CS/MATH111 ASSIGNMENT 5

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Problem 1: In class you proved the following corollary from Euler's formula: If G is a connected planar graph with $n \geq 3$ and no cycles of length 3, then $m \leq 2n - 4$. Can you generalize this corollary? In other words, find and prove the inequality if G has no cycles of length l.

Solution 1:

Proof: According to Euler's Formula, we know that the sum of the degrees of the faces is equal to twice the number of edges. Each face must have a degree $\geq (l+1)$. We want to prove that the inequality has no cycles of length l.

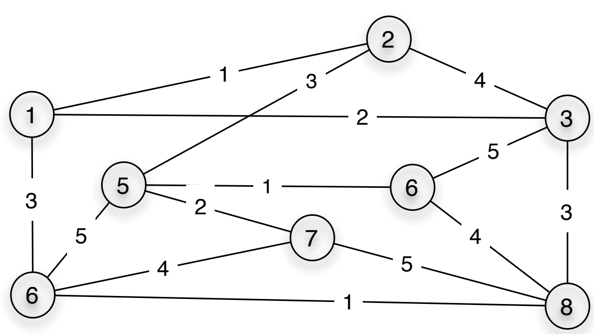
$2m \geq (l+1)f$ in which m is the number of edges and f is the number of faces. According to Euler's Formula, $f = m - n + 2$, thus we can assume $(l+1)f = (n+1)(m-n+2)$.

When we combine this with $2m \geq (l+1)f$, we get

$$\begin{gathered}
2m \geq (l+1)(m-n+2) \\
2m \geq lm - ln + 2l - n + 2 \\
m - lm \geq -ln + 2l - n + 2 \\
-m(l-1) \geq -n(l+1) + 2(l+1) \\
-m(l-1) \geq (2-n)(l+1) \\
m(l-1) \leq -(2-n)(l+1) \\
m \leq \frac{(n-2)(l+1)}{(l-1)}
\end{gathered}$$

Problem 2: An *edge coloring* of a graph is an assignment of colors to edges such that any two edges that share an endpoint have different colors. (It can be proved that if the maximum vertex degree $D \geq 1$, then G can be edge-colored with at most $2D - 1$ colors.)

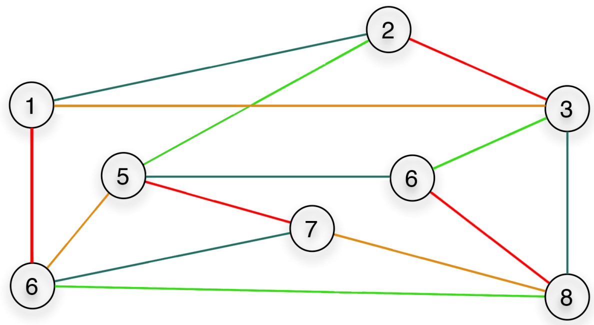
Here is an example of an edge coloring of a graph with 5 colors (colors represented by numbers):



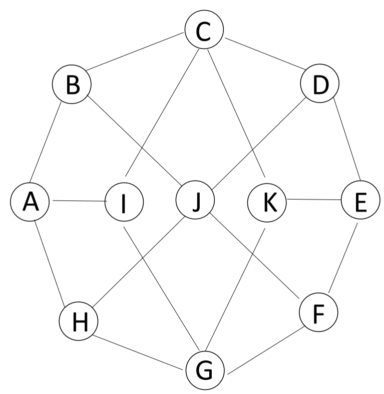
For the graph above, find an edge coloring with at most 4 colors.

Solution 2:

In the graph below, I used four colors: dark green, bright green, orange, and red.

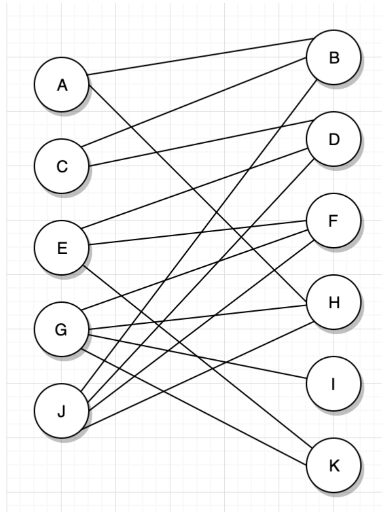


Problem 3: Let G be the graph below.



