Game Theory

# Subgame Perfect Equilibrium

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For a finite-horizon extensive-form game, we define Nash equilibrium and then introduce another solution concept, called **subgame perfect equilibrium**.

**Remark 1.** All extensive-form games in this note are of perfect recall. □

## 1 Nash Equilibrium

We have define Nash equilibrium for normal-form games. Hence, to define it for finite-horizon extensive-form games, we first transform the extensive-form game into a normal-form game and then define Nash equilibrium for that normal-form game.

**Example 1.** Consider the following extensive-form game:

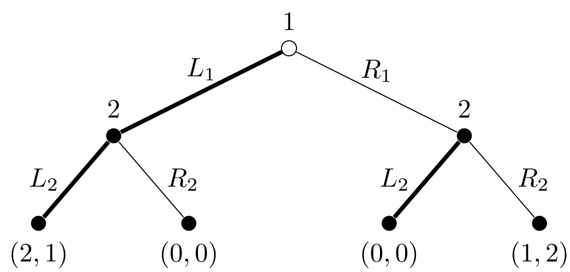


Figure 1: a Nash equilibrium

Player 1's pure-strategy space is $S_1 = \{L_1, R_1\}$, and player 2's pure-strategy space is $S_2 = \{(L_2, L_2), (L_2, R_2), (R_2, L_2), (R_2, R_2)\}$.

Consider the strategy profile illustrated in bold edges in Figure 1.

1. Player $i$'s mixed strategy is a probability distribution over $S_i$. Hence, the mixed-strategy profile $\sigma^* = (\sigma_1^*, \sigma_2^*)$ that corresponds to the bold edges is:
   - $\sigma_1^* = (1, 0)$, which assigns probability 1 to $L_1$.
   - $\sigma_2^* = (1, 0, 0, 0)$, which assigns probability 1 to $(L_2, L_2)$.

2. Player $i$'s behavioral strategy assigns her information set $h_i$ to the distribution over her actions $A(h_i)$. Hence, the behavioral-strategy profile $\beta^* = (\beta_1^*, \beta_2^*)$ that corresponds to the bold edges is:
   - $\beta_1^*(L_1 \mid \{\varnothing\}) = 1$.
   - $\beta_2^*(L_2 \mid \{L_1\}) = \beta_2^*(L_2 \mid \{R_1\}) = 1$.

This strategy profile, whether in mixed strategies or in behavioral strategies, is a Nash equilibrium of the extensive-form game, because neither player has an incentive to deviate. For example, if player 2 plays a strategy $\beta_2^{**}$ such that $\beta_2^{**}(L_2 \mid \{L_1\}) = 1$ and $\beta_2^{**}(R_2 \mid \{R_1\}) = 1$, he still has the same payoff because the information set $\{R_1\}$ is never reached along player 1's strategy $\beta_1^*$. $\square$

### 1.1 Nash Equilibrium

Generalizing the idea presented in Example 1, we define Nash equilibrium for extensive-form games.

**Definition 1.** In a finite-horizon extensive-form game $\Gamma$, a mixed-strategy profile $\sigma^*$ is a **Nash equilibrium** if for each $i \in I$ and for player $i$'s every mixed strategy $\sigma_i$,

$$u_i(\sigma_i^*, \sigma_{-i}^*) \geq u_i(\sigma_i, \sigma_{-i}^*).$$

where for a given mixed-strategy profile $\sigma$, we denote by $u_i(\sigma) \equiv \mathbb{E}_{O(\sigma)}[u_i(z)]$ player $i$'s expected payoff when outcome $z$ is distributed according to the outcome $O(\sigma)$.

**Definition 2.** In a finite-horizon extensive-form game $\Gamma$, a behavioral-strategy profile $\beta^*$ is a **Nash equilibrium** if for each $i \in I$ and for player $i$'s every behavioral strategy $\beta_i$,

$$u_i(\beta_i^*, \beta_{-i}^*) \geq u_i(\beta_i, \beta_{-i}^*),$$

where for a given behavioral-strategy profile $\beta$, we denote by $u_i(\beta) \equiv \mathbb{E}_{O(\beta)}[u_i(z)]$ player $i$'s expected payoff when outcome $z$ is distributed according to the outcome $O(\beta)$.

