Problem Set 1

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The Model

Assume that the economic model generating the data for potential outcomes is of the form:

$$
\begin{aligned}
Y_1 &= \alpha + \varphi + U_1, \\
Y_0 &= \alpha + U_0,
\end{aligned}
$$

where U_1 and U_0 represent the unobservables in the potential outcome equations and φ represents the benefit associated with the treatment ($D = 1$). Individuals decide whether or not to receive the treatment ($D = 1$ or $D = 0$) based on a latent variable I:

$$I = Z\gamma + V,$$

where Z and V represent observables and unobservables, respectively. Thus, we can define a binary variable D indicating treatment status,

$$D = \mathbb{1}\left[I \geq 0\right].$$

Finally, we assume that the error terms in the model are not independent even conditioning on the observables, i.e. $U_1 \not\perp\!\!\!\perp U_0 \not\perp\!\!\!\perp V \mid Z$, but $(U_1, U_0, V) \perp\!\!\!\perp Z$.

1. Treatment Parameters

Let $\beta = Y_1 - Y_0 = \varphi + U_1 - U_0$ such that, in regression notation, $Y = \alpha + \beta D + \varepsilon$ where $\varepsilon = U_0$.

Average Treatment Effect.

$$
\begin{aligned}
ATE &= E\left[\beta\right] \\
&= E\left[\varphi + U_1 - U_0\right] \\
&= \varphi
\end{aligned}
$$

Treatment on the Treated.

$$
\begin{aligned}
TT &= E\left[\beta \mid D = 1\right] \\
&= E\left[\varphi + U_1 - U_0 \mid Z\gamma + V \geq 0\right] \\
&= \varphi + E\left[U_1 - U_0 \mid V \geq -Z\gamma\right]
\end{aligned}
$$

Treatment on the Untreated.

$$
\begin{aligned}
TUT &= E\left[\beta \mid D = 0\right] \\
&= E\left[\varphi + U_1 - U_0 \mid Z\gamma + V < 0\right] \\
&= \varphi + E\left[U_1 - U_0 \mid V < -Z\gamma\right]
\end{aligned}
$$

Marginal Treatment Effect.

$$
\begin{aligned}
MTE &= E\left[\beta \mid I = 0,\ V = v\right] \\
&= E\left[\varphi + U_1 - U_0 \mid I = 0,\ V = v\right] \\
&= \varphi + E\left[U_1 - U_0 \mid Z\gamma = -V,\ V = v\right]
\end{aligned}
$$

Instrumental Variables.

$$\hat{\beta}_{\mathrm{IV}}(J(Z)) = \frac{\mathrm{Cov}(J(Z), Y)}{\mathrm{Cov}(J(Z), D)} \xrightarrow{p} \beta$$

My solutions to the first problem set in Advanced Microeconometrics (ECO – 20513).

Ordinary Least Squares.

$$\hat{\beta}_{\text{OLS}} = \frac{\text{Cov}(Y, D)}{\text{Var}(D)} \implies \hat{\beta}_{\text{OLS}} = \left(D^T D\right)^{-1} D^T Y$$

Local Average Treatment Effect.

$$\begin{aligned}
LATE &= E\left[\beta \mid D(z) = 0, D(z') = 1\right] \\
&= E\left[\beta \mid z\gamma < -V \leq z'\gamma\right] \\
&= \varphi + E\left[U_1 - U_0 \mid -z'\gamma \leq V < -z\gamma\right]
\end{aligned}$$

2. Some closed form expressions

Suppose that the error terms in the model have the following structure:

$$\begin{aligned}
U_1 &= \sigma_1 \epsilon, \\
U_0 &= \sigma_0 \epsilon, \\
V &= \sigma_V^* \epsilon, \\
\epsilon &\sim \mathcal{N}(0, 1).
\end{aligned}$$

