CS/MATH111 ASSIGNMENT 4

Problem 1: Give the asymptotic value (using the Θ-notation) for the number of letters that will be printed by the algorithms below. In each algorithm the argument n is a positive integer. Your solution needs to consist of an appropriate recurrence equation and its solution. You also need to give a brief justification for the recurrence.

Part (a)

(i) **Algorithm** PrintAs (n : **integer**)
    if $n < 4$
        print("A")
    else
        PrintAs($\lceil n/3 \rceil$)
        PrintAs($\lceil n/3 \rceil$)
        PrintAs($\lceil n/3 \rceil$)
        PrintAs($\lceil n/3 \rceil$)
        PrintAs($\lceil n/3 \rceil$)
        for $i \leftarrow 1$ **to** $4n^2$ **do** print("A")

(ii) **Algorithm** PrintBs (n : **integer**)
    if $n < 2$
        print("B")
    else
        for $j \leftarrow 1$ **to** 8 **do** PrintBs($\lfloor n/2 \rfloor$)
        for $i \leftarrow 1$ **to** $10n^3$ **do** print("B")

(iii) **Algorithm** PrintCs (n : **integer**)
    if $n < 3$
        print("C")
    else
        PrintCs($\lceil n/2 \rceil$)
        PrintCs($\lceil n/2 \rceil$)
        PrintCs($\lceil n/2 \rceil$)
        PrintCs($\lceil n/2 \rceil$)
        for $i \leftarrow 1$ **to** 20 **do** print("C")

Part (b)

(iv) **Algorithm** PrintDs (n : **integer**)
    if $n < 2$
        print("D")
    else
        for $j \leftarrow 1$ **to** 4 **do** PrintDs($\lfloor n/2 \rfloor$)
        for $i \leftarrow 1$ **to** $10n^k$ **do** print("D"),

where k is a nonnegative integer.

Solution 1: (a) For all these recurrences, we ignore the floor/ceiling because we are interested in the asymptotic solution.

	Recurrence	Solution
(i)	$A(n) = 5 \cdot A(n/3) + 4n^2$	$a = 5, b = 3, d = 2$ Since $a < b^d$, using Master Theorem, the solution is $A(n) = \Theta(n^2)$.
(ii)	$B(n) = 8 \cdot B(n/2) + 10n^3$	$a = 8, b = 2, d = 3$ Since $a = b^d$, using Master Theorem, the solution is $B(n) = \Theta(n^3 \log n)$.
(iii)	$C(n) = 4 \cdot C(n/2) + 20$	$a = 4, b = 2, d = 0$ Since $a > b^d$, using Master Theorem, the solution is $C(n) = \Theta(n^{\log_2 4}) = \Theta(n^2)$.
(iv)	$D(n) = 4 \cdot D(n/2) + 10n^k$	$a = 4, b = 2, d = k$ Using Master Theorem, the solution is: $D(n) = \begin{cases} \Theta(n^{\log_2 4}) = \Theta(n^2) \text{ if } k < 2, \\ \Theta(n^2 \log n) \text{ if } k = 2, \\ \Theta(n^k) \text{ if } k > 2. \end{cases}$

Problem 2: Bill is buying his wife a bouquet of carnations, roses, tulips, daisies, and lilies. The bouquet will have 28 flowers, with

- at most 6 carnations,
- between 3 and 7 roses,
- between 2 and 11 tulips,
- at least 4 daisies, and
- at least 1 lily.

How many different combinations of flowers satisfy these requirements? You need to use the counting method for integer partitions and show your work.

Solution 2: The number of bouquets is equal to the number of solutions of the following equation:

$$\begin{aligned} c + r + t + d + l &= 28 \\ 0 \leq c &\leq 6 \\ 3 \leq r &\leq 7 \\ 2 \leq t &\leq 11 \\ 4 &\leq d \\ 1 &\leq l \end{aligned}$$

After substitutions (removing lower bounds), this is equivalent to counting the number of solutions of

$$\begin{aligned} c' + r' + t' + d' + l' &= 18 \\ c' &\le 6 \\ r' &\le 4 \\ t' &\le 9 \end{aligned}$$

(All values are required to be nonnegative integers. The values of d' and l' have no upper bound constraints. So $d' \le 18$ and $l' \le 18$.) This number of solutions, using the notation from class, is given by

$$S(c' \le 6 \wedge r' \le 4 \wedge t' \le 9 \wedge d' \le 18 \wedge l' \le 18) = S_t - S(c' \ge 7 \vee r' \ge 5 \vee t' \ge 10 \vee d' \ge 19 \vee l' \ge 19) = S_t - S(c' \ge 7 \vee r' \ge 5 \vee t' \ge 10).$$

We have $S_t = \binom{22}{4} = 7315$. To compute $S(c' \ge 7 \vee r' \ge 5 \vee t' \ge 10)$, we apply the inclusion-exclusion principle:

$$\begin{aligned} S(c' \ge 7 \vee r' \ge 5 \vee t' \ge 10) &= S(c' \ge 7) + S(r' \ge 5) + S(t' \ge 10) \\ &\quad - S(c' \ge 7 \wedge r' \ge 5) - S(c' \ge 7 \wedge t' \ge 10) - S(r' \ge 5 \wedge t' \ge 10) \\ &\quad + S(c' \ge 7 \wedge r' \ge 5 \wedge t' \ge 10) \\ &= \binom{15}{4} + \binom{17}{4} + \binom{12}{4} - \binom{10}{4} - \binom{5}{4} - \binom{7}{4} + 0 \\ &= 1365 + 2380 + 495 - 210 - 5 - 35 = 3990. \end{aligned}$$

So

$$S_t - S(d' \ge 7 \vee c' \ge 10 \vee t' \ge 7) = 7315 - 3990 = 3325.$$

Problem 3: We have three sets P, Q, R with the following properties:

(a) $|P| = 3|Q|$ and $|R| = |P|$,

(b) $|P \cap Q| = 11$, $|Q \cap R| = 14$, $|P \cap R| = 2|Q|$,

(c) $7 \le |P \cap Q \cap R| \le 17$,

(d) $|P \cup Q \cup R| = 100$.

Use the inclusion-exclusion principle to determine the number of elements in P. Show your work. (Hint: You may get an equation with two unknowns, but one of them has only a few possible values.)

Solution 3: From the inclusion-exclusion formula, we know that

$$|P \cup Q \cup R| = |P| + |Q| + |R| - |P \cap Q| - |P \cap R| - |Q \cap R| + |P \cap Q \cap R|.$$

$$100 = a + 3a + 3a - 11 - 14 - 2a + b$$

which simplifies to

$$5a + b = 125.$$

This would have infinitely many solutions, but we know that both a and b are non-negative integers and that $7 \le b \le 17$. So there are only 11 possibilities. Rather than trying them all, note that we need $b \equiv 0$

(mod 5), because both, $5a$ and 125, are multiples of 5. The only two numbers in the range $7 \leq d \leq 17$ that satisfy this condition are $b = 10$ and $b = 15$.

We still need to verify whether these values of b actually make sense. For $b = 15$, we obtain that $|P \cap Q \cap R| > |P \cap Q|$, which is impossible.

For $b = 10$, we get $a = 23$, so $|Q| = 23$ and $|P| = 69$. This solution is feasible, that is, there are sets with these cardinalities that satisfy all constraints. Thus the final answer is $|P| = 69$.

Submission. To submit the homework, you need to upload the pdf file into ilearn and Gradescope.