The following result is a corollary of Kuhn's Theorem (the outcome equivalence between mixed strategies and behavioral strategies in extensive-form games with perfect recall).

**Corollary 1.** *In a finite-horizon extensive-form game $\Gamma$ of perfect recall, Definitions 1 and 2 are equivalent.*

**Existence** The following result is a corollary of Nash's Existence Theorem. Recall that a finite-horizon extensive-form game $\Gamma$ is finite if the set $X$ of nodes is finite.

**Corollary 2.** *In a finite extensive-form game $\Gamma$, there exists a Nash equilibrium.*

## 2 Subgame Perfect Equilibrium

**Example 2.** A Nash equilibrium does not require that a player be rational at her *every* information set. For example, consider a (behavioral) Nash equilibrium $\beta^*$ in Example 1:

$\beta_1^*(L_1 \mid \{\varnothing\}) = 1$ and $\beta_2^*(L_2 \mid \{L_1\}) = \beta_2^*(L_2 \mid \{R_1\}) = 1$. As discussed above, even if player 2 deviates to the strategy $\beta_2^{**}$ illustrated in bold edges in Figure 2, he still has the same payoff.

But, what would he really choose if he *were* at the information set $\{R_1\}$? He would choose action $R_2$, which is more profitable than action $L_2$ (conditional on that information set).

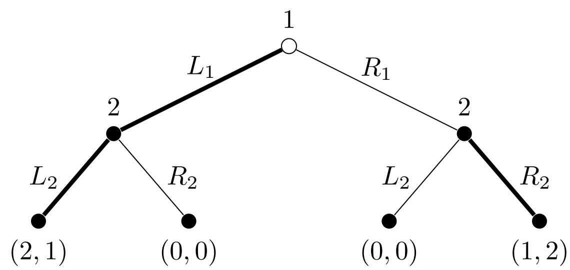


Figure 2: a subgame perfect equilibrium

If player 2 is rational at all information sets, whether or not they are reached with positive probability along player 1's strategy, then player 2 will choose $\beta_2^{**}$. Predicting this behavior, player 1 will still choose $\beta_1^*$. The strategy profile $\beta^{**} = (\beta_1^*, \beta_2^{**})$ is called a subgame perfect equilibrium.

The Nash equilibrium $\beta^*$ includes player 2's **empty threat**, which is not credible. The subgame perfect equilibrium $\beta^{**}$ does not include any empty threat. □

**Remark 2.** For a finite-horizon extensive-form game $\Gamma$ and for a mixed- or behavioral strategy profile, a node $x$ or an information set $h$ is called **on-path** if it is reached with non-zero probability according to the strategy profile, and is called **off-path** otherwise.

Given the jargon, the subgame perfect equilibrium $\beta^{**}$ requires that player 2 be rational even at the off-path information set $\{R_2\}$. □

## 2.1 Subgames

To formalize the idea presented in Example 2, we need to define subgames.

**Definition 3.** In a finite-horizon extensive-form game $\Gamma = \langle I, X, P, (u_i, H_i)_i, \pi \rangle$, the **subgame that follows a node** $x \in X \setminus Z$ is the (finite-horizon) extensive-form game $\Gamma|_x = \langle I, X|_x, P|_x, (u_i|_x, H_i|_x)_i, \pi|_x \rangle$ such that:

1. The (unique) information set containing the node $x$ is a singleton.
2. $X|_x$ is the set of all nodes $y \in X$ such that $x \to y$, plus the node $x$ itself.
   - $z \in X|_x$ is a terminal node if there exists no $y \in X|_x$ such that $z \to y$.
   - $Z|_x$ is the set of all terminal nodes $z \in X|_x$.
3. $P|_x : X|_x \to I$ is the function that assigns to node $y \in X|_x$ a player moving at node $y$.

