Solving and Estimating
an Incomplete Information Entry Game

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1 Setup

Suppose there are two firms, A and B, that simultaneously decide whether to enter in the market $m \in \{1, \ldots, M\}$ or not. If firm $i \in \{A, B\}$ decides to enter, its profits are given by

$$\Pi_{i,m} = X_{i,m}\beta - D_{j,m}\alpha + \nu_m + u_{i,m}, \quad \forall\ i, j \in \{A, B\},\ i \neq j$$

where $X_{i,m}$ is an observable characteristic of firm i that boosts profits in market m, $D_{j,m} \in \{0, 1\}$ is j's decision of entering the market, ν_m is a market fixed effect, and $u_{i,m}$ captures unobserved idiosyncratic shocks to profits.

On the other hand, if firm i decides not to enter market m, its profits are zero.

Notice that $u_{i,m}$ is observed by i but not by its opponent j. Consequently, the firms are playing an incomplete information entry game described below.

		Firm B	
		Enter	Do not enter
Firm A	Enter	$(\Pi_{A,m}, \Pi_{B,m})$	$(X_{A,m}\beta + \nu_m + u_{A,m}, 0)$
	Do not enter	$(0, X_{B,m}\beta + \nu_m + u_{B,m})$	$(0, 0)$

Thus, the expected profits of firm A if it decides to enter market m are

$$\begin{aligned} \mathrm{E}\left(\Pi_{A,m}\right) = \left(D_{A,m}\right) & \left(p(D_{B,m} = 1)\left(X_{A,m}\beta - \alpha + \nu_m + u_{A,m}\right)\right. \\ & + \left.\left(1 - p(D_{B,m} = 1)\right)\left(X_{A,m}\beta - \alpha + u_{A,m}\right)\right) \\ & + \left(1 - D_{A,m}\right) \cdot 0 \end{aligned}$$

where, $\forall\ i, j \in \{A, B\}$, $i \neq j$, the condition for firm i to enter the market in terms of its expected profits is defined as follows:

$$\begin{aligned} D_{i,m} = \mathbf{1}\,&(p(D_{j,m} = 1)\,(X_{i,m}\beta - \alpha + \nu_m + u_{i,m}) \\ &+ (1 - p(D_{j,m} = 1))\,(X_{i,m}\beta + \nu_m + u_{i,m}) > 0)\,. \end{aligned}$$

Namely, the probability that firm A enters market m is

$$\begin{aligned} p\,(D_{A,m} = 1) &= p\,(X_{A,m}\beta - \alpha p\,(D_{B,m} = 1) + \nu_m + u_{A,m} > 0) \\ &= p\,(u_{A,m} > -\,(X_{A,m}\beta - \alpha p\,(D_{B,m} = 1) + \nu_m)) \\ &= \Phi\,(X_{A,m}\beta - \alpha p\,(D_{B,m} = 1) + \nu_m) \end{aligned}$$

where $\Phi = \mathcal{N}(0, 1)$. Recall that we assume $u_{i,m} \sim \mathcal{N}(0, 1)$, $\forall\ i, j \in \{A, B\}$, $i \neq j$.

Similarly,

$$p\,(D_{B,m} = 1) = \Phi\,(X_{B,m}\beta - \alpha p\,(D_{A,m} = 1) + \nu_m)\,.$$

Henceforth, let $p_{i,m} = p(D_{i,m} = 1)$.

2 Simulations

We propose the following parametrization to simulate our model:

$$\begin{aligned} &\alpha = 2 && X_{A,m} \sim \mathcal{U}(0, 1) \\ &\beta = 0.2 && X_{B,m} \sim \mathcal{U}(0.1, 1.4) \\ &\nu_m = 0.9 \times \mathbf{1}\{m \leq 100\} && u_{A,m}, u_{B,m} \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0, 1) \end{aligned}$$

With this setup, we obtain the following entry probabilities for markets 1, 101, 201, 301, and 401.

m	$p_{A,m}$	$p_{B,m}$
1	0.6019	0.4001
101	0.3172	0.2766
201	0.3375	0.2805
301	0.3812	0.2503

m	$p_{A,m}$	$p_{B,m}$
401	0.2590	0.3362

Thus, firm A enters 165 markets while firm B enters 179. However, recall that Bayesian Nash equilibrium allows for ex-post regret where profits are negative succeeding the decision of entry. In our case, firm A regrets 41 times while B regrets 48—24.85% and 26.82%, respectively.

3 Maximum Likelihood

In pursuit of our estimation objectives for $\theta = (\alpha, \beta, \gamma)$ such that γ appears in $\nu_m := \gamma \times \mathbf{1}\{m \leq 100\}$—i.e., we also want to estimate a common market fixed effect for the first 100 markets—we define our log-likelihood function as shown below.

$$\begin{aligned}\log \mathcal{L}_m(\theta \mid \cdot) = \sum_{m=1}^{M} &[D_{A,m} D_{B,m} \log(p_{A,m} p_{B,m}) \\ &+ D_{A,m}(1 - D_{B,m}) \log(p_{A,m}(1 - p_{B,m})) \\ &+ (1 - D_{A,m}) D_{B,m} \log((1 - p_{A,m}) p_{B,m}) \\ &+ (1 - D_{A,m})(1 - D_{B,m}) \log((1 - p_{A,m})(1 - p_{B,m}))]\end{aligned}$$

The maximum likelihood estimates, as well as their standard errors, are shown in the following table.

Parameter	Estimate	SE
α	2.0347	0.1813
β	0.1915	0.0470
γ	0.7421	0.1865

4 GMM

With the generalized method of moments (GMM), we may want to consider up to the fourth moment to ensure over-identification. This way, we can describe an algorithm that performs GMM estimation as follows:

1. Compute $\theta^{(0)} = \arg\min_\theta \bar{g}(\theta)' \bar{g}(\theta)$
2. Compute the heteroskedasticity and auto-correlation consistent matrix $\hat{\Omega}\left(\theta^{(0)}\right)$ like the one proposed by Newey & West (1987)
3. Compute $\theta^{(0)} = \arg\min_\theta \bar{g}(\theta)' \left[\hat{\Omega}\left(\theta^{(0)}\right)\right]^{-1} \bar{g}(\theta)$

4. If $\left\|\theta^{(0)} - \theta^{(1)}\right\| <$ tol, stop. Else, $\theta^{(0)} = \theta^{(1)}$ and repeat from 2
5. Define $\hat{\theta} = \theta^{(1)}$

References

Newey, Whitney K. & West, Kenneth D. 1987. A simple, positive semi-definite, heteroskedasticity and autocorrelation consistent covariance matrix. *Econometrica* 55(3). 703–708. http://www.jstor.org/stable/1913610.