UID: ______________________ **NAME** ______________________ **!!**

1	12	
2	16	
3	4	
4	4	
5	4	
Total	40	

CS170A – Mathematical Models and Methods for Computer Science
Winter 2016
HOMEWORK #1 DUE WEDNESDAY, January 27, 11:55pm

1. **Kinds of Matrices** For each of the following matrices, Determine (true/false) whether each of the following matrices is: invertible, Hermitian, or unitary, where $\theta = \pi/4$, and $i = \sqrt{-1}$.

matrix	invertible?		Hermitian?		unitary?	
$\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$	**True** ☐	**False** ☐	**True** ☐	**False** ☐	**True** ☐	**False** ☐
$\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$	**True** ☐	**False** ☐	**True** ☐	**False** ☐	**True** ☐	**False** ☐
$\begin{pmatrix} \sin\theta & \cos\theta \\ \cos\theta & -\sin\theta \end{pmatrix}$	**True** ☐	**False** ☐	**True** ☐	**False** ☐	**True** ☐	**False** ☐
$\begin{pmatrix} \sin\theta & -\sin\theta \\ -\sin\theta & \sin\theta \end{pmatrix}$	**True** ☐	**False** ☐	**True** ☐	**False** ☐	**True** ☐	**False** ☐

2. **Matrix Algebra and Eigenstructure**

For each of the following equations, mark whether the equation is True (valid for all specified matrices A) or False. Assume that $A^\top$ denotes the transpose of A, A' denotes the hermitian transpose (conjugate transpose) of A, and $i = \sqrt{-1}$.

Assume for real θ that 2×2 rotation matrices have form $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$, and reflections have form $\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$.

True ☐ **False** ☐ Every product of two 2×2 rotation matrices is orthogonal, and has determinant $+1$.

True ☐ **False** ☐ If $\mathbf{v}$ is a 2×1 vector and Q is a 2×2 rotation matrix, then $||Q\,\mathbf{v}||^2 = ||\mathbf{v}||^2 = v_1^2 + v_2^2$.

True ☐ **False** ☐ If $\mathbf{v}$ is a $n \times 1$ vector and Q is an $n \times n$ orthogonal matrix, then $||Q\,\mathbf{v}||^2 = ||\mathbf{v}||^2 = \sum_i v_i^2$.

True ☐ **False** ☐ If $A = (a_{ij})$ is real symmetric, with squared Frobenius norm $||A||_F^2 = \sum_{ij} a_{ij}^2$, and Q is orthogonal, then $||Q\,A\,Q'\,||_F^2 = ||A||_F^2$.

True ☐ **False** ☐ If $A = Q\ L\ Q'$, where Q orthogonal and L is diagonal, but L has *complex values* on the diagonal, then A a normal matrix.

True ☐ **False** ☐ If A is either normal or real symmetric or (real) orthogonal, then so is A^2.

True ☐ **False** ☐ If the eigenvalues of a real symmetric matrix A are all real and positive, then A is invertible.

True ☐ **False** ☐ If A is real symmetric and $\det(A) > 1$, then A is invertible and $\det(A^{-1}) < 1$.

True ☐ **False** ☐ If $A = Q\ L\ Q'$, where L is diagonal, and Q is orthogonal, then $A^2 = Q\ L^2\ Q'$.

True ☐ **False** ☐ If A is a real symmetric matrix and $A = Q\,L\,Q'$, and $B = Q\,L^{1/2}\,Q'$ where $L^{1/2}$ is the diagonal matrix whose entries are square roots of corresponding entries in L, then $B^2 = A$.

True ☐ **False** ☐ If A is a real symmetric 2×2 matrix and $\operatorname{trace}(A) > \det(A) > 0$, then A is positive definite.

True ☐ **False** ☐ If U and V are orthogonal, and $U\,V = V\,U$, then $U\,V$ is normal.

True ☐ **False** ☐ For all real θ and ϕ, the product $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\phi & \sin\phi \\ \sin\phi & -\cos\phi \end{pmatrix}$ is a reflection.

True ☐ **False** ☐ If A has the eigenvalue decomposition $A = Q\ L\ Q'$, where L is the diagonal matrix of eigenvalues of A and Q is a real orthogonal matrix, and A is invertible, then A is positive definite.

True ☐ **False** ☐ For all real θ, if $R(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$ and $U = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, then $R(\theta)\,U = U\,R(-\theta)$.

True ☐ **False** ☐ Let $R(\theta)$ and U be as in the previous question. For all real θ, if $A = R(-\theta)\,U$, then A is a real symmetric matrix and $A = R(-\theta/2)\ U\ R(\theta/2)$ is an eigendecomposition of A.

3. **Interesting Matrices**
Consider the following MATLAB functions:

```
function L = luis(n)
   % assume n is a positive integer
   f = @(x) factorial(x)
   i = (1:n)' * ((1:n)>0)
   j = i'
   L = f(i+j-2) ./ (f(i-1) .* f(j-1))
```

```
function C = puzzle(sequence)
   % assume sequence is a 1D array of integers
   n = length(sequence)
   i = (1:n)' * ((1:n)>0)
   j = i'
   K = i - j + 1
   K_mod_n =  (K .* (K>0)) + ((K+n) .* (K<=0))
   f = @(x) sequence( x )
   C = f( K_mod_n )
```

What matrix does the invocation `luis(3)` produce?