- $X_i|_x = \{y \in X|_x : P|_x(y) = i\}$ is the set of player $i$'s nodes.
- $A|_x(y) = \{y' \in X|_x : y \twoheadrightarrow y'\}$ is the set of actions for player $P|_x(y)$ at node $x \in X|_x$.

4. $u_i|_x : Z|_x \to \mathbb{R}$ is player $i$'s payoff function such that $u_i|_x(z) = u_i(z)$ for each $z \in Z|_x$.

5. $H_i|_x$ is player $i$'s partition of the set $X_i|_x$ such that:
   - $h_i \in H_i|_x$ is called player $i$'s **information set**.
   - $A|_x(y) = A|_x(y')$ for each $y, y' \in h_i$, which is also denoted by $A|_x(h_i)$.
   - A subgame never "cuts" any information set:

     *for each $y \in X|_x$ if $y \in h_i$ then for each $y' \in h_i$, $y' \in X|_x$.*

6. $\pi|_x$ is a function that assigns to node $y \in H_0|_x$ a distribution $\pi|_x \in \Delta(A|_x(y))$.
   - $0 \in I$ denotes the player called **nature**, who moves at random.
   - $\pi|_x(y) \in \Delta(A|_x(y))$ is nature's random choice of action $y' \in A|_x(y)$ at node $y \in X_0|_x$.
   - $\pi|_x(y) = \pi(y)$ for each $y \in X_0|_x$.

When the node $x$ of a subgame $\Gamma|_x$ is obvious from the context or needs not to be specified, we often write $\Gamma' = \langle I, X', P', (u'_i, H'_i)_i, \pi' \rangle$ instead of $\Gamma|_x$.

**Remark 3.** For any finite-horizon extensive-form game $\Gamma$, $\Gamma$ itself is a subgame of $\Gamma$. □

**Remark 4.** If there is no randomness in an original extensive-form game $\Gamma$, we omit nature's play $\pi$, denoting the game by $\Gamma' = \langle I, X', P', (u'_i, H'_i)_{i \in I} \rangle$. □

**Example 3.** Consider the following teamwork-game example:

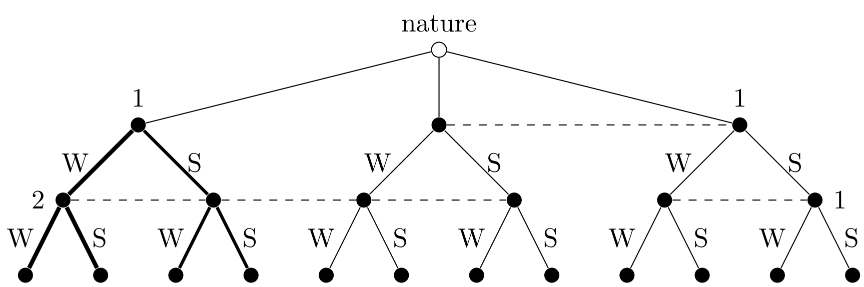


Figure 3: the part highlighted in bold in the game tree is not a subgame

The part in bold edges is not a subgame, because it "cuts" player 2's information set. Indeed, this extensive-form game has only one subgame, which is the whole game itself. □

**Strategies in Subgames** For a finite-horizon extensive-form game $\Gamma$, let $\beta_i$ be player $i$'s behavioral strategy. By restricting it to a subgame $\Gamma'$, we define the corresponding strategy, denoted $\beta_i|_{\Gamma'}$.

**Recap (Games of Perfect/Imperfect Information):** Recall the following definitions:

**Definition 6.** A finite-horizon extensive-form game $\Gamma$ is **finite** if the set $X$ of nodes is finite.

**Definition 7.** A finite-horizon extensive-form game $\Gamma$ is of **perfect information** if each information set $h_i \in H_i$ is a singleton for each $i \in I$, and is of **imperfect information** otherwise.

