Graphical Economics
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General Equilibrium

Fall 2022

Introduction

General Equilibrium Theory

- Long history and deep mathematical grounding
- We attempt to explain supply, demand, and prices
- Arrow–Debreu Model is **central**

Setting and Background

We consider economies consisting of:

- A set $[\ell] := \{1, \ldots, \ell\}$ of **divisible goods**
- A set $[m] := \{1, \ldots, m\}$ of **agents embedded as nodes** in some graph $G = ([m], E)$, whose edges E describe who may trade with whom
- A **bundle of goods** $\mathbf{e}^i \in \mathbb{R}_+^\ell$ that agent $i \in [m]$ enters the market with
- A **utility function** $u_i : \mathbb{R}_+^\ell \to \mathbb{R}_+$ that encodes agent i's preferences over bundles of goods

Graphical Economy

A **graphical economy** is an undirected graph G over agents $[m]$ with neighbor relation $\simeq$, utilities $\{u_i : \mathbb{R}_+^\ell \to \mathbb{R}_+\}_{i\in[m]}$, and endowments $\{\mathbf{e}^i \in \mathbb{R}_+^\ell\}_{i\in[m]}$, where ℓ is an integer denoting the number of goods being traded.

For ease of exposition, G is assumed undirected — all results can be easily extended to directed graphs.

To discuss equilibria in a graphical economy, we also need:

- Local price vectors $\mathbf{p}^i \in \mathbb{R}^\ell_+$ for each agent $i \in [m]$
- The bundle of goods $\mathbf{x}^{ij} \in \mathbb{R}^\ell_+$ agent i purchases from agent j for consumption
  - To enforce the condition that trade must traverse edges, $\mathbf{x}^{ij} = 0$ for $j \not\sim i$

Agent i buys an amount x_k^{ij} of good k from agent j for consumption

$$i \neq j \in [m]$$
$$k \in [\ell]$$

The Arrow–Debreu (AD) Exchange Model

The graphical economies are generalizations of AD which retain AD as a special case.

AD Equilibrium

An **AD Equilibrium** is a pair $(\mathbf{p}, \mathbf{x})$ of a set of price vectors $\mathbf{p}$ and set of consumption plans $\mathbf{x}$ such that, if the underlying graph is complete, we have $\mathbf{p}^i = \mathbf{p}^j$ for all $i, j \in [m]$, and the following conditions are satisfied:

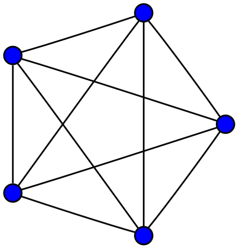
- *Market Clearing.*

$$\sum_{i,j \in [m]} \mathbf{x}^{ij} = \sum_{i \in [m]} \mathbf{e}^i$$

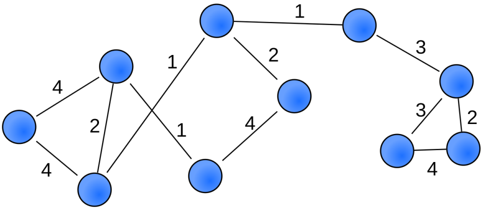
- *Individual Rationality.* For all agents $i \in [m]$, setting $\hat{\mathbf{x}}^i = \mathbf{x}^i$ maximizes their utility $u_i \left(\sum_{j \simeq i} \hat{\mathbf{x}}^{ij} \right)$ over all $\hat{\mathbf{x}}^i \in \mathbb{R}_+^\ell$ satisfying

$$\sum_{j \simeq i} \mathbf{p}^j \cdot \hat{\mathbf{x}}^{ij} \leq \mathbf{p}^i \cdot \mathbf{e}^i$$

- Agents sell endowments at market prices
- They spend profits on goods maximizing their utility
- There is a single global market
    - Single price for each good
    - Every pair of agents can trade
    - Markets are cleared when the demand of all agents is equal to the supply of all agents



