Game Theory

Nash Equilibrium

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In this note, we study possibly the most famous solution concept of **Nash equilibrium** in a normal-form game (Nash, 1950). We also discuss the justifications based on a self-enforcing agreement.

1 Pure-Strategy Nash Equilibrium

Example 1. Consider the following prisoners' dilemma:

	C	D
C	$-1, -1$	$-3, 0$
D	$0, -3$	$-2, -2$

Table 1: prisoners' dilemma

Suppose that players 1 and 2 have agreed on playing an action profile. We examine whether this agreement is *stable* or not, by asking whether there exists at least one player who wants to deviate from that action profile.

Suppose that players 1 and 2 have agreed on action profile (D, D). We will show that they are willing to play it. Because player 1 thinks that player 2 will play D, player 1 is willing to play D. By the same logic, player 2 is willing to play D. That is, they are willing to play as they have agreed. Such a strategy profile that they are willing to follow is called a (pure-strategy) Nash equilibrium.

Suppose that players 1 and 2 have agreed on action profile (D, C). We will show that they are not willing to play it. Because player 1 thinks that player 2 will play C, player 1 is willing to play D. However, because player 2 thinks that player 1 will play D, player 2 should deviate to playing D. Hence, they are not willing to play as they have agreed. □

1.1 Pure-Strategy Nash Equilibrium

The concept of Nash equilibrium captures the *stability* of the agreement of play in a normal-form game. Nash equilibrium does not attempt to examine the process by which the stability is reached.

Definition 1. In a normal-form game G, a pure-strategy profile $a^* \in A$ is a **pure-strategy**

Nash equilibrium if for each $i \in I$ and each $a_i \in A_i$,

$$u_i(a_i^*, a_{-i}^*) \geq u_i(a_i, a_{-i}^*).$$

In a pure-strategy Nash equilibrium, every player chooses an optimal (pure) strategy given the opponents' (pure) strategies. In other words, we say that a pure strategy profile $a \in A$ is a pure-strategy Nash equilibrium if **no player has a profitable deviation**.

Multiplicity There may exist multiple pure-strategy Nash equilibria.

Example 2. Consider the following **coordination game**:

	L	R
U	$1, 1$	$-2, 0$
D	$0, -2$	$0, 0$

Table 2: A coordination game.

In this game, the row player chooses either Up or Down, denoted U or D, while the column player chooses either Left or Right, denoted L or R.

This game has two pure-strategy Nash equilibria, (U, L) and (D, R). Both players are strictly better-off at (U, L) than at (D, R)—that is, (U, L) Pareto-dominates (D, R). $\square$

Non-Existence There may not exist a pure-strategy Nash equilibrium.

Example 3. Consider the following **matching pennies game**:

	H	T
H	$+1, -1$	$-1, +1$
T	$-1, +1$	$+1, -1$

Table 3: a matching pennies

In this game, each player has a penny and secretly turn the penny to Heads or Tails, denoted H and T. Then, the players reveal their choices simultaneously. If the pennies match then the row player gets the both pennies while the column player loses his penny; if the pennies do not match then the column player gets the both pennies while the row player loses her penny.

This game has no pure-strategy Nash equilibrium. $\square$

2 Mixed-Strategy Nash Equilibrium

In Example 3, player i may want to randomize her choice of action. She may choose actions H and T with probabilities $\sigma_i(H)$ and $\sigma_i(T)$ respectively. This randomization makes sense if her "strategy" σ_i is a probability distribution over the set $A_i = \{H, T\}$. That is, $\sigma_i(H) \geq 0$ and $\sigma_i(T) \geq 0$ as well as $\sigma_i(H) + \sigma_i(T) = 1$.

2.1 Mixed Strategies

Definition 2. In a normal-form game $G = \langle I, (A_i, u_i)_{i \in I} \rangle$, we say that:

1. $\Sigma_i = \Delta(A_i)$ is the set of player i's **(mixed) strategies**.[1]
   - $\Sigma = \prod_i \Sigma_i$ is the set of **(mixed) strategy profiles**.
   - $\sigma = (\sigma_i)_i \in \Sigma$ is a (mixed) strategy profile that specifies each player i's strategy σ_i.

