Graphical Economics
Review

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1 Lecture Notes

The following pages will summarize the work of Kakade et al. (2004) without assuming any prior knowledge other than that obtained during the General Equilibrium course lectured in Fall 2022 by Xinyang Wang.[1]

1.1 Graph Theory

Definition 1 (Graph)**.** A **graph** is a tuple $G = (V, E)$ where V is a (finite) set of *vertices* (also called *nodes*) and E is a finite collection of *edges*. The set E contains elements from the union of the one and two element subsets of V. That is, each edge is either a one or two element subset of V. E is a subset of $V^{(2)} = \{\{x, y\} : x, y \in V,\ x \neq y\}$.

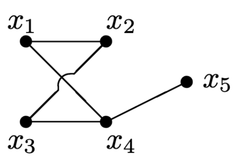

Figure 1: $V = \{x_1, x_2, x_3, x_4, x_5\}$, $E = \{x_1x_2, x_1x_4, x_2x_3, x_3x_4, x_4x_5\}$.

*Remark. We call $V = V(G)$ the **vertex set** of G and $E = E(G)$ the **edge set** of G.*

Definition 2 (Order)**.** The **order** of G is $|G| = |V(G)|$.

Definition 3 (Size)**.** The **size** of G is $e(G) = |E(G)|$.

[1] This may include, but is not limited to, general equilibrium, graph theory, and computer science.

Definition 4 (Self-Loop)**.** If $G = (V, E)$ is a graph, $v \in V$, and $e = \{v\}$, then edge e is called a **self-loop**. That is, any edge that is a single element subset of V is called a self-loop.

Remark. Self-loops are equivalent to reflexive binary relations.

Although we need not know all of the following types of graphs, we might find them interesting for future reference.

Definition 5 (Empty Graph)**.** An **empty graph**, E_n, is a graph $G = (V, E)$ where $V = \{x_1, \ldots, x_n\}$ and $E = \varnothing$. Thus, $|G| = n$ and $e(G) = 0$.

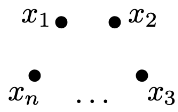

Figure 2: Empty Graph

Definition 6 (Complete Graph)**.** A **complete graph**, K_n, is a graph $G = (V, E)$ where $V = \{x_1, \ldots, x_n\}$ and $E = V^{(2)}$. Thus, $|G| = n$ and $e(G) = \binom{n}{2}$.

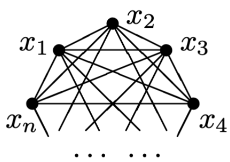

Figure 3: Complete Graph

Definition 7 (Path)**.** A **path**, P_n, of length n is a graph $G = (V, E)$ where $V = \{x_1, \ldots, x_{n+1}\}$ and $E = \{x_i x_{i+1} : 1 \leq i \leq n\}$. Thus, $|G| = n + 1$ and $e(G) = n$.

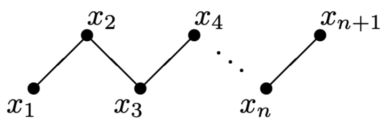

Figure 4: Path

Definition 8 (Cycle)**.** A **cycle**, C_n, of length n is a graph $G = (V, E)$ where $V = \{x_1, \ldots, x_n\}$ and $E = \{x_i x_{i+1} : 1 \leq i \leq n - 1\} \cup \{x_n x_1\}$. Thus, $|G| = n$ and $e(G) = n$.

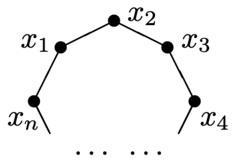

Figure 5: Cycle

Definition 9 (Connectivity)**.** In an undirected graph G, two vertices x and y are called **connected** if G contains a *path* from x to y. Otherwise, they are called **disconnected**.

A graph is said to be **connected** if every pair of vertices in the graph is connected. This means that there is a path between every pair of vertices.

Definition 10 (Graph Isomorphism)**.** Graphs $G = (V, E)$ and $H = (V', E')$ are **isomorphic** if there is a bijection $f : V \to V'$ such that $xy \in E$ if and only if $f(x)f(y) \in E'$.

Definition 11 (Subgraph)**.** We say that $H = (V', E')$ is a subgraph of $G = (V, E)$ if $V' \subset V$ and $E' \subset E$.

Remark. C_n *is a subgraph of* K_n.

Definition 12 (Neighborhood)**.** The **neighborhood** of x is $\Gamma(x) = \{y \in V : xy \in E\}$ and the **degree** of x is $d(x) = |\Gamma(x)|$.

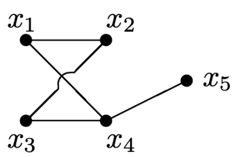

Figure 6: $\Gamma(x_4) = \{x_1, x_3, x_5\}$, $d(x_4) = 3$.

Definition 13 (Digraph)**.** A **directed graph**, or **digraph**, consists of a set V of *vertices* (or *nodes*) together with a set E of ordered pairs of elements of V called *edges* (or *arcs*). The vertex a is called the **initial vertex** of the edge (a, b), and the vertex b is called the **terminal vertex** of this edge.

