MarketCap Model for Proof-of-Deposit*

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November 17, 2021

1 Present Value Model

Based off prior analysis by the cLabs Team [1], we model the marketcap of Celo at a certain time $m(t)$ as the sum of all discounted future demand for Celo. In other words, we are calculating its present value:

$$m(t) = \sum_{s \in S} \int_{t}^{\infty} demand'_s(\tau) \cdot discount_t(\tau) \cdot dilution(\tau)\, d\tau \tag{1}$$

Where:

(i) $demand'_s(\tau)$ is the rate at which demand for Celo expands or contracts at a certain time τ for source s . We will be detailing our analysis of the rates for the different sources of demand S in this paper.

(ii) $discount_t(\tau)$ is the discount factor applied to a future value at time τ. We use a exponential decay function with rate r:

$$discount_t(\tau) = e^{-r(\tau - t)} \tag{2}$$

Note that this function is offset by t so that $discount_t(t) = 1$ (i.e. present value is relative to t).

(iii) $dilution(\tau)$ is another discount factor applied to a future value at time τ. This discount factor comes from an increased supply of Celo. We use a exponential decay function with rate ω with a lower bound $0 \leq \alpha \leq 1$ which represents how quickly the existing supply is diluted and the maximum dilution respectively:

$$dilution(\tau) = (1 - \alpha)e^{-\omega\tau} + \alpha \tag{3}$$

For example, $\alpha = 0.2$, $\omega = 0.1$, and $\tau = 5$ year, means that the initial supply of Celo will be 68.5% of the supply in the 5th year. Ultimately, as $\tau \to \infty$, the initial supply will be 20% of the total supply (i.e. the total supply is 5x that of the initial supply).

Note that in this paper we consider τ and t as units of years.

*Cambridge Cryptographic's techniques are the object of the following patent applications: EP20275088.1, GB2016187.3, GB2016186.5

†Special thanks to my wife Yuqing Zheng without whom this analysis would have been intractable.

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2 Sources of Demand

We consider 4 sources of demand for Celo

(i) $demand_{cash}(\tau)$. The *cash* demand is the demand that users have for CUSD to use as cash in their everyday transactions. Celo's stablecoin reserve mechanism is such that every \$1 of demand for CUSD will create a \$1 of demand for Celo itself.

(ii) $demand_{fees}(\tau)$. The $fees$ demand is the demand that users have for Celo in order to pay their transaction fees when transacting their CUSD cash.

(iii) $demand_{interest\_fees}(\tau)$. Our proof-of-deposit scheme is such that block rewards comprising transaction fees and newly minted Celo are distributed pro-rata amongst users who deposit/lock their CUSD in our scheme. This is essentially paying interest on a deposit. Specifically, the $interest\_fees$ demand is the demand for CUSD that results from fees being paid as interest on deposits of CUSD.

(iv) $demand_{interest\_mint}(\tau)$. Our proof-of-deposit scheme is such that block rewards comprising transaction fees and newly minted Celo are distributed pro-rata amongst users who deposit/lock their CUSD in our scheme. This is essentially paying interest on a deposit. Specifically, the $interest\_mint$ demand is the demand for CUSD that results from newly minted Celo being paid as interest on deposits of CUSD.

2.1 Demand from Cash

We model the demand for CUSD as cash as a simple exponential growth function with rate g and initial demand S_0:

$$demand_{cash}(\tau) = S_0 e^{g\tau} \tag{4}$$

With this model in hand, we can take its derivative to calculate the rate at which demand for CUSD as cash expands or contracts at a certain time τ:

$$demand'_{cash}(\tau) = gS_0 e^{g\tau} \tag{5}$$

For example, if $S_0 = \$10MM$, $g = 0.25$ and $\tau = 5$ years, then the demand for CUSD (i.e. how much CUSD is used as cash) at that time is $\$34.9MM$ and the instantaneous rate at which it expands is $\$8.7MM$ per year.

2.2 Demand from Fees

We model the rate at which demand, for Celo to pay fees, expands or contracts to be a function of how much cash there is at a certain time $demand_{cash}(\tau)$, how many times per year each \$1 of CUSD as cash is transacted v (i.e. velocity of money), and what percentage of transacted value is charged as fees f:

$$\begin{aligned} demand'_{fees}(\tau) &= demand_{cash}(\tau) \cdot vf & (6) \\ &= S_0 e^{g\tau} \cdot vf & (7) \end{aligned}$$

For example, if $S_0 = \$10MM$, $g = 0.25$, $\tau = 5$ years, $v = 10$, and $f = 0.2\%$, then the instantaneous rate at which demand for Celo to pay fees expands is $\$0.7MM$ per year.