We state the following theorem:

**Theorem 1** (Kuhn 1953)**.** *Every finite extensive-form game of perfect information has a subgame perfect equilibrium.*

**Proof.** See Osborne & Rubinstein (1994, p.99). ■

**Definition 4.** In a finite-horizon extensive-form game $\Gamma$, let $\beta_i$ be player $i$'s behavioral strategy. The restriction of $\beta_i$ to a subgame $\Gamma'$, denoted $\beta_i|_{\Gamma'}$ is the unique behavioral strategy such that for each $h_i \in H_i'$,

$$\beta_i|_{\Gamma'}(h_i) = \beta_i(h_i).$$

Let us write $\beta|_{\Gamma'} = (\beta_i|_{\Gamma'})_i$ for the profile of these restricted strategies $\beta_i|_{\Gamma'}$.

### 2.2 Subgame Perfect Equilibrium

We now formalize the concept of subgame perfect equilibrium. In Example 2, player 2 chooses his optimal strategy in his every information set and player 1 chooses her optimal strategy given player 2's (future) move. This is rephrased as all players playing their optimal strategies in the sense of Nash equilibrium, in *all* subgames.

**Definition 5.** In a finite-horizon extensive-form game $\Gamma$, a behavioral strategy profile $\beta^*$ is a **subgame perfect equilibrium** if for each subgame $\Gamma'$ of $\Gamma$, the strategy profile $\beta^*|_{\Gamma'}$ is a Nash equilibrium of $\Gamma'$.

**Remark 5.** Every subgame perfect equilibrium in a finite-horizon extensive-form game $\Gamma$ is a Nash equilibrium. This is because the whole game $\Gamma$ itself is a subgame. In other words, subgame perfect equilibrium is a refinement of Nash equilibrium. □

## 3 Subgame Perfect Equilibrium under Perfect Information

For a finite-horizon extensive-form game of perfect information, we can find a subgame perfect equilibrium by **backward induction**. We omit the detailed procedure of backward induction in this note, while discussing it in class.

**Example 4** (Centipede Game)**.** Consider the following extensive-form game:

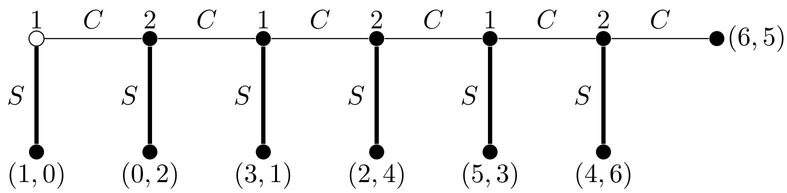


Figure 4: a centipede game

In words, each player decides whether to continue ($C$) or stop ($S$) at the corresponding (singleton) information sets. Note that this game is of perfect information.

By backward induction, we find a unique subgame perfect equilibrium, which each player chooses $S$ at her every information set, as indicated in the bold edges in Figure 4. $\square$

**Coffee Break ☕.** In Example 4, backward induction implies that each player always chooses $S$. Nevertheless, if player 2 found that play reached his first node, he would know that player 1 must have made an "irrational" choice $C$. Then, should we still assume that player 2 believes that player 1 is rational at her next node? It *is* possible that player 2 believes, at his node, that player 1 will make another irrational choice $C$ at her next node; given this belief, it *is* rational that player 2 chooses $C$ as well.

Should we really use backward induction? If so, how can we justify it? One possible idea is to define a player's rationality at node $x$ in terms of what happens *after* node $x$. In this formulation of rationality, common knowledge of rationality implies the solution by backward induction (Aumann, 1995). $\square$

**Example 5** (Stackelberg Game)**.** The Stackelberg game is a sequential version of the Cournot game. It proceeds as follows. First, firm 1 chooses quantity $q_1 \geq 0$. Second, firm 2 observes this choice $q_1$ and then chooses quantity $q_2 \geq 0$. The price of the product depends on the total quantity $Q = q_1 + q_2$, and the inverse demand function $P$ is such that $P(Q) = \max\{1 - Q, 0\}$. Each firm $i$'s production cost is zero regardless of $q_i$, and it maximizes its profit $P(Q)q_i$. Note that this game is of perfect information.

