“Ancient Greek” Trigonometric Formulae:

Trigonometry Done Differently!

Dr. James MacLean

for the New Calculus Channel

July 8, 2025

An Initial Take

Many of us will think that:

- Trigonometry is ‘solved’.
- Angles are easy.
- Punching in numbers into the calculator is easy to get your ‘$\sin(x)$’ or ‘$\arcsin(x)$’.
- Using these ideas is intuitive enough!
- And there’s no real reason to even rethink these things beyond high-school or is there?

Why?

Now why is this presentation even on something that is regarded as trivial or well established? It's simple:

To Think Differently;

and to develop an alternative view of Trigonometry.

It must also be said that this is not a competition either. We're here to see what can be done with regards to Trigonometry.

What is Trigonometry?

Trigonometry:

is the investigation of the relationship between angles and their corresponding side lengths of associated triangles.

We'll now list some basics we should take note of to investigate a new approach to Trigonometry!

Walking in with a Fresh Face:

Some Useful Definitions from Euclid

We'll remind ourselves of ancient definitions to be considered for this presentation:

1. Bk.I.Def.07 - A plane surface is one which lies evenly with straight-lines on itself.
2. Bk.I.Def.08 - And a plane angle is the inclination of the lines to one another, when two lines in a plane meet one another, and are not lying in a straight-line.
3. Bk.I.Def.09 - And when the lines containing the angle are straight then the angle is called rectilinear.
4. Bk.I.Def.10 - And when a straight-line stood upon (another) straight-line makes adjacent angles equal to one another, each of the equal angles is a right-angle, and the former straight-line is called a perpendicular to that upon which it stands.
5. Bk.I.Def.11 - An obtuse angle is one greater than a right-angle.
6. Bk.I.Def.12 - And an acute angle is one less than a right-angle.

Before moving on, let's summarise what a plane angle is!

What is a Plane Angle?

Part 1

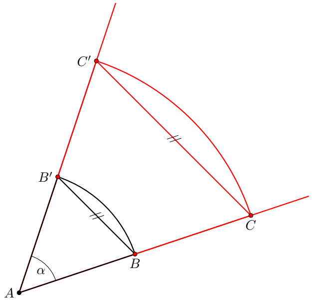
Plane Angle:

An angle, in its primitive definition, arises from the arc between any two radii emanating from a shared point which is the center of the circle. The unique measurement of an angle is determined by the ratio of the arc length to the radius, with the greatest possible angle achieved when the arc encompasses the entire circumference.

- Angles subtended on the same arc are of equal measure regardless of circle size because the ratios are always in proportion. (This is a laborious proof and beyond the scope of the video).
- The figure overleaf demonstrate what one refers to as ‘plane-angle’.

What is a Plane Angle?

Part 2



The general concept of plane-angle is based on the idea that the proportion of *Radius* : *Arclength* remains constant between two lines. In the above:

$$\alpha = AB : BB' = AC : CC'$$

Despite the fact that $AB < AC$ and $BB' < CC'$, these arcs still refer to the same angle α. We can inherit this idea for the upcoming remainder of the presentation. This proportion is also doubly confirmed by the lines BB' and CC' being parallel.

Walking in with a Fresh Face:

Other Prerequisites

Additional and useful prerequisites we will take advantage of are:

- Geometry.
- Elementary Algebra.
- Coordinate-Geometry with rectilinear coordinates, specifically the Cartesian-Plane.

Additional notes:

- Crucially, the Cartesian-Plane beautifully combines Geometry and Algebra into one seamless package that provides both numerical and algebraic feedback with clear demonstration of results.
- Also take note that the Cartesian-Plane leverages the Euclidean definitions cited earlier, most notably: the **right-angle** as in Bk.I.Def.10. Two perpendicular lines make for great convenience in plotting!
- And those are the necessary ingredients to move forward and begin setting some goals!

Necessary Goals:

Outcomes to be achieved

The central desires of Trigonometry is to develop a method wherein:

- The lengths of the triangle segments do not affect the associated and *unique* angle. In short, similar triangles will have the same angles as one another. This is both a geometric and algebraic requirement.
- Angles should be easily identifiable, comprehensible and easy to use.
- There is sufficient flexibility for additional adaptations (more on that near the end).

Setting Up the Cartesian-Plane:

Part 1:

We require an additional necessary Algebraic property on top of all the normal Elementary Algebraic properties we're used, namely the '*ordered-pair*'.

Definition of the Ordered-Pair

An ordered-pair is a list of two algebraic symbols which act as placeholders for geometrical magnitudes and/or numbers describing such magnitudes. This is expressed in the form:

$$(x, y)$$

It is read from left to right as: "x comma y". Two ordered-pairs are equal if and only if:

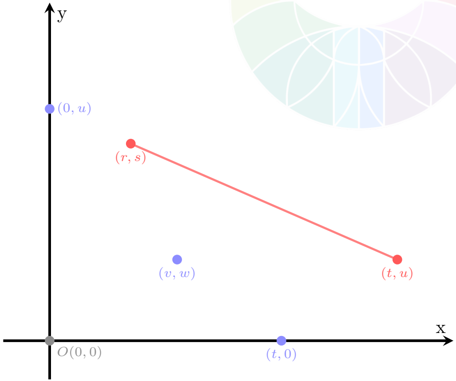
$$(x, y) = (u, v)$$

providing that $x = u$ and $y = v$. Ordered-Pairs may also be given symbolic names appended to them as: $P(u, v)$ and referred to as just their symbols if the context is clear.

