Game Theory

Sequential Equilibrium

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In this note, we study the notion of **sequential equilibrium** for finite extensive-form games (Kreps & Wilson, 1982). It is stronger than the notion of (weak) perfect Bayesian equilibrium, since it imposes additional restrictions on off-path beliefs of the perfect Bayesian equilibrium.

1 Example

Example 1. Consider the game of Figure 1. Since each player has two actions and one information set, we denote player i's strategy by probability $s_i \in [0,1]$ of choosing action L_i. There is one non-singleton information set $\{L_1L_2, L_1R_2\}$ for player 3, denoted h_3. We denote the belief $\mu(\cdot \mid h_3)$ by probability $\nu \in [0,1]$ on the left node. Let $s = (s_1, s_2, s_3)$ be a strategy profile, and let (s, ν) be an assessment.

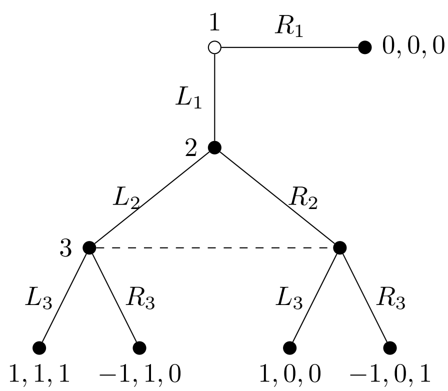

Figure 1: sequential equilibrium

There are two weak perfect Bayesian equilibria $(s, \nu) = (1, 1, 1, 1), (0, 1, 0, 0)$, as discussed in the previous note.

We argue that the former is plausible but the latter is not, by requiring "robustness to small perturbation." First, we consider the weak perfect Bayesian equilibrium $(s, \nu) = (1, 1, 1, 1)$. We perturb the equilibrium play such that player 1 may choose R_1 with probability ϵ and player 2 may choose R_2 with probability ϵ. Hence, player 3 assigns to the left node probability

$$\frac{\mathbb{P}(\text{play reaches node } L_1L_2)}{\mathbb{P}(\text{play reaches information set } h_3)} = \frac{(1-\epsilon)^2}{(1-\epsilon)^2 + (1-\epsilon)\epsilon} = 1 - \epsilon.$$

It is close to his equilibrium belief $\nu = 1$ for any small ϵ. Hence, the belief $\nu = 1$ is robust to the perturbation. The "robust" equilibrium $(s, \nu) = (1, 1, 1, 1)$ is called a **sequential equilibrium**.

Second, we consider the weak perfect Bayesian equilibrium $(s, \nu) = (0, 1, 0, 0)$. We perturb the equilibrium play such that player 1 may choose L_1 with probability ϵ and player 2 may choose R_2 with probability ϵ. Then, information set h_3 has non-zero probability; hence, by Bayes' rule, player 3 assigns to the left node probability

$$\frac{\mathbb{P}(\text{play reaches node } L_1 L_2)}{\mathbb{P}(\text{play reaches information set } h_3)} = \frac{\epsilon(1-\epsilon)}{\epsilon(1-\epsilon)+\epsilon^2} = 1-\epsilon.$$

It is far from his equilibrium belief $\nu = 0$ for any small ϵ. Hence, the belief $\nu = 0$ is not robust to the perturbation. Indeed, this equilibrium is not robust to *any* perturbation. □

Coffee Break ☕. In the literature, the notion of sequential equilibrium has been proposed, followed by the notion of (weak) perfect Bayesian equilibrium as a weaker but tractable one. □

2 Sequential Equilibrium

As in the notion of weak perfect Bayesian equilibrium, we require sequential rationality for sequential equilibrium.

Definition 1. In a finite extensive-form game Γ with perfect recall, an assessment (β^*, μ^*) is **sequentially rational** if for each $i \in N$, each β_i, and each $h_i \in H_i$, it holds that

$$\mathbb{E}_{\beta_i^*, \beta_{-i}^*, \mu^*}[u_i(z) \mid h_i] \geq \mathbb{E}_{\beta_i, \beta_{-i}^*, \mu^*}[u_i(z) \mid h_i].$$

A player updates her belief when observing an event. However, Bayes' rule applies only to on-path information sets. For the notion of sequential equilibrium, we perturb players' strategies, so that all information sets on-path.

