Game Theory

Extensive-Form Representation: Revisited

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We introduce finite- or infinite-horizon **extensive-form representation** as well as relevant notions.

1 Extensive-Form Representation

Unlike a finite-horizon extensive-form game, an infinite-horizon extensive-form game may not end at a terminal node. It suffices to modify the definition of the finite-horizon extensive-form game such that the game may last "forever."

Example 1. Two players play rock-paper-scissors until one wins or loses. That is, if they draw (i.e., choose the same action) then they play rock-paper-scissors again and again until one wins or loses (i.e., choose different actions). Each player discounts payoffs with a discount factor $\delta \in (0,1)$; then, a player's payoff is $+\delta^{n-1}$ if she wins at the n-th play and is $-\delta^{n-1}$ if she loses at the n-th play.

This is not enough to define this possibly infinitely long-lasting game. It may end at a terminal node (at which either wins or loses), or it may never reach a terminal node. Each player, therefore, must have a payoff not only over the set of terminal nodes but also over the set of infinitely long paths of nodes. □

1.1 Extensive-Form Representation

Here is the formal definition of extensive-form representation.

Definition 1. An **extensive-form game**, or the **extensive-form representation** of a game, is a tuple $\Gamma = \langle I, X, P, (u_i, H_i)_i, \pi \rangle$ such that:

1. I is the set of players.
   - I may or may not include player 0, who is called **nature**.
2. X is the set of nodes such that:
   - $\rightarrow \subset X \times X$ is a binary relation that is transitive and asymmetric.[1,2]
     - It defines the "direction" between nodes.
     - There exists an **initial node** $\varnothing \in X$ such that for each $x \in X$, $\varnothing \rightarrow x$.
   - $z \in X$ is a **terminal node** if there exists no $x \in X$ such that $z \rightarrow x$.

[1] Transitivity: for each $x, y, z \in X$, if $x \rightarrow y$ and $y \rightarrow z$ then $x \rightarrow z$.
[2] Asymmetry: for each $x, y \in X$, if $x \rightarrow y$ then $y \not\rightarrow x$.

  - Z is the set of all terminal nodes.
- $\twoheadrightarrow$ is such that $x \twoheadrightarrow y$ if $x \to y$ but there exists no $x' \in X$ such that $x \to x' \to y$.
  - $y \in X \setminus \{\varnothing\}$ has a unique $x \in X$ such that $x \twoheadrightarrow y$.
- $\varnothing \twoheadrightarrow x_1 \twoheadrightarrow x_2 \twoheadrightarrow \cdots$ is a path (from $\varnothing$).
  - A path may or may not have a last element.
  - A last element, if any, of a path is a terminal node.
  - $\bar{Z}$ is the set of all paths that have no last elements (i.e., have infinite lengths).

3. $P : X \setminus Z \to I$ is the function that assigns to node $x \in X \setminus Z$ a player moving at node x.
   - $X_i = \{x \in X : P(x) = i\}$ is the set of player i's nodes.
   - $A(x) = \{y \in X : x \twoheadrightarrow y\}$ is the set of actions for player $P(x)$.
4. $u_i : Z \cup \bar{Z} \to \mathbb{R}$ is player i's payoff function.
5. H_i is player i's **partition** of the set X_i such that:
   - $h_i \in H_i$ is called player i's **information set**.
   - $A(x) = A(x')$ for each $x, x' \in h_i$, which is also denoted by $A(h_i)$.
6. π is a function that assigns to node $x \in X_0$ a distribution $\pi(x) \in \Delta(A(x))$.
   - $0 \in I$ denotes the player called nature, who moves at random.
   - $\pi(x) \in \Delta(A(x))$ is nature's random choice of action $y \in A(x)$ at node $x \in X_0$.

Remark 1. If an extensive-form game Γ is of perfect information, we omit players' information partitions H_i. Similarly, if an extensive-form game Γ has no (random) move by nature, we omit nature's play π. □

1.2 Relevant Notions of Extensive-Form Representation

Definition 2. An extensive-form game Γ is of **finite-horizon** if the set $\bar{Z}$ is an empty set and is of **infinite-horizon** otherwise.

Many definitions from finite-horizon extensive-form games translate into the infinite-horizon ones in a straightforward way. For example, we can define the notions of **perfect/imperfect information** and **perfect recall**, as well as **pure-, mixed-, or behavioral strategies** in the obvious way. Moreover, we can define a Nash equilibrium in the same way, with a caveat that a distribution, induced by a strategy profile, is defined over both terminal nodes and infinite paths of play.

Remark 2. Kuhn's Theorem assumes finite extensive-form games (with the set X of nodes finite). Since every finite extensive-form games is of finite-horizon, we cannot immediately apply Kuhn's Theorem to infinite extensive-form games (in particular, infinite-horizon ones). However, Kuhn's Theorem is generalizable to infinite extensive-form games that satisfy mild conditions (Aumann, 1964). □

We can define a **subgame** $\Gamma(x)$ that follows a non-terminal node $x \in X \setminus Z$ in the obvious way. We also define a **subgame perfect equilibrium** as a strategy profile that is a Nash equilibrium when restricted to each subgame of an original extensive-form game Γ. Note that a finite-horizon extensive-form game may have infinitely many subgames.

Remark 3. It may be difficult to find a subgame perfect equilibrium in an infinite-horizon extensive-form game. For example, we can solve a finite-horizon extensive-form game of perfect information by backward induction (at least in principle). This is possible because in a finite-horizon extensive-form game, every path ends with a terminal node—that is, the "last" move is well-defined.

However, an infinite-horizon extensive-form game may not has the "last" nodes. Then, how can we find a subgame perfect equilibrium in an infinite-horizon extensive-form game? What is worse, it can be a daunting task to verify directly whether a strategy profile is a subgame perfect equilibrium from the definition. This is because we would have to show that, for every subgame, each player has no profitable deviation among all her possible strategies (in that subgame). □

References

Aumann, R. J. (1964). Mixed and behavior strategies in infinite extensive games. In M. Dresher, L. S. Shapley, & A. W. Tucker (Eds.), *Advances in game theory* (pp. 627–650). Princeton University Press.