So, we can say that

$$\begin{aligned}
TT &= \varphi + E\left[U_1 - U_0 \mid V \geq -Z\gamma\right] \\
&= \varphi + E\left[\epsilon(\sigma_1 - \sigma_0) \mid \epsilon \geq \frac{-Z\gamma}{\sigma_V^*}\right] \\
&= \varphi + (\sigma_1 - \sigma_0) E\left[\epsilon \mid \epsilon \geq \frac{-Z\gamma}{\sigma_V^*}\right] \\
&= \varphi + (\sigma_1 - \sigma_0) \frac{\phi\left(-Z\gamma/\sigma_V^*\right)}{1 - \Phi\left(-Z\gamma/\sigma_V^*\right)},
\end{aligned}$$

and

$$\begin{aligned}
TUT &= \varphi + E\left[U_1 - U_0 \mid V < -Z\gamma\right] \\
&= \varphi + E\left[\epsilon(\sigma_1 - \sigma_0) \mid \epsilon < \frac{-Z\gamma}{\sigma_V^*}\right] \\
&= \varphi + (\sigma_1 - \sigma_0) E\left[\epsilon \mid \epsilon < \frac{-Z\gamma}{\sigma_V^*}\right] \\
&= \varphi - (\sigma_1 - \sigma_0) \frac{\phi\left(-Z\gamma/\sigma_V^*\right)}{\Phi\left(-Z\gamma/\sigma_V^*\right)} \\
&= \varphi + (\sigma_0 - \sigma_1) \frac{\phi\left(-Z\gamma/\sigma_V^*\right)}{\Phi\left(-Z\gamma/\sigma_V^*\right)}.
\end{aligned}$$

3. Parametrization 1

Now, to add more structure to the problem, suppose that $Z = (1, Z_1, Z_2)$, and $\gamma = (\gamma_0, \gamma_1, \gamma_2)$. Also, suppose that

$$\begin{aligned}
\gamma_0 &= 0.2, & \gamma_1 &= 0.3, & \gamma_2 &= 0.1, \\
\sigma_1 &= 0.012, & \sigma_0 &= 0.05, & \sigma_V^* &= 1, \\
\alpha &= 0.02, & \varphi &= 0.2.
\end{aligned}$$

