ECO III (Intermediate Microeconomic Theory)

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Topics

1. Consumer choice.

2. Producer choice.

3. Competitive markets in partial equilibrium.

CONSUMER CHOICE

Preferences

- Let A be the consumer's set of alternatives. For example, A could be:
  - A set of baskets of goods.
  - A set of football teams.
- The consumer's **preferences** over her set of alternatives enable her to compare every two alternatives.
- **Definition:** We say that the consumer's preferences are **complete** if she is able to rank every two elements of A.
  - That is, if $a, b \in A$, the consumer is able to say whether she prefers a to b, whether she prefers b to a, or whether she is indifferent.
- **Definition:** We say that the consumer's preferences are **transitive** if when she prefers a to b and b to c, she also prefers a to c.
- **Definition:** We say that the consumer's preferences are **rational** if they are complete and transitive.

Utility representation of preferences

- Working directly with preferences can be difficult.
  - For instance, in order to find the consumer's optimal choice from her set of alternatives we would have to compare every pair of alternatives.
- Fortunately, sometimes preferences can be represented by a **utility function**.
- **Definition:** A utility function, $U : A \to \mathbb{R}$, represents the consumer's preferences if, for any pair $a, b \in A$, a is weakly preferred to b if and only if $U(a) \geq U(b)$.
- **Important observations:**
  1. Multiple utility functions can represent the same preferences.
     - For instance, if $A = \{a, b\}$ and $u(a) = 1$, $u(b) = 0$, and $v(a) = 1,000,000$, $v(b) = 2$, then $u(\cdot)$ and $v(\cdot)$ represent the same preferences.
     - If $g : \mathbb{R} \to \mathbb{R}$ is strictly increasing, $\hat{u}(a) \equiv g(u(a))$ represents the same preferences as $u(a)$.
  2. Utility functions contain no **cardinal** information (i.e. the mere levels of utility do not tell us anything).
     - If person A reports that his utility of eating a steak is 10 and person B reports that his utility of eating a steak is 5, can we say who likes it more? **No**.

When do preferences have a utility representation?

- **Note:** If some preferences can be represented by a utility function, they must be complete and transitive (rational).
- Can all rational preferences be represented by a utility function? **NO**.
  - You will see an example in more advanced courses.
- However, if A is finite and preferences are rational, then they can be represented by a utility function (this is easy to see).
- More generally, if preferences are rational and **continuous**, then they can be represented by a continuous utility function (Utility Representation Theorem).
  - A formal definition of continuity or proof of the theorem is beyond the scope of this course.
  - Intuitively, continuity says that if a is preferred to b and a' is arbitrarily close to a, then a' must also be preferred to b.

Workhorse

- Throughout this course we will work with choice problems in which $A = \mathbb{R}^N$ and where preferences can be represented by continuous utility functions.
  - **One interpretation:**
    - There are N goods that can be consumed.
    - The nth coordinate, x_n, of the consumption basket, $(x_1, x_2, ..., x_N) \in \mathbb{R}^N$, represents the amount of good n specified by the basket.
- For most of the course we will actually consider cases where $N = 2$.
  - Why? Because we can DRAW to develop intuition!

Indifference curves

- Let $A = \mathbb{R}^2$ and $U(x, y)$ be the consumer's utility function.
- The consumer's **indifference curve** given a utility level $\bar{U}$ is given by the set of points $\{(x, y) \in \mathbb{R}^2 \mid U(x, y) = \bar{U}\}$.
- In words, the consumer's indifference curve given a level of utility $\bar{U}$ describes the set of pairs $(x, y) \in \mathbb{R}^2$ that give the consumer a utility $\bar{U}$.
- By considering all possible values of $\bar{U}$ we obtain the whole map of indifference curves.
- **General properties:**
  1. The consumer's map of indifference curves (including the direction of growth) **contains all the information** about her preferences.
     - All the utility functions that represent the same preferences generate the same map of indifference curves.
  2. Different indifference curves **cannot intersect**. Otherwise, the utility function would assign two different values to the same basket of goods.
  3. Indifference curves are **dense** (the utility function is defined over all possible baskets of goods).

