Static Constrained Optimization

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1 Static optimization

Derived from the word economy, '*economization*' reflects the importance of optimization in the study of economics. After all, economics studies agents' behavior aiming to achieve certain ends given scarce means which have alternative uses[1]. It is then natural to think that the persecution of such ends is done in a fashion that maximizes the usefulness of those resources. In fact, this kind of behavior isn't unfamiliar to any of us as consumers that, for example at a grocery store, choose from two indistinct products the one that is cheaper. That is actually the solution of a very simple optimization problem. For this reason, it is fundamental to learn the basic tools of optimization theory to support our study of economics.

Two basic mathematical theorems guide the methodology to apply static optimization: the Lagrange theorem and the Kuhn-Tucker theorem. This is in opposition with dynamic optimization where current decisions influence future outcomes. At a very intuitive level, static optimization is about the characterization of a problem where an objective function is maximized through the choice of specific variables that are nevertheless subject to certain constraint sets or functions.

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[1]This is a popular definition of economic science introduced by Robbins (1935).

1.1 Lagrange Theorem

Theorem 1. *(Lagrange) Let $x^* \in \mathcal{R}^n$ be an optimal solution of*

$$\max_x \{f(x) : g(x) = \mathit{0}\} \tag{P-L}$$

where both f and g are continuous and differentiable functions and the rank of $Dg(x^*) = c \leq n$. Then there exist unique Lagrange multipliers $\lambda^* \in \mathbb{R}^c$

$$Df(x^*) + \lambda^{*T} Dg(x^*) = \mathit{0}$$

Proof. This can be found in any math textbook and should be left as an exercise[2]. ☐

The first comment is that this theorem provide us with a very simple method to reduce our constrained maximization problem to one that is unconstrained. Essentially, it says that is possible to build an auxiliary objective function, called the **Lagrangian function**, using some auxiliary variables: **Lagrange multipliers**. By equating the first derivative of that function with respect with the maximization variables and these Lagrangian multipliers to 0, we get critical optimization points. That is, for the Lagrangian:

$$\mathcal{L}(x, \lambda) = f(x) + \lambda^T g(x)$$

and, for a x^* optimum, it follows that:

$$Df(x^*) + \lambda^{*T} Dg(x^*) = \underline{0}$$

Where the last equations are also known as the **first order conditions**.

One second comment is related with the fact that this theorem only provides **necessary conditions** for a optimum. That is, it is possible to have a $\dot{x}$ such that for $\dot{\lambda}$

$$Df(\dot{x}) + \dot{\lambda}^T Dg(\dot{x}) = \underline{0}$$

is true, but $\dot{x}$ is not, nevertheless, a maximum. In order to have also sufficient conditions for a maximum we make use of the following theorem:

Theorem 2. *(sufficient conditions for a strict local maximum) For some λ^*, let x^* satisfy the Lagrange condition. Assume that the matrix of second partial derivatives of the*

[2] For example in Sundaram (1996).

Lagrangian with respect to x when evaluated at (x^*, λ^*):

$$D_x^2\mathcal{L}\left(x^*, \lambda^*\right) = D^2 f(x^*) + \lambda^{*T} D^2 g(x^*)$$

is negative definite subject to the constraint $Dg(x^*)h = \underline{0}$, that is:

$$h^T D_x^2\mathcal{L}\left(x^*, \lambda^*\right) h < 0 \forall h \in \mathbb{R}^n \ st \ \ Dg(x^*)h = 0$$

Then x^* is a strict local maximizer of f subject to $g(x) = \underline{0}$

Proof. This can be found in any math textbook and should be left as an exercise [3]. □

Now that we have both necessary and sufficient conditions, lets apply these theorems with a specific example.

Example 1. Solve the following maximization problem:

$$\begin{aligned} \max_{x_1, x_2} \quad & f(x_1, x_2) = x_1^{2/3} \cdot x_2^{1/3} \\ subject\,to \quad & g(x_1, x_2) = 0 \Leftrightarrow 4 - 3x_1 - x_2 = 0 \end{aligned}$$

The first step to solve this is to construct a Lagrangian:

$$\mathcal{L}\left(x_1, x_2, \lambda\right) = x_1^{2/3} \cdot x_2^{1/3} + \lambda\left(4 - 3x_1 - x_2\right)$$

The first order conditions are just the first derivative of the previous function equated to 0

$$\begin{aligned} x_1 : \ & \frac{\partial \mathcal{L}}{\partial x_1} = \frac{2}{3} x_1^{-1/3} x_2^{1/3} - \lambda 3 = 0 \\ x_2 : \ & \frac{\partial \mathcal{L}}{\partial x_2} = \frac{1}{3} x_2^{-2/3} x_1^{2/3} - \lambda = 0 \\ \lambda : \ & \frac{\partial \mathcal{L}}{\partial \lambda} = 4 - 3x_1 - x_2 = 0 \end{aligned}$$