Setting Up the Cartesian-Plane:

Part 2

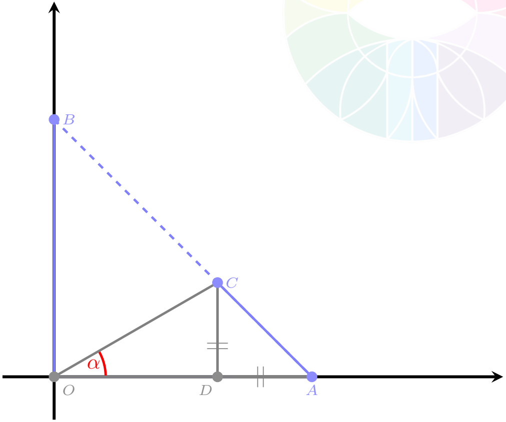
- Begin with two perpendicular lines to use as reference coordinate-axes. Let the vertical and horizontal axes be denoted y and x respectively (Bk.I.Def.10.).
- Let their intersection be named the origin O indicated by the ordered-pair $(0, 0)$.
- Three other types of ordered-pairs can be inferred: one on the x-axis $(t, 0)$, one on the y-axis $(0, u)$ and any other (v, w) which is not on either axis, but on the plane.
- Any two arbitrary ordered-pairs (r, s) and (t, u) will admit a distance between them given by Pythagoras's Theorem in Cartesian Coordinates: $d = \sqrt{(r-t)^2 + (s-u)^2}$.



A ‘Radical’ Approach:

Part 1: The Core Idea

- Begin with an isosceles triangle $\triangle ABO$ on the Cartesian-Plane with points O, A and B.
- Consider a point C on the line segment BA.
- Produce the vertical line segment from CD where D is on the line segment OA.
- Produce line segments OC, OD and DA. Notice the scaled isosceles triangle $\triangle ACD$ that is similar to $\triangle ABO$.
- There is an unique plane-angle α associated with $\triangle COD$.
- Notice that CA has an unique length associated directly with angle α.
- This what we will use as a concept to develop the ‘Radical Angle Unit’ measure.
- Overleaf, we will use Algebra and specific coordinates to generate trigonometric expressions.



A ‘Radical’ Approach:

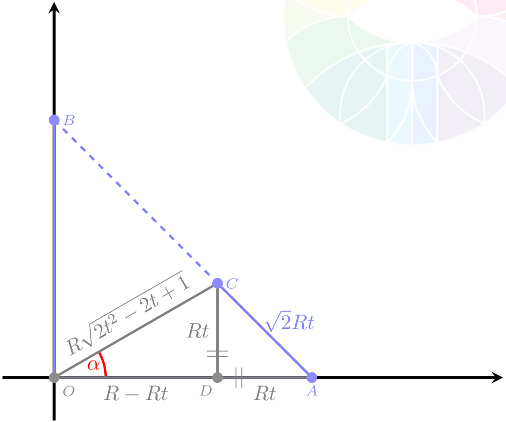
Part 2: Further Algebra

To establish trigonometric relations, we leverage Algebra on the Plane:

- Note the point $O(0,0)$, $A(R,0)$ and $B(0,R)$.
- The equation of the line of segment AB is: $y = R - x$.
- The point D on the segment OA can be parameterised as: $D(R - Rt, 0)$ where t is an algebraic parameter.
- Thus, by substituting in $R - Rt$ into the line, we have $C(R - Rt, Rt)$.
- The lengths of each segment in Algebra are:

$$OA = OB = R, \quad OC = R\sqrt{1 - 2t + 2t^2},$$
$$OD = R - Rt, \quad BA = \sqrt{2}R, \quad BC = \sqrt{2}R(1 - t),$$
$$CD = Rt, \quad DA = Rt$$

- Now we introduce the ‘Radical’ Trigonometry overleaf.



A 'Radical' Approach:

Part 3: Initial Definitions and Restrictions

Definition of Radical Angle Unit

Let the name given to the **Radical Unit Angle (RAU)** be denoted **s** such that:

$$s = \sqrt{2}Rt$$

where the parameter t is subject to the inequality:

$$0 \leq t \leq 1$$

which implies that the **RAU** has the inequality of:

$$0 \leq s \leq \sqrt{2}R$$

Some key notes:

- The name of 'Radical Angle Unit' comes from the radical symbol $\sqrt{}$.
- It does not matter what value R takes, the parameter t is always as described.
- One can select any non-zero value for R and this will not affect the angle relations developed (see slide 7).

Overleaf, the radical sine relation is derived.