By backward induction, we find a subgame perfect equilibrium. First, we study firm 2's decision. At the information set where firm 1 has chosen $q_1$, firm 2 solves the following problem:

$$\max_{q_2} \quad \max\{1 - q_1 - q_2, 0\}q_2.$$

This problem has a unique solution:

$$q_2^*(q_1) = \begin{cases} \frac{1-q_1}{2} & \text{if } q_1 \leq 1 \\ 0 & \text{if } q_1 > 1. \end{cases}$$

Second, we study firm 1's decision. Predicting firm 2's choice $q_2^*(q_1)$, firm 1 solves the following problem:

$$\max_{q_1} \quad \max\{1 - q_1 - q_2^*(q_1), 0\} q_1.$$

This problem has a unique solution $q_1^* = \frac{1}{2}$. In short, the subgame perfect equilibrium is that firm 1 chooses $q_1 = \frac{1}{2}$ and firm 2 chooses $q_2 = q_2^*(q_1)$ at the information set where firm 1 has chosen $q_1$. On the equilibrium path, firms 1 and 2 choose $q_1 = \frac{1}{2}$ and $q_2 = \frac{1}{4}$ respectively. □

**Example 6** (Rule or Discretion)**.** Policy can be conducted by rule or discretion. We consider a simple model to study how a government should choose a policy (Barro & Gordon, 1983).

There are two players, Government (G) and People (P). G chooses an inflation rate $\pi$, balancing the benefit and cost of inflation. In particular, G solves the following problem:

$$\max_{\pi} \quad -\frac{c}{2}\pi^2 + (\pi - \pi^e),$$

where $\pi^e$ is the expected inflation rate, which will be chosen by P, and $c > 0$ is a constant. The first term $-\frac{c}{2}\pi^2$ represents the cost of inflation, and the second term $\pi - \pi^e$ represents the benefit of inflation.[1] Suppose that P chooses $\pi^e$ to maximize their payoff $-(\pi - \pi^e)^2$.

We consider the following two scenarios:

1. We examine the discretion scenario. Specifically, we consider a simultaneous-move game, in which G and P choose $\pi$ and $\pi^e$ respectively. This game has a unique Nash equilibrium $\pi = \pi^e = \frac{1}{c}$, with G's payoff $-\frac{1}{2c}$.

2. We examine the rule scenario. Specifically, we consider a sequential-move game, in which G announces $\pi$ and then P chooses $\pi^e$ after learning G's announcement. This game has a unique subgame perfect equilibrium: $\pi = 0$ and $\pi^e = \pi$ for any $\pi$. Hence, $\pi = \pi^e = 0$ on the equilibrium path, with G's payoff 0.

G's payoff is higher when it sets the inflation rate $\pi$ by rule than by discretion.

However, we note that when announcing an inflation rate $\pi$, G is assumed to **commit** to the announced choice. To see this, consider the case in which G cannot commit to the announcement. Then, even if G announces an inflation rate 0, it may actually implement another rate $\pi \neq 0$. Even if P believes G's announcement and chooses an expected inflation rate $\pi^e = 0$, G still has an incentive to increase its payoff by setting an actual inflation rate $\pi = \frac{1}{c}$.[2] As in this example, after G makes a policy announcement, G often has an incentive to change the announced policy. This is called a **time-inconsistency** problem. □

[1] For example, inflation lowers an unemployment rate in the short-run (the Phillips curve).

[2] What if P does not believe G's announcement? Because P predicts G's deviation, P will choose $\pi^e = \frac{1}{c}$. This game is reduced to the discretion scenario.

# 4 Subgame Perfect Equilibrium under Imperfect Information

We will see two examples of subgame perfect equilibrium under Imperfect Information.

## 4.1 Rabin's (1988) Example

**Example 7.** Consider the extensive-form game illustrated in Figure 5. Player 1 has two information sets $h_1 = \{\varnothing\}$ and $h_1' = \{R_1L_2L_3, R_1L_2R_3\}$. Player 2 has one information set $h_2 = \{R_1\}$. Player 3 has one information set $h_3 = \{R_1L_2\}$.

