Notes on the BISICLES control problem

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BISICLES requires two fields, the rate factor A and the sliding coefficient C, in addition to the ice thickness H and the bedrock topography r in order to solve the stress balance equation. Neither of these are directly observed, but we do typically have observations of either the ice velocity $\vec{u}_{\mathrm{obs}}$, or speed, and of thinning rate. We adopt a control method similar to those reported elsewhere [1, 2, 3, 4], that is, a gradient based optimization method which makes use of the model adjoint equations. We don't seek A itself, but rather take A to be a function of an imposed temperature field T and seek a multiplier, ϕ of the effective viscosity (which contains a factor $A^{1/n}(T)$). The optmization problem we solve is inevitably ill-posed, so we employ Tikhnonov regularization, introducing a bias toward smoothly varying solutions.

For simplicity, the method will be described for a 1D ice sheet, and the 2D equivalents given in the appendix.

1 Model equations

We have a stress balance equation

$$\frac{\partial}{\partial x}\left(\phi H\bar{\mu}\frac{\partial u}{\partial x}\right) - Cu = \rho g H \frac{\partial s}{\partial x} \tag{1}$$

and a mass transport equation

$$\frac{\partial H}{\partial t} + \frac{\partial}{\partial x}\left(\bar{u}H\right) - M = 0 \tag{2}$$

plus boundary conditions. Here, $H\bar{\mu}$ is the vertically integrated effective viscosity, which is computed from A and u through Glen's flow low. u is the velocity at the base of the ice, and $\bar{u}$ is the verticall averaged velocity. We will assume that $u \approx \bar{u}$ for now. M is the total mass source (ie surface accumulation and ablation plus sub-shelf melting etc).

2 Objective function and gradient without regularization

We first consider the C and ϕ which minimise the mismatch between the model speed and observed speed. In that case, our objective function J would be, absent any kind of regularization,

$$J_1 = \int_{x_0}^{x_1} \frac{1}{2\sigma^2}\left(|u| - |u_{\mathrm{obs}}|\right)^2 \,\mathrm{d}x. \tag{3}$$

σ^2 is the variance in the error of $|u_{\mathrm{obs}}|$, which we assume has spatially uncorrelated Gaussian statistics.

To make use of a gradient-based optimization method, we need to compute the functional (Gateaux) derivatives $\frac{\delta J}{\delta C}$, and $\frac{\delta J}{\delta \phi}$ so to that end we add a term to J,

$$J_2 = \int_{x_0}^{x_1} \lambda \left\{ \frac{\partial}{\partial x}\left(\phi H\bar{\mu}\frac{\partial u}{\partial x}\right) - Cu - \rho g H\frac{\partial s}{\partial x} \right\} \mathrm{d}x. \tag{4}$$

where λ is an undetermined Lagrange multiplier. We then seek solutions to

$$\begin{pmatrix} \frac{\delta J}{\delta \phi} \\ \frac{\delta J}{\delta C} \\ \frac{\delta J}{\delta u} \\ \frac{\delta J}{\delta \lambda} \end{pmatrix} = 0. \tag{5}$$

The final row of 5 is satisfied when provided that u is a solution to the stress balance equation. The left hand sides of the first two rows of (5) can be written:

$$\begin{pmatrix} \frac{\delta J}{\delta \phi} \\ \frac{\delta J}{\delta C} \end{pmatrix} = \begin{pmatrix} -\bar{\mu} H \frac{\partial u}{\partial x} \frac{\partial \lambda}{\partial x} \\ -\lambda u \end{pmatrix} \tag{6}$$

where λ is the solution to the last row. As for that third row, if we neglect the dependence of μ on u, we have the *adjoint equation*

$$\frac{\partial}{\partial x}\left(\phi H \bar{\mu}(u) \frac{\partial \lambda}{\partial x}\right) - C\lambda = \frac{1}{\sigma^2}\left(\frac{|u_{obs}|}{u} - 1\right) u \tag{7}$$

plus its boundary conditions

$$\lambda(x_0) = \lambda(x_1) = 0 \tag{8}$$

Equation (7) is linear in λ and has the happy property of having the same left hand side as the stress-balance equation (1), but with λ swapped for u. In other words, the stress balance equation is self-adjoint if (and only if) we neglect the dependence of μ on u, and we will be able to use the same methods to solve (7) as we use to solve (1)