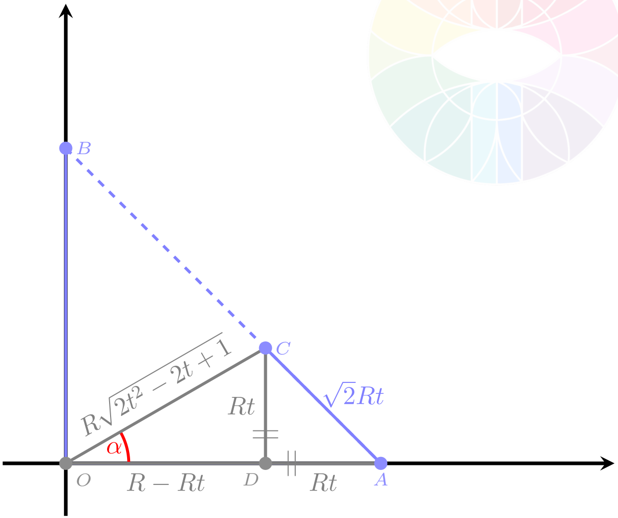
A ‘Radical’ Approach:

Part 4: The Radical Sine Relation

- Denote the plane-angle as α.
- Let the RSIN(t) relation be introduced as the measure of opposite to hypotenuse using the parameter t:
- $$\mathrm{RSIN}(t) \stackrel{\mathrm{def}}{=} \frac{opp}{hyp}$$

  which in Algebra is:
- $$\mathrm{RSIN}(t) \stackrel{\mathrm{def}}{=} \frac{Rt}{R\sqrt{1-2t+2t^2}}$$
- Notice the common factor of R cancels to yield:

$$\mathrm{RSIN}(t) \stackrel{\mathrm{def}}{=} \frac{t}{\sqrt{1-2t+2t^2}}$$



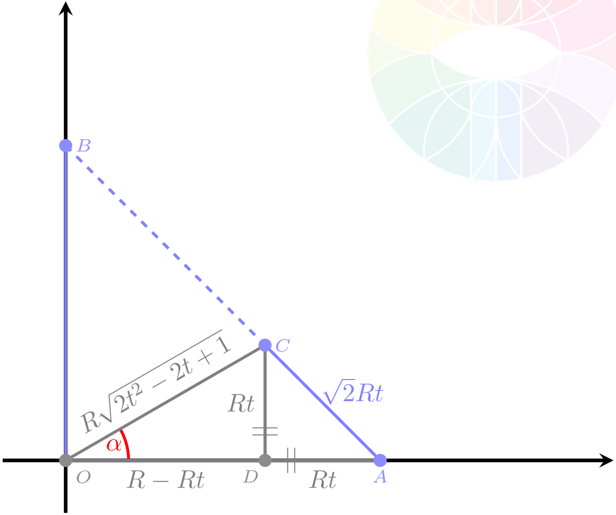
A ‘Radical’ Approach:

Part 5: Trigonometric Relations

The trigonometric ratios are exactly the same as one would do in Trigonometry already:

- $\mathrm{RSIN}(t) \stackrel{\text{def}}{=} \dfrac{t}{\sqrt{1-2t+2t^2}}$
- $\mathrm{RCOS}(t) \stackrel{\text{def}}{=} \dfrac{1-t}{\sqrt{1-2t+2t^2}}$
- $\mathrm{RTAN}(t) \stackrel{\text{def}}{=} \dfrac{t}{1-t}$
- $\mathrm{RCSC}(t) \stackrel{\text{def}}{=} \dfrac{\sqrt{1-2t+2t^2}}{t}$
- $\mathrm{RSEC}(t) \stackrel{\text{def}}{=} \dfrac{\sqrt{1-2t+2t^2}}{1-t}$
- $\mathrm{RCOT}(t) \stackrel{\text{def}}{=} \dfrac{1-t}{t}$

Notice that the value of R always algebraically cancels for all trigonometric ratios.



Crucial Notes:
A Quick Pause

There are some very important things to take note of:

- All of the trigonometric definitions are 'closed-form'. This means that they are written as a finite combination of algebraic operations, namely: $-, +, /, x, ()^{p/q}$. This means as long as one has an algorithm in place for approximating a square root, then all is go!
- This is very different from Newton Series methods of computation which is not closed-form. There is always a next term no matter what. Recall the definition:

$$\sin\alpha \approx \sum_{k=0}^{n} \frac{(-1)^k}{(2k+1)!}\alpha^{2k+1}$$

- Also it needs to be said, that I am not ignoring commonly used methods of $\sin\alpha$ approximation using things such as Chebyshev Polynomials.

A Simple Comparison:

To hint at some computational ideas, but not delve too deep, we will informally investigate Newton's Sine Series versus RSIN(t) for $t = 0.5$ which is equivalent to $\sin \frac{\pi}{4} = \frac{1}{\sqrt{2}}$. Test conditions for the test:

- Use of Matlab's inbuilt value for π for Newton's Series.
- Use of Heron's Method to approximate the square root of a value:

