Game Theory

Iterated Strict Dominance

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A player often has a strategy that does strictly worse than another strategy of hers, regardless of the opponents' strategies. Such a strategy is called **strictly dominated**. No player will play strictly dominated strategies. Assume that all players will never play strictly dominated strategies, that they know they will never play the strictly dominated strategies, that they know they know they will never play the strictly dominated strategies, and so on.[1] *What are all the strategies that they could potentially play based only on this assumption?*

1 Strict Dominance

Example 1. Consider the following prisoners' dilemma:

	C	D
C	$1, 1$	$-1, 2$
D	$2, -1$	$0, 0$

Table 1: prisoners' dilemma

For player 1, C is strictly less profitable than D regardless of player 2's (mixed) strategies. In this case, we say that C is strictly dominated by D (for player 1). □

1.1 Strict Dominance

We will now generalize this example to obtain the notion of strict dominance.

Definition 1. In a normal-form game G, player i's strategy $\sigma_i \in \Sigma_i$ is **strictly dominated** by her strategy $\sigma_i' \in \Sigma_i$ if for player $-i$'s every strategy profile $\sigma_{-i} \in \Sigma_{-i}$

$$u_i(\sigma_i', \sigma_{-i}) > u_i(\sigma_i, \sigma_{-i}).$$

Moreover, σ_i is **strictly dominated** if it is strictly dominated by some σ_i'.

Whenever a strategy is strictly dominated, there exists some strategy that does strictly better *regardless of opponents' strategies*. This implies that a rational player—who maximizes her (expected) payoff—will never use strictly dominated strategies.

Definition 2. In a normal-form game G, player i's strategy $\sigma_i \in \Sigma_i$ is **strictly dominant** if her every strategy $\sigma_i' \in \Sigma_i \setminus \{\sigma_i\}$ is strictly dominated by σ_i.

[1] This assumption is called common knowledge of their never playing strictly dominated strategies.

Properties of Strictly Dominated Strategies We will now see important properties of strict dominance. First, we can simplify the definition.

Proposition 1. *In a normal-form game G, player i's strategy $\sigma_i \in \Sigma_i$ is strictly dominated by her strategy $\sigma_i' \in \Sigma_i$ if and only if for player $-i$'s every action profile $a_{-i} \in A_{-i}$,*

$$u_i(\sigma_i', a_{-i}) > u_i(\sigma_i, a_{-i}).$$

Proof. The "only if" part is trivial. The "if" part is also immediate since expect payoffs are linear in opponents' mixed strategies. ■

Second, even if player i's strategy σ_i is not strictly dominated by any of her pure strategy a_i, it is still possible that σ_i is strictly dominated by some of her mixed strategy σ_i'. That is, to check if σ_i is strictly dominated, it is not sufficient to examine all her *pure* strategies. Here is an example to illustrate this point.

Example 2. Consider the following normal-form game:

	L	R
U	$3, 0$	$-1, 0$
M	$0, 0$	$0, 0$
D	$-1, 0$	$3, 0$

Table 2: M is strictly dominated

No pure strategy is strictly dominated by any pure strategy. However, M is strictly dominated by $\frac{1}{2}U \oplus \frac{1}{2}D$, which randomizes actions U and D with equal probabilities. □

Third, if a pure strategy a_i is strictly dominated then every mixed strategy σ_i that plays action a_i with non-zero probability is also strictly dominated.

Proposition 2. *In a normal-form game G, if player i's pure strategy $a_i \in A_i$ is strictly dominated then her every mixed strategy $\sigma_i \in \Sigma_i$ such that $\sigma_i(a_i) > 0$ is also strictly dominated.*

Proof. Immediate. ■

1.2 Iterated Strict Dominance

Example 3. Consider the following normal-form game:

	L	M	R
U	$2, 2$	$1, 1$	$4, 0$
D	$1, 2$	$4, 1$	$3, 5$

Table 3: iterated deletion of strictly dominated strategies

Neither U nor D is strictly dominated for player 1, while M is strictly dominated for player 2 and will never be played. Player 1 (she) also thinks that if player 2 (he) is rational, then he never plays M, and thus she can delete it from her thinking. After deleting it, player 1 realizes that D is now strictly dominated for her.