[3] See note 2

which, solving for (x_1, x_2, λ) yields:

$$x_1 = 8/9$$
$$x_2 = 4/3$$
$$\lambda = \frac{1}{3}\left(\frac{2}{3}\right)^{2/3}$$

Now that we have a critical point, we should check for the second order conditions:

$$D_x^2 \mathcal{L}\left(x_1, x_2, \lambda\right) = \begin{cases} -\frac{2}{9} x_1^{-4/3} x_2^{1/3} \\ -\frac{2}{9} x_2^{-5/3} x_1^{2/3} \end{cases}$$

But note that $D_x^2 \mathcal{L}\left(8/9, 4/3, \frac{1}{3}\left(\frac{2}{3}\right)^{2/3}\right) < \underline{0}$ so we can confirm that $(x_1^*, x_2^*, \lambda^*) = \left(8/9, 4/3, \frac{1}{3}\left(\frac{2}{3}\right)^{2/3}\right)$ is indeed a local maximum of the problem.

1.2 Kuhn-Tucker theorem

Before introducing the Kuhn-Tucker theorem, which deals with static maximization problems with inequality constraints, it is useful to introduce the notion of complementary slack conditions. In general, let a constrained maximization problem have the following form:

$$\max_x \left\{ f(x) : \, g(x) \geq \underline{0} \right\}$$

It is clear that this problem collapses into the Lagrangian problem if we restrict the constraint for those that are binding, that is, if they are all observed in equality: $g^j(x) = 0$. Then, for all $c \leq n$ linearly independent constrains, we can still associate our usual Lagrange multiplier to each constraint. However, if at the optimum that constraint is slack, then it should have no influence in the problem, implying that the Lagrange multiplier is zero. To summarize either the constraint is binding and the Lagrangian multiplier is larger than zero (that is, it changes the decision of the policy maker); or the constraint is slack (and therefore doesn't influence the decision of the policy maker), implying a zero Lagrange multiplier[4]. Mathematically this can be represented by the following:

$$\lambda_j \geq 0, \, g^j(x) \geq 0, \, \, and \, \lambda_j g^j(x) = 0 \, , \, \forall j = 1, ..., c$$

[4] A third case occurs when both the constraint and the multiplier are equal to zero.

And these are known as **complementary slack conditions**: either the multiplier is equal to zero, or the constrained is equal to zero, or both are equal to zero. Without loss of generality, at the optimum, let the first b constrains be binding. Then the following theorem applies.

Theorem 3. *(Kuhn-Tucker) Let $x^* \in \mathcal{R}^n$ be an optimal solution of*

$$\max_x \{f(x) : g(x) \geq \underline{0}\} \tag{P-KT}$$

where both f and g are continuous and differentiable functions and the rank of $Dg^b(x^*)$ equals $b \leq n$. Then there exist nonnegative Lagrange multipliers $\lambda^* \in \mathbb{R}^c_+$

$$\begin{aligned} &Df(x^*) + \lambda^{*T} Dg(x^*) = \underline{0} \\ &\forall j = 1, ..., c,\ g^j(x) \geq 0 \textit{ and } g^j(x) = 0 \textit{ if } \lambda_j > 0 \end{aligned}$$

Proof. This can be found in any math textbook and should be left as an exercise [5]. □

As before, this theorem provides only necessary conditions for a maximum. Sufficient conditions are achieved by imposing some additional structure to the problem.

Theorem 4. *(sufficient conditions for a global maximum) Given the problem (P-KT), if f is **pseudoconcave** and all constraints **quasiconcave**, then a (x^*, λ^*) satisfying the Lagrange and complimentary slackness conditions is also a global maximum.*

Proof. This can be found in any math textbook and should be left as an exercise. □

The previous two theorems are applied in the following example.

Example 2. Solve the following maximization problem:

$$\begin{aligned} \max_{x_1, x_2} \quad & f(x_1, x_2) = \log x_1 - \frac{\gamma}{2} x_2^2 - \zeta x_2 \\ \textit{subject to} \quad & g_1(x_1, x_2) = 0 \Leftrightarrow w x_2 + a - x_1 = 0 \\ & g_2(x_1, x_2) \geq 0 \Leftrightarrow x_2 \geq 0 \\ & g_3(x_1, x_2) \geq 0 \Leftrightarrow x_2 \leq 1 \end{aligned}$$

Note that all the constraints and objective function are concave so, in order to find the solution of the problem, we only have to solve for the first order conditions and Lagrange

