MARKOV PROCESSES AND RECURSIVE REPRESENTATION

Can we reformulate the neoclassical growth model with uncertainty in a recursive language? Which are the state variables?

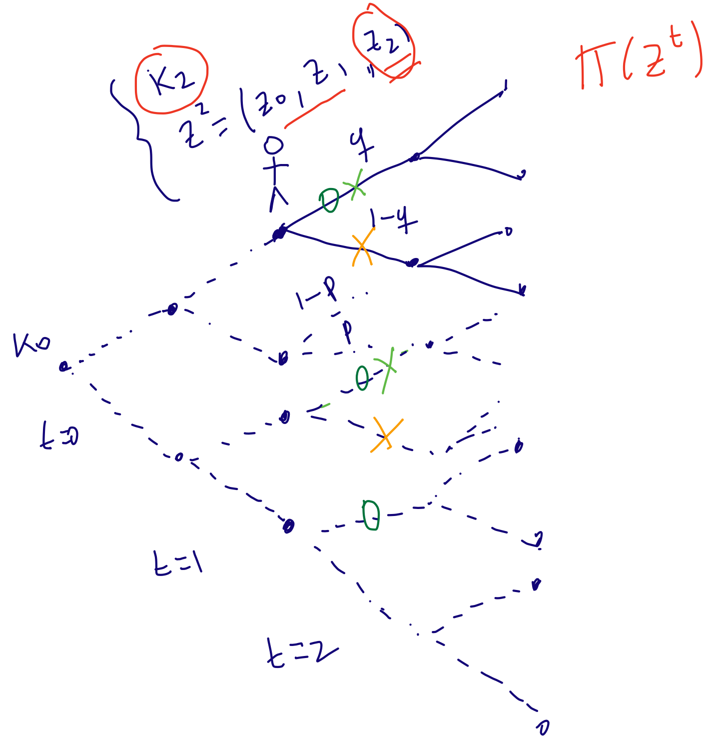
- Individual and aggregate capital

- History of the shocks?

In general, the probability distribution of the shock for the next period depends of all the history of past and current realizations

$$\text{Prob}\,(z_{t+1}) \equiv \text{Prob}\left(z_{t+1} \,\middle|\, \cancel{z^t}\right)$$

unless that the shock follows a *Markovian process*



`First Order Markovian Process`

A first order Markovian process satisfies:

$$\text{Prob}\left(z_{t+1}\left|z^t\right.\right) = \text{Prob}\left(z_{t+1}\left|z_t\right.\right)$$

An i.i.d. shock can be seen as an special case of a Markovian process

Throughout different applications, we will use a first order discrete Markovian process, characterized by the following state-space

$$Z = \left(Z_1, ..., Z_q\right) \quad q = 2$$

and the stationary transition matrix

$$\Pi = \begin{bmatrix} \pi\left(Z_1, Z_1\right) & \pi\left(Z_1, Z_2\right) & .... & \pi\left(Z_1, Z_q\right) \\ \pi\left(Z_2, Z_1\right) & \pi\left(Z_2, Z_2\right) & .... & \pi\left(Z_2, Z_q\right) \\ .... & .... & .... & .... \\ \pi\left(Z_q, Z_1\right) & \pi\left(Z_q, Z_2\right) & .... & \pi\left(Z_q, Z_q\right) \end{bmatrix}_{q \times q}$$

where

$$\pi\left(Z_i, Z_j\right) = \text{Prob}\left(z_{t+1} = Z_j \left|z_t = Z_i\right.\right)$$

and $\sum_{j=1}^{q} \pi\left(Z_i, Z_j\right) = 1$

tomorrow

$$\Pi = \text{today} \begin{array}{c|cc} & L & H \\ \hline L & p & 1-p \\ H & 1-q & q \end{array} \xrightarrow{\text{i.i.d}} \begin{bmatrix} p & 1-p \\ p & 1-p \end{bmatrix} \quad q = 1-p$$

A Markovian process is asymptotically stationary if, given any π_0, $\lim_{t\to\infty} \pi_t = \pi^*$