In an **undirected graph** the edges are bidirectional, with no direction associated with them. Hence, the graph can be traversed in either direction.

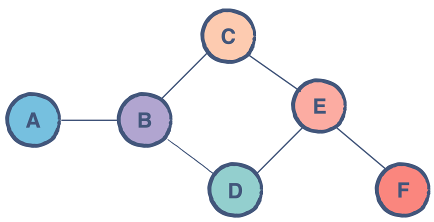

Figure 7: Undirected Graph

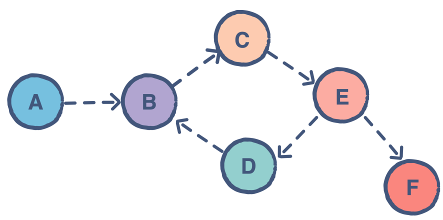

Figure 8: Directed Graph

1.2 Layout

We consider economies consisting of a set $[\ell] := \{1, \ldots, \ell\}$ of **divisible goods** and a set $[m] := \{1, \ldots, m\}$ of **agents** embedded as nodes in some graph $G = ([m], E)$, whose edges E describe who may trade with whom.

Remark. For ease of exposition, G is assumed undirected; all results can be easily extended to directed graphs.

In lieu of E, the reflexive symmetric binary relation $\simeq$ can be used on $[m]$, so that for any two agents $i, j \in [m]$, the presence of an edge between them is denoted by $i \simeq j$.

Remark. For any $i, j \in [m]$, $i \sim j$ when $i \simeq j$ and $i \neq j$.

In the economy, each agent $i \in [m]$ has an endowment of goods $\mathbf{e}^i \in \mathbb{R}^\ell_+$ and a utility function $u_i : \mathbb{R}^\ell_+ \to \mathbb{R}_+$.

Definition 14 (Graphical Economy). A **graphical economy** is an undirected graph G over agents $[m]$ with neighbor relation $\simeq$, utilities $\{u_i : \mathbb{R}^\ell_+ \to \mathbb{R}_+\}_{i\in[m]}$, and endowments $\{\mathbf{e}^i \in \mathbb{R}^\ell_+\}_{i\in[m]}$, where ℓ is an integer denoting the number of goods being traded.

To discuss equilibria in a graphical economy, we also need:

- local price vectors $\mathbf{p}^i \in \mathbb{R}^\ell_+$ for each agent $i \in [m]$, and
- the bundle of goods $\mathbf{x}^{ij} \in \mathbb{R}^\ell_+$ agent i purchases from agent j for consumption.

To enforce the condition that trade must traverse edges, $\mathbf{x}^{ij} = 0$ for $j \not\simeq i$.

1.3 The Arrow–Debreu Exchange Model

The Arrow & Debreu (1954) (AD) exchange economy, also known as the Walrasian model, is extremely well studied due to its central role in general equilibrium theory. The graphical economies are generalizations of AD which retain AD as a special case.

Definition 15 (AD Equilibrium). An **AD Equilibrium** is a pair $(\mathbf{p}, \mathbf{x})$ of a set of price vectors $\mathbf{p}$ and set of consumption plans $\mathbf{x}$ such that, if the underlying graph is complete, we have $\mathbf{p}^i = \mathbf{p}^j$ for all $i, j \in [m]$, and the following conditions are satisfied:

- *Market Clearing.*

$$\sum_{i,j\in[m]} \mathbf{x}^{ij} = \sum_{i\in[m]} \mathbf{e}^i$$

- *Individual Rationality.* For all agents $i \in [m]$, setting $\hat{\mathbf{x}}^i = \mathbf{x}^i$ maximizes their utility $u_i\left(\sum_{j \simeq i} \hat{\mathbf{x}}^{ij}\right)$ over all $\hat{\mathbf{x}}^i \in \mathbb{R}^{\ell}_{+}$ satisfying

$$\sum_{j \simeq i} \mathbf{p}^j \cdot \hat{\mathbf{x}}^{ij} \leq \mathbf{p}^i \cdot \mathbf{e}^i$$

1.4 The Kakade, Kearns, Ortiz Exchange Model

By moving from the complete graph to general graphs, and imposing a local clearing condition instead of a global one, we arrive at the notion of equilibria for graphical economies introduced by Kakade et al. (2004).