If player 2 knows not only that player 1 is rational, but also that player 1 knows that player 2 is rational, then player 2 thinks that player 1 would delete M from her thinking and thus would not play D. Hence, he also deletes D from his thinking. Since he now thinks U will be played, R is strictly dominated and deleted. Consequently, (U, L) is left.

In the above reasoning, we say that D and R are iteratively strictly dominated, respectively, for players 1 and 2. We also say that U and L survive the iterated deletion of strictly dominated strategies. □

Remark 1. In an introductory course in game theory, you might have iteratively deleted strictly dominated "rows" and "columns" mechanically—without considering the rationale behind the process. However, behind the process there is the assumption that players know about their rationality, they know they know about their rationality, they know they know they know about their rationality, and so on. □

Iterated Strict Dominance: Case of Pure Strategies We formalize the idea illustrated in Example 3.

Definition 3. In a normal-form game G, for each $i \in I$ and each $k \in \mathbb{N}$, let $\mathrm{ND}_i^0 = A_i$ and

$$\mathrm{ND}_i^k = \mathrm{ND}_i^{k-1} \setminus \underbrace{\left\{ a_i \in \mathrm{ND}_i^{k-1} : \exists\, \sigma_i' \in \Delta\left(\mathrm{ND}_i^{k-1}\right) \quad \forall\, a_{-i} \in \mathrm{ND}_{-i}^{k-1} \quad u_i(\sigma_i', a_{-i}) > u_i(a_i, a_{-i}) \right\}}_{\text{pure strategies that are strictly dominated}}.$$

Let the set of player i's **pure strategies that survive iterated deletion of strictly dominated strategies** be such that:

$$\mathrm{ND}_i^\infty = \bigcap_{k=0}^{\infty} \mathrm{ND}_i^k.$$

We revisit the rationale behind the iterated deletion process. Every step asks "what are actions that will never be played by rational players?" Then, each player will conclude that no (rational) player will ever play pure strategies that are strictly dominated. Since each player will expect that no (rational) player will play such pure strategies. Furthermore, it will be known that players arrive at this conclusion. That justifies the elimination of these pure strategies from the game. Players will iterate this process until no further pure strategies can be eliminated.

To better understand this concept, we formalize Example 3.

Example 4. Consider the normal-form game of Example 3. Then, the above iteration process yields the following sets:

$$\begin{array}{ll} \mathrm{ND}_1^0 = \{U, D\} & \mathrm{ND}_2^0 = \{L, M, R\} \\ \mathrm{ND}_1^1 = \mathrm{ND}_1^0 & \mathrm{ND}_2^1 = \{L, R\} \\ \mathrm{ND}_1^2 = \{U\} & \mathrm{ND}_2^2 = \mathrm{ND}_2^1 \\ \mathrm{ND}_1^3 = \mathrm{ND}_1^2 & \mathrm{ND}_2^3 = \{L\} \\ \mathrm{ND}_1^4 = \mathrm{ND}_1^3 & \mathrm{ND}_2^4 = \mathrm{ND}_2^3 \end{array}$$

By induction, $\mathrm{ND}_1^\infty = \{U\}$ and $\mathrm{ND}_2^\infty = \{L\}$.

Like this game, a normal-form game is called **dominance solvable** if ND_i^∞ is a singleton for each $i \in I$. For a dominance-solvable game, we can make a sharp prediction about play under weak assumptions. □

Existence We will now discuss the existence of strategies that survive iterated deletion of strictly dominated strategies.

Proposition 3. *In a finite normal-form game G, $\mathrm{ND}_i^\infty \neq \varnothing$ for each $i \in I$.*

Proof. Immediate. ■

The finiteness assumption is necessary for the existence, as illustrated below:

Example 5. Consider the following infinite normal-form game:

	1	2	3	4	5	$\cdots$
1	$0, 0$	$-1, 1$	$-2, 1$	$-3, 1$	$-4, 1$	$\cdots$
2	$1, -1$	$0, 0$	$-1, 2$	$-2, 2$	$-3, 2$	$\cdots$
3	$1, -2$	$2, -1$	$0, 0$	$-1, 3$	$-2, 3$	$\cdots$
4	$1, -3$	$2, -2$	$3, -1$	$0, 0$	$-1, 4$	$\cdots$
5	$1, -4$	$2, -3$	$3, -2$	$4, -1$	$0, 0$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Table 4: $\mathrm{ND}_i^\infty = \varnothing$.