Marginal rate of substitution

- The **marginal rate of substitution** between good x and good y at $(x_0, y_0$ is the rate at which the consumer is willing to substitute good x for good y when she has the basket (x_0, y_0).

- The MRS is the slope of the consumer's indifference curve at (x_0, y_0).

- Two ways of calculating the MRS at (x_0, y_0).

  1. If possible, using the indifference curve that goes through (x_0, y_0), write y as a function of x and $\bar{U}$ (say, $y = h(x, \bar{U})$). Then,
  $$MRS = \frac{\partial h(x_0, \bar{U})}{\partial x}.$$

  2. Use the implicit function theorem. That is, if $U(x, y)$ is differentiable, $U_y(x_0, y_0) \neq 0$ and $y(x)$ is defined implicitly as a function of x by $U(x, y) = \bar{U}$, then:
  $$MRS = \frac{dy(x_0)}{dx} = -\frac{U_x(x_0, y_0)}{U_y(x_0, y_0)}.$$

- **Exercise:** Using both techniques, compute the MRS at (x_0, y_0) if $u(x, y) = x^{\alpha} y^{1-\alpha}$ with $\alpha \in (0, 1)$. Verify that both techniques are indeed equivalent.

Intuition behind the implicit function Theorem

- We start at a point (x_0, y_0) that belongs to the indifference curve, $U(x, y) = \bar{U}$.
- If x increases by a small amount dx, by how much should y increase to keep the indifference?
  - The change in x increases utility be approximately $dxU_x(x_0, y_0)$.
  - Each unit of increase in y increases utility by approximately $U_y(x_0, y_0)$.
  - Thus, to keep the indifference y must increase by approximately $dy = -dx\frac{U_x(x_0,y_0)}{U_y(x_0,y_0)}$.
  - The approximate rate of change needed to accommodate the increase in x is then
  $$\frac{dy}{dx} = -\frac{U_x(x_0, y_0)}{U_y(x_0, y_0)}.$$
  - This approximation becomes exact as dx approaches zero.

Marginal utility

- Let $A = \mathbb{R}^N$ and $U(x)$ be the utility function ($x = (x_1, ..., x_N)$).
- The **marginal utility** of the commodity x_i at a basket x_0 gives the rate at which utility increases (or decreases) as x_i is increased when starting from x_0.
- Formally, $UMGx_i = U_{x_i}(x_0)$, where $U_{x_i}(x_0)$ denotes the partial derivative of U with respect to x_i at $x = x_0$.
- We say that a commodity x_i is a **good** if $U_{x_i}(x) \geq 0$ for all x.
- We say that a commodity x_i is a **bad** if $U_{x_i}(x) \leq 0$ for all x.
- The consumers marginal utility of consuming commodity i tells us whether it is a good or a bad.
- Any other property about preferences that could potentially be inferred by looking only at the marginal utility is not robust to monotone transformations of the utility function.
  - **Example:** $U(x, y) = x^{\frac{1}{2}} y^{\frac{1}{2}}$ and $V(x, y) = x^{\frac{3}{2}} y^{\frac{3}{2}}$ represent the same preferences. However, the marginal utility of x is decreasing in x with U, and increasing in x with V.

Particular properties of preferences: Monotonicity

- **Definition:** A utility function $U : \mathbb{R}^2 \to \mathbb{R}$ is **monotone** (increasing) if for all $(x, y), (x', y') \in \mathbb{R}^2$ such that $x > x'$ and $y > y'$ we have $U(x, y) > U(x', y')$.
- **Definition:** A utility function $U : \mathbb{R}^2 \to \mathbb{R}$ is **strictly monotone** (increasing) if for all $(x, y) \neq (x', y') \in \mathbb{R}^2$, $x \geq x'$ and $y \geq y'$ implies that $U(x, y) > U(x', y')$.
- These properties are essential to preferences. That is, if U and V are utility functions that represent the same preferences and U is monotone (or strictly monotone), then V must also be monotone (or strictly monotone).

