CPT Lecture Notes 2: Real Analysis

Levent Ülkü

August 25, 2019

The following is a list of concepts and results from Real Analysis which we will use in our class.

Let X be a set.

A function $d : X \times X \to \mathbb{R}$ is a metric on X if for all $x, y, z \in X$ the following conditions hold:
(1) $d(x, y) \geq 0$.
(2) $d(x, y) = 0$ if and only if $x = y$.
(3) $d(x, y) = d(y, x)$.
(4) $d(x, y) \leq d(x, z) + d(z, y)$.

A metric space is a pair (X, d) where X is a set and d is a metric on X.

Let (X, d) be a metric space.

For any $x \in X$ and $\varepsilon > 0$, $B(x, \varepsilon) = \{y \in X : d(x, y) < \varepsilon\}$ is the open ball centered at x with radius ε.

Let $S \subseteq X$.

x is an interior point of S if $B(x, \varepsilon) \subset S$ for some $\varepsilon > 0$. The set of all interior points of S is the interior of S, denoted $int(S)$. Note: $int(S) \subseteq S$.

S is open if $S = int(S)$. Note: $X, \emptyset, int(S)$ and $B(x, \varepsilon)$ are open.

Let (X, d) be a metric space and $S \subseteq X$.

x is a closure point of S if $B(x, \varepsilon) \cap S \neq \emptyset$ for every $\varepsilon > 0$. The set of all closure points of S is the closure of S, denoted $cl(S)$. Note: $S \subseteq cl(S)$.

S is closed if $S = cl(S)$. Note: X, $\emptyset$, $cl(S)$ are closed

Theorem: S is closed if and only if $X \backslash S$ is open.

Let (X, d) be a metric space and $S \subseteq X$.

x is a boundary point of S if $B(x, \varepsilon) \cap S \neq \emptyset$ and $B(x, \varepsilon) \cap (X \backslash S) \neq \emptyset$ for every $\varepsilon > 0$. The set of all boundary points of S is the boundary of S, denoted ∂S.

Note:
(1) $\partial S = \partial(X \backslash S)$.
(2) $\partial S \subseteq cl(S)$.
(3) S is closed if and only if $\partial S \subseteq S$.
(4) ∂S is a closed set.

Let (X, d) be a metric space and $S \subseteq X$.

d is a metric on S and therefore (S, d) is a metric space as well, usually referred to as a metric subspace of (X, d).

Theorem: Let (S, d) be a metric subspace of (X, d).
(1) $A \subseteq S$ is open in (S, d) if and only if there exists an open set U in (X, d) such that $A = S \cap U$.
(2) $A \subseteq S$ is closed in (S, d) if and only if there exists a closed set U in (X, d) such that $A = S \cap U$.

Let (X, d) be a metric space.

A sequence in (X, d) is a map $f : \mathbb{N} \to X$, usually denoted $\{x_k\}$. A sequence $\{x_k\}$ is convergent if the following property holds: there exists some $x \in X$ such that for every $\varepsilon > 0$, there exists some $l \in \mathbb{N}$ such that for every $k > l$, $x_k \in B(x, \varepsilon)$. Such x is called the limit of $\{x_k\}$, denoted $\lim x_k$. If $\lim x_k$ exists, it is unique.

Let $\{x_k\}$ be a sequence. A subsequence of $\{x_k\}$ is a sequence obtained by deleting some (possibly none, possibly infinitely many) members of $\{x_k\}$. Put differently, let $\{k_n\}$ be a nondecreasing sequence of integers. Then $\{x_{k_n}\}$ is a subsequence of $\{x_k\}$.

Theorem: $\{x_k\}$ is convergent with limit x if and only if every subsequence of $\{x_k\}$ is convergent with limit x.

Hence, if two subsequences of $\{x_k\}$ have different limits, then $\{x_k\}$ is not convergent.

Let (X, d) be a metric space and $S \subseteq X$.

It can be shown that $x \in cl(S)$ if and only if there exists a sequence $\{x_k\}$ such that $x_k \in S$ for all k, and $\lim x_k = x$.

Theorem: S is closed if and only if every convergent sequence in S converges to an element of S.

Let (X, d) be a metric space.