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right)$$

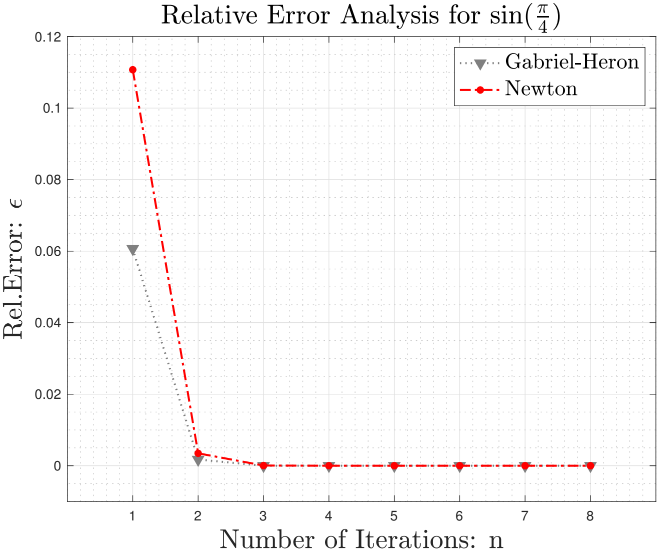
  where $a = 1 - 2t + 2t^2 = 0.5$ for this example.
- A reasonable initial guess for Heron's Method to start: $x_0 = 1$.
- The amount of terms n used in Newton's Series matches the number of iterations used in Heron's Method.
- Relative Error is computed as:

$$\text{Rel.Error} \stackrel{\text{def}}{=} \epsilon = \left\| \frac{\sin(\frac{\pi}{4}) - I(n)}{\sin(\frac{\pi}{4})} \right\|$$

  where $I(n)$ stands for the value computed per iteration n in either method.
- We use Matlab's VPA toolbox to allow for up to 32 digits of precision.

Comparison Plot:

Relative Error



Comparison Table:

Data Comparison

Matlab's 32 digit value for:

$$\sin\frac{\pi}{4} = 0.70710678118654752440084436210485$$

Table: Convergence comparison between Gabriel-Heron and Newton methods. All computations carried up to 32 digits of precision, but only 16 digits shown. Coloured in digits represent correct values when compared to Matlab's value.

Iterations	Gabriel-Heron	Newton
1	0.7500000000000000	0.7853981633974483
2	0.7083333333333333	0.7046526512091675
3	0.7071078431372549	0.7071430457793602
4	0.7071067811873449	0.7071064695751780
5	0.7071067811865475	0.7071067829368671
6	0.7071067811865475	0.7071067811796194
7	0.7071067811865475	0.7071067811865678
8	0.7071067811865475	0.7071067811865475

Gabriel-Heron converges to Matlab's 32 digit value in 5 iterations whereas it takes Newton's 8 iterations.

Before We Advance:

Another Pause

- Both methods converge very quickly, but Gabriel-Heron converges faster and thus, produces better accuracy for less computational effort.
- We were nice for the Newton Series by assuming a pre-calculated and very good estimation of π thanks to Matlab. Computing π is not a joke and is often taken for granted.
- Being realistic, there is not much in this race for practical purposes, but there is a 'winner' so-to-speak.
- Moving on, we look at the relationship between **RAU** and the circle!

Parameterising the Circle:

Part 1

Given the Algebraic Curve equation for the circle in the Cartesian-Plane:

$$x^2 + y^2 = 1 \tag{1}$$

is it possible to parameterise this circle using any of the trigonometric formulae from earlier?

Parameterising the Circle:
Part 2

Recall the definition of $\text{RTAN}(t)$ as:

$$\text{RTAN}(t) = \frac{t}{1-t}$$

There is a point C which marks the end of the hypotenuse, namely:

$$C(1-t, t)$$

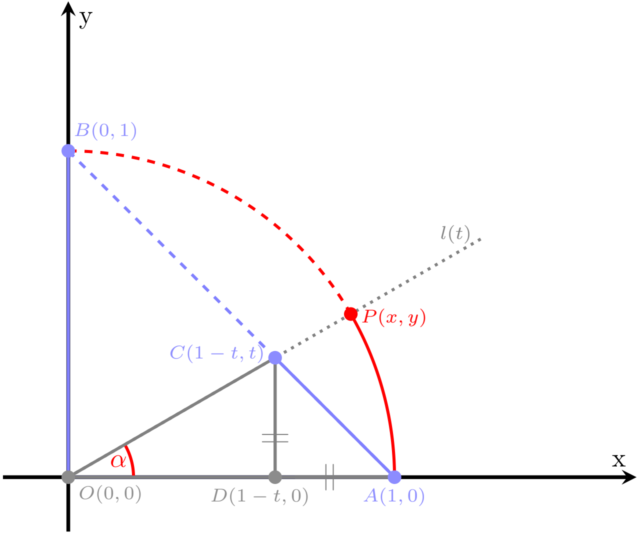
Then using the origin point $O(0,0)$, the equation of the line passing through points O, C is:

$$l(t) = \left(\frac{t}{1-t}\right)x$$

This line will intersect the circle at another point $P(x, y)$. See overleaf.