Particular properties of preferences: Quasi-concavity

- **Definition:** A set $A \subseteq \mathbb{R}^N$ is convex if for all $x, y \in A$ and $\alpha \in (0,1)$, $\alpha x + (1-\alpha)y \in A$.
  - Intuitively, if $x, y \in A$, the straight line that joins them together must be contained in A.
- **Definition:** Let $F : A \to \mathbb{R}$. The **upper-contour set** of F given a level $\bar{U}$ is $OC(\bar{U}) = \{a \in A | F(a) \geq \bar{U}\}$.
- **Definition:** A function $U : \mathbb{R}^N \to R$ is **quasi-concave** if for every $\bar{U}$, the its upper-contour set given $\bar{U}$ is convex.
- Equivalently, U is quasi-concave if, for every $x, y \in \mathbb{R}^N$ and $\alpha \in (0,1)$, $U(\alpha x + (1-\alpha)y) \geq min\{U(x), U(y)\}$.
- **Definition:** A function $U : \mathbb{R}^N \to R$ is **strictly quasi-concave** if, for every $x, y \in \mathbb{R}^N$ and $\alpha \in (0,1)$, $U(\alpha x + (1-\alpha)y) > min\{U(x), U(y)\}$.

Quasi-concavity: Intuition

- Quasi-concavity is a property that is essential to preferences.
- Suppose that $A = \mathbb{R}^2_+$ and that both commodities are goods.
- Then U is quasi-concave if and only if it generates indifference curves convex with respect to the origin (i.e. with decreasing marginal rate of substitution).
- This means that in this context preferences are quasi-concave if and only if the consumer is more willing to substitute x for y the more units of x she has to begin with. Thus, it is a rather intuitive property.
- These are good news because we will see that quasi-concavity is also a very useful mathematical property.

Particular properties of preferences: Homothetisity

- **Definition:** A utility function $U : \mathbb{R}^N \to \mathbb{R}$ is **homothetic** if $U(x) \geq U(y)$ implies that, for all $\lambda > 0$, $U(\lambda x) \geq U(\lambda y)$.
- **Intuition:** the scale of consumption does not change relative preferences.
- **Note:** If preferences are homothetic, indifference curves must have the same marginal rate of substitution along each ray from the origin.
  - Thus, if this does not hold, preferences are not homothetic.
  - The intuition behind the result was given in class.
  - The converse is also true when the utility function is monotone.

When are preferences homothetic?

- **Definition:** The function $F : \mathbb{R}^N \to \mathbb{R}$ is homogeneous of degree $k > 0$ if, for all $\lambda \neq 0$ and $x \in \mathbb{R}^N$, $F(\lambda x) = \lambda^k F(x)$.

- **Proposition:** If a utility function is homogeneous of degree k, then it is homothetic.

- **Proof:** Suppose $U(x)$ is homogenous of degree k, and take any $x, x \in \mathbb{R}^N$ such that $U(x) \geq U(x')$, and any $\lambda > 0$. Then it must be the case that $\lambda^k U(x) \geq \lambda^k U(x')$. By the homogeneity of U it then follows that $U(\lambda x) \geq U(\lambda x')$, as we needed to show.

- **Note:** The converse is not true. There are homothetic functions that are not homogeneous.

  - **Example:** $U(x, y) = xy + 5$ is clearly not homogeneous but it is homothetic, as it represents the same preferences as $V(x, y) = xy$, which is homogeneous of degree 2 and therefore homothetic.

  - **Note:** Homotheticity is preserved under monotone transformations, but homogeneity is not.

Examples I

- The following utility functions are all defined over $\mathbb{R}^2_+$.
- For each one make sure you understand how to draw the map of indifference curves and why the properties hold or not.

1. **Cobb-Douglas:** $U(x, y) = x^\alpha y^\beta$, with $\alpha, \beta > 0$.
    - Both commodities are goods.
    - Preferences are monotone (but not strictly).
    - Preferences are quasi-concave (but not strictly).
    - Preferences are homothetic.
    - $MRS(x, y) = -\frac{\alpha}{\beta}\frac{y}{x}$.

