Game Theory

# Alternating-Offer Bargaining

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The ultimatum bargaining, presented in the previous note, has to end even if an offer is rejected. But why don't players continue to bargain? In this note, we study the model of **alternating-offer bargaining**, in which players continue to bargain until they reach an agreement (Rubinstein, 1982).

## 1 Alternating-Offer Bargaining

We study a bargaining problem between two players. Like the ultimatum bargaining, a player makes an offer, while the other player responds by accepting it or rejecting it. Unlike the ultimatum bargaining, the alternating-offer bargaining continues even after an offer is rejected. Every period an offer is rejected, the responder must submit a counter-offer in the next period. In this way, the two players alternate offers until an offer is accepted.

### 1.1 Model

Two players, denoted 1 and 2, bargain over a pie of size 1. Given a discount factor $\delta \in (0,1)$, the bargaining proceeds according to the following protocol:

**Period 1:** Player 1 makes an offer $s_1 \in [0,1]$, where player 1 receives a share $s_1$ and player 2 a share $1 - s_1$. After observing offer $s_1$, player 2 decides whether to accept or reject it. If player 2 accepts the offer then the pie is divided according to the offer, resulting in payoffs $(s_1, 1 - s_1)$; if player 2 rejects the offer then play proceeds to period 2.

**Period 2:** Player 2 makes an offer $s_2 \in [0,1]$, where player 1 receives a share $s_2$ and player 2 a share $1 - s_2$. After observing offer $s_2$, player 1 decides whether to accept or reject it. If player 1 accepts the offer then the pie is divided according to the offer, resulting in payoffs $(\delta s_2, \delta(1 - s_2))$; if player 1 rejects the offer then play proceeds to period 3.

**Period $t \geq 3$:** Player $i$ ($i = 1$ if $t$ is odd and $i = 2$ if $t$ is even) makes an offer $s_t \in [0,1]$, where player 1 receives a share $s_t$ and player 2 a share $1 - s_t$. After observing offer $s_t$, player $-i$ decides whether to accept or reject it. If player $-i$ accepts the offer then the pie is divided according to the offer, resulting in payoffs $(\delta^{t-1} s_t, \delta^{t-1}(1 - s_t))$; if player $-i$ rejects the offer then play proceeds to period $t + 1$.

This extensive-form game is of infinite-horizon since if a responder always rejects an offer, the game lasts "forever." Assume that each player $i$ receives payoff 0 if the game lasts forever.

The discount factor $\delta$ is a parameter that measures players' patience. A high discount factor $\delta$ means that they are patient and do not discount future payoffs much compared to current ones; in contrast, a low discount factor $\delta$ means that they are impatient and discount future payoffs much compared to current ones.

The alternating-offer bargaining is represented by the following game tree:

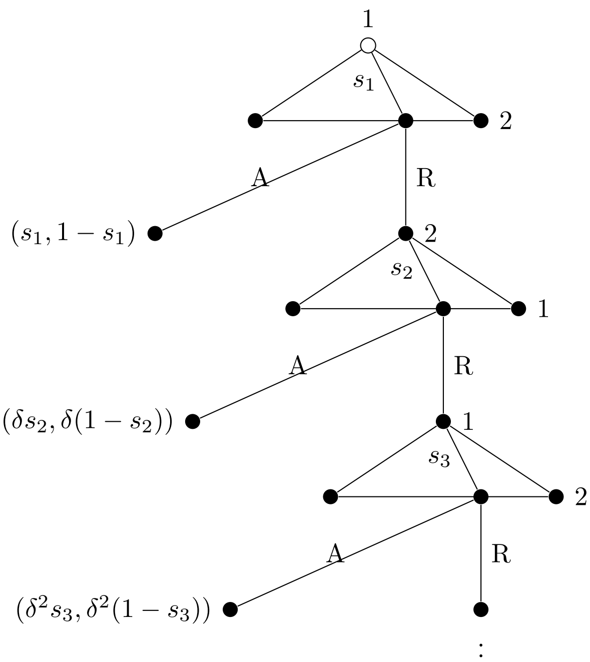


Figure 1: the alternating-offer bargaining

## 1.2 Equilibrium

This alternating-offer bargaining has a unique subgame perfect equilibrium.

**Theorem 1.** *The alternating-offer bargaining has a unique subgame perfect equilibrium, and it is such that:*

**Period $t = 1, 3, \ldots$:**

- *Player 1 makes an offer $s_t = \frac{1}{1+\delta}$.*
- *Player 2 accepts an offer $s_t$ if $s_t \leq \frac{1}{1+\delta}$ and rejects it otherwise.*

**Period $t = 2, 4, \ldots$:**

- *Player 2 makes an offer $s_t = \frac{\delta}{1+\delta}$.*
- *Player 1 accepts an offer $s_t$ if $s_t \geq \frac{\delta}{1+\delta}$ and rejects it otherwise.*

**Remark 1.** The equilibrium outcome is that at period $t = 1$, player 1 makes an offer $s_1 = \frac{1}{1+\delta}$ and player 2 accepts it, resulting in payoffs $(\frac{1}{1+\delta}, \frac{\delta}{1+\delta})$. □

The alternating-offer bargaining model has prominent features as follows:

1. In the **unique** subgame perfect equilibrium, players reach an **immediate agreement**.
   - The (unique) equilibrium is efficient in the sense that there is no delay in reaching the agreement and thus the sum of the two players' payoffs are maximized.
2. A proposer always gains more than the responder.
   - Being a proposer can be interpreted as having a more bargaining power.
3. The equilibrium share converges to the equal share as $\delta \to 1$.
   - The bargaining power from being a proposer is related to how much discount a delay. In the limit of no time discounting, the bargaining power vanishes.

