

Fresnel 積分

Thm. (Fresnel 積分)

$$\int_{-\infty}^{\infty} \sin x^2 dx = \sqrt{\frac{\pi}{2}} \quad \int_{-\infty}^{\infty} \cos x^2 dx = \sqrt{\frac{\pi}{2}}$$

$$\int_{-\infty}^{\infty} e^{ix^2} dx = \int_{-\infty}^{\infty} \sin x^2 dx + i \int_{-\infty}^{\infty} \cos x^2 dx$$

$$C_R : z(t) = Re^{it} \quad (t : 0 \rightarrow \frac{\pi}{4}), \quad L_R : z(t) = \frac{(1+i)}{\sqrt{2}} t \quad (t : R \rightarrow 0)$$

$$\Gamma_R = [0, R] \cup C_R \cup L_R$$

とおくと, e^{iz^2} は Γ_R の内部及び近傍で正則なので, Cauchy の積分定理により,

$$0 = \oint_{\Gamma_R} e^{iz^2} dz = \int_0^R e^{ix^2} dx + \int_{C_R} e^{iz^2} dz + \int_{L_R} e^{iz^2} dz$$

L_R 上の積分は

$$\begin{aligned} \int_{L_R} e^{iz^2} dz &= \int_{L_R} e^{iz^2} dz = \int_R^0 \frac{(1+i)}{\sqrt{2}} e^{i\left(\frac{(1+i)}{\sqrt{2}} t\right)^2} dt = -\frac{(1+i)}{\sqrt{2}} \int_0^R (1+i) e^{-t^2} dt \\ &\rightarrow -\frac{(1+i)}{\sqrt{2}} \frac{\sqrt{\pi}}{2} \quad (R \rightarrow \infty) \end{aligned}$$

C_R 上の積分は

$$\begin{aligned} \left| \int_{C_R} e^{iz^2} dz \right| &= \left| \int_0^{\frac{\pi}{4}} e^{i(Re^{it})^2} Rie^{it} dt \right| \leq \int_0^{\frac{\pi}{4}} \left| e^{iR^2 e^{2it}} Rie^{it} \right| dt \\ &\leq R \int_0^{\frac{\pi}{4}} \left| e^{iR^2(\cos 2t + i \sin 2t)} \right| dt = R \int_0^{\frac{\pi}{4}} e^{-R^2 \sin 2t} dt \\ &\leq R \int_0^{\frac{\pi}{4}} e^{-R^2 \frac{2}{\pi} 2t} dt \leq \frac{R\pi}{4R^2} (1 - e^{-R^2}) \rightarrow 0 \quad (R \rightarrow \infty) \\ \therefore \int_{C_R} e^{iz^2} dz &\rightarrow 0 \quad (R \rightarrow \infty) \end{aligned}$$

但し, $\frac{2}{\pi}\theta \leq \sin \theta \leq \theta$ ($0 \leq \theta \leq \frac{\pi}{2}$) を用いた.

以上より,

$$\begin{aligned} \int_0^R e^{ix^2} dx &= \int_{L_R} e^{iz^2} dz - \int_{C_R} e^{iz^2} dz \rightarrow \frac{(1+i)}{\sqrt{2}} \frac{\sqrt{\pi}}{2} \quad (R \rightarrow \infty) \\ \therefore \int_{-\infty}^{\infty} e^{ix^2} dx &= 2 \lim_{R \rightarrow \infty} \int_0^R e^{ix^2} dx = (1+i) \sqrt{\frac{\pi}{2}} \\ \therefore \int_{-\infty}^{\infty} \sin x^2 dx &= \sqrt{\frac{\pi}{2}} \quad \int_{-\infty}^{\infty} \cos x^2 dx = \sqrt{\frac{\pi}{2}} \end{aligned}$$