Examples II

2. **Perfect substitutes:** $U(x,y) = \alpha x + \beta y$, with $\alpha, \beta > 0$.
   - Both commodities are goods.
   - Preferences are strictly monotone.
   - Preferences are quasi-concave (but not strictly).
   - Preferences are homothetic.
   - $MRS(x,y) = -\frac{\alpha}{\beta}$.
3. **Perfect complements:** $U(x,y) = min\{\alpha x, \beta y\}$, with $\alpha, \beta > 0$.
   - Both commodities are goods.
   - Preferences are monotone (but not strictly).
   - Preferences are quasi-concave (but not strictly).
   - Preferences are homothetic.
   - $MRS(x,y) = 0$ if $\alpha x > \beta y$, $MRS(x,y) = -\infty$ if $\alpha x < \beta y$, and is not defined otherwise.

Examples III

4. **CES (constant elasticity of substitution):**
   $U(x,y) = \left(\alpha x^{\frac{r-1}{r}} + \beta y^{\frac{r-1}{r}}\right)^{\frac{r}{r-1}}$, with $\alpha, \beta > 0$, and $r \in (0,1) \bigcup (1,\infty)$.

   - Both commodities are goods. $Umgx = \left(\alpha x^{\frac{r-1}{r}} + \beta y^{\frac{r-1}{r}}\right)^{\frac{1}{r-1}} \alpha x^{-\frac{1}{r}}$.
   - Preferences are strictly monotone.
   - Preferences are strictly quasi-concave.
   - Preferences are homothetic.
   - $MRS(x,y) = -\frac{\alpha}{\beta}\left(\frac{y}{x}\right)^{\frac{1}{r}}$.
   - **Note:** CES preferences generalize the previous three cases.
   - r measures the degree of substitutability of the two goods.
     - As $r \to 0$, CES represents the same preferences as $U(x,y) = min\{x,y\}$.
     - As $r \to 1$, CES represents the same preferences as $U(x,y) = x^{\alpha}y^{\beta}$.
     - As $r \to \infty$, CES represents the same preferences as $U(x,y) = \alpha x + \beta y$.

Examples IV

5. **Quasi-linear:** $U(x,y) = x + v(y)$, with $v'(y) > 0$.
   - Both commodities are goods.
   - Preferences are strictly monotone.
   - Preferences are strictly quasi-concave if $v(y)$ is strictly concave.
   - Preferences are not homothetic unless $v(y)$ is linear.
   - $MRS(x,y) = -\frac{1}{v'(y)}$.
6. $U(x,y) = x - v(y)$, with $v'(y) > 0$.
   - x is a good and y is a bad.
   - Preferences are not monotone.
   - Preferences are strictly quasi-concave if $v(y)$ is strictly convex.
   - Preferences are not homothetic unless $v(y)$ is linear.
   - $MRS(x,y) = \frac{1}{v'(y)}$.

Examples V

7. $U(x, y) = -x^\alpha y^\beta$, with $\alpha, \beta > 0$.
    - Both commodities are bads.
    - Preferences are monotone (decreasing).
    - Preferences are not quasi-concave.
    - Preferences are homothetic.
    - $MRS(x, y) = -\frac{\alpha y}{\beta x}$.
8. $U(x, y) = (x - 1)^2 + (y - 1)^2$.
    - x and y are nor goods nor bads.
    - Preferences are not monotone.
    - Preferences are not quasi-concave.
    - Preferences are not homothetic.
    - $MRS(x, y) = -\frac{x-1}{y-1}$.

Budget constraints

- We are done with our study of preferences.
- The set of alternatives from which an agent can choose is restricted by her **budget constraint**. The nature of this constraint depends on the context.
- **For example:**

1. **Marshallian budget constraint:** A consumer has income I to consume N different goods (x_n is the amount of good n consumed). The price of good n is linear and equal to p_n. The constraint implied by this description is:

$$\sum_{n=1}^{N} p_n x_n \leq I.$$

2. **Walrasian budget constraint:** The situation is the same as above, except that instead of income, the consumer has an endowment ω_n of each good n. The constraint implied by this description is:

$$\sum_{n=1}^{N} p_n x_n \leq \sum_{n=1}^{N} p_n \omega_n.$$

3. Typically, we also add the restrictions that the consumption of each good must be **non-negative**. That is, $x_n \geq 0$ for all n.