## 1.3 Stationary Equilibrium

**Definition 1.** In the alternating-offer bargaining, a strategy for player $i$ is **stationary** if all of the following hold:

1. At any period $t$ when making an offer, player $i$ makes the same offer (independent of period $t$).
2. At any period $t$ when responding to an offer, player $i$ responds with the same rule (independent of period $t$).

A subgame perfect equilibrium is **stationary** if the equilibrium strategy for each player is stationary.

In light of Definition 1, the (unique) subgame perfect equilibrium of Theorem 1 is stationary. Indeed, any offer $s_t$ or any response rule is independent of time index $t$. It is important to note that players' strategy spaces are not restricted to the set of stationary strategies. In other words, each player is allowed to choose any strategy, whether stationary or non-stationary, but in the equilibrium, both players choose to play the stationary strategies.

**Remark 2.** The concept of stationarity is defined not only for the alternating-offer bargaining but also for many other (infinite-horizon) extensive-form games. As the detailed specification of stationary strategies depends on the detail of a game under consideration, we do not attempt to give a general definition of stationary strategies. That being said, the key idea behind stationarity is that a strategy depends on a "state" of play but is independent of time index $t$ itself. $\square$

**Recap (Infinity and Recursivity):** Here is a math quiz: What is $x$?

$$x = \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \cdots}}}}.$$

Once we find a "nested $x$" on the right-hand side, we find $x = \sqrt{2+x}$ and then $x = 2$.[a]

A **recursive** structure is a powerful tool to examine an infinite structure. Indeed, we will find a recursive structure from the (infinite-horizon) alternating bargaining structure under the stationarity requirement. (By the way, did you notice a recursive structure in the definition of Nash equilibrium?)

[a] This is not rigorous but enough to sketch the recursive structure hidden in the right-hand side.

## 1.4 Proof of Theorem 1

Our proof consists of two steps.

1. We find a stationary equilibrium, which coincides with the equilibrium prescribed in Theorem 1. By the **one-shot deviation principle**, it suffices to show that no player has a profitable deviation from her equilibrium strategy in a single period.

2. We show that there is no other subgame perfect equilibrium.

**Proof of Theorem 1.** In the first step, we find a stationary subgame perfect equilibrium. By stationarity, player 1's equilibrium offers $s_1, s_3, s_5, \ldots$ at period $t = 1, 3, 5, \ldots$ are equal to each other. In particular, $s_1 = s_3$. It must be that:

**Period 2:** Player 2's equilibrium offer $s_2$ is such that player 1 is indifferent between accepting and rejecting it. Hence, $s_2 = \delta s_3$.

**Period 1:** Player 1's equilibrium offer $s_1$ is such that player 2 is indifferent between accepting and rejecting it. Hence, $1 - s_1 = \delta(1 - s_2)$.

Hence, $1 - s_1 = \delta(1 - s_2) = \delta(1 - \delta s_3)$. Since $s_1 = s_3$, it follows that

$$s_1 = \frac{1}{1+\delta}.$$

Hence, $s_1 = s_3 = s_5 = \cdots = \frac{1}{1+\delta}$. Also, the response rule for player 2 (at period $t = 1, 3, 5, \ldots$) is to accept an offer $s_t$ if $s_t \leq \frac{1}{1+\delta}$ and to reject it otherwise. Similarly, we find player 2's equilibrium offers $s_2 = s_4 = s_6 = \cdots = \frac{\delta}{1+\delta}$. Also, the response rule for player 1 (at period $t = 2, 4, 6, \ldots$) is to accept an offer $s_t$ if $s_t \geq \frac{\delta}{1+\delta}$ and to reject it otherwise. Summarizing the argument so far, we obtain the candidate for a stationary subgame perfect equilibrium:

**Period $t = 1, 3, \ldots$:**

- Player 1 makes an offer $s_t = \frac{1}{1+\delta}$.

- Player 2 accepts an offer $s_t$ if $s_t \leq \frac{1}{1+\delta}$ and rejects it otherwise.

**Period $t = 2, 4, \ldots$:**

- Player 2 makes an offer $s_t = \frac{\delta}{1+\delta}$.
- Player 1 accepts an offer $s_t$ if $s_t \geq \frac{\delta}{1+\delta}$ and rejects it otherwise.

