CS111 ASSIGNMENT 3

Problem 1: We want to tile an $n \times 1$ strip with tiles of two types: 1×1 tiles that are dark-blue, light-blue, and red, and 2×1 green tiles. Give a formula for the number of such tilings T_n, considering that blue tiles cannot be next to each other. Your solution must include a recurrence equation (with initial conditions!), and a full justification. You do not need to solve it.

Solution 1: The number of the tilings T_n satisfy the recurrence:

$$\begin{aligned} T_n &= T_{n-1} + 3T_{n-2} + 2T_{n-3} \\ T_0 &= 1 \\ T_1 &= 3 \\ T_2 &= 6 \end{aligned}$$

(Explanation/justification - in class)

Problem 2: Solve the following recurrence equation:

$$\begin{aligned} f_n &= f_{n-1} + 4f_{n-2} + 2f_{n-3} \\ f_0 &= 0 \\ f_1 &= 1 \\ f_2 &= 2 \end{aligned}$$

Show your work (all steps: the characteristic polynomial and its roots, the general solution, using the initial conditions to compute the final solution.)

Solution 2: The characteristic equation is $x^3 - x^2 - 4x - 2 = 0$. The roots are $r_1 = -1$, $r_2 = 1 + \sqrt{3}$ and $r_3 = 1 - \sqrt{3}$. So the general solution is

$$S_n = \alpha_1(-1)^n + \alpha_2(1+\sqrt{3})^n + \alpha_3(1-\sqrt{3})^n.$$

We now plug it into the initial conditions:

$$\begin{aligned} \alpha_1(-1)^0 + \alpha_2(1+\sqrt{3})^0 + \alpha_3(1-\sqrt{3})^0 = 0 \\ \alpha_1(-1)^1 + \alpha_2(1+\sqrt{3})^1 + \alpha_3(1-\sqrt{3})^1 = 1 \\ \alpha_1(-1)^2 + \alpha_2(1+\sqrt{3})^2 + \alpha_3(1-\sqrt{3})^2 = 2 \end{aligned}$$

which simplifies to

$$\begin{aligned} \alpha_1 + \alpha_2 + \alpha_3 = 0 \\ -\alpha_1 + (1+\sqrt{3})\alpha_2 + (1-\sqrt{3})\alpha_3 = 1 \\ \alpha_1 + (4+2\sqrt{3})\alpha_2 + (4-2\sqrt{3})\alpha_3 = 2 \end{aligned}$$

Solving this, we obtain $\alpha_1 = 0$, $\alpha_2 = \frac{\sqrt{3}}{6}$ and $\alpha_3 = -\frac{\sqrt{3}}{6}$. So the final solution is

$$S_n = \tfrac{\sqrt{3}}{6}(1+\sqrt{3})^n - \tfrac{\sqrt{3}}{6}(1-\sqrt{3})^n.$$

Problem 3: Solve the following recurrence equation:

$$\begin{aligned} f_n &= 13f_{n-2} + 12f_{n-3} + 2n + 1 \\ f_0 &= 0 \\ f_1 &= 1 \\ f_2 &= 1 \end{aligned}$$

Show your work (all steps: the associated homogeneous equation, the characteristic polynomial and its roots, the general solution of the homogeneous equation, computing a particular solution, the general solution of the non-homogeneous equation, using the initial conditions to compute the final solution.)

Solution 3: We start with the associated homogeneous equation, $f_n = 13f_{n-2} + 12f_{n-3}$. The characteristic equation is $x^3 - 13x - 12 = 0$. It's a cubic equation. The candidates for integer roots are -12, -6, -4, -3, -2, -1, 1, 2, 3, 4, 6, 12. Plugging these in, we obtain that -3, -1 and 4 satisfy this equation, so we have all three roots: $r_1 = -3$, $r_2 = -1$ and $r_3 = 4$. Thus the general solution of the associated homogeneous equation is

$$f'_n = \alpha_1(-3)^n + \alpha_2(-1)^n + \alpha_3 4^n.$$

Next, we want to find a particular solution of the non-homogeneous equation. Since the non-homogeneous term is $q(n) = 2n + 1$, a linear function, we try a linear function $f''_n = \beta_1 n + \beta_2$. (Notice, that 1 is not a solution of the characteristic equation.) Plugging this into the equation, we get

$$\beta_1 n + \beta_2 = 13[\beta_1(n-2) + \beta_2] + 12[\beta_1(n-3) + \beta_2] + 2n + 1$$

which reduces to

$$(-24\beta_1 - 2)n + (62\beta_1 - 24\beta_2 - 1) = 0.$$

Setting both, $-24\beta_1 - 2$ and $62\beta_1 - 24\beta_2 - 1$ to 0, and solving these equations simultaneously, yields $\beta_1 = -\frac{1}{12}$ and $\beta_2 = -\frac{37}{144}$. So our particular solution is $f''_n = -\frac{1}{12}n - \frac{37}{144}$.

This gives us the general solution of the original non-homogeneous equation:

$$f_n = \alpha_1(-3)^n + \alpha_2(-1)^n + \alpha_3 4^n - \frac{1}{12}n - \frac{37}{144}.$$

Using the initial conditions, we obtain three equations:

$$\begin{aligned} \alpha_1 + \alpha_2 + \alpha_3 &= \frac{37}{144} \\ -3\alpha_1 - \alpha_2 + 4\alpha_3 &= \frac{193}{144} \\ 9\alpha_1 + \alpha_2 + 16\alpha_3 &= \frac{205}{144} \end{aligned}$$

Solving these system of equations, we get $\alpha_1 = -\frac{29}{112}$, $\alpha_2 = \frac{3}{10}$, and $\alpha_3 = \frac{68}{315}$. So the final solution is

$$f_n = -\frac{29}{112}(-3)^n + \frac{3}{10}(-1)^n + \frac{68}{315}4^n - \frac{1}{12}n - \frac{37}{144}.$$

Problem 4: Find a particular solution of the recurrence equation:

$$t_n = 4t_{n-1} + t_{n-2} + 3 \cdot 2^n$$

Show your work.

Solution 4:

The non-homogeneous term is $q(n) = 3 \cdot 2^n$. The associated homogeneous recurrence is $t_n = 4t_{n-1} + t_{n-2}$, giving us the characteristic equation $x^2 - 4x - 1 = 0$. Since 2 is not a solution of the characteristic equation, we try $f''_n = \beta \cdot 2^n$. Plugging this into the equation, we get

$$\beta \cdot 2^n = 4\beta \cdot 2^{n-1} + \beta \cdot 2^{n-2} + 3 \cdot 2^n,$$

which can be divided by 2^{n-2} and simplified as follows:

$$\begin{aligned} 4\beta &= 8\beta + \beta + 12, \\ \beta &= -\frac{12}{5}. \end{aligned}$$

Finally

$$f''_n = -\frac{12}{5} \cdot 2^n.$$