Player i's (mixed) strategy $\sigma_i \in \Sigma_i$ is classified as follows:

1. σ_i is a **pure strategy** if it assigns probability 1 to a single action.

2. σ_i is a **strictly mixed strategy** if it is not a pure strategy.

Remark 1. A (mixed) strategy σ_i may be a pure or strictly mixed strategy. □

Independent Randomization When players choose a mixed strategy profile $\sigma = (\sigma_i)_{i \in I}$, their randomization is assumed to be *independent*. Player i's payoff from the mixed strategy profile σ, denoted $u_i(\sigma)$, is the expectation of her payoffs:[2]

$$u_i(\sigma) = \sum_{a \in A} \left(\prod_{i \in I} \sigma_i(a_i) \right) u_i(a). \tag{1}$$

Remark 2. This independence is represented by the notation $\Sigma = \prod_i \Delta(A_i)$. The order of $\prod_i$ and Δ is important. If we interchange the order—that is, if $\Delta(\prod_i A_i)$—then an element $\sigma \in \Delta(A)$ allows players's randomization to be correlated. Such strategies that allow for correlated randomization are called correlated strategies, which we will study later. □

Mixed Extension With the concept of mixed strategies and the (expected) payoff functions, we can extend a normal-form game such that players can play mixed strategies.

Definition 3. For a normal-form game $G = \langle I, (A_i, u_i)_{i \in I} \rangle$, the **mixed extension** is a normal-form game $G' = \langle I, (\Sigma_i, u_i)_{i \in I} \rangle$, where player i's payoff function u_i is extended in the expected payoff fashion (1).

[1] For a set X, let $\Delta(X)$ be the set of probability distributions over the set X.

[2] For the sake of simplicity, we assume that the set of players and the set of each player's actions are countable. It is straightforward to extend this notion to the uncountable case.

2.2 Mixed-Strategy Nash Equilibrium

With the concept of mixed extension, we can formalize mixed-strategy Nash equilibria.

Definition 4. In a normal-form game G, a **(mixed-strategy) Nash equilibrium** $\sigma^* \in \Sigma$ is a pure-strategy Nash equilibrium of its mixed extension G'.[3]

Equivalent Definition There is an equivalent definition of Nash equilibrium. To see it, we introduce the concept of best-response correspondence.

Definition 5. In a normal-form game G, player i's **best-response correspondence** is a correspondence $B_i : \Sigma_{-i} \to 2^{\Sigma_i}$ such that for each $\sigma_{-i} \in \Sigma_{-i}$, $B_i(\sigma_{-i})$ is the set of his optimal strategies when players $-i$ play strategies σ_{-i}. That is,

$$B_i(\sigma_{-i}) = \underset{\sigma_i' \in \Sigma_i}{\operatorname{argmax}} \; u_i(\sigma_i', \sigma_{-i}).$$

This definition immediately gives an equivalent definition of Nash equilibrium.

Definition 6. In a normal-form game G, a **(mixed-strategy) Nash equilibrium** $\sigma^* \in \Sigma$ is a mixed-strategy profile such that for each $i \in I$,

$$\sigma_i^* \in B_i(\sigma_{-i}^*).$$

This definition of Nash equilibrium reformulates a Nash equilibrium as an intersection of all players' best-response correspondences. This interpretation turns out to be useful in computing Nash equilibria and in proving the existence of Nash equilibria.

2.3 How to Compute Mixed-Strategy Nash Equilibria

Example 4. Penalty kicks in soccer involve two players: a kicker (K) and a goalkeeper (G). As shots by soccer professionals are fast, we assume that both K and G must move simultaneously. Since it is rare to kick at the middle of the goal, we assume that K chooses to which Left (L) or Right (R) to kick and that G also chooses to which Left (L) or Right (R) to jump. That is, let $A_K = A_G = \{L, R\}$ be the sets of actions. When they play action profile (a_K, a_G), K's payoff is probability $\pi(a_K, a_G)$ that he makes a goal and G's payoff is $1 - \pi(a_K, a_G)$.[4] To specify the payoffs, we use the real data of professional soccer games. For example, we estimate the goal probability $\pi(L, R)$ to be the likelihood that K makes a goal when he kicked to the left and G

[3]That is, in a normal-form game G, a (mixed) strategy profile $\sigma^* \in \Sigma$ is a (mixed-strategy) Nash equilibrium if for each $i \in I$ and each $\sigma_i \in \Sigma_i$, $u_i(\sigma_i^*, \sigma_{-i}^*) \geq u_i(\sigma_i, \sigma_{-i}^*)$.