We are now in a position to use a gradient-based optimization method to mimimise J. BISICLES uses a (nonlinear) conjugate gradient method, but other methods ought to work too. Whichever one we choose, we will need to compute the functional derivatives of J, which we can do for a given C and ϕ as follows:

1. Solve (1) for u given C and ϕ in order to compute $\bar{\mu}$.
2. Solve the adjoint equation (7) for λ given C,ϕ and $\bar{\mu}$.
3. Compute the functional derivatives using (6).

3 Objective function and gradient with regularization

The optimization problem of the previous section is ill-posed in at least two senses. First, we are seeking two scalar fields (C and ϕ) given one scalar field of data ($|u_{obs}|$). Second, even if we were just seeking C, say, we might expect the problem to be ill-posed - imagine for example that we added a single, tiny sticky spot to C, then we would expect that to make little difference to the field u[1]

To ameliorate this ill-posedness, we can add some penalty functions to the objective function. To bias in favour of smooth C and ϕ, we can add

$$J_3 = \alpha_C^2 \left[\int_{x_0}^{x_1} \left(\frac{\partial C}{\partial x}\right)^2 dx + (C(x_0) - C_0) + (C(x_1) - C_0) \right] \tag{9}$$

$$J_4 = \alpha_\phi^2 \left[\int_{x_0}^{x_1} \left(\frac{\partial \phi}{\partial x}\right)^2 dx + (\phi(x_0) - \phi_0) + (\phi(x_1) - \phi_0) \right] \tag{10}$$

[1]SLC: do some meaningful analysis here, it shouldn't be difficult to show that $u(C)$ is compact.

C_0,C_1,ϕ_0 and ϕ_1 are boundary data. BISICLES boundaries tend to be far from the action, at ice divides and in the ocean, so we set $\phi_0 = \phi_1 = 1$ and C_0, C_1 to be some large value (on land), or zero (for floating ice). The coefficients α^2_C and α^2_ϕ could be determined by some systematic means (e.g. cross validation, finding a critical point in the $L-$curve) but they can also be regarded as a characteristic length scales of variation in C and ϕ, and imposed on that basis. The larger they are, the worse the fit between model and data will be.[2]

Adding J_3 and J_4 to the objective function, (6) becomes:

$$\left(\begin{array}{c} \frac{\delta J}{\delta \phi} \\ \frac{\delta J}{\delta C} \end{array} \right) = \left(\begin{array}{l} -\bar{\mu} H \frac{\partial u}{\partial x} \frac{\partial \lambda}{\partial x^2} - \alpha^2_\phi \frac{\partial^2 \phi}{\partial x^2} \\ -\lambda u - \alpha^2_C \frac{\partial^2 C}{\partial^2 x} \end{array} \right). \tag{11}$$

In effect, we are solving a pair of Poisson equations in ϕ and C with Robin boundary conditions.

Appendix

References

[1] I Joughin et al. Basal conditions for Pine Island and Thwaites Glaciers, West Antarctica, determined using satellite and airborne data. *Journal of Glaciology*, 55:245, 2009.

[2] I Joughin, B E Smith, and D M Holland. Sensitivity of 21st century sea level to ocean-induced thinning of Pine Island Glacier, Antarctica. *Geophysical Research Letters*, 37, 2010.

[3] D R MacAyeal. A tutorial on the use of control methods in ice-sheet modeling. *Journal of Glaciology*, 39:91, 1993.

[4] M Morlighem et al. Spatial patterns of basal drag inferred using control methods from a fullStokes and simpler models for Pine Island Glacier, West Antarctica. *Geophysical Resarch Letters*, 37:L14502, 2010.

[2] We probably should support some systematic means of choosing these parameters, but we also make use of the iterative regularization provided by CG, so needs a bit of thought.