Game Theory

Dominance

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Suppose that a player has an action that yields strictly less payoffs than another (mixed) strategy of hers, regardless of opponents' strategies. Such an action is called **strictly dominated**. Assume that all players will never play strictly dominated actions; in addition, assume (i) they know they will never play strictly dominated actions, (ii) they know they know they will never play strictly dominated actions, (iii) they know they know they know they will never play strictly dominated actions, and so on.[1] Then, what are all the strategies that they potentially play based only on this assumption?

1 Strict Dominance

Example 1. Consider the following prisoners' dilemma:

	C	D
C	$1, 1$	$-1, 2$
D	$2, -1$	$0, 0$

Table 1: prisoners' dilemma

Player 1's payoff when taking action C is strictly less than her payoff when taking action D regardless of player 2's (mixed) strategies. We say that action C is strictly dominated by action D (for player 1). □

1.1 Strict Dominance

We will now generalize this example to obtain the notion of strict dominance.

Definition 1. In a normal-form game G, player i's action $a_i \in A_i$ is **strictly dominated** by her (mixed) strategy $\sigma_i \in \Sigma_i$ if for player $-i$'s every (mixed) strategy profile $\sigma_{-i} \in \Sigma_{-i}$,

$$u_i(\sigma_i, \sigma_{-i}) > u_i(a_i, \sigma_{-i}).$$

Action a_i is **strictly dominated** if it is strictly dominated by some strategy σ_i.

When an action, or a pure strategy, is strictly dominated for a player, she has a (mixed) strategy that does strictly better *regardless of opponents' strategies*. This implies that a rational player—who maximizes her (expected) payoff—will never use strictly dominated actions.

[1] This assumption is called common knowledge of their never playing strictly dominated actions.

Properties of Strictly Dominated Actions We will now see important properties of strict dominance. First, we can simplify the definition.

Proposition 1. *In a normal-form game G, player i's action $a_i \in A_i$ is strictly dominated by her (mixed) strategy $\sigma_i \in \Sigma_i$ if and only if for player $-i$'s every action profile $a_{-i} \in A_{-i}$,*

$$u_i(\sigma_i, a_{-i}) > u_i(a_i, a_{-i}).$$

Proof. The "only if" part is trivial. The "if" part is also immediate since expect payoffs are linear in opponents' mixed strategies. ■

Second, even if player i's action a_i is not strictly dominated by any action a_i', it is still possible that a_i is strictly dominated by some mixed strategy σ_i. That is, to check if a_i is strictly dominated, it is not sufficient to examine all her *pure* strategies. Here is an example to illustrate this point.

Example 2. Consider the following normal-form game:

	L	R
U	$3,0$	$-1,0$
M	$0,0$	$0,0$
D	$-1,0$	$3,0$

Table 2: M is strictly dominated

For player 1, no action is strictly dominated by any other actions. However, M is strictly dominated by $\frac{1}{2}U \oplus \frac{1}{2}D$, which randomizes U and D with equal probabilities. □

1.2 Iterated Strict Dominance

Example 3. Consider the following normal-form game:

	L	M	R
U	$2,2$	$1,1$	$4,0$
D	$1,2$	$4,1$	$3,5$

Table 3: iterated deletion of strictly dominated actions

Neither U nor D is strictly dominated for player 1, while M is strictly dominated for player 2 and will never be played. Player 1 (she) also thinks that if player 2 (he) is rational, then he never plays M, and thus she can delete it from her thinking. After deleting it, player 1 realizes that D is now strictly dominated for her.

If player 2 knows not only that player 1 is rational, but also that player 1 knows that player 2 is rational, then player 2 thinks that player 1 would delete M from her thinking and thus would not play D. Hence, he also deletes D from his thinking. Since he now thinks U will be played, R is strictly dominated and deleted. Consequently, (U, L) is left.

In the above reasoning, we say that D and R are iteratively strictly dominated, respectively, for players 1 and 2. We also say that U and L survive the iterated deletion of strictly dominated actions. $\square$

Remark 1. In an introductory course in game theory, you might have iteratively deleted strictly dominated "rows" and "columns" mechanically—without considering the rationale behind the process. However, behind the process there is the assumption that players know about their rationality, they know they know about their rationality, they know they know they know about their rationality, and so on. $\square$

Iterated Strict Dominance We formalize the idea illustrated in Example 3.