[5] See note 2

multipliers. The first step to find those is to construct a Lagrangian function:

$$\mathcal{L}(x_1, x_2, \lambda, \phi_1, \phi_2) = \log x_1 - \frac{\gamma}{2}x_2^2 - \zeta x_2 + \lambda(wx_2 + a - x_1) + \phi_1 x_2 + \phi_2(1 - x_2)$$

where λ, ϕ_1 and ϕ_2 are Lagrange multipliers. This imply the following first order conditions:

$$\begin{aligned} x_1 &: \frac{\partial \mathcal{L}}{\partial x_1} = \frac{1}{x_1} - \lambda = 0 \\ x_2 &: \frac{\partial \mathcal{L}}{\partial x_2} = -\gamma x_2 - \zeta + \lambda w + \phi_1 - \phi_2 = 0 \\ \lambda &: \frac{\partial \mathcal{L}}{\partial \lambda} = wx_2 + a - x_1 = 0 \end{aligned}$$

and complementary slack conditions:

$$\begin{aligned} \phi_1 x_2 &= 0 \; and \; \phi_1 \geq 0 \\ \phi_2(1 - x_2) &= 0 \; and \; \phi_2 \geq 0 \end{aligned}$$

To solve for this system we need to analyze 3 situations:

1) $0 < x_2 < 1$: Given the complementary slackness conditions it follows that $\phi_1 = \phi_2 = 0$. But then:

$$\frac{1}{x_1}w = \gamma x_2 + \zeta$$

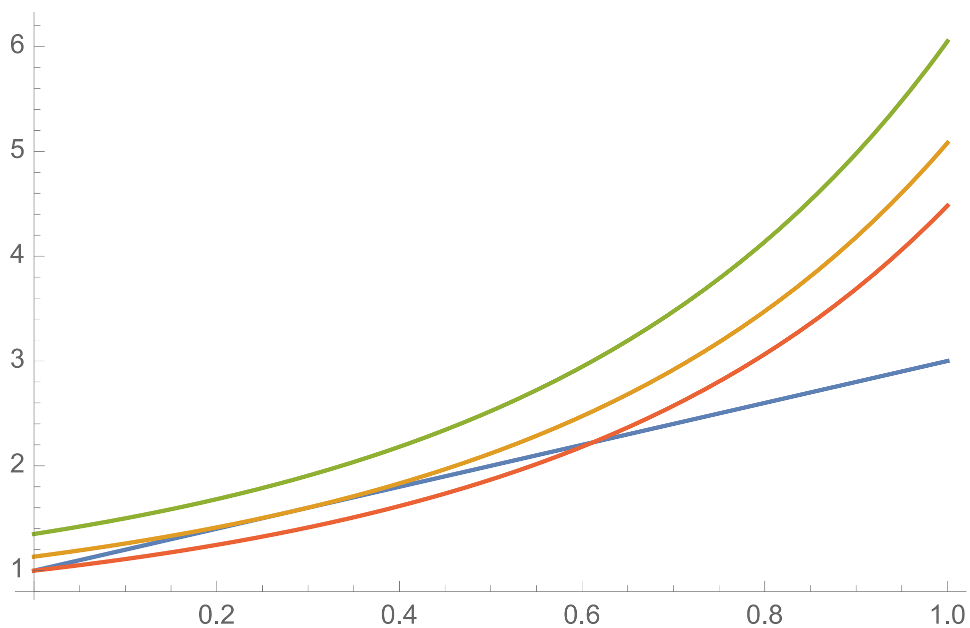
which implies that

$$\begin{aligned} x_2 &= \frac{1}{x_1} \cdot \frac{w}{\gamma} - \frac{\zeta}{\gamma} \\ \Rightarrow x_1 &= \frac{(a - w\zeta/\gamma) \pm \sqrt{(a - w\zeta/\gamma)^2 + 4w^2/\gamma}}{2} \end{aligned}$$

So, for a particular parameter choice of, say, $a = 1, w = 2, \gamma = 1, \zeta = 1$, the solution would be:

$$x_1 = 1.56$$
$$x_2 = 0.28$$
$$\lambda = 1/1.56$$
$$\phi_1 = 0$$
$$\phi_2 = 0$$

The graphical representation of this equilibrium can be found in the following figure, at the intersections (as a tangency) between the yellow line (the objective) and the blue line (the constraint). Note that the equilibrium is characterized by $0 < x_2 < 1$. This is an example of an interior solution precisely because the optimum is characterized by a tangency.



