NAME: SID:

Problem 1: (a) Complete the statement of the Master Theorem by filling in the blanks.

Assume that $a \geq$ ________ , $b >$ ________, $c >$ ________ and $d \geq$ ________, and that $T(n)$ satisfies the recurrence $T(n) = aT(n/b) + cn^d$. Then

$$T(n) = \begin{cases} \_\_\_\_\_\_\_\_\_\_ & \text{if} \quad \_\_\_\_\_\_\_\_\_\_ \\ \_\_\_\_\_\_\_\_\_\_ & \text{if} \quad \_\_\_\_\_\_\_\_\_\_ \\ \_\_\_\_\_\_\_\_\_\_ & \text{if} \quad \_\_\_\_\_\_\_\_\_\_ \end{cases}$$

(b) Give the asymptotic value (using the Θ-notation) for the number of letters that will be printed by the algorithms below. In each algorithm the argument n is a positive integer. Your solution needs to consist of an appropriate recurrence equation and its solution.

(i) **Algorithm** PrintAs (n : **integer**)
```
    if n < 4
        print("A")
    else
        PrintAs(⌈n/3⌉)
        PrintAs(⌈n/3⌉)
        for i ← 1 to 4n do print("A")
```

Solution: $f(n) = 2f(n/3) + 4n, \qquad f(n) = \Theta(n)$

(ii) **Algorithm** PrintBs (n : **integer**)
```
    if n < 2
        print("B")
    else
        for j ← 1 to 9 do PrintBs(⌊n/2⌋)
        for i ← 1 to 10n^4 do print("B")
```

Solution: $f(n) = 9f(n/2) + 10n^4, \qquad f(n) = \Theta(n^4)$

(iii) **Algorithm** PrintCs (n : **integer**)
```
    if n < 3
        print("C")
    else
        PrintCs(⌈n/2⌉)
        PrintCs(⌈n/2⌉)
        PrintCs(⌈n/2⌉)
        PrintCs(⌈n/2⌉)
        for i ← 1 to 20n^2 do print("C")
```

Solution: $f(n) = 4f(n/2) + 20n^2, \qquad f(n) = \Theta(n^2 \log n)$

Problem 2: (b) We have a group of 49 students, including 20 English majors, 25 Music majors, and 31 Biology majors (and no other majors). The numbers of double majors of each type: English-Music, English-Biology, and Music-Biology are all equal. The number of triple majors is 6. How many students are double majors?

Solution. Let E, M and B represent the groups of students of English, Music, and Biology majors respectively. Let also x be the number of double-majors of each type. Then, using inclusion-exclusion principle, we have:

$$|E \cup M \cup B| = |E| + |M| + |B| - |E \cap M| - |E \cap B| - |M \cap B| + |E \cap M \cap B|.$$

Plugging in the numbers gives

$$49 = 20 + 25 + 31 - x - x - x + 6,$$

so $x = 11$. So the total number of double majors is $3x = 33$.

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Problem 3: (i) Find a particular solution of the recurrence $A_n = 3A_{n-1} + 10A_{n-2} + 2n$. Show your work.

The inhomogeneous term is a linear function, and 1 is not a solution of the characteristic equation of the associated homogeneous equation, so a particular solution is of the form $\beta_1 n + \beta_2$. Plugging this in, we get an equation

$$\beta_1 n + \beta_2 = 3(\beta_1(n-1) + \beta_2) + 10[\beta_1(n-2) + \beta_2] + 2n$$

that simplifies to

$$(12\beta_1 + 2)n + (-23\beta_1 + 12\beta_2) = 0.$$

Since this equation must hold for all n, we have $12\beta_1 + 2 = 0$ and $-23\beta_1 + 12\beta_2 = 0$. So $\beta_1 = -\frac{1}{6}$ and $\beta_2 = -\frac{23}{72}$. This gives us the following particular solution:

$$A''_n = -\tfrac{1}{6}n - \tfrac{23}{72}$$

ii) Determine the *general solution* of the recurrence equation

$$f_n = 3f_{n-1} + 10f_{n-2} + 2 \cdot 3^n.$$

(a) Characteristic equation and its solution:

$$x^2 - 3x - 10 = 0$$

The roots are 5 and -2.