A sequence $\{x_k\}$ in X is bounded if it is contained in a ball with finite radius, i.e., if for some $\varepsilon > 0$ and $y \in X$, $x_k \in B(y, \varepsilon)$ for every k.

Theorem: (Bolzano-Weierstrass) Every bounded sequence in $\mathbb{R}^m$ has a convergent subsequence.

Let (X, d) be a metric space.

A sequence $\{x_k\}$ in X is a Cauchy sequence if for every $\varepsilon > 0$ there exists a positive integer k_ε such that whenever $k, l > k_\varepsilon$, $d(x_k, x_l) < \varepsilon$.

Theorem: If $\{x_k\}$ is convergent, then $\{x_k\}$ is Cauchy. If $\{x_k\}$ is Cauchy, then $\{x_k\}$ is bounded.

Let (X, d) be a metric space.

(X, d) is complete if every Cauchy sequence in X converges to an element of X.

Theorem: Let (X, d) be complete and $S \subseteq X$. (S, d) is complete if and only if S is a closed subset of X.

Let (X, d) be a metric space and $S \subseteq X$.

A collection of open sets O_i , $i \in I$, is an open cover for S if $S \subset \bigcup_{i \in I} O_i$.

Let O_i, $i \in I$, be an open cover for S. If I_0 is a finite subset of I and $S \subset \bigcup_{i \in I_0} O_i$, then the collection O_i, $i \in I_0$, is a finite subcover of S.

S is compact if every open cover of S has a finite subcover.

Theorem: A compact set is bounded and closed.

Theorem: (Sequential compactness) S is compact if and only if every sequence in S has a subsequence that converges to a point in S.

Theorem: (Heine-Borel) A subset of $\mathbb{R}^m$ is compact if and only if it is closed and bounded.

Let (X, d) and (Y, σ) be metric spaces.

A function $f : X \to Y$ is continuous at $x \in X$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that whenever $x' \in B(x, \delta)$, $f(x') \in B(f(x), \varepsilon)$.

f is continuous if it is continuous at x for every $x \in X$.

Theorem: f is continuous at x if and only if for every sequence $\{x_k\}$ in X, whenever $\lim x_k = x$, $\lim f(x_k) = f(x)$.

Let (X, d) and (Y, σ) be metric spaces and $f : X \to Y$.

If $S \subseteq X$, then $f(S) = \{f(x) \in Y : x \in S\}$. If $S \subseteq Y$, then $f^{-1}(S) = \{x \in X : f(x) \in S\}$.

Theorem: The following are equivalent:
(1) f is continuous.
(2) $f^{-1}(S)$ is open if S is open.
(3) $f^{-1}(S)$ is closed if S is closed.

Let (X, d) be a metric space.

A function $f : X \to \mathbb{R}$ is upper semicontinuous if $\{x \in X : f(x) \geq \alpha\}$ is closed for every $\alpha \in \mathbb{R}$.

A function $f : X \to \mathbb{R}$ is lower semicontinuous if $\{x \in X : f(x) \leq \alpha\}$ is closed for every $\alpha \in \mathbb{R}$.

Theorem: $f : X \to \mathbb{R}$ is continuous if and only if f is upper semicontinuous and lower semicontinuous.

Let (X, d) be a metric space and $S \subseteq X$.

A function $f : S \longrightarrow \mathbb{R}$ has a maximizer if for some $x^* \in S$, $f(x) \leq f(x^*)$ for every $x \in S$. Similarly, f has a minimizer if for some $x_* \in S$, $f(x) \geq f(x_*)$ for every $x \in S$.

Theorem: (Weierstrass) Let S be compact and $f : S \longrightarrow \mathbb{R}$ be continuous. Then f has a maximizer and a minimizer.

Theorem: Let S be compact and $f : S \longrightarrow \mathbb{R}$ be upper semi-continuous. Then f has a maximizer.

Theorem: Let S be compact and $f : S \longrightarrow \mathbb{R}$ be lower semi-continuous. Then f has a minimizer.

Let (X, d) be a metric space.

Theorem: (Separation) If S and T are disjoint and closed subsets of X, then there exist disjoint and open sets U and V such that $U \supset S$ and $V \supset T$.