NEOCLASSICAL GROWTH MODEL WITH UNCERTAINTY

Adding uncertainty to the neoclassical growth model:

- Technology shocks
$$Y_t = \theta_t F(K_t, L_t)$$
where θ_t is a random variable

- Shocks to preferences (ex: β_t stochastic discount factor)

- Policy shocks (ex: g_t o M_t random)

Uncertainty affects the behavior of consumers:

- Agents know current and past realizations (history) of the shocks - but not future realizations

- However, they know the stochastic process

- Agents choose contingent plans (that depend on the history of the shocks) for each variable

- Objective: maximize expected utility
$$E_0 \sum_{t=0}^{\infty} \beta^t u(c_t)$$

Notation

- z : random variable (shock)
- Z : set of possible realizations of z
- $z_t \in Z$: realization of shock in period t
- $z^t = (z_0, z_1, z_2, ...z_t)$: history of realizations of the shock up to t
- Z^t : set of all possible stories up to the period t
- $Z^t \Big| z^{t-s}$: set of all possible stories up to the period t that begin with z^{t-s}

If $Z = (Z_1, Z_2, ..., Z_n)$ is finite, z is a discrete random variable. Then, given $z_0 \in Z$, we define

- $\pi\left(z^t\right)$: probability of observing story z^t in period t, with

$$0 \leq \pi\left(z^t\right) \leq 1 \qquad \sum_{z^t \in Z^t} \pi\left(z^t\right) = 1$$

- Expectation: $E_0 x_t\left(z^t\right) = \sum_{z^t \in Z^t} \pi\left(z^t\right) x_t\left(z^t\right)$
- Conditional expectation: $E_{t-s} x_t\left(z^t\right) = \sum_{z^t \in Z^t | z^{t-s}} \frac{\pi\left(z^t\right)}{\pi\left(z^{t-s}\right)} x_t\left(z^t\right)$

If $Z \subseteq R$ is infinite, z is a continuous random variable. Then we will work with a density function $\phi\left(z^t\right)$

Model with Discrete Technology Shocks

- Stochastic production function:

$$Y_t = e^{z_t} F(K_t, L_t)$$

where F statisfies common assumptions. With $L_t = 1$, we rewrite

$$Y_t = F(K_t, 1) = e^{z_t} f(K_t)$$

- Expected utility:

$$E_0 \sum_{t=0}^{\infty} \beta^t u\left[c_t\left(z^t\right)\right] = \sum_{t=0}^{\infty} \sum_{z^t \in Z^t} \beta^t \pi\left(z^t\right) u\left[c_t\left(z^t\right)\right]$$

Stochastic Competitive General Equilibrium

A stochastic competitive equilibrium for this economy is a set of contingent plans for the quantities $c_t\left(z^t\right)$, $i_t\left(z^t\right)$, $k_{t+1}\left(z^t\right)$, $Y_t\left(z^t\right)$, $K_t\left(z^t\right)$ and prices $w_t\left(z^t\right)$, $r_t\left(z^t\right)$ such that:

i) Given $k_0 > 0$, z_0, $w_t\left(z^t\right)$, $r_t\left(z^t\right)$ and the process for z, the contingent plans $c_t\left(z^t\right)$, $i_t\left(z^t\right)$, and $k_{t+1}\left(z^t\right)$ solve the problem of the household:

$$\max \quad \sum_{t=0}^{\infty} \sum_{z^t \in Z^t} \beta^t \pi\left(z^t\right) u\left[c_t\left(z^t\right)\right]$$

$$s.t. \qquad c_t\left(z^t\right) + i_t\left(z^t\right) = w_t\left(z^t\right) + r_t\left(z^t\right) k_t\left(z^{t-1}\right) \qquad \forall z^t, \forall t$$

$$k_{t+1}\left(z^t\right) = (1-\delta) k_t\left(z^{t-1}\right) + i_t\left(z^t\right) \qquad \forall z^t, \forall t$$

ii) For each story z^t at each period t, given $w_t\left(z^t\right)$ and $r_t\left(z^t\right)$, the values $Y_t\left(z^t\right)$ and $K_t\left(z^t\right)$ solves the firm problem:

$$\begin{array}{ll} \max & Y_t\left(z^t\right) - w_t\left(z^t\right) - r_t\left(z^t\right) K_t\left(z^t\right) \\ s.t. & Y_t\left(z^t\right) = e^{z_t} f\left[K_t\left(z^t\right)\right] \end{array}$$

iii) For each story z^t at each period t, markets clear:

$$Y_t\left(z^t\right) = c_t\left(z^t\right) + i_t\left(z^t\right)$$

$$K_t\left(z^t\right) = k_t\left(z^{t-1}\right)$$

`Social Planner's Problem`

Given $k_0 > 0$, z_0 and the stochastic process for z, the social planner chooses contingent plans for the quantities $c_t\left(z^t\right)$, $i_t\left(z^t\right)$ and $k_{t+1}\left(z^t\right)$ solving:

$$\max \quad \sum_{t=0}^{\infty} \sum_{z^t \in Z^t} \beta^t \pi\left(z^t\right) u\left[c_t\left(z^t\right)\right]$$

$$s.t. \quad c_t\left(z^t\right) = e^{z_t} f\left[k_t\left(z^{t-1}\right)\right] - i_t\left(z^t\right) \qquad \forall z^t, \forall t$$

$$k_{t+1}\left(z^t\right) = (1-\delta) k_t\left(z^{t-1}\right) + i_t\left(z^t\right) \qquad \forall z^t, \forall t$$

Even with uncertainty, if there are no distortions or externalities, the Welfare Theorems hold

The Lagrangian for this problem is given by:

$$L = \sum_{t=0}^{\infty} \sum_{z^t \in Z^t} \left\{ \beta^t \pi\left(z^t\right) u\left[c_t\left(z^t\right)\right] \right.$$
$$-\lambda_{1t}\left(z^t\right)\left[c_t\left(z^t\right) + i_t\left(z^t\right) - e^{z_t} f\left[k_t\left(z^{t-1}\right)\right]\right]$$
$$\left. -\lambda_{2t}\left(z^t\right)\left[k_{t+1}\left(z^t\right) - (1-\delta) k_t\left(z^{t-1}\right) - i_t\left(z^t\right)\right]\right\}$$

with first order conditions:

$$\frac{\partial L}{\partial c_t\left(z^t\right)} = \beta^t \pi\left(z^t\right) u'\left[c_t\left(z^t\right)\right] - \lambda_{1t}\left(z^t\right) = 0$$

$$\frac{\partial L}{\partial i_t\left(z^t\right)} = -\lambda_{1t}\left(z^t\right) + \lambda_{2t}\left(z^t\right) = 0$$

$$\frac{\partial L}{\partial k_{t+1}\left(z^t\right)} = \sum_{z^{t+1} \in Z^{t+1} | z^t} \lambda_{1t+1}\left(z^{t+1}\right) e^{z_{t+1}} f'\left[k_{t+1}\left(z^t\right)\right]$$
$$-\lambda_{2t}\left(z^t\right) + \sum_{z^{t+1} \in Z^{t+1} | z^t} \lambda_{2t+1}\left(z^{t+1}\right)(1-\delta) = 0$$

The solution is characterized by:

- Euler equation (stochastic):

$$u'\left[c_t\left(z^t\right)\right] = \beta E_t\left\{u'\left[c_{t+1}\left(z^{t+1}\right)\right]\left(e^{z_{t+1}}f'\left[k_{t+1}\left(z^t\right)\right] + (1-\delta)\right)\right\}$$

$$\text{with:} \quad E_t x\left(z^{t+1}\right) = \sum_{z^{t+1}\in Z^{t+1}|z^t} \frac{\pi\left(z^{t+1}\right)}{\pi\left(z^t\right)} x\left(z^{t+1}\right)$$

- Feasibility condition:

$$c_t\left(z^t\right) = e^{z_t}f\left[k_t\left(z^{t-1}\right)\right] - k_{t+1}\left(z^t\right) + (1-\delta)\,k_t\left(z^{t-1}\right)$$

- Transversality condition:

$$\lim_{t\to\infty} E_0\beta^t u'\left[c_t\left(z^t\right)\right]k_{t+1}\left(z^t\right) = 0$$

`Stationary Equilibrium`

- Solvng the equilibrium, we obtain contingent plans for all the variables (quantities, prices)

- Given the stochastic processes for the shocks, these plans define processes for the variables - difficult to characterize