*We would like something more **local***



The Kakade, Kearns, Ortiz (KKO) Exchange Model

KKO Equilibrium

An **KKO Equilibrium** is a pair $(\mathbf{p}, \mathbf{x})$ of prices $\mathbf{p} \in \mathbb{R}^{m \times \ell}$ and consumption plans $\mathbf{x} \in \mathbb{R}^{m \times m \times \ell}$ such that the following conditions are satisfied for all $i \in [m]$:

- *Local Clearing.*

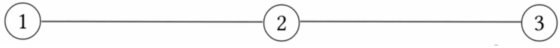
$$\sum_{j \simeq i} \mathbf{x}^{ji} = \mathbf{e}^i$$

- *Individual Rationality.* Setting $\hat{\mathbf{x}}^i = \mathbf{x}^i$ maximizes their utility $u_i\left(\sum_{j \simeq i} \hat{\mathbf{x}}^{ij}\right)$ over all $\hat{\mathbf{x}}^i \in \mathbb{R}_+^{m \times \ell}$ satisfying

$$\sum_{j \simeq i} \mathbf{p}^j \cdot \hat{\mathbf{x}}^{ij} \leq \mathbf{p}^i \cdot \mathbf{e}^i$$

- Network of local markets
    - Each agent sets its own price for each good
    - Pairs of agents can trade if connected by an edge
    - Markets are cleared when the demand of each agent is equal to the supply of each agent
- Captures underlying trade structures of economies
- Can be too local in some cases

Local prices



$$\mathbf{e}^1 = (1,2) \qquad \mathbf{e}^2 = (1,1) \qquad \mathbf{e}^3 = (2,1)$$

$$\mathbf{u}^1 = (1,0) \qquad \mathbf{u}^2 = (1,1) \qquad \mathbf{u}^3 = (0,1)$$

$$\mathbf{p}^1 = (2,1) \qquad \mathbf{p}^2 = (2,2) \qquad \mathbf{p}^3 = (1,2)$$

There is ***no*** *graphical equilibrium in which the prices for both goods is the same from all consumers, so price variations are essential for equilibrium*

Graphical Equilibrium Existence

Assumptions

A-1. For all consumers i, the utility function u_i satisfies the following three properties:

- continuity
- strict monotonicity
- quasi-concavity

A-2. (Non-Zero Endowments) For each consumer i and good k, $e^i_k > 0$

Graphical Quasi-Equilibrium

An **graphical quasi-equilibrium** is a set of globally normalized prices (i.e., $\sum_{i,k} p^{ik} = 1$) and a set of consumption plans in which the local markets clear for each consumer i, with wealth $\mathbf{w}^i = \mathbf{p}^i \cdot \mathbf{e}^i$, one of the following conditions is met

- *Individual Rationality.* If consumer i has positive wealth, then i is utility-maximizing
- *Individual Quasi-Rationality.* If i has no wealth, then the plan $\mathbf{x}^i$ is budget constrained (and does not necessarily maximize utility).

Results

Lemma 1

In any graphical economy in which **A-1** holds, there exists a graphical quasi-equilibrium.

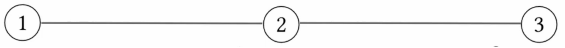
Lemma 2

If the graph of a graphical economy is connected and if **A-1** and **A-2** hold, then for any quasi-equilibrium set of prices $\{\mathbf{p}^i\}$, it holds that every consumer has non-zero wealth.

Theorem 1

For any graphical economy in which **A-1** and **A-2** hold, there exists a graphical equilibrium.

Limitations



$$\mathbf{e}^1 = (1,0) \qquad \mathbf{e}^2 = (0,0) \qquad \mathbf{e}^3 = (0,1)$$
$$\mathbf{u}^1 = (0,1) \qquad \mathbf{u}^2 = (1,1) \qquad \mathbf{u}^3 = (1,0)$$

- Agent 2 has no endowment, thus no profit
- If agent 1 or 3 profit, they try to spend profit on unavailable goods
- If agent 1 or 3 do not profit:
  - They must have 0 prices
  - Agent 2 consumes an infinite amount of their good

Challenging Learning Problems and Issues

- Rational Learning in Graphical Games
- No-Regret Learning in Graphical Games
- Learning in Traditional AD Economies
- Learning in Graphical Economics