Next, we show that this candidate is indeed a subgame perfect equilibrium. To this end, we need the **one-shot deviation principle** (Appendix A). Here, we take it for granted and see how to use it. Consider any period $t = 1, 3, \ldots$, when player 1 makes an offer. She has no profitable deviation. That is, she has no offer from which she receives more than $\frac{1}{1+\delta}$. To see this, it suffices to show that she has no offer from which she receives more than $\frac{1}{1+\delta}$, *provided that players 1 and 2 follow the candidate strategies after player 1 makes an offer at period $t$.* If player 1 makes an offer $s_t > \frac{1}{1+\delta}$ then player 2 will reject it and in the next period, he will make an offer $s_{t+1} = \frac{\delta}{1+\delta}$, which player 1 will accept to receive payoff $\frac{\delta^2}{1+\delta}$ in terms of period $t$; however, since $\frac{\delta^2}{1+\delta} < \frac{1}{1+\delta}$, player 1 finds that this deviation is unprofitable. Player 2 also has no profitable deviation when responding at period $t$. Hence, neither player has a profitable deviation in any period $t = 1, 3, \ldots$. By the same logic, neither player has a profitable deviation in any period $t = 2, 4, \ldots$. Therefore, we conclude that the above candidate is a subgame perfect equilibrium.

In the second step, we show that there exists no other subgame perfect equilibrium. To this end, let $\bar{v} \in [0, 1]$ denote the supremum equilibrium share that player 1 can obtain across periods at which player 1 makes an offer, and let $\underline{v} \in [0, 1]$ denote the infimum equilibrium share that player 1 can obtain across periods at which player 1 makes an offer.

- In any period $t = 2, 4, \ldots$ (when player 2 makes an offer), player 1 will accept any offer $s_t > \delta\bar{v}_1$ and reject any offer $s_t < \delta\underline{v}_1$. Hence, starting from this period, player 2's share must be between $1 - \delta\bar{v}_1$ and $1 - \delta\underline{v}_1$.

- In any period $t = 1, 3, \ldots$ (when player 1 makes an offer), player 2 will never accept an offer $1 - s_t < \delta(1 - \delta\bar{v}_1)$; hence, $\bar{v}_1 \leq 1 - \delta(1 - \delta\bar{v}_1)$. Player 2 will accept an offer $1 - s_t \geq \delta(1 - \delta\underline{v}_1)$; hence, $\underline{v}_1 \geq 1 - \delta(1 - \delta\underline{v}_1)$.

  From these inequalities, it follows that

$$\underline{v}_1 \geq \frac{1}{1+\delta} \geq \bar{v}_1.$$

Since $\bar{v}_1 \geq \underline{v}_1$, it follows that $\bar{v}_1 = \underline{v}_1 = \frac{1}{1+\delta}$, which completes the proof. ∎

# A One-Shot Deviation Principle

It is often difficult to verify whether a given strategy profile $\beta^*$ is a subgame perfect equilibrium becausee we have to show that, for every subgame, each player has no profitable deviation among all her possible strategies (in that subgame). The **one-shot deviation principle** simplifies this task. It states that in an extensive-form game (with mild conditions), to check if player $i$ has a profitable deviation, it suffices to check for her strategies that coincides with the strategy $\beta_i^*$ at all information sets except one.

**Definition 2.** In the alternating-offer bargaining, player $i$'s strategy $\beta_i$ is a **one-shot deviation** from her strategy $\beta_i^*$ if there exists a unique node $x \in X_i$ (at which player $i$ moves) such that $\beta_i(x) \neq \beta_i^*(x)$ but for every $x' \in X_i \setminus \{x\}$,

$$\beta_i(x') = \beta_i^*(x').$$

Now we state the one-shot deviation principle for the alternating-offer bargaining.[1]

**Theorem 2** (One-Shot Deviation Principle in Alternating-Offer Bargaining)**.** *In the alternating-offer bargaining, a strategy profile $\beta^*$ is a subgame perfect equilibrium if and only if for each $i \in I$ and each $x \in X_i$ and for player $i$'s every one-shot deviation $\beta_i$ from her strategy $\beta_i^*$,*

$$u_i|_x(\beta_i^*|_x, \beta_{-i}^*|_x) \geq u_i|_x(\beta_i|_x, \beta_{-i}^*|_x), \tag{1}$$

*where for a given strategy profile $\beta$, we write $u_i|_x(\beta|_x) \equiv \mathbb{E}_{O(\beta|_x)}[u_i|_x(z)]$ for player $i$'s expected payoff when outcome $z$ is distributed according to the outcome $O(\beta|_x)$.*

**Proof.** Omitted. ■

**Recap (One-Shot Deviation Principle in a Nutshell):** We keep in mind the following:

> Given players $-i$'s strategy profile $\beta_{-i}$, if player $i$ has a profitable deviation from a strategy $\beta_i^*$, then she has a profitable one-shot deviation from that strategy $\beta_i^*$. That is, to show that she has no profitable deviation, it suffices to show that she has no profitable one-shot deviation.

In an extensive-form game of perfect information, a strategy profile $\beta^*$ is a subgame perfect equilibrium if and only if no player has a profitable one-shot deviation at any node.

## References


Rubinstein, A. (1982). Perfect equilibrium in a bargaining model. *Econometrica*, *50*(1), 97–109.


[1] The one-shot deviation principle itself holds true for a much more general setting.