CS/MATH111 ASSIGNMENT 1

due Tuesday, January 22, 11:50PM

Solution 1: (a) The internal loop prints $(j+2)^2$ words for each $i = 1, 2, ..., 2n+3$, so the total number of words printed will be

$$Z(n) = \sum_{i=1}^{2n+3} (j+2)^2.$$

(b) The formula for the sum of squares of k first integers is $\sum_{i=1}^{k} i^2 = \frac{1}{6}k(k+1)(2k+1)$, and the formula for the sum of k first integers is $\sum_{i=1}^{k} i = \frac{1}{2}k(k+1)$ So the above summation can be simplified as follows:

$$\begin{aligned} Z(n) = \sum_{j=1}^{2n+3} (j+2)^2 &= \sum_{j=2}^{2n+3} (j^2 + 4j + 4) \\ &= \sum_{j=1}^{2n+3} j^2 + \sum_{j=1}^{2n+3} 4j + \sum_{j=1}^{2n+3} 4 \\ &= \tfrac{1}{6}(2n+3)(2n+4)[2(2n+3)+1] + \tfrac{1}{2}(2n+3)(2n+4) + 4(2n+3) \\ &= \tfrac{1}{3}(2n^2+7n+6)(4n+7) + 4(2n^2+7n+6) + 8n + 12 \\ &= \tfrac{1}{3}[(2n^2+7n+6)(4n+19) + 3(8n+12)] \\ &= \tfrac{1}{3}(8n^3 + 66n^2 + 181n + 150). \end{aligned}$$

(c) $Z(n) = \Omega(n^3)$, since $\frac{1}{3}(8n^3 + 66n^2 + 181n + 150) \geq n^3 = \Omega(n^3)$.

$Z(n) = O(n^3)$, since $\frac{1}{3}(8n^3 + 66n^2 + 181n + 150) \leq \frac{181}{3}(n^3 + n^2 + n + 1) \leq \frac{181}{3}(n^3 + n^3 + n^3 + n^3) = \frac{724}{3}(n^3) = O(n^3)$. We conclude that $Z(n) = \Theta(n^3)$.

Solution 2: Following the hint, we first show that $B_n \geq \frac{1}{2} \cdot 2.3^n$ for all $n \geq 0$. The proof is by induction on n.

Base case. If $n = 0$, we have $B_0 = 1 \geq \frac{1}{2} \cdot 2.3^0$, so the inequality holds. If $n = 1$, $B_1 = 2 \geq \frac{1}{2} \cdot 2.3^1$, so the inequality holds as well.

In the inductive step, assume that we have some $k \geq 2$ and that the inequality holds for any $n < k$. We want to prove that it also holds for k, that is $B_k \geq \frac{1}{2} \cdot 2.3^k$.

Using the formula for B_k, the inductive assumption for $n = k-1$ and $n = k-2$, and then some algebra, we have

$$\begin{aligned} B_k &= B_{k-1} + 3B_{k-2} \\ &\geq \tfrac{1}{2} \cdot 2.3^{k-1} + 3 \cdot \tfrac{1}{2} \cdot 2.3^{k-2} \\ &= \tfrac{1}{2} \cdot 2.3^{k-2}(2.3+3) \\ &= \tfrac{1}{2} \cdot 2.3^{k-2} \cdot 5.3 \\ &\geq \tfrac{1}{2} \cdot 2.3^{k-2} \cdot 5.29 \\ &= \tfrac{1}{2} \cdot 2.3^{k-2} \cdot 2.3^2 \\ &= \tfrac{1}{2} \cdot 2.3^k. \end{aligned}$$

This sequence of inequalities and equations implies that $B_k \geq \frac{1}{2} \cdot 2.3^k$, which is exactly what we needed to prove to complete the inductive step. We thus have proved that $B_n \geq \frac{1}{2} \cdot 2.3^n$ for all $n \geq 0$. This implies that $B_n = \Omega(2.3^n)$.

Next, we show the other inequality, $B_n \leq 2.4^n$ for $n \geq 0$. Again, we proceed by induction on n.

Base case. When $n = 0$, $B_0 = 1 \leq 2.4^0$, so the inequality holds. For $n = 1$, we have $B_1 = 2 \leq 2.4^1$, and the inequality holds as well.

In the inductive step, assume that we have some $k \geq 2$ and that the inequality holds for any $n < k$. We want to prove that it also holds for k, that is $B_k \leq 2.4^k$.

Using the formula for B_k, the inductive assumption for $n = k-1$ and $n = k-2$, and then some algebra, we have

$$\begin{aligned}
B_k &= B_{k-1} + 3B_{k-2} \\
&\leq 2.4^{k-1} + 3 \cdot 2.4^{k-2} \\
&= 2.4^{k-2}(2.4 + 3) \\
&= 2.4^{k-2} \cdot 5.4 \\
&\leq 2.4^{k-2} \cdot 5.76 \\
&= 2.4^{k-2} \cdot 2.4^2 \\
&= \tfrac{1}{2} \cdot 2.4^k.
\end{aligned}$$

This sequence of inequalities and equations implies that $B_k \leq 2.4^k$, which is exactly what we needed to prove the inductive claim.