[4]This kind of game is called a zero-sum game because for every action profile $a = (a_K, a_G)$, the sum of the two players' payoffs is zero: $u_K(a) + u_G(a) = 0$.

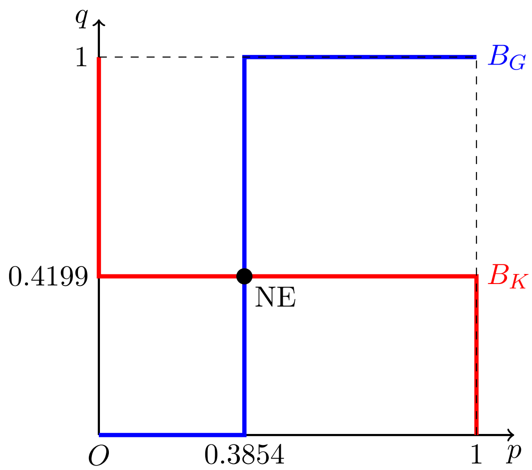

Figure 1: best-response correspondences in the penalty kick game

jumped to the right:

$$\pi(L, R) = \frac{\#\text{goals when } K \text{ kicked to the left and } G \text{ jumped to the right}}{\#\text{penalty kicks such that } K \text{ kicked to the left and } G \text{ jumped to the right}}.$$

The payoffs are summarized in Table 4.

$K \backslash G$	L	R
L	$58.30, 41.70$	$94.97, 5.03$
R	$92.92, 7.08$	$69.92, 30.08$

Table 4: Palacios-Huerta's (2003) penalty kick game

This penalty kick game has a unique Nash equilibrium, and it is a strictly mixed-strategy Nash equilibrium. Since each player i has two actions, to define her strategy, it suffices to specify the probability of playing $a_i = L$ (which implies that she plays $a_i = R$ with the remaining probability). To find the Nash equilibrium, we find K's best response. Letting q be the probability that G plays $a_G = L$, we find K's best response B_K to be such that:[5]

$$B_K(q) = \begin{cases} \{1\} & \text{if } q < 0.4199 \\ [0, 1] & \text{if } q = 0.4199 \\ \{0\} & \text{if } q > 0.4199. \end{cases}$$

Similarly, we can find G's best response B_G. These two best responses are illustrated in Figure 1. The Nash equilibrium is, by Definition 6, the intersection of the two best responses. We summarize the Nash equilibrium in Table 5 (row 1).

[5] Here "0.4199" is an approximation of $\frac{835}{1989}$.

	$a_G = L$	$a_G = R$	$a_K = L$	$a_K = R$
theory	0.4199	0.5801	0.3854	0.6146
data	0.4231	0.5769	0.3998	0.6002

Table 5: Nash equilibrium versus data

In Table 5, we compare the Nash equilibrium (row 1) with the real data of soccer professionals' play (row 2). The game theory prediction fits the data very well (or the soccer professionals are game theory masters)! □

Remark 3. You may be puzzled by the aspect of mixed-strategy Nash equilibria that each player is indifferent between all actions to which he assigns non-zero probabilities, but he randomizes them *in order to* make other players indifferent. In Example 4, it seems that G randomizes his actions *in order to* make K indifferent between L and R. This puzzle is justified by Harsanyi's Purification Theorem. □

Coffee Break ☕. Many studies examine and compare how people play a game in the field and in the laboratory. Interestingly, many of them suggest that the field play is often different from the laboratory play. For example, Walker & Wooders (2001) study the field play by tennis professionals and Chiappori et al. (2002) soccer professionals. More recent studies compare professionals' play in the field and their play in the laboratory (e.g., Palacios-Huerta & Volij, 2008; Levitt et al., 2010; Wooders, 2010). □

Another Way to Compute Mixed-Strategy Nash Equilibria In a Nash equilibria σ^*, each player i is indifferent between his all actions that are played with non-zero probabilities. This property turns out to be useful in computing Nash equilibria.

