Midterm exam

Empirical Industrial Organization

Instituto Tecnológico Autónomo de México

Carlos Lezama

This exam is inspired by Luco (2019).

Individuals in their working age make savings decisions for retirement. We approach the worker's decision problem in a static manner, i.e., the worker makes a one-time decision.

One Level Decision

The worker must choose one of J pension fund administrators (PFA) to manage her retirement savings. The worker is mandated to save 10% of her salary, y_i. Each PFA charges a percentage fee, p_j, over the worker's salary. PFAs differ in their return on investment, R_j. Finally, ε_{ij} is an i.i.d. Type I Extreme Value preference shock. With all this in hand, we can write down the indirect utility that worker i obtains from enrolling in PFA j as

$$u_{ij} = \alpha_i \left(y_i - 0.1y_i - p_j y_i\right) + \beta_i R_j + \varepsilon_{ij}, \tag{1}$$

where α_i and β_i denote random coefficients. Moreover, let $\gamma_i = (\alpha_i, \beta_i)$, and $\gamma = (\alpha, \beta)$. We assume,

$$\gamma_i = \gamma + \Gamma D_i + \nu_i, \tag{2}$$

where D_i is a $d \times 1$ vector of demographic variables, Γ is a $2 \times d$ matrix of coefficients that measure how taste varies with demographics, and ν_i is a 2×1 vector of unobserved individual characteristics determining taste, where $\nu_i \sim F_\nu(\cdot)$, with $F_\nu(\cdot)$ a distribution function.

Following Train (2009, 3.1), we can write the probability that worker i chooses PFA j, s_{ij}, as follows:

$$\begin{aligned} s_{ij} &= P(u_{ij} \geq u_{ik}, \ \forall j \neq k) && (3) \\ &= P(\varepsilon_{ij} - \varepsilon_{ik} \geq \omega_{ik} - \omega_{ij}) && (4) \\ &= \frac{\exp(\omega_{ij})}{\sum\limits_{j \in J} \exp(\omega_{ik})} && (5) \end{aligned}$$

where $\omega_{ij} = \beta_i R_j - \alpha_i p_j y_i$.

Additionally, we can express the price elasticity of the demand for PFA j, η_{j,p_j}, and the cross return elasticity of the demand for PFA j with respect to the return of PFA k (with $j \neq k$), η_{j,R_k}, like so

$$\begin{aligned} \eta_{j,p_j} &= \frac{p_j}{s_{ij}} \cdot \frac{\partial s_{ij}}{\partial p_j} && (6) \\ &= p_j(1 - s_{ij}) \cdot \frac{\partial \omega_{ij}}{\partial p_j} && (7) \\ &= -\alpha_i p_j y_i (1 - s_{ij}), && (8) \\ & && (9) \\ \eta_{j,R_k} &= \frac{R_k}{s_{ij}} \cdot \frac{\partial s_{ij}}{\partial R_k} && (10) \\ &= -R_k s_{ik} \cdot \frac{\partial \omega_{ik}}{\partial R_k} && (11) \\ &= -\beta_i R_k s_{ik}. && (12) \end{aligned}$$

as derived in Train (2009, 3.6).

Subsequently, with this setup, we can describe the log-likelihood of the model, $\mathcal{L}(\cdot)$, such that

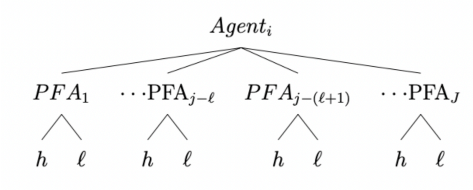
$$\mathcal{L}(\cdot) = \sum_i \sum_j \mathbf{1}_{ij} \log(s_{ij}) \tag{13}$$

$$= \sum_i \sum_j \left(\frac{\exp(\omega_{ij})}{\sum\limits_{j \in J} \exp(\omega_{ik})} \right). \tag{14}$$

Two Level Decision

Now, suppose that each PFA offers two portfolios that workers can choose where to invest their savings in: a high-return, high-risk portfolio, h; and low-return, low-risk portfolio, l.

