

## 5-) [20 Points] Growing Bacteria Species

Consider two different species of bacteria, Species X and Species Y, growing in a petri dish. The lifetime of each bacterium is a random variable. The joint density functions of these variables is given by:

$$f(x, y) = \begin{cases} cxy^2 & \text{if } x \geq 0, y \geq 0, x + y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

And, the random variable Z that corresponds the evolution probability of the species denoted as:

$$Z = \begin{cases} \max\{X, Y\} & 0 \leq x \leq \frac{1}{2}, 0 \leq y \leq \frac{1}{2}, x + y \leq 1 \\ 1 & \{x \geq \frac{1}{2}, y \geq 0, x + y \leq 1\} \cup \{x \geq 0, y \geq \frac{1}{2}, x + y \leq 1\} \\ 0 & \text{otherwise} \end{cases}$$

a-) Find c.

b-) Find **CDF** of Z and draw it.

c-) Find **PDF** of Z and draw it.

d-) Find **E[Z]**.

a-)

$$\int_{x=0}^1 \int_{y=0}^{1-x} cxy^2 dy dx = 1$$

$$= \int_0^1 \left( cxy^3 \Big|_0^{1-x} \right) dx = \int_0^1 c \cdot x \cdot \frac{(1-x)^3}{3} dx$$

$$= c \left( \frac{x^2}{6} - \frac{1}{3}x^3 + \frac{x^4}{4} - \frac{x^5}{5} \right) \Big|_0^1 = \frac{c}{60} = 1$$

$$c = 60$$

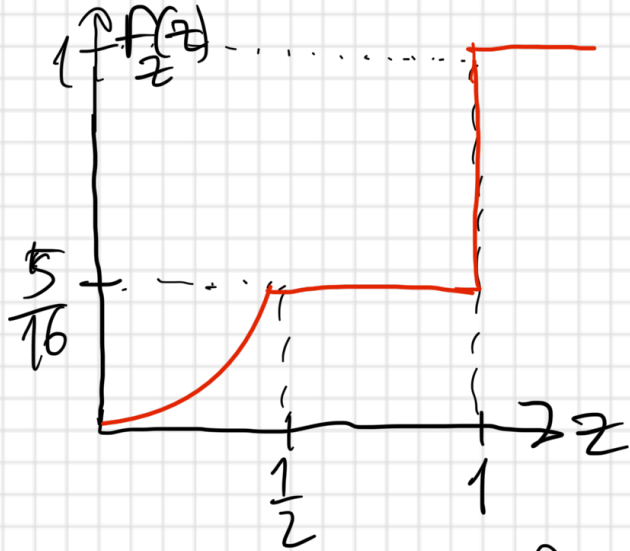
b-)  $F_Z(z) = P[\max\{X, Y\} \leq z]$  where  $z \leq \frac{1}{2}$

$$= 60 \int_0^z \int_0^z xy^2 dx dy = 60 \frac{z^2}{2} \cdot \frac{z^3}{3} = 10z^5$$

$$F_Z\left(\frac{1}{2}\right) = \frac{5}{16}$$

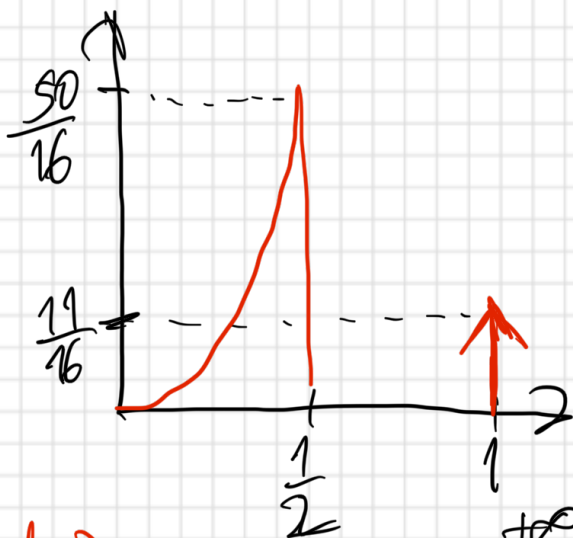
$$F_Z(1) = 1$$

$$\rightarrow F_Z = \begin{cases} 0 & ; z < 0 \\ 10z^5 & ; 0 \leq z < \frac{1}{2} \\ \frac{5}{16} & ; \frac{1}{2} \leq z < 1 \\ 1 & ; z \geq 1 \end{cases}$$



c-)  $f_z(z) = \frac{df_z(z)}{dz}$

$$= \begin{cases} 50z^4; & 0 \leq z \leq \frac{1}{2} \\ \frac{11}{16}\delta(z-1); & z=1 \\ 0; & \text{otherwise} \end{cases}$$



d-)  $E\{z\} = \int_{-\infty}^{\infty} z \cdot f_z(z) dz$

$$= \int_0^{\frac{1}{2}} 50z^5 dz + \frac{11}{16} \int_{-\infty}^{\infty} z \cdot \delta(z-1) dz$$

$$= \frac{50}{6} z^{\frac{6}{2}} \Big|_0^{\frac{1}{2}} + \frac{11}{16} = \frac{50}{6.64} + \frac{11}{16} = \frac{157}{192}$$