(b) General solution of the homogeneous equation:

$$f'_n = \alpha_1 \cdot 5^n + \alpha_2 \cdot (-2)^n$$

(c) Compute particular solution of the inhomogeneous equation:

Since the non-homogeneous term is $2 \cdot 3^n$, and 3 is not a solution of the char. equation, $f''_n = \beta \cdot 3^n$. Plugging it into the equation and simplifying, we get

$$\begin{aligned} \beta 3^n &= 3 \cdot \beta 3^{n-1} + 10 \cdot \beta 3^{n-2} + 2 \cdot 3^n \\ 9\beta &= 9\beta + 10\beta + 18 \\ \beta &= -\frac{9}{5} \end{aligned}$$

So $f''_n = -\frac{9}{5} \cdot 3^n$.

(d) General solution of the inhomogeneous equation:

$$f'_n = \alpha_1 \cdot 5^n + \alpha_2 \cdot (-2)^n - \frac{9}{5} \cdot 3^n$$

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Problem 4: Let $G = (V, E)$ be a connected undirected graph with the number of vertices $n = 12$.

(a) If G is the complete graph, find the number of edges in G. Justify.

$11 + 10 + ... + 1 = \frac{(11+1)\cdot 11}{2} = 66$

(b) Can G be (properly) colored with 8 colors if the degree of each vertex is 6? Justify.

Yes. Since the max vertex degree is 6, the graph can be colored with 7 or more colors.

(c) Find the number of edges in G if the degree of each vertex is 7. Justify.

We can use Handshaking Lemma: The total vertex degree of G is $7 \cdot 12 = 72$, and it is equal to $2|E|$. Thus, the number of edges $|E| = 36$.

(d) Does G have a Hamiltonian cycle if

(i) the degree of each vertex is 6? Justify.

Yes, use Dirac's Theorem.

(ii) the degree of each vertex is 3? Justify.

Dirac's(Ore's) Theorems give sufficient conditions, which are not satisfied here. Thus, we do not know.

(e) The maximum vertex degree of G is 8. Does G have an Euler Tour?

We don't know. In order to answer this question, we need to know, whether there is a vertex, whose degree is odd. However, we only know the max vertex degree.

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Problem 5: (a) Complete the statement of the Master Theorem by filling in the blanks.

Assume that $a \geq$ _______ , $b >$ _______, $c >$ _______ and $d \geq$ _______, and that $T(n)$ satisfies the recurrence $T(n) = aT(n/b) + cn^d$. Then

$$T(n) \quad = \quad \begin{cases} \underline{\hspace{3cm}} & \text{if} \quad \underline{\hspace{3cm}} \\ \underline{\hspace{3cm}} & \text{if} \quad \underline{\hspace{3cm}} \\ \underline{\hspace{3cm}} & \text{if} \quad \underline{\hspace{3cm}} \end{cases}$$

(b) Give the asymptotic value (using the Θ-notation) for the number of letters that will be printed by the algorithms below. In each algorithm the argument n is a positive integer. Your solution needs to consist of an appropriate recurrence equation and its solution.

(i) **Algorithm** PrintCs (n : **integer**)
  if $n < 3$
    print("C")
  else
    PrintCs($\lceil n/2 \rceil$)
    PrintCs($\lceil n/2 \rceil$)
    PrintCs($\lceil n/2 \rceil$)
    PrintCs($\lceil n/2 \rceil$)
    for $i \leftarrow 1$ **to** $20n^2$ **do** print("C")

Solution: $f(n) = 4f(n/2) + 20n^2$, $f(n) = \Theta(n^2 \log n)$

(ii) **Algorithm** PrintAs (n : **integer**)
  if $n < 4$
    print("A")
  else
    PrintAs($\lceil n/3 \rceil$)
    PrintAs($\lceil n/3 \rceil$)
    for $i \leftarrow 1$ **to** $4n$ **do** print("A")

Solution: $f(n) = 2f(n/3) + 4n$, $f(n) = \Theta(n)$

(iii) **Algorithm** PrintBs (n : **integer**)
  if $n < 2$
    print("B")
  else
    for $j \leftarrow 1$ **to** 9 **do** PrintBs($\lfloor n/2 \rfloor$)
    for $i \leftarrow 1$ **to** $10n^4$ **do** print("B")

Solution: $f(n) = 9f(n/2) + 10n^4$, $f(n) = \Theta(n^4)$

Problem 6: (b) We have a group of 55 students, including 19 English majors, 24 Music majors, and 33 Biology majors (and no other majors). The numbers of double majors of each type: English-Music, English-Biology, and Music-Biology are all equal. The number of triple majors is 9 . How many students are double majors?