Proposition 1. *In a normal-form game G, if σ^* is a Nash equilibrium then for all $a_i, a_i' \in A_i$ such that $\sigma_i^*(a_i), \sigma_i^*(a_i') > 0$,*

$$u_i(\sigma_i^*, \sigma_{-i}^*) = u_i(a_i, \sigma_{-i}^*) = u_i(a_i', \sigma_{-i}^*).$$

Proof. Suppose, for a contradiction, that there exist $a_i, a_i' \in A_i$ such that $\sigma_i^*(a_i), \sigma_i^*(a_i') > 0$ but $u_i(a_i, \sigma_{-i}^*) > u_i(a_i', \sigma_{-i}^*)$. Then, player i would increase his payoff by increasing the probability of playing a_i. This contradicts the assumption that σ_i^* is optimal given σ_{-i}^*. ■

Example 5. We compute the Nash equilibrium, using Proposition 1. We continue to write q for the probability that G plays $a_G = L$. Then, K's payoff from playing $a_K = L$ is $58.30q + 94.97(1-q)$, while K's payoff from playing $a_K = R$ is $92.92q + 69.92(1-q)$. By Proposition 1, these payoffs must equal at the Nash equilibrium. Hence, $q = 0.4199$ at the Nash equilibrium. Similarly, we can compute the probability that K plays $a_K = L$. □

Recap (Kakutani's Fixed-Point Theorem):

Definition 8. Let $X, Y \subset \mathbb{R}^n$ be the sets for $n \in \mathbb{N}$. Let $F : X \to 2^Y$ be a correspondence.

1. F is **non-empty-valued** if for each $x \in X$, $F(x)$ is non-empty.
2. F is **convex-valued** if for each $x \in X$, $F(x)$ is convex.
3. F has a **closed graph** if $\mathrm{Gr}(F)$ is a closed set in $X \times Y$ (with respect to the relative topology), where $\mathrm{Gr}(F)$ is called the **graph** of F such that:

$$\mathrm{Gr}(F) = \{(x, y) \in X \times Y : y \in F(x)\}.$$

Theorem 2 (Kakutani's Fixed-Point Theorem)**.** *Let $X \subset \mathbb{R}^n$ be a non-empty, compact, convex subset for $n \in \mathbb{N}$. Let $F : X \to 2^X$ be a non-empty-valued, convex-valued correspondence with a closed graph. Then, there exists some $x^* \in X$ such that $x^* \in F(x^*)$.*

Remark 4. We note that if F is a (single-valued) function, then the closed graph property coincides exactly with the definition of continuity. In this respect, the closed graph property is the natural extension of the continuity to correspondences. □

Remark 5. In general, the closed graph property is **not** equivalent to the notion of upper hemicontinuity. However, the closed graph property and upper hemicontinuity are equivalent if Y is compact and F is closed-valued, which will be satisfied in this course. □

3 Nash's Existence Theorem

As seen above, there may not exist a pure-strategy Nash equilibrium (Example 3). Now we ask when there exists a (mixed-strategy) Nash equilibrium.

Definition 7. A normal-form game G is **finite** if each of the sets I and A is finite, and is **infinite** otherwise.

For example, all games of Examples 1 to 4 are finite games, while standard Cournot game and Bertrand game are infinite games.

Theorem 1 (Nash's Existence Theorem)**.** *Every finite normal-form game G has a Nash equilibrium (in mixed-strategies).*

Nash Equilibria as Fixed Points Define a correspondence $B : \Sigma \to 2^\Sigma$ such that for each $\sigma \in \Sigma$, $B(\sigma) = (B_i(\sigma_{-i}))_{i \in I}$. In this view, a Nash equilibrium $\sigma^* \in \Sigma$ is a fixed point $\sigma^* \in B(\sigma^*)$ of this correspondence B. Hence, the proof of Theorem 1 is reduced to the existence of such a fixed point.