Parameterising the Circle:

Part 3



Parameterising the Circle:

Part 4

To determine the line's point of intersection with the circle, substitute the line equation $l(t)$ into the circle equation:

$$x^2 + y^2 = 1 \longrightarrow x^2 + \left[\left(\frac{t}{1-t}\right)x\right]^2 = 1$$

$$(1-t)^2x^2 + t^2x^2 = (1-t)^2 \longrightarrow x^2 = \frac{(1-t)^2}{(1-t)^2 + t^2}$$

Hence the x solutions are:

$$x = \pm\frac{1-t}{\sqrt{2t^2 - 2t + 1}}$$

As for the y solutions, substitute the above into the line equation:

$$y = \frac{t}{1-t}\left(\pm\frac{1-t}{\sqrt{2t^2-2t+1}}\right) \rightarrow \boxed{y = \pm\frac{t}{\sqrt{2t^2-2t+1}}}$$

Point P corresponds with the positive solutions, thus:

$$P\left(\frac{1-t}{\sqrt{2t^2-2t+1}}, \frac{t}{\sqrt{2t^2-2t+1}}\right)$$

Parameterising the Circle:

Part 5

If we just focus on the two positive solutions, we now have an effective parameterisation for the circle in the positive quadrant as the ordered pair:

$$\boxed{\gamma_1(t) \stackrel{\text{def}}{=} \left(\frac{1-t}{\sqrt{2t^2-2t+1}}, \frac{t}{\sqrt{2t^2-2t+1}} \right)}$$

Let's check a few t values of interest, $t = 0, 0.5, 1$:

$$\gamma_1(0) = \left(\frac{1-0}{\sqrt{2\cdot 0^2 - 2\cdot 0 + 1}}, \frac{0}{\sqrt{2\cdot 0^2 - 2\cdot 0 + 1}} \right) = (1, 0)$$

$$\gamma_1(0.5) = \left(\frac{1-0.5}{\sqrt{2\cdot 0.5^2 - 2\cdot 0.5 + 1}}, \frac{0.5}{\sqrt{2\cdot 0.5^2 - 2\cdot 0.5 + 1}} \right) = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right)$$

$$\gamma_1(1) = \left(\frac{1-1}{\sqrt{2\cdot 1^2 - 2\cdot 1 + 1}}, \frac{1}{\sqrt{2\cdot 1^2 - 2\cdot 1 + 1}} \right) = (0, 1)$$

Which are what one would expect!

Connecting with Radian Measure:

Part 1

For those who are very observant, this parameterisation is equivalent to the usual parameterisation using $\sin\alpha$ and $\cos\alpha$, namely:

$$v(\alpha) \stackrel{\text{def}}{=} (\cos\alpha, \sin\alpha) \equiv \gamma_1(t) \stackrel{\text{def}}{=} \left(\frac{1-t}{\sqrt{2t^2-2t+1}}, \frac{t}{\sqrt{2t^2-2t+1}}\right)$$

Hence:

$$\cos\alpha \equiv \frac{1-t}{\sqrt{2t^2-2t+1}} \quad \text{and} \quad \sin\alpha \equiv \frac{t}{\sqrt{2t^2-2t+1}}$$

As such, there is a direct way to convert between the two!

Conversion from a known Ratio to RAU Parameter

First Example: Part I

Given some known $\sin\alpha$ ratio, one can find the corresponding t for the $\mathrm{RSIN}(t)$ that produces the same ratio. Let $k = \sin\alpha = \mathrm{RSIN}(t)$, then by the equivalence shown earlier:

$$\begin{aligned} \mathrm{RSIN}(t) &= \frac{t}{\sqrt{1-2t+2t^2}} \\ k &= \frac{t}{\sqrt{1-2t+2t^2}} \\ k^2 &= \frac{t^2}{1-2t+2t^2} \\ 0 &= (2k^2-1)t^2 - 2k^2t + k^2 \end{aligned}$$

Using the quadratic formula:

$$\boxed{t = \frac{k^2 \pm k\sqrt{1-k^2}}{2k^2-1}}$$

And thus, we have a conversion from a known $\sin\alpha$ ratio to its equivalent parameter t in **RAU**. Now for an example overleaf.

Conversion from a known Ratio to RAU Parameter

First Example: Part 2

Let $k = \sin(\frac{\pi}{6}) = 0.5$, then:

$$
\begin{aligned}
t &= \frac{k^2 \pm k\sqrt{1-k^2}}{2k^2-1} \\
&= \frac{(0.5)^2 \pm (0.5)\sqrt{1-(0.5)^2}}{2\,(0.5)^2 - 1} \\
&= -\frac{1}{2} \mp \frac{\sqrt{3}}{2}
\end{aligned}
$$

- Since $0 \leq t \leq 1$, $t = -\frac{1}{2} + \frac{\sqrt{3}}{2} \approx 0.366025$ is the only viable solution. Hence we now have our parameter t in **RAU**.
- Also note, that what has just been done here is an **Inverse Operation**, i.e. we have gone from a known ratio to the **RAU** parameter. We will list these formulae later!
- Also note, that we take the $-ve$ general solution as that is the one that corresponds to the positive quadrant we have been working in.
- But... there is a minor elephant in the room.

Conversion from a known ratio to RAU Parameter

First Example: Part III

For those who are very observant, one might point out:

Inverse Undefined?

What if $k = \frac{1}{\sqrt{2}}$? Is the inverse now undefined?!?! You're dividing by 0!!! Terrible formula!!!

Undefined Inverse?

Not a problem!