- There is NO stationary equilibrium in a strict sense, the variables move permanently

- We can, however, have an equilibrium in which all variables follow stationary stochastic processes - all their moments (mean, variance, etc.) are constant over time

Sequential Markets and Arrow-Debreu Markets

So far we have worked with *sequential markets*:

- In each period a new market is opened, in which goods and production factors of the current period are exchanged
- In the deterministic model there are as many markets as periods
- With uncertainty, there is a market for every history or state in the world
- Therefore, we also have a budget constraint for each period or state of the world

An alternative market structure is the one proposed by *Arrow-Debreu*:

- There is only one market, which opens in the initial period ($t = 0$)
- In this market, promises to deliver goods or productive factors are exchanged in any future period for each state of the world
- There is only one budget constraint
- We interpret the contingent plans as baskets of Arrow-Debreu goods

$p_t\left(z^t\right)$: price in period 0 of one unit of the only good delivered at period t if the history of the shocks is z^t

We normalize $p_0\left(z_0\right) = 1$

An Arrow-Debreu equilibrium is a set of baskets $c_t\left(z^t\right)$, $i_t\left(z^t\right)$, $k_{t+1}\left(z^t\right)$, $Y_t\left(z^t\right)$, $K_t\left(z^t\right)$ and prices $p_t\left(z^t\right)$, $w_t\left(z^t\right)$, $r_t\left(z^t\right)$ such that:

i) Given $k_0 > 0$, z_0, $p_t\left(z^t\right)$, $w_t\left(z^t\right)$, $r_t\left(z^t\right)$ and the process for z, the baskets $c_t\left(z^t\right)$, $i_t\left(z^t\right)$ and $k_{t+1}\left(z^t\right)$ solve:

$$\max \qquad \sum_{t=0}^{\infty} \sum_{z^t \in Z^t} \beta^t \pi\left(z^t\right) u\left[c_t\left(z^t\right)\right]$$

$$s.t. \qquad \sum_{t=0}^{\infty} \sum_{z^t \in Z^t} p_t\left(z^t\right)\left[c_t\left(z^t\right) + i_t\left(z^t\right)\right]$$

$$= \sum_{t=0}^{\infty} \sum_{z^t \in Z^t} p_t\left(z^t\right)\left[w_t\left(z^t\right) + r_t\left(z^t\right) k_t\left(z^{t-1}\right)\right]$$

$$k_{t+1}\left(z^t\right) = (1-\delta) k_t\left(z^{t-1}\right) + i_t\left(z^t\right) \qquad \forall z^t, \forall t$$

ii) Given $w_t\left(z^t\right)$ and $r_t\left(z^t\right)$, the baskets $Y_t\left(z^t\right)$ and $K_t\left(z^t\right)$ solve:

$$\max \quad \sum_{t=0}^{\infty}\sum_{z^t\in Z^t} p_t\left(z^t\right)\left[Y_t\left(z^t\right) - w_t\left(z^t\right) - r_t\left(z^t\right)K_t\left(z^t\right)\right]$$

$$s.t. \qquad Y_t\left(z^t\right) = e^{z_t} f\left[K_t\left(z^t\right)\right] \qquad \forall z^t, \forall t$$

iii) For each story z^t at period t, markets clear:

$$Y_t\left(z^t\right) = c_t\left(z^t\right) + i_t\left(z^t\right)$$

$$K_t\left(z^t\right) = k_t\left(z^{t-1}\right)$$

Solving the household problem:

$$\begin{aligned} L = \sum_{t=0}^{\infty}\sum_{z^t\in Z^t} &\left\{\beta^t \pi\left(z^t\right) u\left[c_t\left(z^t\right)\right]\right. \\ &-\lambda_1 p_t\left(z^t\right)\left[c_t\left(z^t\right) + i_t\left(z^t\right) - w_t\left(z^t\right) - r_t\left(z^t\right)k_t\left(z^{t-1}\right)\right] \\ &\left.-\lambda_{2t}\left(z^t\right)\left[k_{t+1}\left(z^t\right) - (1-\delta)k_t\left(z^{t-1}\right) - i_t\left(z^t\right)\right]\right\} \end{aligned}$$

with first order conditions:

$$\frac{\partial L}{\partial c_t\left(z^t\right)} = \beta^t \pi\left(z^t\right) u'\left[c_t\left(z^t\right)\right] - \lambda_1 p_t\left(z^t\right) = 0$$

$$\frac{\partial L}{\partial i_t\left(z^t\right)} = -\lambda_1 p_t\left(z^t\right) + \lambda_{2t}\left(z^t\right) = 0$$

$$\begin{aligned}\frac{\partial L}{\partial k_{t+1}\left(z^t\right)} = & \sum_{z^{t+1}\in Z^{t+1}|z^t} \lambda_1 p_{t+1}\left(z^{t+1}\right) r_{t+1}\left(z^{t+1}\right) \\ & -\lambda_{2t}\left(z^t\right) + \sum_{z^{t+1}\in Z^{t+1}|z^t} \lambda_{2t+1}\left(z^{t+1}\right)(1-\delta) = 0\end{aligned}$$

The Arrow-Debreu equilibrium is characterized by:

- Stochastic Euler Equation:
$$u'\left[c_t\left(z^t\right)\right] = \beta E_t\left\{u'\left[c_{t+1}\left(z^{t+1}\right)\right]\left(r_{t+1}\left(z^{t+1}\right) + (1-\delta)\right)\right\}$$

- Feasibility Condition:
$$c_t\left(z^t\right) = e^{z_t} f\left[k_t\left(z^{t-1}\right)\right] - k_{t+1}\left(z^t\right) + (1-\delta) k_t\left(z^{t-1}\right)$$

- Arrow-Debreu Prices:
$$\frac{p_t\left(z^t\right)}{p_{t+1}\left(z^{t+1}\right)} = \frac{\pi\left(z^t\right) u'\left[c_t\left(z^t\right)\right]}{\beta\pi\left(z^{t+1}\right) u'\left[c_{t+1}\left(z^{t+1}\right)\right]}$$

Using the price normalization $p_0(z_0) = 1$

$$p_t\left(z^t\right) = \beta^t \frac{u'\left[c_t\left(z^t\right)\right]}{u'\left[c_0\left(z_o\right)\right]} \pi\left(z^t\right)$$

- Factor prices:

$$w_t\left(z^t\right) = e^{z_t} f\left[K_t\left(z^t\right)\right] - e^{z_t} f'\left[K_t\left(z^t\right)\right] K_t\left(z^t\right)$$

$$r_t\left(z^t\right) = e^{z_t} f'\left[K_t\left(z^t\right)\right]$$

- Transversality condition:

$$\lim_{t\to\infty} \sum_{z^t \in Z^t} p_t\left(z^t\right) k_{t+1}\left(z^t\right) = 0$$

`Complete Markets`

A structure of sequential and complete markets requieres the existence of assets for each posible state of the world

- $b_{t+1}\left(z^{t+1}\right)$: contingent bond purchased in period t, with return:

$$\begin{cases} 1, \text{ if } z^{t+1} \text{ ocure} \\ 0, \text{ in other case} \end{cases}$$

- $q_t\left(z^{t+1}\right)$: price of the contingent bond in period t

Given that the contingent bonds are traded between all consumers, the net supply is equal zero

A competitive equilibrium with complete markets is a set of contingent plans $c_t\left(z^t\right)$, $i_t\left(z^t\right)$, $b_{t+1}\left(z^{t+1}\right)$, $k_{t+1}\left(z^t\right)$, $Y_t\left(z^t\right)$, $K_t\left(z^t\right)$ and prices $q_t\left(z^{t+1}\right)$,$w_t\left(z^t\right)$, $r_t\left(z^t\right)$ such that

i) Given $k_0 > 0$, z_0, $b_0 = 0$, $q_t\left(z^{t+1}\right)$, $w_t\left(z^t\right)$, $r_t\left(z^t\right)$ and the process for z, the plans $c_t\left(z^t\right)$, $i_t\left(z^t\right)$, $b_{t+1}\left(z^{t+1}\right)$ and $k_{t+1}\left(z^t\right)$ solve:

$$\max \quad \sum_{t=0}^{\infty} \sum_{z^t \in Z^t} \beta^t \pi\left(z^t\right) u\left[c_t\left(z^t\right)\right]$$