Result: if all the elements of Π are strictly positive, the corresponding process is asymptotically stationary and converges to a unique invariant distribution

Defining the un~~i~~nconditional probability distribution

$$\pi_t = \begin{bmatrix} \pi_{1t} \\ \pi_{2t} \\ .... \\ \pi_{qt} \end{bmatrix} = \begin{bmatrix} \text{Prob}\,(z_t = Z_1) \\ \text{Prob}\,(z_t = Z_2) \\ .... \\ \text{Prob}\,(z_t = Z_q) \end{bmatrix}$$

the vector π_t evolves according to:

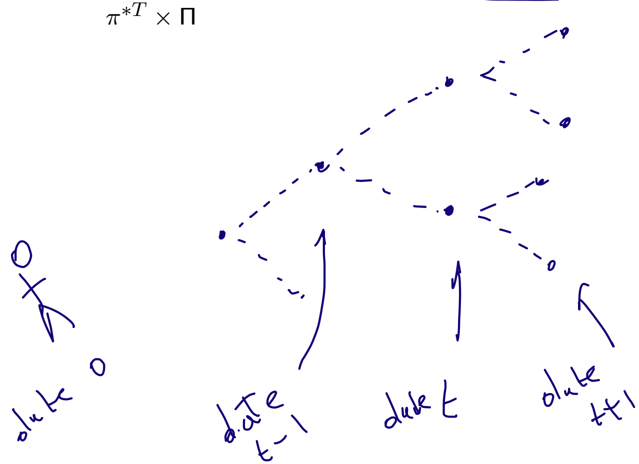
$$\pi_{t+1}^T = \pi_t^T \times \Pi$$

invariant distribution (π^*)

The probability distribution π^* is *invariant* if $\pi^{*T} = \pi^{*T} \times \Pi$

$$\begin{bmatrix} \pi_{t+1}^L & \underline{\pi_{t+1}^H} \end{bmatrix} = \begin{bmatrix} \pi_t^L & \pi_t^H \end{bmatrix} \begin{bmatrix} p & 1-p \\ 1-q & q \end{bmatrix}$$

$$= \begin{bmatrix} p\pi_t^L + (1-q)\pi_t^H & \underline{(1-p)\pi_t^L} + \underline{q\pi_t^H} \end{bmatrix}$$



$$\begin{bmatrix} Pr(z_t = L) \to \pi_t^L \end{bmatrix}$$

$$\pi_t = \begin{bmatrix} \cdots \\ Pr(z_t = H) \to \pi_t^H \end{bmatrix}$$

$$\pi_{t+1} = \begin{bmatrix} Pr(z_{t+1} = L) \to \pi_{t+1}^L \\ Pr(z_{t+1} = H) \to \pi_{t+1}^H \end{bmatrix}$$

In other cases, we will work with continuous Markovian process, such as the $AR(1)$:

$$z' = \mu + \rho z + \varepsilon'$$

$$\varepsilon' \sim N\left(0, \sigma^2\right)$$

$\rho \in (0,1)$

where ε is a random variable, normally distributed with zero mean and variance σ^2

If $|\rho| < 1$, this process is stationary therefore the conditional density function

$$g\left(z' \,|z\right) = \frac{1}{\sigma\left(2\pi\right)^{1/2}} \exp\left\{-\frac{\left(z' - \mu - \rho z\right)^2}{2\sigma^2}\right\}$$

and its principal moments (mean, variance) are constant over time

$$\begin{cases} \text{mean}(z) = \dfrac{\mu}{1-\rho} \\ \text{var}(z) = \dfrac{\sigma^2}{1-\rho^2} \end{cases}$$

consumer states: (k, K, z) $\rightarrow$ prices

Stochastic Recursive Competitive Equilibrium

A Stochastic Recursive Competitive Equilibrium is a set of functions $v(k,K,z)$, $c(k,K,z)$, $i(k,K,z)$, and $k'(k,K,z)$, prices $w(K,z)$ and $r(K,z)$ and aggregate law of motion $\Gamma(K,z)$ such that:

i) For each triple (k,K,z), given the functions w, r and Γ, the value function $v(k,K,z)$ solves the household's Bellman equation:

$$v(k,K,z) = \max_{c,i,k'} \left\{ u(c) + \beta E_z v\left(k', K', z'\right) \right\}$$

$$\begin{aligned} s.t. \quad c+i &= w(K,z) + r(K,z)\,k \\ k' &= (1-\delta)\,k + i \\ K' &= \Gamma(K,z) \end{aligned}$$

and $c(k,K,z)$, $i(k,K,z)$, $k'(k,K,z)$ are optimal decision rules

ii) For each pair (K,z), prices satisfies marginal conditions:

$$r(K,z) = e^z f'(K)$$

$$w(K,z) = e^z f(K) - e^z f'(K)\,K$$

iii) For each pair (K,z), markets clear:

$$e^z f(K) = c(K,K,z) + i(K,K,z)$$

iv) For each pair (K,z), the aggregate law of motion is consistent with agents individual decisions:

$$\underbrace{\Gamma(K,z)}_{K'} = k'(K,K,z)$$

The expected value function for the next period is defined as:

- If z follows a first order discrete Markovian process with a state space $Z = \left(Z_1, ..., Z_q\right)$ and transition matrix Π, then

$$E_z v\left(k', K', z'\right) = \sum_{j=1}^{q} \pi\left(z, Z_j\right) v\left(k', K', Z_j\right)$$

- If z follows a $AR(1)$ process with conditional density function $g\left(z' \,|z\right)$, then

$$E_z v\left(k', K', z'\right) = \int_Z v\left(k', K', z'\right) g\left(z' \,|z\right) dz'$$

$$\Pi = \begin{array}{ccc} & L & H \\ L & \multicolumn{2}{c}{\multirow{2}{*}{$\begin{bmatrix} p & 1-p \\ 1-q & q \end{bmatrix}$}} \\ H & & \end{array}$$

$$E_{z=L}\, v(k', K', z') = p\, v(k', K', L) + (1-p)\, v(k', K', H)$$

$$E_{z=H}\, v(k', K', z') = (1-q)\, v(k', K', L) + q\, v(k', K', H)$$

Social Planner's Problem

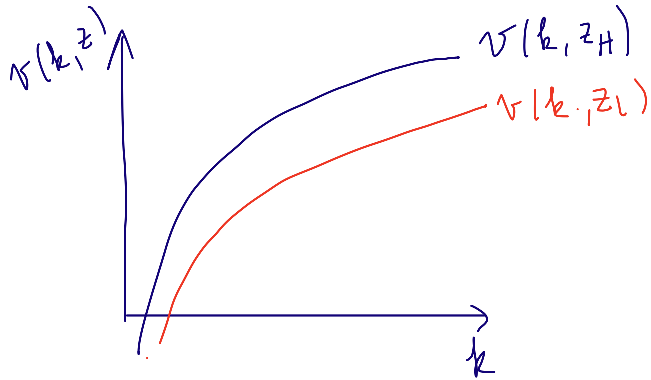
A benevolent social planner chooses functions $v(k,z)$, $c(k,z)$, $i(k,z)$, $k'(k,z)$ to solve the Bellman equation:

$$v(k,z) = \max_{c,i,k'} \left\{ u(c) + \beta E_z v\left(k', z'\right) \right\}$$

$$\begin{aligned} s.t. \quad c + i &= e^z f(k) \\ k' &= (1-\delta) k + i \end{aligned}$$

As before, if the Welfare Theorems hold, the solution to this problem is equivalent to the competitive equilibrium

Note that the corresponding contingent plans can be found using the optimal decision rule: $k_{t+1}\left(z^t\right) = k'\left(k'\left(k'(\ldots), z_{t-1}\right), z_t\right)$

