THE NEOCLASSICAL GROWTH MODEL

The Baseline Model

Dynamic General Equilibrium model with two types of agents:

- A large number of identical households that live an infinite number of periods (e.g. representative household)

- A large number of identical firms that produces the only good in the economy (e.g. representative firm)

The representative household (of size L_t) is characterized by:

- Preferences: intertemporal utility function
$$U = \sum_{t=0}^{\infty} \beta^t u\left(\frac{C_t}{L_t}\right)$$
u : period utility function, satisfying $u' > 0$, $u'' < 0$ and
$$\lim_{c \to 0} u'(c) = \infty$$
$\beta \in (0,1)$: discount factor

- Endowments: households own all the capital and labor in the economy, which they rent to firms

Budget constraint:

$$C_t + I_t = w_t L_t + r_t K_t + \Pi_t$$

The price of the unique good (numeraire) is normalized to $p_t = 1$, $\forall t$

$\Rightarrow w_t$ and r_t are relative prices expressed in units of the only good

The household's stock of capital follows the law of motion:

$$K_{t+1} = (1 - \delta)K_t + I_t$$

The number of workers (size of the household) grows a the constant rate n:

$$L_{t+1} = (1 + n)L_t$$

The representative firm is characterized by:

- Technology: aggregate production function

$$Y_t = F(K_t, L_t)$$

  satisfying: (i) constant returns to scale , (ii) concavity: $F_K, F_L > 0$, $F_{KK}, F_{LL} < 0$ y $F_{KL} > 0$, and (iii) Inada conditions:

$$\lim_{K \to 0} F_K(K, L) = \infty \qquad \lim_{K \to \infty} F_K(K, L) = 0$$

$$\lim_{L \to 0} F_L(K, L) = \infty \qquad \lim_{L \to \infty} F_L(K, L) = 0$$

- Objective: maximize profits

$$\Pi_t = Y_t - w_t L_t - r_t K_t$$

Model in intensive form:

(variables c_t, i_t, k_t, y_t in units of the good *per worker*)

$$u\left(\frac{C_t}{L_t}\right) = u(c_t)$$

$$\frac{K_{t+1}}{L_t} = \frac{L_{t+1}}{L_t}\frac{K_{t+1}}{L_{t+1}} = (1+n)\,k_{t+1}$$

$$y_t = \frac{Y_t}{L_t} = F\left(\frac{K_t}{L_t}, 1\right) = f(k_t)$$

with $f' > 0$, $f'' < 0$ and

$$\lim_{k\to 0} f'(k) = \infty \qquad \lim_{k\to\infty} f'(k) = 0$$

`Competitive General Equilibrium`

A *Competitive General Equilibrium* (CGE) for this economy is a set of sequences for the quantities c_t, i_t, y_t and k_{t+1} and the prices w_t and r_t such that:

i) Given $k_0 > 0$, w_t and r_t, the sequences c_t, i_t and k_{t+1} solve the problem of the representative household:

$$\max \qquad \sum_{t=0}^{\infty} \beta^t u(c_t)$$

$$\begin{array}{lrcll} s.t. & c_t + i_t & = & w_t + r_t k_t & \forall t \\ & (1+n)\,k_{t+1} & = & (1-\delta)\,k_t + i_t & \forall t \\ & c_t, k_{t+1} & \geq & 0 & \end{array}$$

ii) In each period t, given w_t and r_t, the values y_t and k_t solve the problem of the representative firm:

$$\begin{array}{ll} \text{max} & y_t - w_t - r_t k_t \\ s.t. & y_t = f(k_t) \end{array}$$

and profits are equal to zero

$$y_t = w_t + r_t k_t$$

iii) In each period t, markets clear:

$$y_t = c_t + i_t$$

Notice that we are automatically assuming that the markets for labor and capital are in equilibrium

The Social Planner Problem

Given $k_0 > 0$, the social planner solves

$$\text{max} \qquad \sum_{t=0}^{\infty} \beta^t u(c_t)$$

$$\begin{array}{lrcll} s.t. & c_t + i_t & = & f(k_t) & \forall t \\ & (1+n) k_{t+1} & = & (1-\delta) k_t + i_t & \forall t \\ & c_t, k_{t+1} & \geq & 0 & \end{array}$$

The sequences c_t, i_t y k_{t+1} resulting from this maximization problem are *Pareto Efficient* $\Rightarrow$ there is no possible way of increasing the utility of one of the families without reducing the utility of another family