It is possible to derive an alternative expression for the inverse of RSIN(t):

$$t = \frac{k^2 - k\sqrt{1-k^2}}{2k^2 - 1}$$

To do this, rationalise the numerator and look for common factors:

$$t = \frac{k^2 - k\sqrt{1-k^2}}{2k^2 - 1} \cdot \frac{k + \sqrt{1-k^2}}{k + \sqrt{1-k^2}}$$

Skipping a bunch of algebraic steps (thanks Matlab):

$$\boxed{t = \frac{k}{k + \sqrt{1-k^2}}}$$

Notice this equation IS NOW DEFINED for $k = \frac{1}{\sqrt{2}}$, but IS NOT for $k = -\frac{1}{\sqrt{2}}$. The key thing to note is that, just because one representation 'has a hole/is undefined' for a particular value, does not mean the curve at that point geometrically has a 'magical' hole in it. Context is key! One can often find an alternative algebraic expression that works.

Algebra vs Geometry

A Quick Note!

Some useful notes for what just happened:

- The rationalisation of the numerator to yield a form which works for $k = \frac{1}{\sqrt{2}}$ is a problem of Algebra and not Geometry.
- This is because Geometry directly works with magnitudes which remain unaffected by their algebraic descriptors.
- The rationalisation as a process, is strictly an algebraic process and not a geometric one.

Now we will introduce the notation for this inverse operation.

Definition of the Inverse of RSIN(t)

Given a known $\sin\alpha$ ratio denoted by k, the parameter t is then determined by the definition:

Definition of ISIN(k):

$$t = \mathrm{ISIN(k)} \stackrel{\text{def}}{=} \frac{k}{k + \sqrt{1 - k^2}}$$

for $0 \leq k \leq 1$.

Notably, keeping k in this inequality prevents complex solutions from arising. Now for a list of the **RAU** trigonometric definitions with their inverses.

List of RAU Formulae and their Inverses

For the usual three formulae

Here are the important trig relations and their inverses listed:

- $\mathrm{RSIN}(t) \stackrel{\text{def}}{=} \dfrac{t}{\sqrt{1-2t+2t^2}} \quad \text{for} \quad 0 \leq t \leq 1$
- $\mathrm{RCOS}(t) \stackrel{\text{def}}{=} \dfrac{1-t}{\sqrt{1-2t+2t^2}} \quad \text{for} \quad 0 \leq t \leq 1$
- $\mathrm{RTAN}(t) \stackrel{\text{def}}{=} \dfrac{t}{1-t} \quad \text{for} \quad 0 \leq t < 1$

and their inverses:

- $\mathrm{ISIN}(k) \stackrel{\text{def}}{=} \dfrac{k}{k+\sqrt{1-k^2}} \quad \text{for} \quad 0 \leq k \leq 1$
- $\mathrm{ICOS}(k) \stackrel{\text{def}}{=} 1 - \dfrac{k}{k+\sqrt{1-k^2}} \quad \text{for} \quad 0 \leq k \leq 1$
- $\mathrm{ITAN}(k) \stackrel{\text{def}}{=} \dfrac{k}{1+k} \quad \text{for} \quad 0 \leq k \leq 1$

There is also a connection between the Radian and **RAU**.

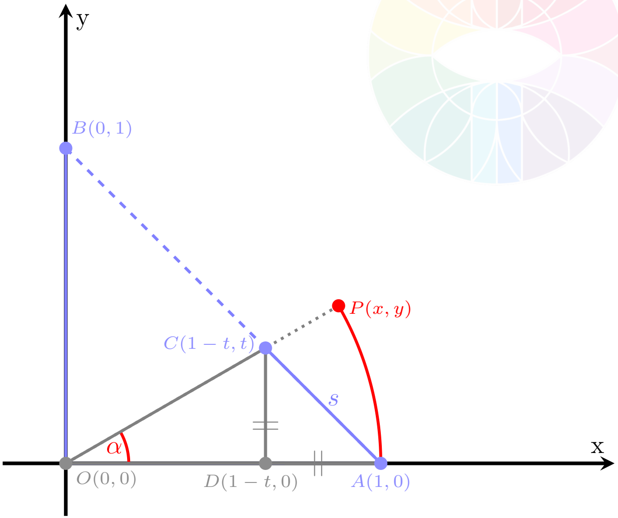
Converting between RAU and Radians

A Useful Formula

- There is a direct relationship between **RAU** measure and the Radian. It is given by the following formula:

$$s(\alpha) = \sqrt{2 - 2\cos\alpha}$$

- Given some known Radian Angle α, one can find the corresponding **RAU** equivalent.
- The exercise to derive this relation is left as an exercise to the reader.



Converting from a Radian Angle to RAU Measure

An Example

Say we know a certain angle in radians $\alpha = \frac{\pi}{6}$ and the radius $R = 1$. What is the corresponding RAU s?