We model the worker's choice of PFA and portfolio as a sequential decision, where individuals first choose the PFA, and then, conditional on the PFA, they choose the portfolio. The following diagram depicts the sequential two-level decision of the worker,



In this context, we model the indirect utility of choosing PFA $j = 1, \ldots, J$ and portfolio $g = h, l$ as,

$$u_{ijg} = \alpha \left(y_i - 0.1 y_i - p_j y_i \right) + \beta R_{jg} + \theta C_{jg} + \varepsilon_{ijg}, \tag{15}$$

where now, for simplicity, α and β are not random but homogeneous coefficients across individuals. The variable R_{jg} denotes the return of portfolio g in PFA j and, similarly, C_{jg} denotes the risk of portfolio g in PFA j. Finally, ε_{ijg} is an i.i.d. GEV preference shock such that the model is a two-level nested logit.

Note that the indirect utility can be rewritten as $u_{ijg} = u_{ij} + u_{ig|j} + \varepsilon_{ijg}$, where

$$u_{ij} = \alpha(y_i - 0.1 y_i - p_i y_i) \tag{16}$$

is the indirect utility associated to choosing a PFA (i.e., first level), and

$$u_{ig|j} = \beta R_{jg} + \theta C_{jg} \tag{17}$$

is the indirect utility associated to choosing a portfolio conditional on a particular PFA (i.e., second level). Note that $u_{ig|j} = u_{g|j}, \forall i$.

Furthermore, as shown in Train (2009, 4.2.3), the probability of choosing PFA j and portfolio g, s_{ijg}, can be expressed as the product of a marginal probability of choosing PFA j and a conditional probability of choosing portfolio g conditional on PFA j. That is, $s_{ijg} = s_{ij} \cdot s_{ig|j}$, where

$$s_{ij} = \frac{\exp\left(u_{ij} + \lambda_j I_{ij}\right)}{\sum\limits_j \exp\left(u_{ij} + \lambda_j I_{ij}\right)} \qquad \text{(upper level model)} \tag{18}$$

and

$$s_{ig|j} = \frac{\exp\left(\frac{u_{ig|j}}{\lambda_j}\right)}{\sum\limits_g \exp\left(\frac{u_{ig|j}}{\lambda_j}\right)} \qquad \text{(lower level model)} \tag{19}$$

with

$$I_{ij} = \log \sum_g \exp\left(\frac{u_{ig|j}}{\lambda_j}\right) \qquad \text{(inclusive value)} \tag{20}$$

Bear in mind that λ_j must be in the $[0, 1]$ interval as a sufficient condition that characterizes the correlation of utilities that a consumer experiences among the products in the same group j.

Finally, we can express the price elasticity of the demand for PFA j, η_{j,p_j}, and the cross price elasticity of the demand for PFA j with respect to the price of PFA k (with $j \neq k$), η_{j,p_k}, like so

$$\begin{aligned}
\eta_{j,p_j} &= \frac{p_j}{s_{ijg}} \cdot \frac{\partial s_{ijg}}{\partial p_j} && (21)\\
&= \frac{p_j}{s_{ijg}} \left(s_{ig|j} \cdot \frac{\partial s_{ij}}{\partial p_j} + s_{ij} \cdot \frac{\partial s_{ig|j}}{\partial p_j} \right) && (22)\\
&= p_j \left(\frac{s_{ig|j}}{s_{ijg}} \right) \frac{\partial s_{ij}}{\partial p_j} && (23)\\
&= p_j \left(\frac{s_{ig|j}(1 - s_{ij}) s_{ij}}{s_{ijg}} \right) \frac{\partial u_{ij}}{\partial p_j} && (24)\\
&= -\alpha p_j y_i (1 - s_{ij}), && (25)\\
& && (26)\\
\eta_{j,p_k} &= \frac{p_k}{s_{ijg}} \cdot \frac{\partial s_{ijg}}{\partial p_k} && (27)\\
&= \frac{p_k}{s_{ijg}} \left(s_{ig|j} \cdot \frac{\partial s_{ij}}{\partial p_k} + s_{ij} \cdot \frac{\partial s_{ig|j}}{\partial p_k} \right) && (28)\\
&= p_k \left(\frac{s_{ig|j}}{s_{ijg}} \right) \frac{\partial s_{ij}}{\partial p_k} && (29)\\
&= -p_k \left(\frac{s_{ig|j} \cdot s_{ij} \cdot s_{ik}}{s_{ijg}} \right) \frac{\partial u_{ik}}{\partial p_k} && (30)\\
&= \alpha p_k y_i s_{ik}. && (31)
\end{aligned}$$