Definition 2. In a finite extensive-form game Γ with perfect recall, player i's strategy β_i is **completely mixed** if it assigns strictly positive probabilities to all actions available at her every information set h_i.

If all players' strategies are completely mixed, then Bayes' rule applies for all information sets. Hence, we have the following concept of consistency:

Definition 3. In a finite extensive-form game Γ with perfect recall, an assessment (β^*, μ^*) is **consistent** if there exists a sequence $(\beta^n, \mu^n)_{n \in \mathbb{N}}$ of assessments such that:

1. β^n is a completely mixed strategy profile.
2. μ^n is the belief system determined by Bayes' rule and the strategy profile β^n.
3. $\beta^n \to \beta^*$ and $\mu^n \to \mu^*$ as $n \to \infty$.[1]

[1] Convergence is in the Euclidian space.

Remark 1. In Definition 3, a completely mixed strategy profile β^n need *not* be rational for each n. In particular, it need not be sequentially rational under the belief system μ^n. $\square$

Sequential Equilibrium Once we have had the concept of sequential rationality and consistency, we can now define sequential equilibrium.

Definition 4. In a finite extensive-form game Γ with perfect recall, an assessment (β^*, μ^*) is a **sequential equilibrium** if the following conditions are satisfied:

1. (β^*, μ^*) is sequentially rational.
2. (β^*, μ^*) is consistent.

Example 2. In Example 1, which has two weak perfect Bayesian equilibria, we show that the weak perfect Bayesian equilibrium $(s, \nu) = (1, 1, 1, 1)$ is a sequential equilibrium. It suffices to find a sequence $(s^n, \nu^n)_n$ of assessments such that:

- $s^n = (s_1^n, s_2^n, s_3^n)$ is a completely mixed strategy profile.
- ν^n is the belief system determined by Bayes' rule and the strategies s^n.
- $s^n \to (1, 1, 1)$ and $\nu^n \to 1$ as $n \to \infty$.

For each n, we have strategies $s_1^n = s_2^n = s_3^n = \frac{n}{n+1}$ with the belief

$$\nu^n = \frac{s_1^n s_2^n}{s_1^n s_2^n + s_1^n(1 - s_2^n)} = s_2^n.$$

Since $s_2^n \to 1$, we have $\nu^n \to 1$.

We show that the weak perfect Bayesian equilibrium $(s, \nu) = (0, 1, 0, 0)$ is not a sequential equilibrium. It suffices to show that for *any* completely mixed strategy profile $s^n \to (0, 1, 0)$, the sequence $(\nu^n)_n$ of the belief system determined by Bayes' rule and the strategies s^n does not converge to 0. Indeed, since $\nu^n = s_2^n$ as above and $s_2^n \to 1$, we have $\nu^n \to 1$. $\square$

3 Examples

3.1 Selten's Horse

Example 3. Consider the game of Figure 2, called Selten's horse.[2] Since each player has two actions and one information set, we denote player i's strategy by probability $s_i \in [0, 1]$ of choosing action L_i. There is one non-singleton information set $\{L_1, R_1L_2\}$ for player 3, denoted h_3. We denote the belief $\mu(\cdot \mid h_3)$ by probability $\nu \in [0, 1]$ on the (left) node L_1. Hence, let (s_1, s_2, s_3, ν) denote an assessment.

[2]This game is named after the game tree's looking like a horse.