- $s(\frac{\pi}{6}) = \sqrt{2 - 2\cos(\frac{\pi}{6})}$
- $s(\frac{\pi}{6}) = \sqrt{2 - 2 \cdot \frac{\sqrt{3}}{2}}$
- $s(\frac{\pi}{6}) = \sqrt{2 - \sqrt{3}}$
- $s(\frac{\pi}{6}) \approx 0.517638$

Then one can calculate the the parameter r for this particular s:

- $t = \dfrac{s}{\sqrt{2}}$
- $t = \dfrac{\sqrt{2 - \sqrt{3}}}{\sqrt{2}} = \sqrt{1 - \dfrac{\sqrt{3}}{2}} \approx 0.366025$

Which can then result in any ratio of one's choosing:

$$\mathrm{RSIN}(0.366025) = \frac{0.366025}{\sqrt{1 - 2(0.366025) + 2(0.366025)^2}} \approx 0.50000$$

Extending the Definition:

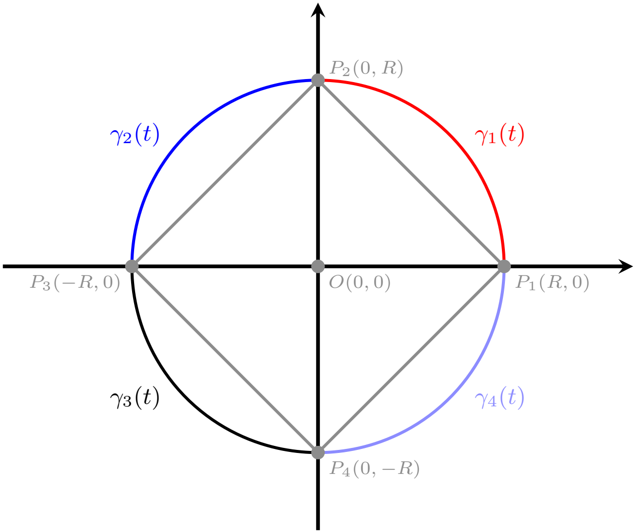
Part 1

There is more! Remember the substitutions carried out earlier for the Algebraic Equation of the circle? The other unused combinations of solutions from the $\pm$ give the other parameterisations for the other quadrants, namely:

$$\gamma_1(t) = \left(\frac{1-t}{\sqrt{2t^2-2t+1}}, \frac{t}{\sqrt{2t^2-2t+1}} \right)$$
$$\gamma_2(t) = \left(\frac{t-1}{\sqrt{2t^2-2t+1}}, \frac{t}{\sqrt{2t^2-2t+1}} \right)$$
$$\gamma_3(t) = \left(\frac{t-1}{\sqrt{2t^2-2t+1}}, \frac{-t}{\sqrt{2t^2-2t+1}} \right)$$
$$\gamma_4(t) = \left(\frac{1-t}{\sqrt{2t^2-2t+1}}, \frac{-t}{\sqrt{2t^2-2t+1}} \right)$$

This is shown overleaf!

Extending the Definition:
Part 2



Extending the Definition:

Periodicity Part 1

The **RAU** trig functions lack the 'periodic' properties we are used to expecting with normal $\sin(x)$ for example, namely: $\text{RSIN}(x+2\pi) \neq \text{RSIN}(x)$. We also know this if we try to differentiate the **RAU** $\text{RSIN}(x)$:

$$
\begin{aligned}
\text{RSIN}(t) &= \frac{t}{\sqrt{1-2t+2t^2}} \\
\frac{d}{dt}(\text{RSIN}(t)) &= \frac{1-t}{\sqrt{(1-2t+2t^2)^3}} \\
\frac{d}{dt}(\text{RSIN}(t)) &= \frac{1-t}{\sqrt{1-2t+2t^2}} \cdot \left(\frac{1}{1-2t+2t^2}\right) \\
\frac{d}{dt}(\text{RSIN}(t)) &= \text{RCOS}(t) \cdot \left(\frac{1}{1-2t+2t^2}\right)
\end{aligned}
$$

Without much imagination, if differentiation is continued, we will not end up with our original $\text{RSIN}(t)$ function as it will appear with common factors of other known quantities.

Extending the Definition:

Periodicity Part 2

Well looks like we're out of luck...

Or are we?

When it comes to implementing $\sin(x)$ on a computer, what is *actually* done?

- CORDIC Algorithm
- Taylor (Newton) Series
- Chebyshev Polynomials
- Lagrange Polynomials
- Lookup Tables
- Combinations of the above!

Extending the Definition:

Periodicity Part 3

What happens on a computer?

- Only the first quadrant is needed to be accurately estimated for a traditional trig function, i.e. for $0 \rightarrow \pi/2$, when implemented on a computer.
- Why is this? Quadrants 2, 3 and 4 can be created from quadrant 1 by mirroring in the x- and y-axes.
- Once you pick your approximation method, you can combine it with a lookup table for important critical values at $x = \left\{0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}\right\}$.
- Since we work in 4 quadrants for x in $0 \rightarrow 2\pi$, if the input x is outside this range, the $\text{mod}(x, 2\pi)$ function can be used to re-map x back into one of the 4 quadrants and proceed as normal.

Extending the Definition:

Periodicity Part 4

Guess What?

We can do the exact same procedure with the **RAU** RSIN(t) formula that is done for any regular sin(x) function implemented in a C/C++ library or your language of preference.

In fact to demonstrate a point, I will implement periodicity in a MATLAB program! Then, to close the presentation, a simulation of a simple damped oscillator using this definition.

