Natural Transformation Notes

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Abstract

These are notes taken in our Functional Programming Study Group

1 Natural Transformations

We work out the example of the determinant interpreted as a natural transformation, which is discussed on page 16 of Mac Lane's "Categories for the Working Mathematician."

Recall that a natural transformation is defined in terms of two categories, (C), (B), along with two funtors, $S, T : (C) \to (B)$. That is, a natural transformation τ, is really

$$\tau = \tau(S, T, C, B) : Ob(C) \to Mor(B).$$

We can interpret the determinant as a natural transformation,

$$\det = \det(GL_n(\cdot), (\cdot)^\times, (CommAlg), (Grps)).$$

Here, $GL_n(\cdot), (\cdot)^\times : (CommAlg) \to (Grps)$ are functors. (Note that n is fixed.).

Then for $\mathbb{R}, \mathbb{C} \in Ob(CommAlg)$, let $f : \mathbb{R} \to \mathbb{C}$ be the standard inclusion map (homomorphism). Then

$$\det_{\mathbb{R}} = GL_n\mathbb{R} \to \mathbb{R}^\times$$
$$\det_{\mathbb{C}} = GL_n\mathbb{C} \to \mathbb{C}^\times$$

is the value of the determinant natural transformation applied to $\mathbb{R}$. To see that this is a natural transformation, we have to see that this map commutes with the standard inclusion homomorphism.