For each $i \in I$ and each $k \in \mathbb{N}$, $\mathrm{ND}_i^0 = \{1, 2, \ldots\}$ and $\mathrm{ND}_i^k = \mathbb{N} \setminus \{1, 2, \ldots, k\}$. It is then immediate that $\mathrm{ND}_i^\infty = \bigcap_{k=0}^\infty \mathrm{ND}_i^k = \varnothing$. □

Order Independence of Iterated Strict Dominance In Definition 3, we delete *all* pure strategies that are strictly dominated at each step. However, there seems no compelling reason why players have to delete all strictly dominated strategies. What would happen if we changed the order in which we delete strictly dominated strategies? Do we still have the same result regardless of the order of deletion? Yes, but we omit the proof. For the proof, see Osborne & Rubinstein (1994, Proposition 61.2 together with the discussion right after the proposition).

Iterated Strict Dominance: Case of Mixed Strategies We consider the set of mixed strategies that survive iterated deletion of strictly dominated strategies. One may jump to the (wrong) conclusion that it is the set $\Delta(\mathrm{ND}_i^\infty)$ of all distributions over ND_i^∞, but this is not true in general. That is, mixed strategies in $\Delta(\mathrm{ND}_i^\infty)$ can be strictly dominated by some strategy in $\Delta(\mathrm{ND}_i^\infty)$.

Example 6. Consider the following normal-form game:

	L	R
U	$1, 0$	$-2, 0$
M	$-2, 0$	$1, 0$
D	$0, 0$	$0, 0$

Table 5: a mixed strategy in $\Delta(\mathrm{ND}_i^\infty)$ can be strictly dominated

For each player, all pure strategies survive iterated deletion of strictly dominated strategies: $\mathrm{ND}_i^\infty = A_i$. Hence, $\frac{1}{2}U \oplus \frac{1}{2}M \in \Delta(\mathrm{ND}_1^\infty)$, but it yields payoff $-\frac{1}{2}$ regardless of player 2's strategy; that is, $\frac{1}{2}U \oplus \frac{1}{2}M$ is strictly dominated by D. □

2 Weak Dominance

So far we have studied strict dominance. We have another concept of weak dominance.

Example 7. Consider the following normal-form game:

	L	R
U	$1, 1$	$1, 0$
D	$0, 1$	$1, 1$

Table 6: D and R are weakly dominated

Neither has strictly dominated strategies. However, it seems strange that player 1 chooses D or player 2 chooses R. For player 1, U is strictly better than D if player 2 chooses L, and U is as good as D if player 2 chooses R. In this case, we say that D is weakly dominated by U (for player 1). Similarly, R is weakly dominated by L (for player 2). □

2.1 Weak Dominance

We can formalize the concept of weak dominance by relaxing the strict inequalities in the definition of strict dominance with weak inequalities.

Definition 4. In a normal-form game G, player i's strategy $\sigma_i \in \Sigma_i$ is **weakly dominated** by her strategy $\sigma_i' \in \Sigma_i$ if and only if the following holds:

1. $u_i(\sigma_i', a_{-i}) \geq u_i(\sigma_i, a_{-i})$ for each $a_{-i} \in A_{-i}$.

2. $u_i(\sigma_i', a_{-i}) > u_i(\sigma_i, a_{-i})$ for some $a_{-i} \in A_{-i}$.