Definition of Periodic RSIN(t):

Periodicity Part 5

- The parameterisations are well-defined in each quadrant of the Cartesian-Plane
- This means one can map how to go from one parameterisation to the next in a periodic manner.
- Why? Because every single other quadrant is simply a reflection of the first quadrant, this means the values in the first quadrant are easily translatable to the others.
- Hence the concept of periodicity can be cleanly applied by allowing for the parameter t to exceed the value of 1 previously.
- How should we do this?

Definition of Periodic RSIN(t):

Periodicity Part 6

- The parameterisations are well-defined in each quadrant of the Cartesian-Plane
- This means one can map how to go from one parameterisation to the next in a periodic manner.
- Why? Because every single other quadrant is simply a reflection of the first quadrant, this means the values in the first quadrant are easily translatable to the others.
- Hence the concept of periodicity can be cleanly applied by allowing for the parameter t to exceed the value of 1 previously.
- How should we do this?

Definition of Periodic RSIN(t):

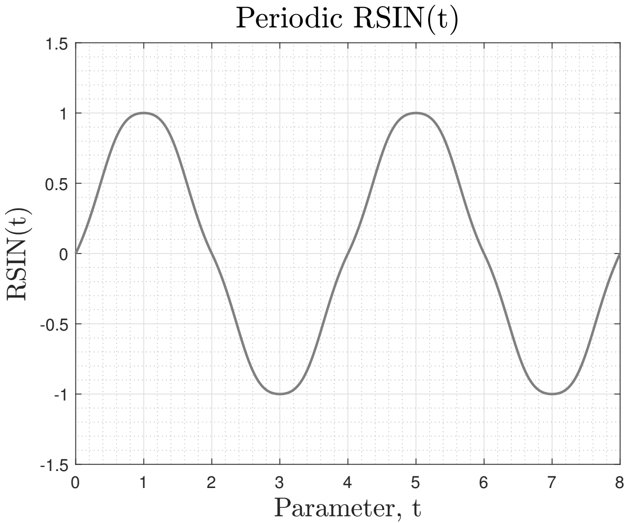
Periodicity Part 7

Modulo Arithmetic!

Since we know there is a parameterisation for each quadrant, all one has to do is:

- Extend the range of the **RAU** parameter t for $0 \leq t \leq 4$ such that:
  - $\gamma_1(t); 0 \leq t \leq 1$
  - $\gamma_2(t); 1 < t \leq 2$
  - $\gamma_3(t); 2 < t \leq 3$
  - $\gamma_4(t); 3 < t \leq 4$
- Use the floor(t) function to determine what quadrant the RSIN(t) is in. This allows for the easy use of the switch case in C/C++.
- Wrap t back to the inequality $0 \leq t \leq 1$ and compute RSIN(t) in this manner using the correct quadrant parameterisation.
- Use the modulo arithmetic mod($t, 4$) to wrap any results back to one of the four quadrants should t exceed $t > 4$.

Periodic RSIN(t) Graphs:
Periodicity Part 8



ODE Example:
ODE

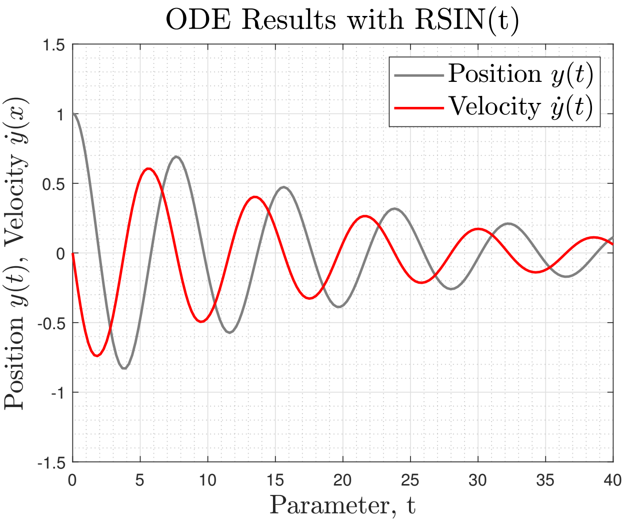
Also, just to further demonstrate the periodic implementation of RSIN(t), consider on the following ODE:

$$\frac{d^2y}{dt^2} + 0.1\frac{dy}{dt} + \mathbf{RSIN}(y) = 0, \quad y(0) = 1, \quad \dot{y}(0) = 0$$

Consider the graph overleaf. (Matlab's ODE45 was used to do the numerical integration)

ODE Graph:

ODE



Some Questions

Question 1:

How might theory have changed in general if this had been a concurrent method (or in place) of Newton's Series method?

Question 2:

Do you, the viewer, think that Newton himself would have valued this method?

Closing Off

Engagement!

Write your comments/thoughts in the comment section please. Tell myself and John what you thought about these ideas presented here today.

Main Message!

The main thing to take away from today, is about perspective and looking at problems differently. If there is anything the viewer should take away from this, is not being static about the way in which things *should* be done or *understood*.

Closing Statement!

Once upon a time it was believed that closed-form formulae for the trigonometric ratios were not possible. If you didn't know, well then now you do!