NAME: SID:

Problem 1: (a) Solve the following system of equations:

$$\begin{cases} x - 3z &= 1 \\ x + y + z &= 10 \\ 2y + z &= 4 \end{cases}$$

Show your work.

Solution. From the first equation, $x = 3z+1$. Plug into the 2nd equation, getting $y+4z = 9$, so $y = 9-4z$, which you plug into the 3rd equation. So the solution is: x = 7, y = 1, and z = 2.

(b) Compute all roots of the equation $2x^3 + 3x^2 - 8x + 3 = 0$. Show your work.

Solution. We try to guess integer roots. Since the constant coefficient is 3, the only candidates for integer roots are $1, -1, 3, -3$. Plugging these in, we find that one of the roots is 1. Then polynomial $2x^3+3x^2-8x+3$ can be factored as $(x-1)(2x^2+5x-3)$, which can be factored further as $(x-1)(x+3)(2x-1)$. Finally, the solutions are: $x_1 = 1$, $x_2 = -3$, $x_3 = \frac{1}{2}$.

Problem 2: Prove by induction that

$$\sum_{j=1}^{n}(-1)^j j^2 = \tfrac{1}{2}(-1)^n n(n+1)$$

for all $n \geq 0$.

Solution. We proceed by induction.
Base case. For $n = 1$ the left-hand side is -1 and the right-hand size is also -1, which proves the base case.
Inductive assumption. Fix some $m \geq 1$ and suppose that the formula holds for $n = m$, that is

$$\textstyle\sum_{j=1}^{m}(-1)^j j^2 = \frac{1}{2}(-1)^m m(m+1).$$

We need to show that it holds for $n = m+1$, that is

$$\textstyle\sum_{j=1}^{m+1}(-1)^j j^2 = \frac{1}{2}(-1)^{m+1}(m+1)(m+2).$$

We start with the left-hand size of this equation and step-by-step derive the right-hand side, by first dividing the sum into the sum of all terms but last, and the last term. Then we apply the inductive assumption, and finish it off with some algebra:

$$\begin{aligned}\textstyle\sum_{j=1}^{m+1}(-1)^j j^2 &= \textstyle\sum_{j=1}^{m}(-1)^j j^2 + (-1)^{m+1}(m+1)^2 \\ &= \tfrac{1}{2}(-1)^m m(m+1) + (-1)^{m+1}(m+1)^2 \\ &= (-1)^m (m+1)\big[\,\tfrac{1}{2}m + (-1)(m+1)\,\big] \\ &= (-1)^m (m+1)\big[-\tfrac{1}{2}m - 1\big] \\ &= \tfrac{1}{2}(-1)^{m+1}(m+1)(m+2).\end{aligned}$$

Problem 3: Let $X = \{2, 3, 5, 6, 7, 11, 19\}$ and $Y = \{7, 8, 9\}$.

(a) The power set of Y is $\mathcal{P}(Y) = \{\emptyset, \{7\}, \{8\}, \{9\}, \{7,8\}, \{7,9\}, \{8,9\}, \{7,8,9\}\}$

(b) The union of X and Y is $X \cup Y = \{2, 3, 5, 6, 7, 8, 9, 11, 19\}$

(c) The number of five-element subsets of X is $\binom{7}{5} = \frac{7!}{5!2!} = \frac{6 \cdot 7}{2} = 21$.

(d) The number of ways to order all elements of X is $7!$

Note: In part (c) you need to give both a correct expression and the numerical value. In part (d) it is sufficient to give only the correct expression; you do not have to calculate the numerical value.

Problem 4: Determine the numerical values of the expressions below:

(a) $1 + 2 + 3 + ... + 60 = \frac{60 \cdot 61}{2} = 1830$

(b) $\sum_{i=0}^{\infty} (1/3)^i = \frac{1}{1 - \frac{1}{3}} = \frac{3}{2}$

(c) $\log_{2^4} 16^3 - \log_5 25 = \frac{12}{4} - 2 = 1$

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Problem 5: (a) Solve the following system of equations:

$$\begin{cases} x - 3z &= 2 \\ x - y + z &= 7 \\ y + 6z &= 5 \end{cases}$$

Show your work.