Definition 2. In a normal-form game G, for each $i \in I$ and each $k \in \mathbb{N}$, let $\mathrm{ND}_i^0 = A_i$ and

$$\mathrm{ND}_i^k = \mathrm{ND}_i^{k-1} \setminus \underbrace{\left\{a_i \in \mathrm{ND}_i^{k-1} : \exists\, \sigma_i' \in \Delta\left(\mathrm{ND}_i^{k-1}\right) \quad \forall\, a_{-i} \in \mathrm{ND}_{-i}^{k-1} \quad u_i(\sigma_i', a_{-i}) > u_i(a_i, a_{-i})\right\}}_{\text{pure strategies that are strictly dominated}}.$$

Let the set ND_i^∞ of player i's actions that survive **iterated deletion of strictly dominated actions** be such that

$$\mathrm{ND}_i^\infty = \bigcap_{k=0}^{\infty} \mathrm{ND}_i^k.$$

We revisit the rationale behind the iterated deletion process. Every step asks "what are actions that will never be played by rational players?" Then, each player will conclude that no (rational) player will ever play actions that are strictly dominated. Since each player will expect that no (rational) player will play such actions. Furthermore, it will be known that players arrive at this conclusion. That justifies the deletion of these actions from the game. Players will iterate this process until no further actions can be deleted.

To better understand this concept, we formalize Example 3.

Example 4. Consider the normal-form game of Example 3. Then, the above iteration process yields the following sets:

$$\begin{array}{ll}
\mathrm{ND}_1^0 = \{U, D\} & \mathrm{ND}_2^0 = \{L, M, R\} \\
\mathrm{ND}_1^1 = \mathrm{ND}_1^0 & \mathrm{ND}_2^1 = \{L, R\} \\
\mathrm{ND}_1^2 = \{U\} & \mathrm{ND}_2^2 = \mathrm{ND}_2^1 \\
\mathrm{ND}_1^3 = \mathrm{ND}_1^2 & \mathrm{ND}_2^3 = \{L\} \\
\mathrm{ND}_1^4 = \mathrm{ND}_1^3 & \mathrm{ND}_2^4 = \mathrm{ND}_2^3
\end{array}$$

By induction, $\mathrm{ND}_1^\infty = \{U\}$ and $\mathrm{ND}_2^\infty = \{L\}$. Like this game, a normal-form game is called dominance solvable if ND_i^∞ is a singleton for each $i \in I$. $\square$

Definition 3. A normal-form game G is **dominance solvable** if the set ND_i^∞ is a singleton for each $i \in I$.

Existence We will now see the existence of actions that survive iterated deletion of strictly dominated actions.

Proposition 2. *In a finite normal-form game G, $\mathrm{ND}_i^\infty \neq \varnothing$ for each $i \in I$.*

Proof. Immediate. ∎

The finiteness assumption is necessary for the existence, as illustrated below:

Example 5. Consider the following infinite normal-form game:

	1	2	3	4	5	$\cdots$
1	$0,0$	$-1,1$	$-2,1$	$-3,1$	$-4,1$	$\cdots$
2	$1,-1$	$0,0$	$-1,2$	$-2,2$	$-3,2$	$\cdots$
3	$1,-2$	$2,-1$	$0,0$	$-1,3$	$-2,3$	$\cdots$
4	$1,-3$	$2,-2$	$3,-1$	$0,0$	$-1,4$	$\cdots$
5	$1,-4$	$2,-3$	$3,-2$	$4,-1$	$0,0$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Table 4: $\mathrm{ND}_i^\infty = \varnothing$.

For each $i \in I$ and each $k \in \mathbb{N}$, $\mathrm{ND}_i^0 = \{1, 2, \ldots\}$ and $\mathrm{ND}_i^k = \mathbb{N} \setminus \{1, 2, \ldots, k\}$. It is then immediate that $\mathrm{ND}_i^\infty = \bigcap_{k=0}^\infty \mathrm{ND}_i^k = \varnothing$. □

Order Independence of Iterated Strict Dominance In Definition 2, we delete *all* actions that are strictly dominated at each round. However, there seems no compelling reason why players have to delete all strictly dominated actions. Even if we change the order of deleting strictly dominated actions, do we still have the same set ND_i^∞ for each $i \in I$? That is, is the set ND_i^∞ independent of the order of deletion? The answer is yes, but we omit the proof here. For the proof, see Osborne & Rubinstein (1994, Proposition 61.2 together with the discussion right after the proposition).