Welfare Theorems: without distortions such as taxes or externalities,

- WT1: Every Competitive Equilibrium is Pareto Efficient
- WT2: For each Pareto Efficient allocation there exists a system of prices that make that allocation a Competitive Equilibrium

Strategy: find the CGE by first solving the social planner problem and then find the prices from the firms' maximization problem

`First Order Conditions`

The Lagrangian (L) for the social planner problem is given by:

$$L = \sum_{t=0}^{\infty} \Big[\beta^t u(c_t) - \lambda_{1t}(c_t + i_t - f(k_t))$$

$$-\lambda_{2t}((1+n)k_{t+1} - (1-\delta)k_t - i_t)]$$

Notice that we are not considering the non-negative restrictions for consumption and investment: $c_t, k_{t+1} \geq 0$ (*Inada conditions* ensure an interior solution*)*

Maximizing L, we obtain the following first order conditions (FOC)

$$\begin{aligned}
\frac{\partial L}{\partial c_t} &= \beta^t u'(c_t) - \lambda_{1t} = 0 \\
\frac{\partial L}{\partial i_t} &= -\lambda_{1t} + \lambda_{2t} = 0 \\
\frac{\partial L}{\partial k_{t+1}} &= \lambda_{1t+1} f'(k_{t+1}) - \lambda_{2t}(1+n) + \lambda_{2t+1}(1-\delta) = 0
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial L}{\partial \lambda_{1t}} &= c_t + i_t - f(k_t) = 0 \\
\frac{\partial L}{\partial \lambda_{2t}} &= (1+n)\, k_{t+1} - (1-\delta)\, k_t - i_t = 0
\end{aligned}$$

In addition, we impose the *transversality condition*

$$\lim_{t\to\infty} \left(\frac{\lambda_{2t}}{\lambda_{20}}\right) k_{t+1} = 0$$

in which λ_{2t} represents the value (shadow price) of an extra unit of capital

The transversality condition ensures that the value of the stock of capital at the "end" of the problem equals zero

Notice that, using the first order conditions, we can rewrite the transversality condition as

$$\lim_{t\to\infty} \beta^t \left(\frac{u'(c_t)}{u'(c_0)}\right) k_{t+1} = 0$$

Solving these equations, we obtain:

- The *Euler Equation*:

$$\frac{u'(c_t)}{\beta u'(c_{t+1})} = \frac{f'(k_{t+1}) + (1-\delta)}{1+n}$$

  The marginal rate of substitution between present consumption and future consumption, adjusted by a discount factor, must be equal to the rate of return of investment for one period

- The *Feasibility Condition*:

$$c_t = f(k_t) - (1+n)\, k_{t+1} + (1-\delta)\, k_t$$

  The consumption is the difference between the total production less the investment in new capital

We have a non-linear system of two equations in difference of first order for c_t and k_t

In addition, we have:

- the initial condition k_0 , and

- the transversality condition

... this system characterizes the Pareto Optimum *(PO)*

We obtain the prices by solving the problem of the representative firm

$$\max \qquad f(k_t) - w_t - r_t k_t$$

from where

$$r_t = f'(k_t)$$

To find w_t we use the zero profit condition, such that

$$w_t = f(k_t) - f'(k_t)k_t$$

The optimal trajectories for c_t and k_t, together with the prices, constitutes a CGE

To verify that we have a CGE, we solve the household problem with Lagrangian:

$$L = \sum_{t=0}^{\infty} \left[\beta^t u(c_t) - \lambda_{1t}(c_t + i_t - w_t - r_t k_t) \right.$$
$$\left. -\lambda_{2t}\left((1+n)k_{t+1} - (1-\delta)k_t - i_t\right)\right]$$

and first order conditions:

$$\frac{\partial L}{\partial c_t} = \beta^t u'(c_t) - \lambda_{1t} = 0 \qquad \frac{\partial L}{\partial i_t} = -\lambda_{1t} + \lambda_{2t} = 0$$

$$\frac{\partial L}{\partial k_{t+1}} = \lambda_{1t+1} r_{t+1} - \lambda_{2t}(1+n) + \lambda_{2t+1}(1-\delta) = 0$$

$$\frac{\partial L}{\partial \lambda_{1t}} = c_t + i_t - w_t - r_t k_t = 0 \qquad \frac{\partial L}{\partial \lambda_{2t}} = (1+n)k_{t+1} - (1-\delta)k_t - i_t = 0$$