Solution. From the first equation, $x = 3z + 2$. Plug into the 2nd equation, getting $-y + 4z = 5$, so $y = 4z - 5$, which you plug into the 3rd equation. This will give $z = 1$. So the solution is: x = 5, y = -1, and z = 1.

(b) Compute all roots of the equation $2x^3 + 5x^2 - 4x - 3 = 0$. Show your work.

Solution. We try to guess integer roots. Since the constant coefficient is -3, the only candidates for integer roots are $1, -1, 3, -3$. Plugging these in, we find that one of the roots is 1. Then polynomial $2x^3+5x^2-4x-3$ can be factored as $(x - 1)(2x^2 + 7x + 3)$, which can be factored further as $(x - 1)(x + 3)(2x + 1)$. Finally, the solutions are: $x_1 = 1$, $x_2 = -3$, $x_3 = -\frac{1}{2}$.

Problem 6: Prove by induction that

$$\sum_{i=1}^{k}(-1)^i i^2 = \tfrac{1}{2}(-1)^k k(k+1)$$

for all $k \geq 1$.

Solution. We proceed by induction.
Base case. For $k = 1$ the left-hand side is -1 and the right-hand size is also -1, which proves the base case.
Inductive assumption. Fix some $m \geq 1$ and suppose that the formula holds for $k = m$, that is

$$\textstyle\sum_{i=1}^{m}(-1)^i i^2 = \tfrac{1}{2}(-1)^m m(m+1).$$

We need to show that it holds for $k = m+1$, that is

$$\textstyle\sum_{i=1}^{m+1}(-1)^i i^2 = \tfrac{1}{2}(-1)^{m+1}(m+1)(m+2).$$

We start with the left-hand size of this equation and step-by-step derive the right-hand side, by first dividing the sum into the sum of all terms but last, and the last term. Then we apply the inductive assumption, and finish it off with some algebra:

$$\begin{aligned}
\textstyle\sum_{i=1}^{m+1}(-1)^i i^2 &= \textstyle\sum_{i=1}^{m}(-1)^i i^2 + (-1)^{m+1}(m+1)^2 \\
&= \tfrac{1}{2}(-1)^m m(m+1) + (-1)^{m+1}(m+1)^2 \\
&= (-1)^m (m+1)\left[\, \tfrac{1}{2}m + (-1)(m+1)\, \right] \\
&= (-1)^m (m+1)\left[\, -\tfrac{1}{2}m - 1\right] \\
&= \tfrac{1}{2}(-1)^{m+1}(m+1)(m+2).
\end{aligned}$$

Problem 7: Let $U = \{A, C, D, E, F, K, L\}$ and $V = \{F, G, H\}$.

(a) The power set of V is $\mathcal{P}(V) == \{\emptyset, \{F\}, \{G\}, \{H\}, \{F, G\}, \{F, H\}, \{G, H\}, \{F, G, H\}\}$

(b) The union of U and V is $U \cup V == \{A, C, D, E, F, G, H, K, L\}$

(c) The number of ways to order all elements of U is 7!

(d) The number of five-element subsets of U is $\binom{7}{5} = \frac{7!}{5!2!} = \frac{6 \cdot 7}{2} = 21$.

Note: In part (d) you need to give both a correct expression and the numerical value. In parts (c) it is sufficient to give only the correct expression; you do not have to calculate the numerical value.

Problem 8: Determine the numerical values of the expressions below:

(a) $\sum_{i=0}^{\infty} (1/5)^i = \frac{1}{1 - \frac{1}{5}} = \frac{5}{4}$

(b) $\log_{2^4} 16^3 - \log_3 9 = \frac{12}{4} - 2 = 1$

(c) $1 + 2 + 3 + ... + 50 = \frac{50 \cdot 51}{2} = 1275$