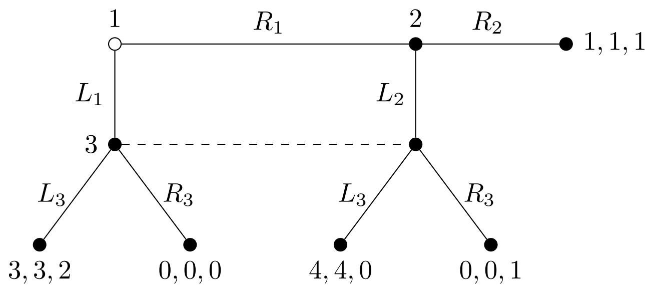

Figure 2: Selten's horse

An assessment $(0, 0, 0, \frac{1}{3})$ is a sequential equilibrium. Player 3's strategy $s_3 = 1$ is sequentially rational because given his belief $\nu = \frac{1}{3}$, he is indifferent between L_3 and R_3. Similarly, strategies $s_1 = 0$ and $s_2 = 0$ are sequentially rational. It remains to show consistency. Consider completely mixed strategies $s_1^n = 1 - \frac{1}{n+2}$ and $s_2^n = 1 - \frac{2}{n+2}$. Then, the belief system is

$$\nu^n = \frac{\frac{1}{n+2}}{\frac{1}{n+2} + \left(1 - \frac{1}{n+2}\right)\frac{2}{n+2}} \to \frac{1}{3},$$

which establishes the consistency. $\square$

4 Sequential Equilibrium versus Weak Perfect Bayesian Equilibrium

The notion of sequential equilibrium is stronger that that of weak perfect Bayesian equilibrium.

Theorem 1. *In a finite extensive-form game Γ with perfect recall, if an assessment (β, μ) is a sequential equilibrium then it is a weak perfect Bayesian equilibrium.*

Proof. It suffices to show that for any consistent belief μ, the belief $\mu(\cdot \mid h)$ for each on-path information set h under β satisfies Bayes' rule. By consistency, there is a sequence $(\beta^n)_n$ of completely mixed strategy profiles such that the corresponding sequence $(\mu^n)_n$ converges to μ. Fix any on-path information set h under β and any node $x \in h$. Then, $\mu^n(x \mid h) = \frac{Q^n(x)}{Q^n(h)}$, where $Q^n(x)$ is the probability of reaching x under β^n and $Q^n(h) = \sum_{x' \in h} Q^n(x')$ is the probability of reaching h under β^n. For each $x' \in h$, $Q^n(x') \to Q(x')$ as $n \to \infty$.[3] Hence, $\frac{Q^n(x)}{Q^n(h)} \to \frac{Q(x)}{Q(h)}$, which is the equilibrium belief $\mu(x \mid h)$. $\blacksquare$

[3] We show this convergence. Each node x corresponds to a unique sequence of nodes $\varnothing \equiv x_0 \to \cdots \to x_m \equiv x$. For each $t = 0, 1, \ldots, m-1$, let $P(x_t)$ be the player who moves at node x_t, and let $H(x_t)$ be the information set including node x_t. Given a strategy profile β^n, the probability of moving from x_t to x_{t+1} is $\beta^n_{P(x_t)}[H(x_t)](x_{t+1})$. Hence, the probability of reaching node x is $Q^n(x) \equiv \prod_{t=0}^{m-1} \beta^n_{P(x_t)}[H(x_t)](x_{t+1})$. Similarly, the probability of reaching x is $Q(x) \equiv \prod_{t=0}^{m-1} \beta_{P(x_t)}[H(x_t)](x_{t+1})$. Since $\beta^n_{P(x_t)}[H(x_t)] \to \beta_{P(x_t)}[H(x_t)]$, $Q^n(x) \to Q(x)$.

4.1 Example

How do we guess assessments that are *likely* to be sequential equilibria? Since all sequential equilibria are weak perfect Bayesian equilibria, we can shortlist the assessments that are likely to be sequential equilibria by finding weak perfect Bayesian equilibria.

Example 4. Consider the game of Figure 3. Since player i has two actions, we denote her strategy by probability $s_i \in [0,1]$ of choosing action A_i. Since player i has one two-element information set h_i, we denote her belief by probability $\nu_i \in [0,1]$ on the left node. Let (s_1, s_2, ν_1, ν_2) be an assessment. Since $\nu_1 = \frac{1}{3}$, it suffices to consider s_1, s_2, and ν_2.