2.3 Demand from Interest (paid by X)

Like Celo's stablecoin mechanism, we assume the APY (interest rate) on deposits of CUSD are held at an equilibrium rate by market forces; for example, if the equilibrium rate is 5% APY, and the APY is currently at 6%, we assume market forces will deposit more CUSD until the effective APY is lowered to 5%. We model the compound interest rate over a period of time $\Delta\tau$ with an equilbirum rate i as follows:

$$compound\_interest\_rate(\Delta\tau) = e^{i\Delta\tau} - 1 \tag{8}$$

For example:

1. If $i = 0.05$ and $\Delta\tau = 1$ year, then the compounded interest rate is 5.1%.
2. If $i = 0.05$ and $\Delta\tau = 5$ year, then the compounded interest rate is 28.4%.

Given this and the amount of interest paid over a period of time $\beta \cdot interest\_from\_X'(\tau) \cdot \Delta\tau$ where $0 \leq \beta \leq 1$ is the portion of interest paid on deposited CUSD (with $1 - \beta$ being the portion paid on staked Celo), we can arrive at a general formula to calculate the demand for CUSD as deposits at a certain time τ:

$$demand_{interest\_X}(\tau) = \lim_{\Delta\tau \to 0} \frac{\beta \cdot interest\_from\_X'(\tau) \cdot \Delta\tau}{compound\_interest\_rate(\Delta\tau)} \tag{9}$$

$$= \lim_{\Delta\tau \to 0} \frac{\beta \cdot interest\_from\_X'(\tau) \cdot \Delta\tau}{e^{i\Delta\tau} - 1} \tag{10}$$

Applying L'Hospital's rule $\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}$:

$$= \lim_{\Delta\tau \to 0} \frac{\beta \cdot interest\_from\_X'(\tau)}{ie^{i\Delta\tau}} \tag{11}$$

$$= \frac{\beta \cdot interest\_from\_X'(\tau)}{i} \tag{12}$$

2.4 Demand from Interest (paid by Fees)

As we already know the rate at which fees expands and contracts, we can use this directly as the rate of interest:

$$interest\_from\_fees'(\tau) = demand'_{fees}(\tau) \tag{13}$$

We then have a model for demand for CUSD as deposit as a result of fees being paid as interest at a certain time τ:

$$demand_{interest\_fees}(\tau) = \frac{\beta \cdot interest\_from\_fees'(\tau)}{i} \tag{14}$$

$$= \frac{\beta \cdot S_0 e^{g\tau} \cdot vf}{i} \tag{15}$$

To calculate the rate at which this demand changes $demand'_{interest\_fees}(\tau)$, we take the derivative:

$$demand'_{interest\_fees}(\tau) = \frac{\beta \cdot S_0 \cdot vf}{i} \cdot ge^{g\tau} \tag{16}$$

2.5 Demand from Interest (paid by Minted Celo)

The rate of interest from minted Celo tokens can be determined from the rate at which the marketcap $m(t)$ is diluted $dilution(\tau)$:

$$interest\_from\_mint'(\tau) = (1 - dilution(\tau)) \cdot m(t) \tag{17}$$

$$= (1 - (1 - \alpha)e^{-\omega\tau} - \alpha) \cdot m(t) \tag{18}$$

$$= (1 - \alpha) \cdot m(t) \cdot (1 - e^{-\omega\tau}) \tag{19}$$

We then have a model for demand for CUSD as deposit as a result of minted Celo being paid as interest at a certain time τ:

$$demand_{interest\_mint}(\tau) = \frac{\beta \cdot interest\_from\_mint'(\tau)}{i} \tag{20}$$

$$= \frac{\beta \cdot (1-\alpha) \cdot m(t) \cdot (1 - e^{-\omega\tau})}{i} \tag{21}$$

To calculate the rate at which this demand changes $demand'_{interest\_mint}(\tau)$, we take the derivative:

$$demand'_{interest\_mint}(\tau) = \frac{\beta \cdot (1-\alpha) \cdot m(t)}{i} \cdot \omega e^{-\omega\tau} \tag{22}$$

3 Present Value of Demand

We now apply the present value formula on each source of demand:

$$PV_s(t) = \int_t^{\infty} demand'_s(\tau) \cdot discount_t(\tau) \cdot dilution(\tau)\, d\tau \tag{23}$$