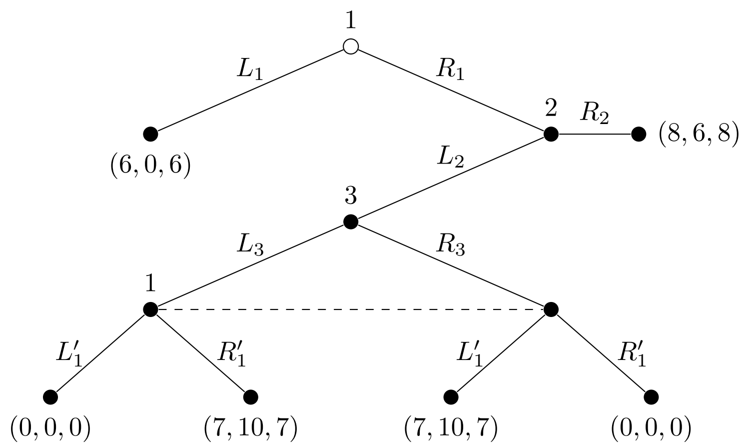


Figure 5: Rabin's (1988) extensive-form game

This extensive-form game has three subgame perfect equilibria $\beta^*$:

1. $\beta_1^*(R_1 \mid h_1) = 1$, $\beta_2^*(L_2 \mid h_2) = 1$, $\beta_3^*(L_3 \mid h_3) = 1$, and $\beta_1^*(R_1' \mid h_1') = 1$.

2. $\beta_1^*(R_1 \mid h_1) = 1$, $\beta_2^*(L_2 \mid h_2) = 1$, $\beta_3^*(R_3 \mid h_3) = 1$, and $\beta_1^*(L_1' \mid h_1') = 1$.

3. $\beta_1^*(R_1 \mid h_1) = 1$, $\beta_2^*(R_2 \mid h_2) = 1$, $\beta_3^*(L_3 \mid h_3) = \frac{1}{2}$, and $\beta_1^*(L_3' \mid h_1') = \frac{1}{2}$.

Note that player 1 chooses $R_1$ in all subgame perfect equilibria.

Is it really "wrong" for player 1 to choose $L_1$? Suppose that player 1 expects that when players 3 and 1 move, they will play a mixed-strategy Nash equilibrium, with payoff profile $(\frac{7}{2}, 5, \frac{7}{2})$. In addition, suppose that player 2 expects that when players 3 and 1 move, they will play a pure-strategy Nash equilibrium, with payoff profile $(7, 10, 7)$. If player 1 knows this expectation of player 2, then she should choose $L_1$ (with payoff 6) because if she chose $R_1$, she would have payoff $\frac{7}{2}$. □

Example 7 illustrates a critical assumption of subgame perfect equilibrium:

> All players not only expect Nash equilibria in all subgames but also expect the *same* Nash equilibria.

However, is it always reasonable to assume that all players have exactly the same prediction about Nash equilibrium play when they cannot communicate in any sense?

## 4.2 Bagwell's (1995) Example

**Example 8.** Consider the following extensive-form game:

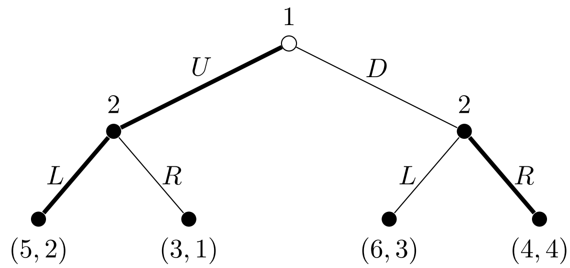


Figure 6: Bagwell's (1995) game of perfect information

In words, player 1 chooses $U$ or $D$, and then player 2 *perfectly* observes player 1's move and chooses $L$ or $R$. This game has a unique subgame perfect equilibrium, where player 1 chooses $U$ and player 2 chooses $L$ and $R$, respectively, at his information sets $U$ and $D$. This equilibrium is illustrated in the bold edges.