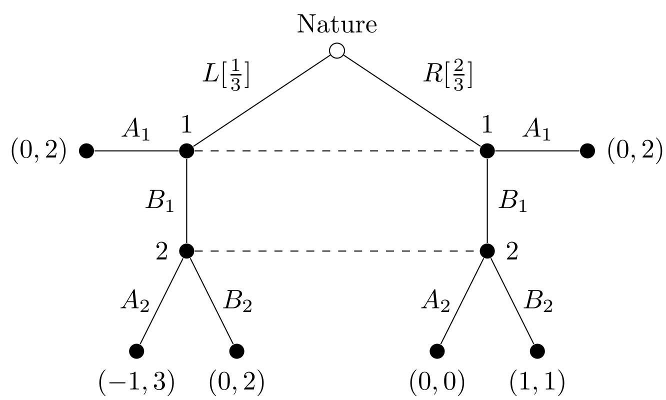

Figure 3: an example

We find all weak perfect Bayesian equilibria. There are two cases to consider:

1. Suppose that h_2 is on the equilibrium path. By Bayes' rule, $\nu_2 = \frac{1}{3}$. Hence, player 2 plays B_2, which makes player 1 play B_1. Hence, we have a weak perfect Bayesian equilibrium $(s_1, s_2, \nu_1, \nu_2) = (0, 0, \frac{1}{3}, \frac{1}{3})$.

2. Suppose that h_2 is off the equilibrium path. This means that player 1 plays $s_1 = 1$. To examine the sequential rationality, we evaluate player 1's payoff from playing B_1. The payoff from playing B_1 is $-\frac{1}{3}s_2 + \frac{2}{3}(1 - s_2) = \frac{2}{3} - s_2$. This payoff cannot exceed zero because otherwise, she would deviate to B_1. Hence, $s_2 \geq \frac{2}{3}$. Then, what belief ν_2 rationalizes player 2's strategy $s_2 \geq \frac{2}{3}$? If he plays A_2 then his payoff is $3\nu_2$, while if he plays B_2 then his payoff is $2\nu_2 + (1 - \nu_2) = 1 + \nu_2$.

   - If $3\nu_2 > 1 + \nu_2$ (i.e., $\nu_2 > \frac{1}{2}$) then he plays $s_2 = 1$. Hence, we have a (weak) perfect Bayesian equilibrium $(s_1, s_2, \nu_1, \nu_2) = (1, 1, \frac{1}{3}, \nu_2)$ for any $\nu_2 > \frac{1}{2}$.
   - If $3\nu_2 = 1 + \nu_2$ (i.e., $\nu_2 = \frac{1}{2}$) then he chooses $s_2 \in [\frac{2}{3}, 1]$. Hence, we have a (weak) perfect Bayesian equilibrium $(s_1, s_2, \nu_1, \nu_2) = (1, s_2, \frac{1}{3}, \frac{1}{2})$ for any $s_2 \in [\frac{2}{3}, 1]$.

Next, we show that the assessment $(s_1, s_2, \nu_1, \nu_2) = (0, 0, \frac{1}{3}, \frac{1}{3})$ is a sequential equilibrium. Since we have established the sequential rationality, it suffices to prove the consistency. For completely mixed strategies $s_1^n = s_2^n = \frac{1}{n}$, it follows that

$$\nu_2^n = \frac{\frac{1}{3}(1-\frac{1}{n})}{\frac{1}{3}(1-\frac{1}{n}) + \frac{2}{3}(1-\frac{1}{n})} = \frac{1}{3},$$

which establishes the consistency.

However, all the other weak perfect Bayesian equilibria (with player 1's strategy $s_1 = 1$) are not sequential equilibria. To see why, consider any perturbed strategies $s_1^n \to 1$. The strategies determine player 2's belief

$$\nu_2^n = \frac{\frac{1}{3}(1-s_1^n)}{\frac{1}{3}(1-s_1^n) + \frac{2}{3}(1-s_1^n)} = \frac{1}{3},$$

which contradicts the requirement $\nu_2 \geq \frac{1}{2}$. $\square$

References

Kreps, D. M., & Wilson, R. (1982). Sequential equilibria. *Econometrica*, *50*(4), 863–894.