By solving, we obtain the Euler equation:

$$\frac{u'(c_t)}{\beta u'(c_{t+1})} = \frac{r_{t+1} + (1-\delta)}{1+n}$$

and the feasibility condition:

$$c_t = w_t + [r_t + (1-\delta)]\, k_t - (1+n)\, k_{t+1}$$

Replacing the prices obtained from the firm's problem:

$$r_{t+1} = f'(k_{t+1}) \qquad w_t + r_t k_t = f(k_t)$$

$\Rightarrow$ The *CGE* and the *PO* are equivalent

`Steady State`

A *Steady State* is a CGE in which all the quantities (per-worker) are constant over time:

$$c_{t+1} = c_t = c^* \qquad k_{t+1} = k_t = k^*$$

Therefore, the absolute quantities C_t, K_t grow at the rate n

From the Euler equation:

$$f'(k^*) = \frac{1+n}{\beta} - (1-\delta)$$

$\Rightarrow$ There exists a unique stock of capital per-worker k^* in the steady state

Stability result: Starting from any $k_0 > 0$, the economy converges in the long run to the unique steady state.

Extensions: Endogenous labor Supply

Until now, the supply of labor is exogenous and does not respond to changes in the real wage

We are going to make this decision endogenous by introducing leisure-consumption decision in the baseline model

The model focuses on the intensive (not the extensive) margin of the labor supply

Augmented intertemporal utility function:

$$U = \sum_{t=0}^{\infty} \beta^t u\left(\frac{C_t}{L_t}, \frac{L_t - L_t^s}{L_t}\right)$$

L_t^s : total labor supply of the representative household

$L_t - L_t^s$: leisure time

Assumptions: $u_1 > 0$, $u_2 > 0$, $u_{11} < 0$, $u_{22} < 0$ y $u_{21} > 0$

In intensive form, we divide all the variables by L_t

- One period utility function:

$$u(c_t, 1 - l_t)$$

- Budget constraint:

$$c_t + i_t = w_t l_t + r_t k_t$$

- Production function:

$$y_t = \frac{Y_t}{L_t} = F\left(\frac{K_t}{L_t}, \frac{L_t^s}{L_t}\right) = F(k_t, l_t)$$

A *CGE* is a set of sequences for the quantities c_t, i_t, l_t, y_t and k_{t+1} and prices w_t and r_t such that:

i) Given $k_0 > 0$, w_t and r_t, the sequences for c_t, l_t, i_t and k_{t+1} solve the household problem:

$$\max \quad \sum_{t=0}^{\infty} \beta^t u(c_t, 1 - l_t)$$

$$\begin{aligned} s.t. \quad c_t + i_t &= w_t l_t + r_t k_t \\ (1+n)\, k_{t+1} &= (1-\delta)\, k_t + i_t \\ c_t, k_{t+1} &\geq 0 \\ 0 &\leq l_t \leq 1 \end{aligned}$$

ii) In each period t, given w_t and r_t, the values y_t, k_t and l_t solve the firm problem:

$$\begin{array}{ll} \max & y_t - w_t l_t - r_t k_t \\ s.t. & y_t = F(k_t, l_t) \end{array}$$

iii) In each period t, markets clear:

$$y_t = c_t + i_t$$

The social planner problem is given by:

$$\max \quad \sum_{t=0}^{\infty} \beta^t u(c_t, 1 - l_t)$$

$$\begin{array}{lrcl} s.t. & c_t + i_t & = & F(k_t, l_t) \\ & (1+n) k_{t+1} & = & (1-\delta) k_t + i_t \end{array}$$

with Lagrangian:

$$\begin{aligned} L = & \sum_{t=0}^{\infty} \left[\beta^t u(c_t, 1 - l_t) - \lambda_{1t} (c_t + i_t - F(k_t, l_t)) \right. \\ & \left. -\lambda_{2t} ((1+n) k_{t+1} - (1-\delta) k_t - i_t) \right] \end{aligned}$$

