Notes on the Chebyshev Inequality

Let X be a random variable with mean $E(X) = \mu$ and variance $\sigma^2 = var(X)$. Then, the Chebyshev inequality states that

$$P(|X - \mu| \geq t) \leq \frac{\sigma^2}{t^2},$$

for any $t > 0$. Other equivalent forms can be written for this inequality, by simple manipulation:

$$P(|X - \mu| < t) > 1 - \frac{\sigma^2}{t^2}$$

$$P(|X - \mu| \geq n\sigma) \leq \frac{1}{n^2}$$

We can also bound probabilities that involve an interval of X that is *not* centered on the mean, μ:

$$P(|X - c| \geq t) \leq \frac{E[(X-c)^2]}{t^2} = \frac{\sigma^2 + (\mu - c)^2}{t^2}$$

$$P(|X - c| < t) > 1 - \frac{E[(X-c)^2]}{t^2}$$

The above inequality is the most general form of the 2-sided Chebyshev: putting $c = \mu$ yields the standard form. Note that the statement that $|X - c| < t$ is the same as $-t + c < X < t + c$. Thus,

$$P(a < X < b) = P(|X - \frac{a+b}{2}| < \frac{b-a}{2}) \geq 1 - \frac{\sigma^2 + (\mu - \frac{a+b}{2})^2}{\left(\frac{b-a}{2}\right)^2},$$

where we have substituted $a = -t + c$ and $b = t + c$.

One-Sided Chebyshev : Using the Markov Inequality, one can also show that for any random variable with mean μ and variance σ^2, and any positve number $a > 0$, the following *one-sided Chebyshev* inequalities hold:

$$P(X \geq \mu + a) \leq \frac{\sigma^2}{\sigma^2 + a^2}$$

$$P(X \leq \mu - a) \leq \frac{\sigma^2}{\sigma^2 + a^2}$$

Example: Roll a single fair die and let X be the outcome. Then, $E(X) = 3.5$ and $var(X) = 35/12$. (Make sure you can compute this!) Suppose we want to compute $p = P(X \geq 6)$.

(a). Exact: We easily see that $p = P(X \geq 6) = P(X = 6) = 1/6 \approx 0.167$.

(b). By Markov inequality, we get:

$$P(X \geq 6) \leq \frac{21/6}{6} \approx 0.583$$

(c). By the usual (two-sided) Chebyshev inequality, we can obtain a stronger bound on p:

$$P(X \geq 6) \leq P(X \geq 6 \text{ OR } X \leq 1) = P(|X - 3.5| \geq 2.5) \leq \frac{35/12}{(2.5)^2} = \frac{7}{15} \approx 0.467$$

(d). By using the one-sided Chebyshev inequality, we can obtain an even stronger bound on p:

$$P(X \geq 6) = P(X \geq 3.5 + 2.5) \leq \frac{35/12}{(35/12) + (2.5)^2} = \frac{7}{22} \approx 0.318$$