$$s.t. \quad c_t\left(z^t\right) + i_t\left(z^t\right) + \sum_{z^{t+1} \in Z^{t+1}|z^t} q_t\left(z^{t+1}\right) b_{t+1}\left(z^{t+1}\right)$$

$$= w_t\left(z^t\right) + r_t\left(z^t\right) k_t\left(z^{t-1}\right) + b_t\left(z^t\right) \qquad \forall z^t, \forall t$$

$$k_{t+1}\left(z^t\right) = (1-\delta)\, k_t\left(z^{t-1}\right) + i_t\left(z^t\right) \qquad \forall z^t, \forall t$$

$$b_{t+1}\left(z^{t+1}\right) > -B \qquad \forall z^{t+1} \in Z^{t+1}, \forall z^t, \forall t$$

ii) For each story z^t at each period t, given $w_t\left(z^t\right)$ and $r_t\left(z^t\right)$, the values $Y_t\left(z^t\right)$ and $K_t\left(z^t\right)$ solve:

$$\begin{array}{ll} \max & Y_t\left(z^t\right) - w_t\left(z^t\right) - r_t\left(z^t\right) K_t\left(z^t\right) \\ s.t. & Y_t\left(z^t\right) = e^{z_t} f\left[K_t\left(z^t\right)\right] \end{array}$$

iii) For each story z^t at each period t, markets clear:

$$Y_t\left(z^t\right) = c_t\left(z^t\right) + i_t\left(z^t\right)$$

$$K_t\left(z^t\right) = k_t\left(z^{t-1}\right)$$

$$b_{t+1}\left(z^{t+1}\right) = 0 \qquad \forall z^{t+1} \in Z^{t+1}\Big| z^t$$

The equilibrium is characterized by:

- Stochastic Euler Equation:

$$u'\left[c_t\left(z^t\right)\right] = \beta E_t\left\{u'\left[c_{t+1}\left(z^{t+1}\right)\right]\left(r_{t+1}\left(z^{t+1}\right) + (1-\delta)\right)\right\}$$

- Feasibility Condition:

$$c_t\left(z^t\right) = e^{z_t} f\left[k_t\left(z^{t-1}\right)\right] - k_{t+1}\left(z^t\right) + (1-\delta) k_t\left(z^{t-1}\right)$$

- Bond Prices:

$$q_t\left(z^{t+1}\right) = \frac{\beta\pi\left(z^{t+1}\right) u'\left[c_{t+1}\left(z^{t+1}\right)\right]}{\pi\left(z^t\right) u'\left[c_t\left(z^t\right)\right]} \qquad \forall z^{t+1} \in Z^{t+1}\Big| z^t$$

- Factor Prices:

$$w_t\left(z^t\right) = e^{z_t} f\left[K_t\left(z^t\right)\right] - e^{z_t} f'\left[K_t\left(z^t\right)\right] K_t\left(z^t\right)$$

$$r_t\left(z^t\right) = e^{z_t} f'\left[K_t\left(z^t\right)\right]$$

- Transversality Condition:

$$\lim_{t\to\infty} \sum_{z^t\in Z^t} \left(\prod_{j=1}^{t} q_{j-1}\left(z^j\right)\right) k_{t+1}\left(z^t\right) = 0$$

$$\lim_{t\to\infty} \sum_{z^t\in Z^t} \left(\prod_{j=1}^{t} q_{j-1}\left(z^j\right)\right) b_t\left(z^t\right) = 0$$

- With complete markets, the Arrow-Debreu and sequential markets equilibria are equivalent:

  i) the contingent plans $c_t\left(z^t\right)$, $i_t\left(z^t\right)$, $k_{t+1}\left(z^t\right)$, $Y_t\left(z^t\right)$, $K_t\left(z^t\right)$ are the same

  ii) the factor prices $w_t\left(z^t\right)$, $r_t\left(z^t\right)$ are the same

  iii) the bond prices satisfies:

$$q_t\left(z^{t+1}\right) = \frac{p_{t+1}\left(z^{t+1}\right)}{p_t\left(z^t\right)} \qquad \forall z^{t+1} \in Z^{t+1}\Big| z^t$$

- Given that we have a unique (representative) agent, the net supply of each financial asset is zero

  In equilibrium, the representative consumer is restricted by market clearing conditions

  Therefore, we could implement the Arrow-Debreu equilibrium with only one asset (capital)