 and 2 repeat the game G of the Bertrand competition with differentiated products. It is straightforward to set up the corresponding repeated game with perfect monitoring.

Theorem 1. *Under Assumption 1, there exists some $\bar{\delta} \in (0,1)$ such that for each $\delta > \bar{\delta}$, there exists a subgame perfect equilibrium with the average payoff vector (π^M, π^M).*

Proof. Consider the following (grim) trigger strategy:

$$\sigma_i(h^t) = \begin{cases} p^M & \text{if player } i \text{ has observed only price profile } (p^M, p^M) \text{ in history } h^t \\ p^N & \text{otherwise.} \end{cases}$$

In words, the trigger strategy is such that (i) in period $t = 1$, player i plays $p_i^1 = p^M$, and (ii) in period $t \geq 2$, player i plays $p_i^t = p^M$ if each has observed only (p^M, p^M) in periods $1, 2, \dots, t-1$ but plays $p_i^t = p^N$ otherwise.

We show that no player has a profitable deviation. By the one-shot deviation principle, it suffices to show that no player has a profitable one-shot deviation. There are two cases to consider:

1. In the cooperation phase, where each player i is prescribed to play $p_i^t = p^M$ at period t, firm i compares the following two cases:

   - If player i plays $p_i^t = p^M$ then her average continuation payoff is
   $$(1-\delta)\Big(\pi^M + \delta\pi^M + \delta^2\pi^M + \cdots\Big) = \pi^M.$$
   - If player i plays $p_i^t \neq p^M$ then her average continuation payoff is
   $$(1-\delta)\left(\max_{p_i^t \neq p^M}\Big\{\pi_i\Big(p_i^t, p^M\Big)\Big\} + \delta\pi^N + \delta^2\pi^N + \cdots\right) \leq (1-\delta)\cdot 2\pi^M + \delta\pi^N,$$
   where $\max\{\pi_i(p_i^t, p^M)\} \leq 2\pi^M$ by Remark 2.

   Player i has no profitable one-shot deviation for any high enough δ.

2. In the punishment phase, where each player i is prescribed to play $p_i^t = p^N$ at period t, firm i has no incentive to deviate, because pricing p^N is the unique (stage-game) best response to firm $-i$'s price p^N. Moreover, period-t behavior will never change the future behavior, which means that pricing p^N at every period is uniquely optimal.

For any high enough δ, the above trigger strategy profile is a subgame perfect equilibrium, with the average payoff vector (π^M, π^M). ■

Remark 3. The punishment in the proof does not involve the minimax strategy against the deviator. Instead of the minimax strategies, it involves the stage-game Nash equilibrium strategy. This kind of punishment is called the **Nash reversion** punishment. □

3 Collusion in Reality

Now we look at the real world, but this time through the lens of theory.

If you were a regulator, how would you "define" collusion? So far, we have (implicitly) assumed that you know market demand and firms' costs. In reality, however, you may not know the demand or costs. Even if you were knowledgeable about particular industries, you still might not be sure about exogenous shocks to the market—for instance, an outbreak of a new infectious disease. Therefore, even if you observe supracompetitive prices, that alone does not mean that the firms are colluding. This is because they might have simply failed to optimize their prices. It is possible, for example, when they misestimate the market demand. As they receive possibly correlated information about the market, they can make similar misestimation.

Then, how would you say that the firms are colluding? Through the lens of economic theory, collusion crucially involves a **reward-punishment scheme** designed to provide the incentives for firms to consistently price above the competitive level (Harrington, 2018). The reward-punishment scheme ensures that the supracompetitive outcomes may be obtained in equilibrium and do not result from a failure to optimize.

References

Harrington, J. E. (2018). Developing competition law for collusion by autonomous artificial agents. *Journal of Competition Law & Economics*, *14*(3), 331–363.