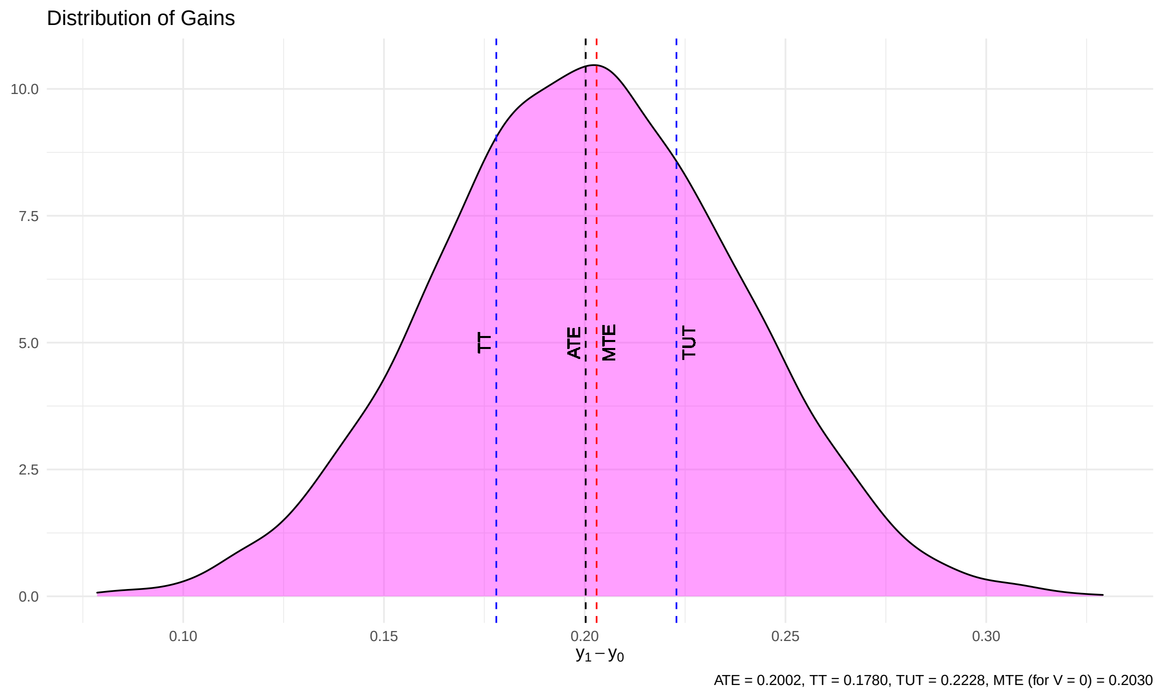
Finally,

$$\begin{aligned}
Z_1 &\sim \mathcal{N}(-1, 9), \\
Z_2 &\sim \mathcal{N}(1, 9),
\end{aligned}$$

where $Z_1 \perp\!\!\!\perp Z_2$. Since we only observe either Y_1 or Y_0, the econometrician observes the outcome described as follows:

$$Y = DY_1 + (1 - D)Y_0.$$

```r
set.seed(1234)
TOL <- 0.01

N <- 5000 # random sample observations

alpha <- 0.02
phi <- 0.2

g <- matrix(c(0.2, 0.3, 0.1))
sigma1 <- 0.012
sigma0 <- 0.05
sigmaV <- 1

eps <- rnorm(N)

U1 <- sigma1 * eps
U0 <- sigma0 * eps
V <- sigmaV * eps

Z <- cbind(rep(1, N),
           rnorm(N, -1, 3),
           rnorm(N, 1, 3))

I <- (Z %*% g) + V
D <- ifelse(I >= 0, 1, 0)

Y1 <- alpha + phi + U1
Y0 <- alpha + U0

Y <- (D * Y1) + ((1 - D) * Y0)

df <- data.frame('beta' = Y1 - Y0,
                 'decision' = D,
                 'y1' = Y1,
                 'y0' = Y0,
                 'net utility' = I)

ATE <- df %>%
    pull(beta) %>%
    mean()

TT <- df %>%
    filter(decision == 1) %>%
    pull(beta) %>%
    mean()

TUT <- df %>%
    filter(decision == 0) %>%
    pull(beta) %>%
    mean()

MTE <- df %>%
    filter(abs(net.utility - V) <= TOL) %>%
    pull(beta) %>%
    mean()
```



4. Parametrization 2

```
LATE <- function(param, old, new) {
    Z.old <- Z
    Z.new <- Z

    Z.old[, param] <- old
    Z.new[, param] <- new

    I.old <- (Z.old %*% g) + V
    D.old <- ifelse(I.old >= 0, 1, 0)

    I.new <- (Z.new %*% g) + V
    D.new <- ifelse(I.new >= 0, 1, 0)

    test <- ifelse((D.old == 0) & (D.new == 1), 1, 0)

    temp.df <- data.frame(
        'beta' = Y1 - Y0,
        'test' = test
    )

    r <- temp.df %>%
        filter((test == 1)) %>%
        pull(beta) %>%
        mean()

    return(r)
}

regress <- function(beta) {
    intercept <- mean(Y) - (beta * mean(D))
    intercept + (beta * D)
}
```