☐ $\begin{pmatrix} 3 & 2 & 1 \\ 2 & 1 & 3 \\ 1 & 3 & 2 \end{pmatrix}$ ☐ $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 6 \end{pmatrix}$ ☐ $\begin{pmatrix} 1 & 3 & 2 \\ 2 & 1 & 3 \\ 3 & 2 & 1 \end{pmatrix}$ ☐ $\begin{pmatrix} 2 & 3 & 1 \\ 3 & 1 & 2 \\ 1 & 2 & 3 \end{pmatrix}$ ☐ $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{pmatrix}$ ☐ $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 4 \end{pmatrix}$

What matrix does the invocation `puzzle(1:3)` produce?

☐ $\begin{pmatrix} 3 & 2 & 1 \\ 2 & 1 & 3 \\ 1 & 3 & 2 \end{pmatrix}$ ☐ $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 6 \end{pmatrix}$ ☐ $\begin{pmatrix} 1 & 3 & 2 \\ 2 & 1 & 3 \\ 3 & 2 & 1 \end{pmatrix}$ ☐ $\begin{pmatrix} 2 & 3 & 1 \\ 3 & 1 & 2 \\ 1 & 2 & 3 \end{pmatrix}$ ☐ $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{pmatrix}$ ☐ $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 4 \end{pmatrix}$

What matrix does the invocation `puzzle((1:3) * 2)` produce?

☐ $\begin{pmatrix} 6 & 4 & 2 \\ 4 & 2 & 6 \\ 2 & 6 & 4 \end{pmatrix}$ ☐ $\begin{pmatrix} 2 & 4 & 6 \\ 4 & 6 & 8 \\ 6 & 8 & 10 \end{pmatrix}$ ☐ $\begin{pmatrix} 2 & 6 & 4 \\ 4 & 2 & 6 \\ 6 & 4 & 2 \end{pmatrix}$ ☐ $\begin{pmatrix} 2 & 3 & 1 \\ 3 & 1 & 2 \\ 1 & 2 & 3 \end{pmatrix}$ ☐ $\begin{pmatrix} 2 & 2 & 2 \\ 2 & 4 & 6 \\ 2 & 6 & 12 \end{pmatrix}$ ☐ $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 4 \end{pmatrix}$

Using the functions above, which of the following yields the $n \times n$ Pascal matrix $P = (P_{ij})$ where $P_{ij} = \frac{(i+j-2)!}{(i-1)!(j-1)!}$?

☐ `pascal = @(n) puzzle(1:n)`
☐ `pascal = @(n) abs( luis(n) )`
☐ `pascal = @(n) 1 + luis(n)`
☐ `pascal = @(n) luis(n)`

4. **Rotations and Linear Transformations**
Suppose that the following MATLAB statements have been executed:

```
Rot = @(t) [ cos(t) -sin(t) ; sin(t) cos(t) ]          %                    |A 0 .. 0|
Q = @(t, w) = @(t, w) blkdiag(Rot(t), Rot(w))          % blkdiag(A,B,...) = |0 B .. 0|
R = Q(rand(), rand())                                  %                    |0 0 ..  |
```

True ☐ **False** ☐ det(R) is always +1
True ☐ **False** ☐ R is always an orthogonal matrix
True ☐ **False** ☐ $|Rv| = |v|$ for all $v \in \mathbb{R}^4$
True ☐ **False** ☐ The linear transformation R always preserves angles. (i.e. $< Rv, Rw > = < v, w >$ for all $v, w \in \mathbb{R}^4$)

5. **Singular Value Decomposition**
Consider the following fact:

The singular values of a matrix A are the positive ***square roots*** *of the eigenvalues of* $A^T A$

Let $A = \begin{pmatrix} 0 & 2 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$. If $A = USV'$, which of the following matrices is the correct S matrix?

☐ $\begin{pmatrix} 0 & 0 \\ 0 & 4 \end{pmatrix}$ ☐ $\begin{pmatrix} 0 & 2 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$ ☐ $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ ☐ $\begin{pmatrix} 4 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$ ☐ $\begin{pmatrix} 2 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$ ☐ $\begin{pmatrix} 4 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

For each the following equations, mark whether the equation is **True** (valid for all specified matrices X) or **False**. Assume that X' denotes the Hermitian (conjugate) transpose of X.

True ☐ **False** ☐ If X is a 2×2 reflection matrix X, then its SVD is unique.

True ☐ **False** ☐ If X has the SVD $X = U\ S\ V'$, where S is the diagonal matrix of singular values σ_i of X, and all the singular values are **nonzero** and **equal**, then X is a constant multiple of a unitary matrix.

True ☐ **False** ☐ If X has the SVD $X = U\ S\ U'$, where S is the diagonal matrix of singular values σ_i of X, and U is a real unitary matrix, then X is real symmetric and U is orthogonal.