Now suppose that player 2 *almost but not perfectly* observes player 1's move. For a small $\epsilon > 0$, if player 1 chooses $U$ then player 2 gets signal $s_U$ with probability $1 - \epsilon$ and signal $s_D$ with probability $\epsilon$. Similarly, if player 1 chooses $D$ then player 2 gets signal $s_D$ with probability $1 - \epsilon$ but signal $s_U$ with probability $\epsilon$. Note that $s_U$ and $s_D$ are highly correlated to player 1's choosing $U$ and $D$ respectively. This extensive-form game is described as follows:[3]

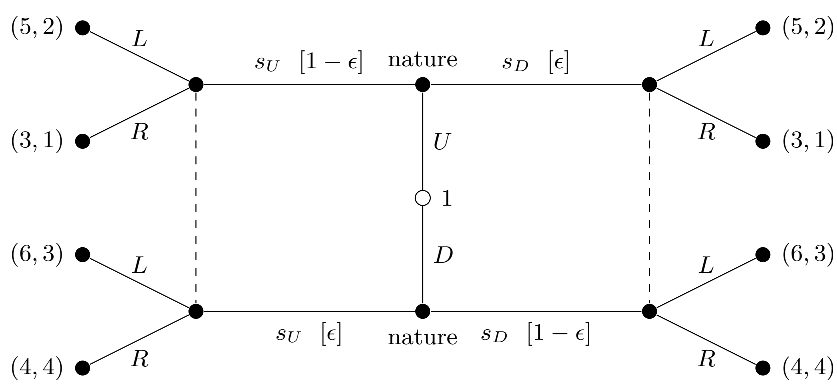


Figure 7: Bagwell's (1995) game of imperfect information

Player 1 has one information set $h_1 = \{\varnothing\}$, while player 2 has two information sets $h_2^U$ and $h_2^D$ at which she observes $s_U$ and $s_D$ respectively.

[3]The game tree does not have to start from the "top" node.

This extensive-form game has only one subgame, or the whole game itself. Hence, every subgame perfect equilibrium is a Nash equilibrium. We study a Nash equilibrium $\beta = (\beta_1, \beta_2)$ in pure strategies:

1. Suppose that player 1 plays $U$ in a pure-strategy Nash equilibrium. Then, player 2 ignores $s_U, s_D$ (because he is sure about $U$) and plays $L$ at both $h_2^U, h_2^D$. However, if (player 1 is sure) player 2 will play $L$, player 1 deviates to play $D$. Hence, player 1 would not play $U$ in any pure-strategy Nash equilibrium.

2. Suppose that player 1 plays $D$ in a pure-strategy Nash equilibrium. Then, player 2 ignores $s_U, s_D$ (because he is sure about $D$) and plays $R$ at both $h_2^U, h_2^D$. Moreover, if (player 1 is sure) player 2 will play $R$, player 1 is willing to play $D$. Hence, a pure-strategy Nash equilibrium is such that player 1 plays $D$ at $h_1$ and player 2 plays $R$ at both $h_2^U, h_2^D$.

In short, this game has a unique Nash equilibrium in pure strategies: $\beta_1(U \mid h_1) = 1$ and $\beta_2(R \mid h_2^U) = \beta_2(R \mid h_2^D) = 1$. □

**Remark 6.** Bagwell's (1995) example illustrates that a small noise $\epsilon$ discontinuously changes the set of (Nash or subgame perfect) equilibria in pure strategies. However, the discontinuity disappears once we allow for mixed strategies. That is, there is a mixed-strategy Nash equilibrium $\beta^\epsilon = (\beta_1^\epsilon, \beta_2^\epsilon)$ such that $\beta_1^\epsilon(U \mid h_1) \to 1$, $\beta_2^\epsilon(L \mid h_2^U) \to 1$, and $\beta_2^\epsilon(R \mid h_2^U) \to 1$ as $\epsilon \to 0$. For any small $\epsilon$, this equilibrium is "close" to the subgame perfect equilibrium in the noise-less game. □

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