2) $x_2 = 0$**:** Say now that $a = 3, w = 2, \gamma = 1, \zeta = 1$. If we were to assume that $\phi_1 = \phi_2 = 0$ the solution would be:

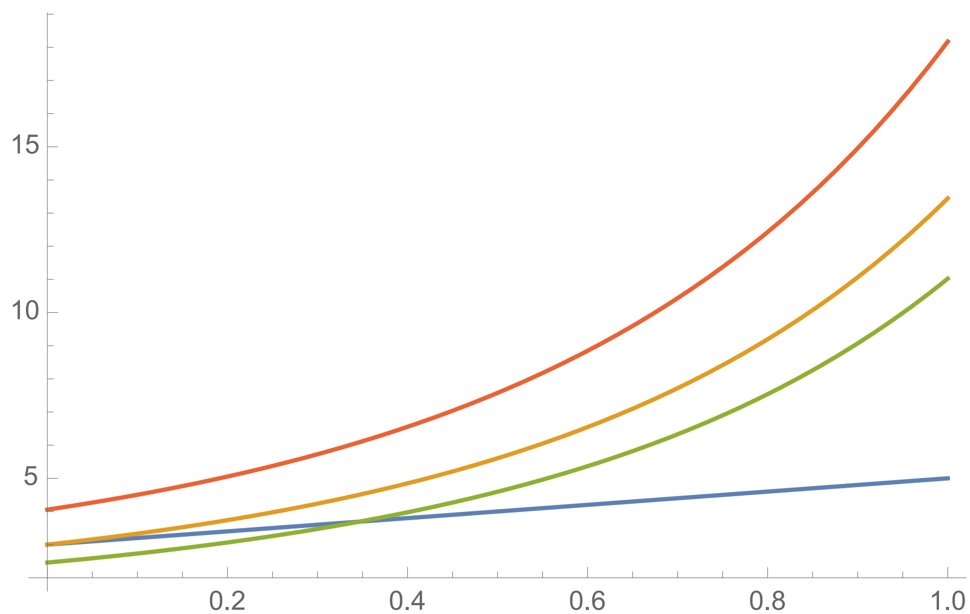
$$x_1 = 2.56$$
$$x_2 = -0.22$$
$$\lambda = 1/2.56$$

which clearly violates the constraint $x_2 \geq 0$. It follows that $\phi_1 > 0$ and $x_2 = 0$ (and clearly

$\phi_2 = 0$). But then the solution is given by:

$$\begin{aligned} x_1 &= 3 \\ x_2 &= 0 \\ \lambda &= 1/3 \\ \phi_1 &= 1/3 \\ \phi_2 &= 0 \end{aligned}$$

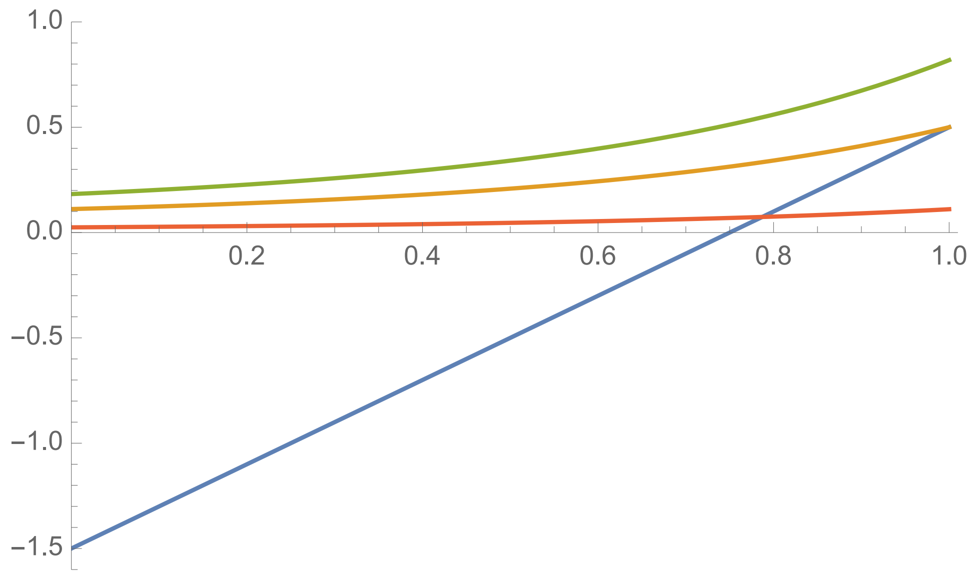
The graphical representation of this equilibrium can be found in the following figure, again, at the intersections between the yellow line (the objective) and the blue line (the constraint). Note that the equilibrium is characterized by $0 = x_2$. We now have an example of a interior solution where tangency no longer holds.



3) $x_2 = 1$**:** Say now that $a = -1.5, w = 2, \gamma = 1, \zeta = 1$. Proceeding in the same fashion as before we can show that the solution is characterized by:

$$\begin{aligned} x_1 &= 0.5 \\ x_2 &= 1 \\ \lambda &= 1/0.5 \\ \phi_1 &= 0 \\ \phi_2 &= 2 \end{aligned}$$

As before, the graphical representation of this equilibrium can be found in the following figure, at the intersections between the yellow line (the objective) and the blue line (the constraint). Note that now the equilibrium is characterized by $x_2 = 1$. This is again a corner solution.



References

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