The consumer's problem

- We begin by considering the problem of a consumer that faces a Marshallian budget constraint (with non-negative consumption).
- In the case with two commodities, x and y, the consumer solves:

$$\max_{x,y} \quad U(x,y),$$

s.t.

$$p_x x + p_y y \leq I,$$
$$x \geq 0, \quad y \geq 0.$$

- Let $V(p_x, p_y, I)$ be the value function of this problem. This function is called the **indirect utility function**.
- The functions $x^*(p_x, p_y, I)$ and $y^*(p_x, p_y, I)$, which solve the consumer's problem, are called the consumer's **demand correspondences**.
- **Observations:**
  - If preferences are monotone, the budget constraint must bind at the optimum.
  - Will the consumer's preferred choice be unique? Not necessarily. It is unique if preferences are strictly quasi-concave.

Solution with smooth, monotone and quasi-concave preferences

- Let $N = 2$. You should understand the intuition behind the following results by drawing the budget constraint and indifference curves.

- **Interior solutions:** (x^*, y^*) is an interior solution if and only if:

  1. $\frac{p_x}{p_y} = \frac{U_x(x^*,y^*)}{U_y(x^*,y^*)}$. That is, relative prices = MRS.

  2. $x^* p_x + y^* p_y = I$. That is, the budget constraint must bind.

- **Corner solutions at $x = 0$:** $(0, y^*)$ is a corner solution if and only if:

  1. $\frac{p_x}{p_y} \geq \frac{U_x(0,y^*)}{U_y(0,y^*)}$. That is, relative prices $\geq$ MRS (in absolute value).

  2. $y^* p_y = I$. That is, the budget constraint must bind.

- **Corner solutions at $y = 0$:** $(x^*, 0)$ is a corner solution if and only if:

  1. $\frac{p_x}{p_y} \leq \frac{U_x(x^*,0)}{U_y(x^*,0)}$. That is, relative prices $\leq$ MRS (in absolute value).

  2. $x^* p_x = I$. That is, the budget constraint must bind.

- **More general solution:** See the optimization note.

Solution using the K-T method I

- The Lagrangian of the problem is:

$$L(x, y, \lambda, \mu_x, \mu_y; p_x, p_y, I) \equiv U(x, y) - \lambda(p_x x + p_y y - I) + \mu_x x + \mu_y y.$$

- $(x^*, y^*, \lambda^*, \mu_x^*, \mu_y^*)$ satisfies the K-T conditions if:

1. $U_x(x^*, y^*) = \lambda^* p_x - \mu_x^*$.
2. $U_y(x^*, y^*) = \lambda^* p_y - \mu_y^*$.
3. $p_x x^* + p_y y^* \leq I$, $\lambda^* \geq 0$, and $\lambda^*(p_x x^* + p_y y^* - I) = 0$.
4. $x^* \geq 0$, $\mu_x^* \geq 0$, and $x^* \mu_x^* = 0$.
5. $y^* \geq 0$, $\mu_y^* \geq 0$, and $y^* \mu_y^* = 0$.

Solution using the K-T method II

- **Recall:**
  - Conditions (1) and (2) are about comparing the slope of the budget constraint with the slope of the indifference curve at the optimum.
  - Conditions (3)-(5) reflect the fact that each constraint either binds (holds with equality) or does not bind (and its multiplier is zero).
  - There are many ways of writing the Lagrangian, but the way you write the K-T conditions must be consistent with your choice.
  - If this is not clear, go back to the optimization note.
- **Simplifying the problem:** The K-T method forces us to consider different cases. However, in many instances, some cases can easily be ruled out.
  1. If preferences are monotone (increasing), then the budget constraint must bind and we must have $p_x x^* + p_y y^* = I$.
  2. If $U(x, y)$ has $MRS = -\infty$ as x approaches zero, then we cannot have a solution where $x^* = 0$ and thus μ_x^* must be zero.
  3. If $U(x, y)$ has $MRS = 0$ as y approaches zero, then we cannot have a solution where $y^* = 0$ and thus μ_y^* must be zero. (2) and (3) are called **Inada** conditions.