A. $LATE(z_1 = -2, z_1' = 1)$.

```
LATE.1 <- LATE(2, -2, 1)
```

$LATE(z_1 = -2, z_1' = 1) = 0.2051.$

B. $LATE(z_2 = 0, z_2' = 2)$**.**

```
LATE.2 <- LATE(3, 0, 2)
```

$LATE(z_2 = 0, z_2' = 2) = 0.1982.$

C. $IV(Z_1)$**.**

```
beta.IV.Z1 <- (cov(Z[, 2], Y) / cov(Z[, 2], D))[1, 1]
Y.IV.Z1 <- regress(beta.IV.Z1)
```

$\hat{\beta}_{\text{IV}}(Z_1) = 0.2045.$

D. $IV(Z_2)$**.**

```
beta.IV.Z2 <- (cov(Z[, 3], Y) / cov(Z[, 3], D))[1, 1]
Y.IV.Z2 <- regress(beta.IV.Z2)
```

$\hat{\beta}_{\text{IV}}(Z_2) = 0.206.$

E. OLS**.**

```
beta.OLS <- (cov(D, Y) / var(D))[1, 1]
Y.OLS <- regress(beta.OLS)
```

$\hat{\beta}_{\text{OLS}} = 0.237.$

5. GMM

Using $Z = (Z_1, Z_2)$ as instruments and $\hat{V}_0 = \dfrac{1}{n}\displaystyle\sum_{i=1}^{n} z_i' z_i$ as the weighting matrix, we can estimate our model by the generalized method of moments.

Hence, we can obtain

$$\hat{\beta}_{\text{GMM}} = \left(D'ZV_0^{-1}Z'D\right)^{-1} D'ZV_0^{-1}Z'Y.$$

Note that we can rewrite $\hat{V}_0 = \dfrac{1}{n} Z'Z$.

Thus,

```
subset.Z <- Z[, -1]

V0 <- (1 / N) * (t(subset.Z) %*% subset.Z)

beta.GMM <-
    (
        solve(t(D) %*% subset.Z %*% solve(V0) %*% t(subset.Z) %*% D) %*%
            t(D) %*% subset.Z %*% solve(V0) %*% t(subset.Z) %*% Y
    )[1, 1]

Y.GMM <- regress(beta.GMM)
```

Estimator	Estimate	Standard Error
$\hat{\beta}_{\text{GMM}}$	0.2087	0.0321
$\hat{\beta}_{\text{IV}}(Z_1)$	0.2045	0.0331
$\hat{\beta}_{\text{IV}}(Z_2)$	0.206	0.0327
$\hat{\beta}_{\text{OLS}}$	0.237	0.0288

6. Likelihood Function

We know that

$$
\begin{aligned}
Pr\left(D=1 \mid Z\right) &= Pr\left(\gamma Z+V \geq 0\right) \\
&= Pr\left(\frac{-Z\gamma}{\sigma_V^*} \leq \epsilon\right) \\
&= 1-Pr\left(\epsilon \leq \frac{-Z\gamma}{\sigma_V^*}\right) \\
&= 1-\Phi\left(\frac{-Z\gamma}{\sigma_V^*}\right),
\end{aligned}
$$

and

$$
\begin{aligned}
Pr\left(D=0 \mid Z\right) &= Pr\left(\gamma Z+V < 0\right) \\
&= Pr\left(\epsilon \leq \frac{-Z\gamma}{\sigma_V^*}\right) \\
&= \Phi\left(\frac{-Z\gamma}{\sigma_V^*}\right).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\mathcal{L} &= \prod_{d_i=0}\left[\Phi\left(\frac{-Z\gamma}{\sigma_v^*}\right)\right]\prod_{d_i=1}\left[1-\Phi\left(\frac{-Z\gamma}{\sigma_v^*}\right)\right] \\
&= \prod_{i}\left[\Phi\left(\frac{-Z\gamma}{\sigma_v^*}\right)\right]^{1-d_i}\left[1-\Phi\left(\frac{-Z\gamma}{\sigma_v^*}\right)\right]^{d_i},
\end{aligned}
$$

and

$$
\begin{aligned}
\ell &= \log\mathcal{L} \\
&= \sum_i (1-d_i)\log\left(\Phi\left(\frac{-Z\gamma}{\sigma_V^*}\right)\right) \\
&+ \sum_i d_i\log\left(1-\Phi\left(\frac{-Z\gamma}{\sigma_V^*}\right)\right) \\
&= \sum_i (1-d_i)\log\left(\Phi\left(\frac{-\gamma_0-z_1\gamma_1-z_2\gamma_2}{\sigma_V^*}\right)\right) \\
&+ \sum_i d_i\log\left(1-\Phi\left(\frac{-\gamma_0-z_1\gamma_1-z_2\gamma_2}{\sigma_V^*}\right)\right).
\end{aligned}
$$

7. Discussion

8. Maximum Likelihood