Remark 2. A weakly dominated strategy may be played in a Nash equilibrium (although a strictly dominated strategy is never played in any Nash equilibrium). In Example 7, the game has a pure-strategy Nash equilibrium (D, R), in which both players use the weakly dominated strategies. □

2.2 Difficulty in Defining Iterated Weak Dominance

If a player has a weakly dominated strategy then she has no reason that she should use it. Therefore, it sounds natural to delete weakly dominated strategies, which leads to the concept of iterated weak dominance. However, we cannot actually have a reasonable definition of iterated weak dominance. This is because of the critical theoretical defect illustrated below.

Example 8. Consider the following normal-form game:

	L	R
U	$1, 1$	$0, 0$
M	$1, 1$	$2, 1$
D	$0, 0$	$2, 1$

Table 7: the order dependence of iterated deletion of weakly dominated strategies

There are two ways to delete weakly dominated strategies. First, if we delete U (that is weakly dominated by M), then we can delete L (that is weakly dominated by R), which leaves R for player 2. Second, if we delete D (that is weakly dominated by M), then we can delete R (that is weakly dominated by L), which leaves L for player 2. Hence, the prediction is sensitive to the order of deletion, which suggests that we cannot have a reasonable definition of iterated weak dominance. □

3 Trembling-Hand Perfect Equilibrium*

Despite the above reasons that make it less compelling to exclude weakly dominated strategies, it is still true that there is no advantage to using a weakly dominated strategy. Therefore, it seems natural not to play weakly dominated strategies.

This intuition is justified if we allow for "'mistakes." In Example 7, player 1 is willing to play D, which is weakly dominated, only if (she thinks) player 2 plays R with probability 1. In other words, if player 2 may "trembles his hand" to take an unintended action with small probability, then player 1 is no longer willing to play D.

Recap (Strategies): In a normal-form game G, player i's strategy $\sigma_i \in \Sigma_i$ is called as follows:

1. σ_i is a **pure strategy** if it assigns probability 1 to a single action.
2. σ_i is a **strictly mixed strategy** if it is not a pure strategy.

The notion of completely mixed strategies (Definition 5) may be confused with the notion of strictly mixed strategies. In general, the two are different but equivalent when a player has binary actions.

Example 9. In Example 7, suppose that player 2 "trembles his hand" so that when he intends to take action a_2, he might take action $a_2' \neq a_2$. That is, he takes a mixed strategy σ_2 that assigns strictly positive probabilities to both actions. Then, player 1 will never take D, which is weakly dominated. Her payoff from U is $\sigma_2(L) + \sigma_2(R) = 1$, while her payoff from D is $\sigma_2(R)$. Since $\sigma_2(R) < 1$, player 1 will never want to play D. □

Roughly speaking, we say that a player "trembles her hand" if she chooses any action with strictly positive (yet possibly small) probabilities.

Definition 5. In a finite normal-form game G, player i's strategy $\sigma_i \in \Sigma_i$ is said to be a **completely mixed strategy** if it assigns strictly positive probabilities to all her actions.

Now we introduce the concept of trembling-hand perfection.

Definition 6. In a finite normal-form game G, a mixed strategy profile σ is a **trembling-hand perfect equilibrium** if there exists a sequence $(\sigma^k)_k$ of completely mixed strategy profiles such that:

1. $\sigma^k \to \sigma$ as $k \to \infty$.
2. σ_i is a best response to σ^k_{-i} for each $i \in I$ and each $k \in \mathbb{N}$.

Example 10. In Example 7, a Nash equilibrium (U, L) is trembling-hand perfect. To show this, it suffices to find a sequence $(\sigma^k)_k$ of completely mixed strategy profiles such that:

1. $\sigma^k_1(U) \to 1$ and $\sigma^k_2(L) \to 1$ as $k \to \infty$.
2. U is a best response to σ^k_2 and L is a best response to σ^k_1 for each $k \in \mathbb{N}$.

For example, let σ^k be such that $\sigma^k_1(U) = \sigma^k_2(L) = 1 - \frac{1}{k+1}$ for each $k \in \mathbb{N}$. It is not difficult to see that this sequence satisfies the desired conditions. In contrast, a Nash equilibrium (D, R) is not trembling-hand perfect, as suggested in Example 9. □

References

Osborne, M. J., & Rubinstein, A. (1994). *A course in game theory.* MIT Press.