The first order conditions for this problem are:

$$
\begin{aligned}
\frac{\partial L}{\partial c_t} &= \beta^t u_1\left(c_t, 1-l_t\right) - \lambda_{1t} = 0 \\
\frac{\partial L}{\partial l_t} &= -\beta^t u_2\left(c_t, 1-l_t\right) + \lambda_{1t} F_L\left(k_t, l_t\right) = 0 \\
\frac{\partial L}{\partial i_t} &= -\lambda_{1t} + \lambda_{2t} = 0 \\
\frac{\partial L}{\partial k_{t+1}} &= \lambda_{1t+1} F_K\left(k_{t+1}, l_{t+1}\right) - \lambda_{2t}(1+n) + \lambda_{2t+1}(1-\delta) = 0
\end{aligned}
$$

plus the standard transversality condition

Combining, we obtain the Euler equation:

$$
\frac{u_1\left(c_t, 1-l_t\right)}{\beta u_1\left(c_{t+1}, 1-l_{t+1}\right)} = \frac{F_K\left(k_{t+1}, l_{t+1}\right) + (1-\delta)}{1+n}
$$

the feasibility condition:

$$
c_t = F\left(k_t, l_t\right) - (1+n)\, k_{t+1} + (1-\delta)\, k_t
$$

and one additional static equation:

$$
u_1\left(c_t, 1-l_t\right) = \frac{u_2\left(c_t, 1-l_t\right)}{F_L\left(k_t, l_t\right)}
$$

This relation implicitly defines a labor supply function that depends positively on the real wage (or F_L)

The steady state, in which all variables per-worker remains constant, is characterized by the following system of equations:

$$\frac{1}{\beta} = \frac{F_K(k^*, l^*) + (1-\delta)}{1+n}$$

$$c^* = F(k^*, l^*) - (n+\delta) k^*$$

$$u_1(c^*, 1-l^*) = \frac{u_2(c^*, 1-l^*)}{F_L(k^*, l^*)}$$

that we can solve for k^*, l^* and c^*

`Exogenous Growth and Technological Changes`

Steady state: the product per-worker remains constant

$\Rightarrow$ In the baseline model there is no long-run growth

Now we are going to introduce exogenous technological growth affecting labor productivity

Production function:

$$F(K_t, A_t L_t)$$

$$A_{t+1} = (1+g) A_t$$

A_t : technological level ($A_0 = 1$)

g : exogenous rate of technical progress

Dividing all variables by $A_t L_t$ in order to express them in *effective units of labor (e.u.l.)*

$$\hat{c}_t = \frac{C_t}{A_t L_t} = \frac{c_t}{A_t}$$

Production function:

$$\hat{y}_t = F\left(\frac{K_t}{A_t L_t}, 1\right) = f(\hat{k}_t)$$

Budget constraint: $\hat{c}_t + \hat{\imath}_t = \hat{w}_t + r_t \hat{k}_t$, where $\hat{w}_t = \frac{w_t}{A_t}$

The law of motion for the capital: $(1+g)(1+n)\hat{k}_{t+1} = (1-\delta)\hat{k}_t + \hat{\imath}_t$

Transforming the utility function in e.u.l. terms is more complicated

Special case:

$$u(c_t) = \frac{c_t^{1-\sigma}}{1-\sigma}$$

Then:

$$\begin{aligned}
\sum_{t=0}^{\infty} \beta^t u(c_t) &= \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\sigma}}{1-\sigma} \\
&= \sum_{t=0}^{\infty} \beta^t A_t^{1-\sigma} \frac{\hat{c}_t^{1-\sigma}}{1-\sigma} \\
&= \sum_{t=0}^{\infty} \hat{\beta}^t u(\hat{c}_t)
\end{aligned}$$

where $\hat{\beta} = \beta(1+g)^{1-\sigma}$

We have redefined the variables of the model such that its structure is similar to the baseline model

$\Rightarrow$ The definition of equilibrium and first order conditions are the same

Euler Equation:

$$\frac{u'(\hat{c}_t)}{\hat{\beta} u'(\hat{c}_{t+1})} = \frac{f'(\hat{k}_{t+1}) + (1-\delta)}{(1+n)(1+g)}$$

Feasibility condition:

$$\hat{c}_t = f(\hat{k}_t) - (1+n)(1+g)\,\hat{k}_{t+1} + (1-\delta)\,\hat{k}_t$$

In the long term, the economy converges to the steady state, where $\hat{k}_t$ and $\hat{c}_t$ remains constant

In this steady state:

$$f'(\hat{k}^*) = \frac{(1+n)(1+g)}{\hat{\beta}} - (1-\delta)$$

$$\hat{c}^* = f(\hat{k}^*) - (\delta + n + g + ng)\,\hat{k}^*$$

Unlike the previous baseline model, in steady state all variables per-worker grow at the same constant rate g (*balanced growth path*)

We have long-run growth, but at a rate that is exogenously given by technological progress and is independent of other parameters