NAME: SID:

Problem 1: (a) Use the Θ-notation to determine the rate of growth of the following functions:

Function	Θ estimate
$n^2 7^{-n} + n5^n + n^3 4^n + 3n^5 \log n^8$	$\boldsymbol{\Theta(n5^n)}$
$3n^2 \log n + 2n \log^5 n + 0.3/(n^3\sqrt{n})$	$\boldsymbol{\Theta(n^2 \log n)}$

(b) For each piece of pseudo-code below, give its asymptotic running time as a function of n. Express this running time using the Θ() notation. Include a brief informal explanation (at most 20 words).

Pseudo-code	Running time	Explanation

for $i \leftarrow 1$ **to** $3n$ **do**
    for $j \leftarrow 1$ **to** $2n$ **do**
        $x \leftarrow x^2$
for $k \leftarrow 1$ **to** $5n$ **do**
    $z \leftarrow x + z$
	$\Theta(n^2)$	The first nested loop makes $3n \cdot 2n = \Theta(n^2)$ iterations. The second loop makes $\Theta(n)$ iterations.

for $i \leftarrow 1$ **to** n **do**
    $j \leftarrow 1$
    while $j < n$ **do**
        $j \leftarrow 3 \cdot j$
        $x \leftarrow 7 \cdot x$
	$\Theta(n \log n)$	The external loop makes n iterations. In the internal loop j triples each time and stops at n, so it makes $\Theta(\log n)$ iterations.

(c) Let $f(x)$ be a polynomial of degree m. Prove that $f(x) = O(x^m)$.

On the slides.

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Problem 2: (a) Compute $7^{10} \pmod 5$ by squaring. Show your work.

$7^{10} \pmod 5 = ... = 4$

(b) Compute $8^{-1} \pmod{11}$ using the method of linear combinations (listing multiples). Show your work.

$8^{-1} \pmod{11} = ... = 7$

(c) Compute $3^{-1} \pmod{13}$ using Fermat's Little Theorem. Show your work.

$3^{-1} \pmod{13} = ... = 9$

(d) Solve:

$$
\begin{aligned}
3x &\equiv 5 \pmod{13}. \\
3^{-1} \cdot 5x &\equiv 5 \cdot 3^{-1} \pmod{13} \\
x &\equiv 5 \cdot 9 \pmod{13} \\
x &\equiv 45 \pmod{13} \\
x &= 6
\end{aligned}
$$

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Problem 3: Part 1. Prove or disprove the following statements. (For each statement tell whether it is true or false and justify your answer.)

(a) For all nonnegative integers n, $n^2 + 3n + 2$ is not a prime number.

False. When $n = 0$, $n^2 + 3n + 2$ is prime.

(b) For all positive integers n, $n^2 + 3n + 2$ is not a prime number.

True. $n^2 + 3n + 2 = (n + 1)(n + 2)$. It is a product of two integers, and each of the factors is greater than 1.

(c) For all nonnegative integers n, $n^2 + 3n + 2$ is an even number.

True. $n^2 + 3n + 2 = (n + 1)(n + 2)$. A product of two consecutive integers is an even number.

Part 2. Using mathematical induction, prove that for all integers $k \geq 0$, $9^k - 2^k$ is divisible by 7.

On the slides.

NAME: SID:

Problem 4: (a) Use the Θ-notation to determine the rate of growth of the following functions:

Function	Θ estimate
$n^3 14^{-n} + n3^n + n^2 7^n + 3n^5 \log n^8$	$\boldsymbol{\Theta(n^2 7^n)}$
$10/(n^2\sqrt{n}) + 3n^2 \log n + 4n \log^5 n$	$\boldsymbol{\Theta(n^2 \log n)}$

(b) For each piece of pseudo-code below, give its asymptotic running time as a function of n. Express this running time using the Θ() notation. Include a brief informal explanation (at most 20 words).

Pseudo-code	Running time	Explanation

for $i \leftarrow 1$ **to** $3n$ **do**
    for $j \leftarrow 1$ **to** $5n$ **do**
        $x \leftarrow x^4$
for $k \leftarrow 1$ **to** $2n$ **do**
    $z \leftarrow x + z$
	$\Theta(n^2)$	The first nested loop makes $3n \cdot 5n = \Theta(n^2)$ iterations. The second loop makes $\Theta(n)$ iterations.

for $i \leftarrow 1$ **to** n **do**
    $j \leftarrow 1$
    while $j < n$ **do**
        $x \leftarrow 7 \cdot x$
        $j \leftarrow 3 \cdot j$
	$\Theta(n \log n)$	The external loop makes n iterations. In the internal loop j triples each time and stops at n, so it makes $\Theta(\log n)$ iterations.

(c) Let $g(x)$ be a polynomial of degree k. Prove that $f(x) = O(x^k)$.

On the slides.

NAME: SID:

Problem 5: (a) Compute $8^9 \pmod 5$ by squaring. Show your work.

$8^9 \pmod 5 = ... = 3$

(b) Compute $6^{-1} \pmod{13}$ using the method of linear combinations (listing multiples). Show your work.

$6^{-1} \pmod{13} = ... = 11$

(c) Compute $5^{-1} \pmod{11}$ using Fermat's Little Theorem. Show your work.

$5^{-1} \pmod{11} = ... = 9$

(d) Solve:

$$
\begin{aligned}
5x &\equiv 7 \pmod{11}. \\
5^{-1} \cdot 5x &\equiv 7 \cdot 5^{-1} \pmod{11} \\
x &\equiv 7 \cdot 9 \pmod{11} \\
x &\equiv 63 \pmod{11} \\
x &= 8
\end{aligned}
$$

NAME: SID:

Problem 6: Part 1. Prove or disprove the following statements. (For each statement tell whether it is true or false and justify your answer.)

(a) For all nonnegative integers n, $n^2 + 5n + 6$ is an even number.

True. $n^2 + 5n + 6 = (n + 2)(n + 3)$. A product of two consecutive integers is an even number.

(b) For all positive integers n, $n^2 + 5n + 6$ is not a prime number.

True. $n^2 + 5n + 6 = (n + 2)(n + 3)$. It is a product of two integers, and each of the factors is greater than 1.

(c) For all nonnegative integers n, $n^2 + 5n + 6$ is not a prime number.

True. See (b).

Part 2. Using mathematical induction, prove that for all integers $k \geq 0$, $7^k - 2^k$ is divisible by 5.

On the slides.