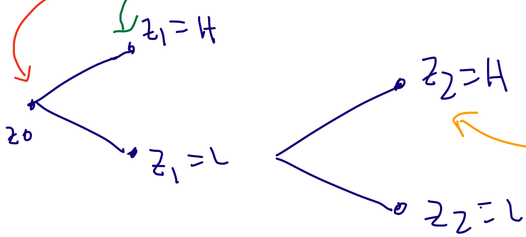


decision rule $k'(k,z)$

$\Rightarrow$ $\underline{k_{t+1}(z^t)}$ contingent plan

(k_0, z_0) given $\quad k_1(k_0, z_0) = k'(k_0, z_0)$

$k_2(k_1, z_1 = H) = k'(\underbrace{k'(k_0, z_0)}, z_1 = H)$



k_1

$$k_3(k_2, z_2 = H)$$

$$= k'\big(k'\big(\underbrace{k'(k_0, z_0)}_{k_1}, z_1 = L\big), z_2 = H\big)$$

Stochastic Dynamic Programming

Consider the following Bellman equation:

$$v(x, z) = \max_{y} \left\{ F(x, z, y) + \beta E_z v\left(y, z'\right) \right\}$$

$$s.t. \quad y \in \Omega(x, z)$$

where z follows a first order Markov process

$x = k \quad y = k'$

state vector (k, z) technology shock

$$F(k, z, k') = u\left[e^{z} f(k) + (1-\delta)k - k'\right]$$

The results of existence, uniqueness and contraction continue to be met under the same conditions for X, F, Ω and β, plus some technical assumptions about the stochastic process z (see *Stokey-Lucas*, Ch. 9)

$$E_z v(k', z') = \sum_{j=1}^{q} \pi(z, z_j)\, v(k', z_j)$$

These technical assumptions are automatically satisfied with a discrete Markovian or an $AR(1)$ processes

$$z' \in \{z_1, z_2, \dots, z_q\}$$

$$\Omega(k, z) = \left[0,\ e^{z} f(k) + (1-\delta)k\right]$$

Value Function Iteration

To implement this method numerically, we assume that the technological shock z follows a first order discrete Markovian process with a state space $Z = \left(Z_{1,} ..., Z_q\right)$ and transition matrix Π

Using the Contraction Theorem, if we start from any function v^0 (for example, $v^0 \left(k, z\right) = 0$) the sequence v^n defined by:

$$v^{n+1}\left(k, z\right) = \max_{k'} \left\{ u\left[e^z f\left(k\right) + \left(1-\delta\right) k - k'\right] + \beta \sum_{j=1}^{q} \pi\left(z, Z_j\right) v^n \left(k', Z_j\right) \right\}$$

$$s.t. \quad k' \in \left[0, e^z f\left(k\right) + \left(1-\delta\right) k\right]$$

converges to the solution v of the social planner, as $n \to \infty$

$AR(1) \xrightarrow{\text{Tauchen}}$ Markov chain

Initial configuration:

- Define a grid $K = \left(K_1, K_2, ..., K_p\right)$ for capital
- Construct a grid M (in 3-d) of $p \times q \times p$ as

return function (cube)

$$M_{i,j,l} \equiv F\left(K_i, Z_j, K_l\right)$$

$$= u\left[\underbrace{e^{Z_j}}_{z} f\left(\underbrace{K_i}_{k}\right) + (1-\delta)\underbrace{K_i}_{k} - \underbrace{K_l}_{k'}\right]$$

M stores the return function evaluated in each possible combination $[k, z, k']$ in the grid

- Eliminate unreachable cells, making:

$$M_{i,j,l} = -10000 \quad \text{if} \quad K_l > e^{Z_j} f\left(K_i\right) + (1-\delta) K_i$$

Algorithm:

1. Propose an initial matrix V^0 of dimension $\overset{k}{p} \times \overset{z}{q}$ (for example $V^0 = 0$) and initialize $s = 0$
2. Given V^s and M, calculate $V_{i,j}^{s+1}$ for all $i, j \in \{1, .., p\} \times \{1, .., q\}$ as:

$$V_{i,j}^{s+1} = \max_{l \in \{1,..,p\}} \left\{ M_{i,j,l} + \beta \sum_{m=1}^{q} \pi\left(Z_j, Z_m\right) V_{l,m}^{s} \right\}$$