So we have proved that $B_n \leq 2.4^n$ for $n \geq 0$. This implies that $B_n = O(2.4^n)$.

Solution 3:

(a) $\frac{1}{2}n^5 + (n^3 - n^2)^2 + 13n = \Theta(n^6)$.

First, $\frac{1}{2}n^5 + (n^3 - n^2)^2 + 13n = \frac{1}{2}n^5 + n^6 - 2n^5 + n^4 + 13n = n^6 - 1\frac{1}{2}n^5 + n^4 + 13n$.

For $n \geq 3$, $\frac{1}{2}n^6 \geq 1\frac{1}{2}n^5$, and so $n^6 - 1\frac{1}{2}n^5 + n^4 + 13n \geq \frac{1}{2}n^6$. This shows that $\frac{1}{2}n^5 + (n^3 - n^2)^2 + 13n = \Omega(n^6)$.

Also, $n^6 - 1\frac{1}{2}n^5 + n^4 + 13n \leq n^6 + n^4 + 13n \leq 13(n^6 + n^4 + n) \leq 13(n^6 + n^6 + n^6) = 39n^6 = O(n^6)$.

Since $\frac{1}{2}n^5 + (n^3 - n^2)^2 + 13n = \Omega(n^6)$ and $\frac{1}{2}n^5 + (n^3 - n^2)^2 + 13n = O(n^6)$, $\frac{1}{2}n^5 + (n^3 - n^2)^2 + 13n = \Theta(n^6)$.

(b) $3 + \frac{2}{n^{-2}} + \frac{1}{n^3 \log^2 n} = \Theta(n^2)$.
$3 + \frac{2}{n^{-2}} + \frac{1}{n^3 \log^2 n} = 2n^2 + 3 + \frac{1}{n^3 \log^2 n}$;

$2n^2 + 3 + \frac{1}{n^3 \log^2 n} \geq 2n^2 = \Omega(n^2)$, so $3 + \frac{2}{n^{-2}} + \frac{1}{n^3 \log^2 n} = \Omega(n^2)$.

Next, the term $\frac{1}{n^3 \log^2 n}$ converges to 0 when n grows to ∞, so $\frac{1}{n^3 \log^2 n} = O(1)$. In addition, any constant function is dominated by any polynomial function. Then $2n^2 + 3 + \frac{1}{n^3 \log^2 n} = O(n^2)$.
We conclude that $\frac{1}{n^3 \log^2 n} = \Theta(n^2)$.

(c) $n(n^2 \log^3 n + 9n^2 \log^5 n) + 15n^4 = \Theta(n^4)$.

$n(n^2 \log^3 n + 9n^2 \log^5 n) + 15n^4 \geq= 15n^4 = \Omega(n^4)$;

$n(n^2 \log^3 n + 9n^2 \log^5 n) + 15n^4 = n^3 \log^3 n + 9n^3 \log^5 n + 15n^4 = n^3(\log^3 n + 9 \log^5 n + 15n)$. Considering that any logarithmic function is dominated by any polynomial function, we have: $n^3(\log^3 n + 9 \log^5 n + 15n) = n^3(O(n) + O(n) + O(n)) = O(n^4)$.

Putting it all together, the expression is $\Theta(n^4)$.

(d) $13n^4 + n2^n + n^3 \log n = \Theta(n2^n)$.

$13n^4 + n2^n + n^3 \log n \geq n2^n = \Omega(n2^n)$.

Any logarithmic function is dominated by any polynomial function, which, in turn, is dominated by any exponential function. So, $13n^4 + n2^n + n^3 \log n = n2^n + n^3(13n + \log n) = n2^n + n^3(O(n) + O(n)) = n2^n + O(n^4) = O(n2^n) + O(2^n) = O(n2^n)$.

Finally, $13n^4 + n2^n + n^3 \log n = \Theta(n2^n)$.

(e) $n3^n + n^3 2^n = \Theta(n3^n)$.

$n3^n + n^3 2^n \geq n3^n = \Omega(n3^n)$.

We know, that $n^2 = O(1.5^n)$. Then $n3^n + n^3 2^n = n(3^n + n^2 2^n) = n(O(3^n) + O(1.5^n)O(2^n)) = O(n3^n)$.

Therefore $n3^n + n^3 2^n = \Theta(n3^n)$.

Submission. To submit the homework, you need to upload the pdf file into ilearn and Gradescope by 11:50PM on Tuesday, January 22.

Reminders. Remember that only LaTeX papers are accepted.