1.3 (Iterated) Strict Dominance for Mixed Strategies*

So far we have defined the concept of (iterated) strict dominance for actions, or pure strategies, but we can extend it for mixed strategies. Indeed, we can generalize Definition 1 for mixed strategies just by replacing action a_i with a mixed strategy. Moreover, Proposition 1 remains true as is. These observations are summarized in the recap box.

Subtleties emerge when we define the set of mixed strategies that survive iterated deletion of strictly dominated strategies. One may jump to the (wrong) conclusion that it is the set

Recap (Strict Dominance for Mixed Strategies): From the straightforward generalization of Definition 1 and Proposition 1, we reach the following definition:

Definition 4. In a normal-form game G, player i's (mixed) strategy $\sigma_i \in \Sigma_i$ is **strictly dominated** by her (mixed) strategy $\sigma_i' \in \Sigma_i$ if for player $-i$'s every action profile $a_{-i} \in A_{-i}$,

$$u_i(\sigma_i', a_{-i}) > u_i(\sigma_i, a_{-i}).$$

$\Delta(\mathrm{ND}_i^\infty)$ of all distributions over ND_i^∞, but this is not true in general. That is, mixed strategies in $\Delta(\mathrm{ND}_i^\infty)$ can be strictly dominated by some strategy in $\Delta(\mathrm{ND}_i^\infty)$.

Example 6. Consider the following normal-form game:

	L	R
U	$1, 0$	$-2, 0$
M	$-2, 0$	$1, 0$
D	$0, 0$	$0, 0$

Table 5: a mixed strategy in $\Delta(\mathrm{ND}_i^\infty)$ is strictly dominated

For each player, all pure strategies survive iterated deletion of strictly dominated actions: $\mathrm{ND}_i^\infty = A_i$. Hence, $\frac{1}{2}U \oplus \frac{1}{2}M \in \Delta(\mathrm{ND}_1^\infty)$, but it yields payoff $-\frac{1}{2}$ regardless of player 2's strategy; that is, $\frac{1}{2}U \oplus \frac{1}{2}M$ is strictly dominated by D. $\square$

2 Weak Dominance

So far we have studied strict dominance. We have another concept of weak dominance.

Example 7. Consider the following normal-form game:

	L	R
U	$1, 1$	$1, 0$
D	$0, 1$	$1, 1$

Table 6: D and R are weakly dominated

Neither player has strictly dominated actions. However, there seems no compelling reasons why player 1 should choose D or player 2 should choose R. For player 1, U is strictly better than D if player 2 chooses L, and U is as good as D if player 2 chooses R. In this case, we say that D is weakly dominated by U (for player 1). Similarly, R is weakly dominated by L (for player 2). $\square$

2.1 Weak Dominance

We can formalize the concept of weak dominance by relaxing the strict inequalities in the definition of strict dominance with weak inequalities.

Definition 5. In a normal-form game G, player i's action $a_i \in A_i$ is **weakly dominated** by her (mixed) strategy $\sigma_i \in \Sigma_i$ if and only if all of the following hold:

1. $u_i(\sigma_i', a_{-i}) \geq u_i(\sigma_i, a_{-i})$ for each $a_{-i} \in A_{-i}$.
2. $u_i(\sigma_i', a_{-i}) > u_i(\sigma_i, a_{-i})$ for some $a_{-i} \in A_{-i}$.

Remark 2. A weakly dominated action may be played in a Nash equilibrium (although a strictly dominated action is never played in any Nash equilibrium). In Example 7, the game has a pure-strategy Nash equilibrium (D, R), in which both players use the weakly dominated actions.