3. Calculate $\left\| V^{s+1} - V^s \right\|$. If the distance is greater than the tolerance criterion, return to step 2 with $s = s + 1$. Otherwise, the algorithm converges

As before, we obtain an approximation to the value function of the planner V in each point of the grid of states:

$$V \approx \begin{bmatrix} v(K_1, Z_1) & v(K_1, Z_2) & ..... & v(K_1, Z_q) \\ v(K_2, Z_1) & v(K_2, Z_2) & ..... & v(K_2, Z_q) \\ ..... & ..... & ..... & ..... \\ v(K_p, Z_1) & v(K_p, Z_2) & ..... & v(K_p, Z_q) \end{bmatrix}$$

We also obtain the corresponding decision rule G, another matrix of $p \times q$ components, with $G_{i,j} \in \{1, ..., p\}$

without uncertainty

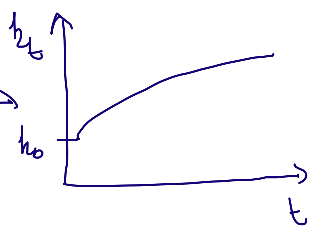
$k' = k'(k)$

given k_0

$k_1 = k'(k_0)$

$k_2 = k'(k_1)$

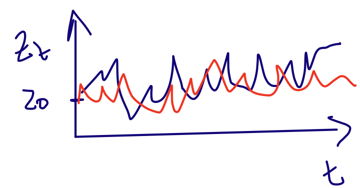
...



$k' = k'(k, z)$

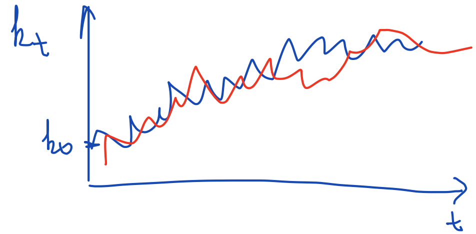
simulation

(k_0, z_0) given



$\{z_t\}_{t=0}^{T} \longrightarrow$ simulated time series for z

$$k_1 = k'(k_0, z_0)$$
$$k_2 = k'(k_1, z_1)$$
$$k_3 = k'(k_2, z_2)$$



Simulation of Optimal Trajectories

The optimal path of capital depends on the history of realizations for z

To obtain a particular time series $k_0, k_1, ..., k_T$ starting from (k_0, z_0) given, we need to simulate the history $z_0, z_1, ..., z_T$ using a random number generator and the information in the transition matrix Π

• Given $z_0 = Z_i$, extract z_1 from $Z = \left(Z_{1,...,} Z_q\right)$ assigning probability π_{ij} to state Z_j

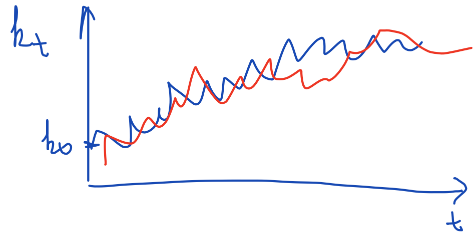
• Recursively, given $z_n = Z_i$, extract z_{n+1} of $Z = \left(Z_{1,...,} Z_q\right)$ assigning probability π_{ij} to state Z_j

• Continue until you complete the history $z_0, z_1, ..., z_T$

Once the history of shocks is known, we recursively compute the optimal path for capital $(k_0, k_1, ..., k_T)$ using the decision rule G

$$\left.\begin{array}{r} k_t = K_j \\ z_t = Z_i \end{array}\right\} \Rightarrow k_{t+1} = K_{G_{i,j}}$$

and then we calculate the corresponding trajectories for other variables



These trajectories represent time series, their main statistics (mean, variance, correlations between them) can be computed and compared with the data

mean $(k_t) = \dots$

variance $(k_t) = \dots$

Corr $(k_t, k_{t-1}) = \dots$

However, these time series and their statistics depend on a particular realization of the shocks

Therefore, it is recommended to do a long simulation ($T = 10,000$ periods, leaving aside the first 1000) so as to approximate the statistics from the invariant distribution (*law of large numbers*)