Proof of Theorem 1. Let $B : \Sigma \to 2^\Sigma$ be a correspondence such that for each $\sigma \in \Sigma$, $B(\sigma) = (B_i(\sigma_{-i}))_{i \in I}$. Then, a Nash equilibrium $\sigma^* \in \Sigma$ is a fixed point $\sigma^* \in B(\sigma^*)$. To show the existence of such a fixed point σ^*, we will rely on Kakutani's Fixed-Point Theorem. To this end, we verify the following properties:

1. Σ_i is non-empty, compact, and convex.

2. B_i is non-empty-valued. To see this, note that u_i is continuous in σ_i and Σ_i is compact.

3. B_i is convex-valued. To see this, let $\sigma_i, \sigma_i' \in B_i(\sigma_{-i})$. By Proposition 1, all $a_i \in \text{supp}(\sigma_i)$ and all $a_i' \in \text{supp}(\sigma_i')$ yield the same payoff.[6] Hence, player i is indifferent to any randomization over $\text{supp}(\sigma_i) \cup \text{supp}(\sigma_i')$.

4. B_i has a closed graph. To see this, let $(\sigma^k, \hat{\sigma}^k) \in \text{Gr}(B)$ such that $\sigma^k \to \sigma$ and $\hat{\sigma}^k \to \hat{\sigma}$ as $k \to \infty$. It suffices to show that $\hat{\sigma}_i \in B_i(\sigma_{-i})$ for each $i \in I$. Note that for each $k \in \mathbb{N}$, $u_i(\hat{\sigma}_i^k, \sigma_{-i}^k) \geq u_i(\sigma_i', \sigma_{-i}^k)$ for each σ_i'. Hence, $\lim_k u_i(\hat{\sigma}_i^k, \sigma_{-i}^k) \geq \lim_k u_i(\sigma_i', \sigma_{-i}^k)$ for each σ_i'. Since u_i is continuous, $u_i(\hat{\sigma}_i, \sigma_{-i}) \geq u_i(\sigma_i', \sigma_{-i})$ for each σ_i'.

Hence, B satisfies all the conditions for Kakutani's Fixed-Point Theorem. There exists a fixed point $\sigma^* \in B(\sigma^*)$. ■

A Generalization of Nash's Existence Theorem*

Theorem 1 assumes that every player has a finite action space. This assumption is used because Kakutani's Fixed-Point Theorem assumes a finite-dimensional Euclidean set $X \subseteq \mathbb{R}^n$.

In this Appendix, we consider the existence of Nash equilibria in normal-form games with continuous action spaces.

Theorem 3. *In a finite-player normal-form game $G = \langle I, (A_i, u_i)_i \rangle$ such that each player i's pure-strategy space A_i is a compact convex subset of a Euclidean space, if each player i's payoff function u_i is continuous in a and quasi-concave in a_i then there exists a pure-strategy Nash equilibrium.*

Remark 6. Theorem 1 is a special case of Theorem 3 because if we identify the pure-strategy space A_i of Theorem 3 as the mixed-strategy space Σ_i of Theorem 1 (which assumes that all action spaces are finite). □

If utility functions are not quasi-concave then Theorem 3 is not necessarily valid. Even in this case, if we allow players to play mixed strategies then we can often guarantee the existence of Nash equilibrium (in mixed strategies). In a normal-form game with continuous action spaces, we define a mixed strategy as a Borel probability measure over the (topological) space A_i of pure strategies.

Theorem 4. *In a finite-player normal-form game $G = \langle I, (A_i, u_i)_i \rangle$ such that every player i's pure-strategy space A_i is a compact subset of a metric space, if every player i's payoff function u_i is continuous in a then there exists a Nash equilibrium (in mixed strategies).*

[6] For a finite set X, the **support** of a distribution $\mu \in \Delta(X)$ is the set $\text{supp}(\mu) = \{x \in X : \mu(x) > 0\}$.

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