Solution. Let E, M and B represent the groups of students of English, Music, and Biology majors respectively. Let also x be the number of double-majors of each type. Then, using inclusion-exclusion principle, we have:

$$|E \cup M \cup B| = |E| + |M| + |B| - |E \cap M| - |E \cap B| - |M \cap B| + |E \cap M \cap B|.$$

Plugging in the numbers gives

$$55 = 19 + 24 + 33 - x - x - x + 9,$$

so $x = 10$. So the total number of double majors is $3x = 30$.

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Problem 7: i) Determine the *general solution* of the recurrence equation

$$f_n = f_{n-1} + 12f_{n-2} + 3 \cdot 2^n.$$

(a) Characteristic equation and its solution:

$$x^2 - x - 12 = 0$$

The roots are 4 and -3.
(b) General solution of the homogeneous equation:

$$f'_n = \alpha_1 \cdot 4^n + \alpha_2 \cdot (-3)^n$$

(c) Compute particular solution of the inhomogeneous equation:
Since the non-homogeneous term is $3 \cdot 2^n$, and 2 is not a solution of the char. equation, $f''_n = \beta 2^n$. Plugging it into the equation and simplifying, we get

$$\begin{aligned}\beta 2^n &= \beta 2^{n-1} + 12 \cdot \beta 2^{n-2} + 2^n \\ 4\beta &= 2\beta + 12\beta + 12 \\ \beta &= -\frac{6}{5}\end{aligned}$$

So $f''_n = -\frac{6}{5} \cdot 2^n$.

(d) General solution of the inhomogeneous equation:

$$f'_n = \alpha_1 \cdot 4^n + \alpha_2 \cdot (-3)^n - \frac{6}{5} \cdot 2^n$$

(ii) Find a particular solution of the recurrence $A_n = A_{n-1} + 12A_{n-2} + 3n$. Show your work.
The inhomogeneous term is a linear function, and 1 is not a solution of the characteristic equation of the associated homogeneous equation, so a particular solution is of the form $\beta_1 n + \beta_2$. Plugging this in, we get an equation

$$\beta_1 n + \beta_2 = (\beta_1(n-1) + \beta_2) + 12[\beta_1(n-2) + \beta_2] + 3n$$

that simplifies to

$$(12\beta_1 + 3)n + (-25\beta_1 + 12\beta_2) = 0.$$

Since this equation must hold for all n, we have $12\beta_1 + 2 = 0$ and $-23\beta_1 + 12\beta_2 = 0$. So $\beta_1 = -\frac{1}{6}$ and $\beta_2 = -\frac{23}{72}$. This gives us the following particular solution:

$$A''_n = -\tfrac{1}{4}n - \tfrac{25}{48}$$

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Problem 8: Let $G = (V, E)$ be a connected undirected graph with the number of vertices $n = 10$.

(a) Does G have a Hamiltonian cycle if

(i) the degree of each vertex is 5? Justify.

Yes, use Dirac's Theorem.

(ii) the degree of each vertex is 3? Justify.

Dirac's(Ore's) Theorems give sufficient conditions, which are not satisfied here. Thus, we do not know.

(b) If G is the complete graph, find the number of edges in G. Justify.

$9 + 8 + ... + 1 = \frac{(9+1) \cdot 9}{2} = 45$

(c) Find the number of edges in G if the degree of each vertex is 6. Justify.

We can use Handshaking Lemma: The total vertex degree of G is $6 \cdot 10 = 60$, and it is equal to $2|E|$. Thus, the number of edges $|E| = 30$.

(d) The maximum vertex degree of G is 6. Does G have an Euler Tour?

We don't know. In order to answer this question, we need to know, whether there is a vertex, whose degree is odd. However, we only know the max vertex degree.

(e) Can G be (properly) colored with 7 colors if the degree of each vertex is 5? Justify.

Yes. Since the max vertex degree is 5, the graph can be colored with 6 or more colors.