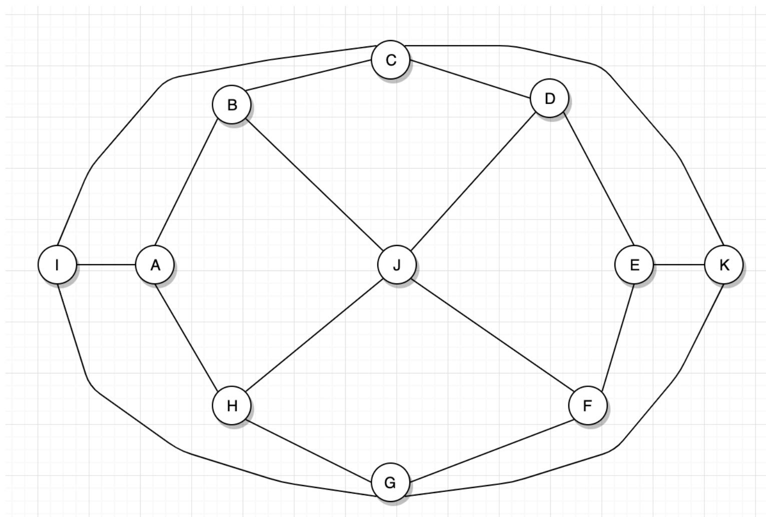
(a) Determine whether it is bipartite. If the graph is bipartite, determine whether it has a perfect matching. Justify your answer.
(b) What is the chromatic number of G? Explain.
(c) Does G have a Hamiltonian Path? Justify.
(d) Is G a planar graph? (You need to either show a planar embedding or prove, that G is nonplanar.)

Solution 3:
(a) G is a bipartite graph because it has no odd cycles and is 2 colorable. However, it does not have perfect matching because after partitioning it, the right side of the graph has more vertices than the left. Therefore, there cannot be perfect matching. G is partitioned in the graph below:

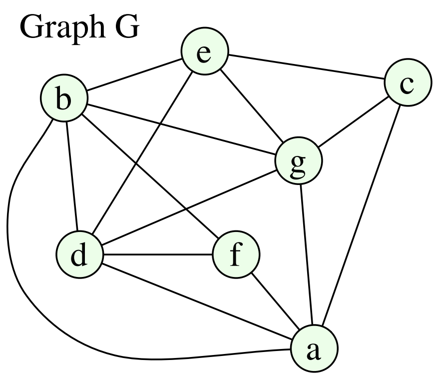


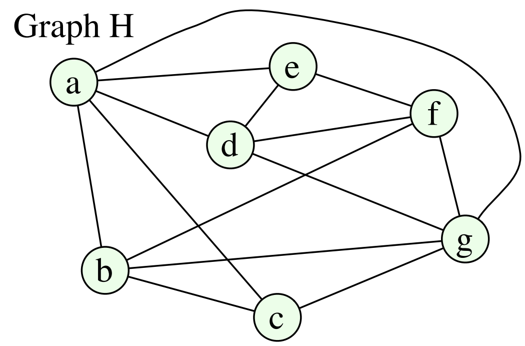
(b) The chromatic number of G is 5. The definition of chromatic number is the least amount of colors needed for a coloring of a graph and is obtained by adding 1 to the maximum vertex degree. 4 is the maximum vertex degree of this graph, and adding one to 4, is 5.

(c) Yes. The Hamiltonian Path of G is $KGFEDJHAICB$.
(d) G is a planar graph, as shown in the embedding below:



Problem 4: Determine which of the following two graphs is/are planar/nonplanar. Justify your answer. (You need to either show a planar embedding or use Kuratowski's theorem.)

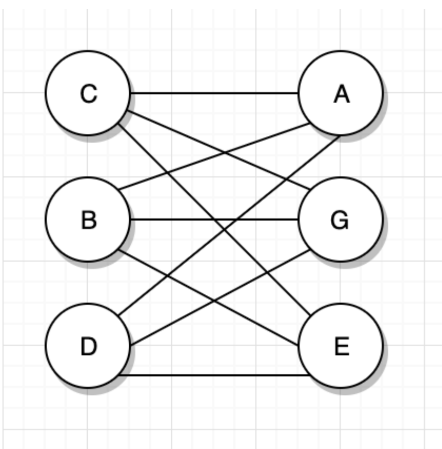


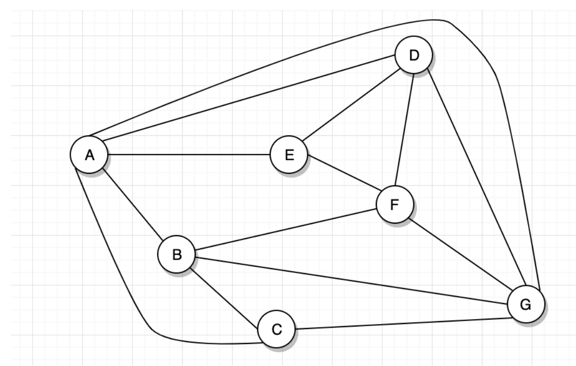


Solution 4:

Graph G has a subgraph that is isomorphic to the $K_{3,3}$ graph, so by Kuratowski's Theorem, it is not planar.

Kuratowski's Theorem does not apply to Graph H, so it is planar.





Submission. To submit the homework, you need to upload the pdf file into ilearn and Gradescope. Pictures should be imported into LaTeX in pdf (see the source file to get an idea of how to do that). You can draw them in any drawing software and export in pdf, or draw by hand and scan.