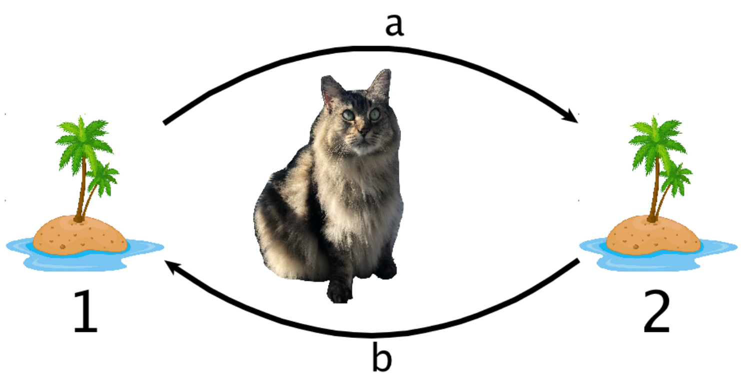
Please work on the following problem in groups of 3-4 and finish as much as you can in class today. These problems will be turned in as part of your Homework 7 assignment, so you should write your own solutions on a separate sheet of paper.

Example 62. *Imagine two islands populated by cats. Always searching for a more comfortable spot, the cats will occasionally swim from one island to the other. As a result, we observe that the rate at which cats leave one island is proportional to the population of that island. Let $x_1(t)$ and $x_2(t)$ be the size of the cat populations of Island 1 and Island 2 at time t.*



(a) Write a system of differential equations that describes the cat populations.

$$x_1'(t) =$$

$$x_2'(t) =$$

(b) For what vectors $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ will the system be in equilibrium?

(c) Notice that if we write $Q = \begin{pmatrix} -a & b \\ a & -b \end{pmatrix}$, the system can be expressed as $\mathbf{x}' = Q\mathbf{x}$. What is $\mathcal{N}(Q)$?

(d) Find the eigenvalues and corresponding eigenvectors of Q.

(e) Find a matrix S and it's inverse S^{-1} such that $D = S^{-1}QS$ is a diagonal matrix.

(f) The matrix exponential *for an $n \times n$ matrix A is defined as*

$$\exp(B) = \sum_{k=0}^{\infty} \frac{A^k}{k!}.$$

For any matrix A and invertible S, we can show that $\exp(SAS^{-1}) = S\exp(A)S^{-1}$. For a diagonal matrix, $D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$, we can show that $\exp(Dt) = \begin{pmatrix} \exp(\lambda_1 t) & 0 \\ 0 & \exp(\lambda_2 t) \end{pmatrix}$. Use these facts to find $M = \exp(Qt)$, where t is a scalar.

(g) *The matrix* M^T *is called a* transition matrix*, and* M^T_{ij} *gives the probability that a cat on island* i *ends up on island* j *at the end of time* t*. Suppose* $a = 1/day$ *and* $b = 2/day$*. If we observe a cat on Island* 1*, what is the probability that that particular cat is on Island* 1 *at time* $t = 4$ *days?*

(h) *If we observe a cat on Island* 2*, what is the probability that that particular cat is on Island* 1 *if we check back one year later?*