Definition 16 (KKO Equilibrium)**.** A **KKO Equilibrium** is a pair $(\mathbf{p}, \mathbf{x})$ of prices $\mathbf{p} \in \mathbb{R}^{m \times \ell}$ and consumption plans $\mathbf{x} \in \mathbb{R}^{m \times m \times \ell}$ such that the following conditions are satisfied for all $i \in [m]$:

- *Local Clearing.*

$$\sum_{j \simeq i} \mathbf{x}^{ji} = \mathbf{e}^i$$

- *Individual Rationality.* Setting $\hat{\mathbf{x}}^i = \mathbf{x}^i$ maximizes their utility $u_i\left(\sum_{j \simeq i} \hat{\mathbf{x}}^{ij}\right)$ over all $\hat{\mathbf{x}}^i \in \mathbb{R}^{m \times \ell}_{+}$ satisfying

$$\sum_{j \simeq i} \mathbf{p}^j \cdot \hat{\mathbf{x}}^{ij} \leq \mathbf{p}^i \cdot \mathbf{e}^i$$

Unlike the AD equilibrium, the KKO model allows asymmetries in trade opportunities to arise from the underlying graph, yielding local price vectors that are generally distinct among agents.

1.5 Graphical Equilibrium Existence

We begin with the assumption on utilities.

A1. For all consumers i, the utility function u_i satisfies the following three properties:

- continuity
- strict monotonicity
- quasi-concavity

The following facts arise from A1 and the consumers' rationality:

1. At equilibrium, the budget constraint inequality for consumer i is saturated, e.g.,in a standard AD economy, a consumer using an equilibrium plan $\mathbf{x}^i$ spends all the money obtained from the sale of the endowment $\mathbf{e}^i$.
2. In any graphical equilibrium, a consumer only purchases a commodity at the cheapest price among the neighboring consumers. Note that the neighboring consumer with the cheapest price may not be unique.

A2 (Non-Zero Endowments)**.** For each consumer i and good k, $e^i_k > 0$.

Essentially, A2 in the AD setting implies that each consumer owns a positive amount of every good in the economy. In the graphical setting, there are effectively $m \times \ell$ goods, but each consumer only has an endowment in ℓ of them.

Definition 17 (Graphical Quasi-Equilibrium)**.** An **graphical quasi-equilibrium** is a set of globally normalized prices (i.e., $\sum_{i,k} p^i_k = 1$) and a set of consumption plans in which the local markets clear for each consumer i, with wealth $\mathbf{w}^i = \mathbf{p}^i \cdot \mathbf{e}^i$, the following condition holds:

- *Individual Rationality.* If consumer i has positive wealth, then i is utility-maximizing.
- *Individual Quasi-Rationality.* If i has no wealth, then the plan $\mathbf{x}^i$ is budget constrained (and does not necessarily maximize utility).

Lemma 1 (Graphical Quasi-Equilibria Existence)**.** In any graphical economy in which A1 holds, there exists a graphical quasi-equilibrium.

Lemma 2. If the graph of a graphical economy is connected and if A1 and A2 hold, then for any quasi-equilibrium set of prices $\{\mathbf{p}^i\}$, it holds that every consumer has non-zero wealth.

Theorem 3 (Graphical Equilibria Existence)**.** For any graphical economy in which A1 and A2 hold, there exists a graphical equilibrium.

Proof of these statements can be found in Kakade et al. (2004).

2 Discussion

> "The graphical economics model suggests a local notion of clearance, directly derived from that of the Arrow & Debreu (1954) model. Rather than asking that the entire (global) market clear in each good, we can ask for the stronger 'provincial' conditions that the local market for each good must clear. For instance, the United States is less concerned that the worldwide production of beef balances worldwide demand than it is that the production of American beef balances worldwide demand for American beef. If this latter condition holds, the American beef industry is doing a good job

> at matching the global demand for their product, even if other countries suffer excess supply or demand." — Kakade et al. (2004)

2.1 Relevance to the Economic Literature

> "Graphical games were introduced in Kearns et al. (2013), where a representation consisting of an undirected graph and a set of local payoff matrices was proposed for multi-player games. [...] These provide an exponentially more succinct representation in cases where the number of players is large, but the degree of the interaction graph is relatively small."
> — Kakade et al. (2004)

This first approach led to a series of papers by several authors that established the computational benefits of this model which approximates Nash equilibria in graphical games with a tree topology.

Inspired by this game–theoretic background, and in the same spirit, Kakade et al. (2004) introduce their model where one of their motivations is to capture the fact that price differences for identical goods can arise due to the network structure of economic interaction. The KKO model allows for *local* markets, where intuitively each agent can set its own prices, and can purchase goods from neighboring agents at their prices.

In order to address concerns about the centralized nature of traditional economic models, several recent works have introduced models of *decentralized* markets. Kakade et al. (2004) introduced a generalization of the AD exchange model to a graphical setting, enabling a robust set of interactions between agents on an arbitrary set of goods which is lacking in other economic models on networks. Unfortunately, the KKO model is "too local" and does not allow for any degree of intermediation between agents.

2.2 Future Directions

In general, this model can allow resale, as shown in Andrade et al. (2021), and even graphical production settings. For instance, a node can extract rent solely from their position in the network—strongly related to degree centrality in graph theory.

Furthermore, in recent applied behavioral economics, these settings have been firmly adopted in studies of wisdom convergence—that seek to prevent the spread of misinformation to the masses—as it is in DeGroot (1974), Golub & Jackson (2010), and, more recently, Banerjee et al. (2019).

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