3.1 PV of Demand from Cash

Substituting into the formula for present value:

$$PV_{cash}(t) = \int_t^{\infty} demand'_{cash}(\tau) \cdot discount_t(\tau) \cdot dilution(\tau)\, d\tau \tag{24}$$

$$= \int_t^{\infty} gS_0 e^{g\tau} \cdot e^{-r(\tau - t)} \cdot ((1-\alpha)e^{-\omega\tau} + \alpha)\, d\tau \tag{25}$$

$$= gS_0 \cdot e^{rt} \cdot \int_t^{\infty} (1-\alpha)e^{(g-r-\omega)\tau} + \alpha e^{(g-r)\tau}\, d\tau \tag{26}$$

$$= gS_0 \cdot e^{rt} \cdot \left[\frac{(1-\alpha)e^{(g-r-\omega)\tau}}{g-r-\omega} + \frac{\alpha e^{(g-r)\tau}}{g-r} \right]_t^{\infty} \tag{27}$$

Under the assumption $g < r$ results in:

$$= gS_0 \cdot e^{rt} \cdot \left(\frac{(1-\alpha)e^{(g-r-\omega)t}}{r+\omega-g} + \frac{\alpha e^{(g-r)t}}{r-g} \right) \tag{28}$$

$$= gS_0 \cdot \left(\frac{(1-\alpha)e^{(g-\omega)t}}{r+\omega-g} + \frac{\alpha e^{gt}}{r-g} \right) \tag{29}$$

3.2 PV of Demand from Fees

Substituting into the formula for present value:

$$PV_{fees}(t) = \int_t^{\infty} demand'_{fees}(\tau) \cdot discount_t(\tau) \cdot dilution(\tau)\, d\tau \tag{30}$$

$$= \int_t^{\infty} S_0 e^{g\tau} \cdot vf \cdot e^{-r(\tau - t)} \cdot ((1-\alpha)e^{-\omega\tau} + \alpha)\, d\tau \tag{31}$$

$$= S_0 \cdot vf \cdot e^{rt} \cdot \int_t^{\infty} (1-\alpha)e^{(g-r-\omega)\tau} + \alpha e^{(g-r)\tau}\, d\tau \tag{32}$$

$$= S_0 \cdot vf \cdot e^{rt} \cdot \left[\frac{(1-\alpha)e^{(g-r-\omega)\tau}}{g-r-\omega} + \frac{\alpha e^{(g-r)\tau}}{g-r} \right]_t^{\infty} \tag{33}$$

Under the assumption $g < r$ results in:

$$= S_0 \cdot vf \cdot e^{rt} \cdot \left(\frac{(1-\alpha)e^{(g-r-\omega)t}}{r+\omega-g} + \frac{\alpha e^{(g-r)t}}{r-g} \right) \tag{34}$$

$$= S_0 \cdot vf \cdot \left(\frac{(1-\alpha)e^{(g-\omega)t}}{r+\omega-g} + \frac{\alpha e^{gt}}{r-g} \right) \tag{35}$$

$$= PV_{cash}(t) \cdot \frac{vf}{g} \tag{36}$$

3.3 PV of Demand from Interest (paid by Fees)

Substituting into the formula for present value:

$$PV_{interest\_fees}(t) = \int_t^\infty demand'_{interest\_fees}(\tau) \cdot discount_t(\tau) \cdot dilution(\tau)\, d\tau \tag{37}$$

$$= \int_t^\infty \frac{\beta \cdot S_0 \cdot vf}{i} \cdot ge^{g\tau} \cdot e^{-r(\tau-t)} \cdot ((1-\alpha)e^{-\omega\tau} + \alpha)\, d\tau \tag{38}$$

$$= \frac{\beta \cdot S_0 \cdot vf}{i} \cdot ge^{rt} \cdot \int_t^\infty (1-\alpha)e^{(g-r-\omega)\tau} + \alpha e^{(g-r)\tau}\, d\tau \tag{39}$$

$$= \frac{\beta \cdot S_0 \cdot vf}{i} \cdot ge^{rt} \cdot \left[\frac{(1-\alpha)e^{(g-r-\omega)\tau}}{g-r-\omega} + \frac{\alpha e^{(g-r)\tau}}{g-r} \right]_t^\infty \tag{40}$$