The envelope theorem in the consumer's maximization problem

- **Note:** Under some mild regularity conditions, if $x^*(p_x, p_y, I)$ and $y^*(p_x, p_y, I)$ solve the consumer's problem, then there exist multipliers $\lambda^*(p_x, p_y, I)$, $\mu_x^*(p_x, p_y, I)$ and $\mu_y^*(p_x, p_y, I)$ such that

$$V(p_x, p_y, I) \equiv L(x^*, y^*, \lambda^*, \mu_x^*, \mu_y^*; p_x, p_y, I).$$

- Interpretation of λ.
  - We have:
  $$\frac{\partial V(p_x, p_y, I)}{\partial I} \equiv \frac{\partial L(x^*, y^*, \lambda^*, \mu_x^*, \mu_y^*; p_x, p_y, I)}{\partial I} \underbrace{\equiv}_{EnvelopeTheorem} \lambda^*$$
  - Thus, λ represents how the consumer's utility changes after a marginal increase in income.
- **Roy's identity** provides a way of computing the demand functions if we only know the indirect utility function.
  - $\frac{\partial V(p_x,p_y,I)}{\partial p_x} \equiv \frac{\partial L(x^*,y^*,\lambda^*,\mu_x^*,\mu_y^*;p_x,p_y,I)}{\partial p_x} \underbrace{\equiv}_{EnvelopeTheorem} -\lambda^* x^*.$
  - Thus, $x^*(p_x, p_y, I) \equiv -\frac{\frac{\partial V(p_x,p_y,I)}{\partial p_x}}{\frac{\partial V(p_x,p_y,I)}{\partial I}}.$

Properties of the Marshallian Demand correspondence

1. Homogenous of degree zero in prices and income. That is,

$$x^*(\lambda p_x, \lambda p_y, \lambda I) \equiv x^*(p_x, p_y, I) \;\; \forall \;\; \lambda > 0.$$

   - **Intuition:** If all prices and income increase by the same proportion, the consumer's budget constraint does not change.

2. If preferences are strictly quasi-concave, $x^*(p_x, p_y, I)$ is a function, since the solution to the problem is unique.

3. The consumer's optimal choice is determined by her preferences. Thus, the solution to the problem is the same for any utility function that represents the same preferences (they all generate the same map of indifference curves).

Properties of the consumer's Indirect Utility function

1. Homogenous of degree zero in prices and income. That is,

$$V(\lambda p_x, \lambda p_y, \lambda I) \equiv V(p_x, p_y, I) \;\; \forall \;\; \lambda > 0.$$

2. Non-decreasing in income and non-increasing in prices.

   - As income increases, the budget set becomes larger, so the value function cannot decrease.
   - As the price of one good increases, the budget set becomes smaller, so the value function cannot increase.

3. The indirect utility function does depend on the utility function chosen to represent preferences, so its level has no meaning.

   - **Example:** By multiplying the utility function by 2 we still represent the same preferences, but the value function will be twice as large.

Elasticities

- **Definition:** The elasticity of a variable, x, with respect to a parameter, a, gives the percent change in x induced by a marginal percent change in a.

- Thus, if we have the function $x(a, b)$ (where b is a vector of other parameters), the elasticity of x with respect to a at (a_0, b_0) is given by

$$\begin{aligned}
\varepsilon_{x,a}(a_0) &\equiv \lim_{a' \to a_0} \frac{\frac{x(a',b_0)-x(a_0,b_0)}{x(a_0,b_0)}}{\frac{a'-a_0}{a_0}} \equiv \lim_{a' \to a_0} \frac{x(a', b_0) - x(a_0, b_0)}{a' - a} \frac{a_0}{x(a_0, b_0)} \\
&\equiv \frac{\partial x(a_0, b_0)}{\partial a} \frac{a_0}{x(a_0, b_0)}.
\end{aligned}$$

Elasticities in the context of the consumer's problem

- Let $x^*(p_x, p_y, I)$ be one of the demands derived from the consumer's problem. Then we have:

1. The **direct-price** elasticity of x^* is $\varepsilon_{x,p_x}(p_x, p_y, I) \equiv \frac{\partial x^*(p_x,p_y,I)}{\partial p_x} \frac{p_x}{x^*(p_x,p_y,I)}$.
    - If $\varepsilon_{x,p_x} > 0$, we say that x is a **giffen** good.
    - If $\varepsilon_{x,p_x} < 0$, we say that x is an **ordinary** good.
    - If $\varepsilon_{x,p_x} = 0$, we say that x is a **neutral** good.
2. The **cross-price** elasticity of $x^*$is $\varepsilon_{x,p_y}(p_x, p_y, I) \equiv \frac{\partial x^*(p_x,p_y,I)}{\partial p_y} \frac{p_y}{x^*(p_x,p_y,I)}$.
    - If $\varepsilon_{x,p_y} > 0$, we say that x and y are **substitutes**.
    - If $\varepsilon_{x,p_y} < 0$, we say that x and y are **complements**.
    - If $\varepsilon_{x,p_y} = 0$, we say that x and y are **independent**.
3. The **income elasticity** of x^* is $\varepsilon_{x,I}(p_x, p_y, I) \equiv \frac{\partial x^*(p_x,p_y,I)}{\partial I} \frac{I}{x^*(p_x,p_y,I)}$.
    - If $\varepsilon_{x,I} > 0$, we say that x is a **normal** good.
    - If $\varepsilon_{x,I} < 0$, we say that x is an **inferior** good.
    - If $\varepsilon_{x,I} = 0$, we say that x is a **neutral** good.

Relationships between elasticities I

1. **Euler:** A way to relate the different elasticities of the same good.

   - **Idea:** Demand is homogeneous of degree zero, so, if we increase prices and income in the same proportion the net change on demand must be zero.
   - Formally, $x^*(\lambda p_x, \lambda p_y, \lambda I) \equiv x^*(p_x, p_y, I)$ for all $\lambda > 0$.
   - Differentiating with respect to λ we obtain:

$$\frac{\partial x^*(\lambda p_x, \lambda p_y, \lambda I)}{\partial p_x} p_x + \frac{\partial x^*(\lambda p_x, \lambda p_y, \lambda I)}{\partial p_y} p_y + \frac{\partial x^*(\lambda p_x, \lambda p_y, \lambda I)}{\partial I} I \equiv 0.$$

   - If $\lambda = 1$ and we divide both sides by x^* we obtain:

$$\varepsilon_{x,p_x} + \varepsilon_{x,p_y} + \varepsilon_{x,I} \equiv 0.$$

Relationships between elasticities II (only when Walras' Law holds)

2. **Cournot:** A way to relate the elasticities of different goods with respect to the same price.

   - **Idea:** If the budget constraint holds with equality, the net change in expenditure derived in response to the change of a price must be zero (income did not change).
   - Formally, $p_x x^*(p_x, p_y, I) + p_y y^*(p_x, p_y, I) \equiv I$.
   - Differentiating with respect to p_x we obtain:

   $$x^*(p_x, p_y, I) + p_x \frac{\partial x^*(p_x, p_y, I)}{\partial p_x} + p_y \frac{\partial y^*(p_x, p_y, I)}{\partial p_x} \equiv 0.$$

   - Multiplying both sides by $\frac{p_x}{I}$, the second term by $\frac{x^*}{x^*}$, and the third term by $\frac{y^*}{y^*}$, and defining $S_x = \frac{p_x x^*}{I}$ and $S_y = \frac{p_y y^*}{I}$:

   $$S_x + S_x \varepsilon_{x,p_x} + S_y \varepsilon_{y,p_x} \equiv 0,$$

Relationships between elasticities III (only when Walras' Law holds)

3. **Engel:** A way to relate the elasticities of different goods with respect to income.

   - **Idea:** If the budget constraint holds with equality, the net change in expenditure derived in response to a change in income must be the same as the change in income.
   - Formally, $p_x x^*(p_x, p_y, I) + p_y y^*(p_x, p_y, I) \equiv I$.
   - Differentiating with respect to I we obtain:
   $$p_x \frac{\partial x^*(p_x, p_y, I)}{\partial I} + p_y \frac{\partial y^*(p_x, p_y, I)}{\partial I} \equiv 1.$$
   - Multiplying the first term by $\frac{x^* I}{x^* I}$, and the second term by $\frac{y^* I}{y^* I}$:
   $$S_x \varepsilon_{x,I} + S_y \varepsilon_{y,I} \equiv 1.$$