Moreover, there exists a normal-form game that has a unique Nash equilibrium consisting of weakly dominated actions for all players. For example, in the standard two-firm Bertrand game with zero marginal cost (presented in the previous note), pricing $p_i = 0$ is weakly dominated for firm i, but a unique Nash equilibrium is such that each firm i sets price $p_i = 0$. □

2.2 Difficulty in Defining Iterated Weak Dominance

If a player has a weakly dominated action then she has no reason that she should use it. Therefore, it sounds natural to delete weakly dominated actions, which leads to the concept of iterated weak dominance. However, we cannot actually have a reasonable definition of iterated weak dominance. This is because of the critical theoretical defect illustrated below.

Example 8. Consider the following normal-form game:

	L	R
U	$1, 1$	$0, 0$
M	$1, 1$	$2, 1$
D	$0, 0$	$2, 1$

Table 7: the order dependence of iterated deletion of weakly dominated actions

There are two ways to delete weakly dominated actions. First, if we delete U (that is weakly dominated by M), then we can delete L (that is weakly dominated by R), which leaves R for player 2. Second, if we delete D (that is weakly dominated by M), then we can delete R (that is weakly dominated by L), which leaves L for player 2. Hence, the prediction is sensitive to the order of deletion, which suggests that we cannot have a reasonable definition of iterated weak dominance. □

Recap (Strategies): In a normal-form game G, player i's strategy $\sigma_i \in \Sigma_i$ is called as follows:

1. σ_i is a **pure strategy** if it assigns probability 1 to a single action.
2. σ_i is a **strictly mixed strategy** if it is not a pure strategy.

The notion of completely mixed strategies (Definition 6) may be confused with the notion of strictly mixed strategies. In general, the two are different but equivalent when a player has binary actions.

3 Trembling-Hand Perfect Equilibrium*

Despite the above reasons that make it less compelling to exclude weakly dominated actions, it is still true that there is no advantage to using a weakly dominated action. Therefore, it seems natural not to play weakly dominated actions.

This intuition is justified if we allow for "'mistakes." In Example 7, player 1 is willing to play D, which is weakly dominated, only if (she thinks) player 2 plays R with probability 1. In other words, if player 2 may "trembles his hand" to take an unintended action with small probability, then player 1 is no longer willing to play D.

Example 9. In Example 7, suppose that player 2 "trembles his hand" so that when he intends to take action a_2, he might take action $a_2' \neq a_2$. That is, he takes a mixed strategy σ_2 that assigns strictly positive probabilities to both actions. Then, player 1 will never take D, which is weakly dominated. Her payoff from U is $\sigma_2(L) + \sigma_2(R) = 1$, while her payoff from D is $\sigma_2(R)$. Since $\sigma_2(R) < 1$, player 1 will never want to play D. □

Roughly speaking, we say that a player "trembles her hand" if she chooses any action with strictly positive (yet possibly small) probabilities.

Definition 6. In a finite normal-form game G, player i's strategy $\sigma_i \in \Sigma_i$ is said to be a **completely mixed strategy** if it assigns strictly positive probabilities to all her actions.

Now we introduce the concept of trembling-hand perfection.

Definition 7. In a finite normal-form game G, a mixed strategy profile σ is a **trembling-hand perfect equilibrium** if there exists a sequence $(\sigma^k)_k$ of completely mixed strategy profiles such that:

1. $\sigma^k \to \sigma$ as $k \to \infty$.
2. σ_i is a best response to σ_{-i}^k for each $i \in I$ and each $k \in \mathbb{N}$.

Example 10. In Example 7, a Nash equilibrium (U, L) is trembling-hand perfect. To show this, it suffices to find a sequence $(\sigma^k)_k$ of completely mixed strategy profiles such that:

1. $\sigma_1^k(U) \to 1$ and $\sigma_2^k(L) \to 1$ as $k \to \infty$.
2. U is a best response to σ_2^k and L is a best response to σ_1^k for each $k \in \mathbb{N}$.

For example, let σ^k be such that $\sigma_1^k(U) = \sigma_2^k(L) = 1 - \frac{1}{k+1}$ for each $k \in \mathbb{N}$. It is not difficult to see that this sequence satisfies the desired conditions. In contrast, a Nash equilibrium (D, R) is not trembling-hand perfect, as suggested in Example 9. $\square$

References

Osborne, M. J., & Rubinstein, A. (1994). *A course in game theory*. MIT Press.