Under the assumption $g < r$ results in:

$$= \frac{\beta \cdot S_0 \cdot vf}{i} \cdot ge^{rt} \cdot \left(\frac{(1-\alpha)e^{(g-r-\omega)t}}{r+\omega-g} + \frac{\alpha e^{(g-r)t}}{r-g} \right) \tag{41}$$

$$= \frac{\beta \cdot S_0 \cdot vf \cdot g}{i} \cdot \left(\frac{(1-\alpha)e^{(g-\omega)t}}{r+\omega-g} + \frac{\alpha e^{gt}}{r-g} \right) \tag{42}$$

$$= PV_{fees}(t) \cdot \frac{\beta g}{i} \tag{43}$$

3.4 PV of Demand from Interest (paid by Minted Celo)

Substituting into the formula for present value:

$$PV_{interest\_mint}(t) = \int_t^\infty demand'_{interest\_mint}(\tau) \cdot discount_t(\tau) \cdot dilution(\tau)\, d\tau \tag{44}$$

$$= \int_t^\infty \frac{\beta \cdot (1-\alpha) \cdot m(t)}{i} \cdot \omega e^{-\omega\tau} \cdot e^{-r(\tau-t)} \cdot ((1-\alpha)e^{-\omega\tau} + \alpha)\, d\tau \tag{45}$$

$$= \frac{\beta \cdot (1-\alpha) \cdot m(t)}{i} \cdot \omega e^{rt} \cdot \int_t^\infty (1-\alpha)e^{(-r-2\omega)\tau} + \alpha e^{(-r-\omega)\tau}\, d\tau \tag{46}$$

$$= \frac{\beta \cdot (1-\alpha) \cdot m(t)}{i} \cdot \omega e^{rt} \cdot \left[\frac{(1-\alpha)e^{(-r-2\omega)\tau}}{-r-2\omega} + \frac{\alpha e^{(-r-\omega)\tau}}{-r-\omega} \right]_t^\infty \tag{47}$$

Under the assumption $g < r$ results in:

$$= \frac{\beta \cdot (1-\alpha) \cdot m(t)}{i} \cdot \omega e^{rt} \cdot \left(\frac{(1-\alpha)e^{(-r-2\omega)t}}{r+2\omega} + \frac{\alpha e^{(-r-\omega)t}}{r+\omega} \right) \tag{48}$$

$$= \frac{\beta \cdot (1-\alpha) \cdot m(t) \cdot \omega}{i} \cdot \left(\frac{(1-\alpha)e^{-2\omega t}}{r+2\omega} + \frac{\alpha e^{-\omega t}}{r+\omega} \right) \tag{49}$$

4 Amplified MarketCap from Paying Minted Celo as Interest on Deposited CUSD

The marketcap of Celo is calculated by:

$$m(t) = PV_{cash}(t) + PV_{fees}(t) + PV_{interest\_fees}(t) + PV_{interest\_mint}(t) \tag{50}$$

If we substitue in $PV_{interest\_mint}(t)$, an amplification effect emerges:

$$m(t) = PV_{cash}(t) + PV_{fees}(t) + PV_{interest\_fees}(t) + \frac{\beta \cdot (1-\alpha) \cdot m(t) \cdot \omega}{i} \cdot \left(\frac{(1-\alpha)e^{-2\omega t}}{r + 2\omega} + \frac{\alpha e^{-\omega t}}{r + \omega} \right) \tag{51}$$

$$m(t) - \frac{\beta \cdot (1-\alpha) \cdot m(t) \cdot \omega}{i} \cdot \left(\frac{(1-\alpha)e^{-2\omega t}}{r + 2\omega} + \frac{\alpha e^{-\omega t}}{r + \omega} \right) = PV_{cash}(t) + PV_{fees}(t) + PV_{interest\_fees}(t) \tag{52}$$

$$m(t) \left(1 - \frac{\beta \cdot (1-\alpha) \cdot \omega}{i} \cdot \left(\frac{(1-\alpha)e^{-2\omega t}}{r + 2\omega} + \frac{\alpha e^{-\omega t}}{r + \omega} \right) \right) = PV_{cash}(t) + PV_{fees}(t) + PV_{interest\_fees}(t) \tag{53}$$

$$m(t) = (PV_{cash}(t) + PV_{fees}(t) + PV_{interest\_fees}(t)) \cdot amplification(t) \tag{54}$$

Where:

$$amplification(t) = \left(1 - \frac{\beta \cdot (1-\alpha) \cdot \omega}{i} \cdot \left(\frac{(1-\alpha)e^{-2\omega t}}{r + 2\omega} + \frac{\alpha e^{-\omega t}}{r + \omega} \right) \right)^{-1} \tag{55}$$

References

[1] cLabs Team. An analysis of the stability characteristics of celo. `https://celo.org/papers/stability`.