MARKOV CHAINS

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1 Exercise 9.7

The eastern bunny plays a game with one of his helpers. There is a bag of 23 blue eggs and 23 red eggs. The rules are that you take two eggs from the bag. If one is red and one is blue then put the red one back and the blue one aside. If the eggs are of the same colour then you put them aside, and put one blue one back in. The game proceeds untill the last one that needs to be put back inside the bag is either blue or red. The eastern bunny bets that it is the red one. What is the chance that the helper will win?

We can model this with a discrete Markov Chain:

The State Variable is: $(x(t)_1, x(t)_2)$ where $x(t)_1$ is the number of blue eggs in the bag at time t and the $x(t)_2$ for the red ones.

The State Space is : $\mathcal{I} = \{(x(t)_1, x(t)_2) | x(t)_1 \in \{0, 1, ..., 34\}$ and $x(t)_2 \in \{1, 3, 5, ..., 23\}\}$. This is because suppose that the helper draw 11 times (red, red), that means every time we would have to add 1 blue egg to the bag, hence the number of blue eggs in the bag can 23 + 11. Then for the red number of eggs in the bag, they only get removed in pairs, as if we both took 2 colours, the red ones are always put back in, hence no red eggs are removed, only if the helper draws two red ones.

For the transition probabilities we have 3 cases:

- Case 1: (BLUE, RED): Red goes back in the bag and the Blue eggs is put aside.
  the state changes to: $(x(t)_1, x(t)_2) \to (x(t)_1 - 1, x(t)_2)$. With probability ? (there are 2 ways to do it, so we would have to multiply the probability by 2, but i dont know what it is)
- Case 2: (BLUE, BLUE): Keep one Blue eggs and put one aside. So it we have a similar transition, but i dont know what the probability is.. ?

- Case 3: (RED, RED): Both aside, but put one blue egg back:

  the state changes to: $(x(t)_1, x(t)_2) \rightarrow (x(t)_1 + 1, x(t)_2 - 2)$. With probability ?

The helper wins when the last egg put back is blue. That can only happen when the last two eggs are either (RED, RED) or (BLUE, BLUE), if he has this that means he would have to put a blue egg back. That means $x(t)_2 = 0$ or $x(t)_2 = 2$. But that is not possible given the state space. Hence the eastern bunny always wins.

2 Exercise 9.8

Here we have to make three distinct Discrete Markov Chains, one without the fertilizer, one with and finally using a fertilizer for when conditions are poor.

Part a

State Variable: Let x(t) be the condition of the soil at time t.

State Space: $\mathcal{I} = \{x(t) \in \{\text{Good, Fair, Poor}\}\}$

Transition Steps:

$$\mathbf{P_{ij}} = \begin{array}{c} \begin{array}{ccc} \text{G} & \text{F} & \text{P} \end{array} \\ \begin{bmatrix} 0.2 & 0.5 & 0.3 \\ 0 & 0.5 & 0.5 \\ 0 & 0 & 1 \end{bmatrix} \end{array} \begin{array}{c} \\ \text{G} \\ \text{F} \\ \text{P} \end{array}$$

Note that $P_{PP} = 1$ is an absorbing state, meaning that in the long run the farmer only has poor quality of soil
Part b

State Variable: Let x(t) be the condition of the soil at time t using the fertilizer.

State Space: $\mathcal{I} = \{x(t) \in \{\text{Good, Fair, Poor}\}\}$

Transition Steps:

$$\mathbf{P_{ij}} = \begin{array}{c} \begin{array}{ccc} \text{G} & \text{F} & \text{P} \end{array} \\ \begin{bmatrix} 0.3 & 0.6 & 0.1 \\ 0.1 & 0.6 & 0.3 \\ 0.05 & 0.4 & 0.55 \end{bmatrix} \end{array} \begin{array}{c} \\ \text{G} \\ \text{F} \\ \text{P} \end{array}$$

Steady States:

$$
\begin{aligned}
\pi_G &= 0.3\pi_G + 0.1\pi_F + 0.05\pi_P \\
\pi_F &= 0.6\pi_G + 0.6\pi_F + 0.4\pi_P \\
\pi_P &= 0.1\pi_G + 0.3\pi_F + 0.55\pi_P \\
1 &= \pi_G + \pi_F + \pi_P
\end{aligned}
$$

Doing Some algebra gives the following results: $\pi_G = \frac{2}{123}, \pi_F = \frac{682}{738}$ and $\pi_P = \frac{22}{369}$.

Part c

State Variable: Let x(t) be the condition of the ground at time t using the fertilizer conditionally on the poor states.

State Space: $\mathcal{I} = \{x(t) \in \{\text{Good, Fair, Poor}\}\}$

Transition Steps:

$$
\mathbf{P_{ij}} = \begin{array}{c} \begin{array}{ccc} \text{G} & \text{F} & \text{P} \end{array} \\ \begin{bmatrix} 0.2 & 0.5 & 0.3 \\ 0 & 0.5 & 0.5 \\ 0.05 & 0.4 & 0.55 \end{bmatrix} \end{array} \begin{array}{c} \\ \text{G} \\ \text{F} \\ \text{P} \end{array}
$$

Steady States:

$$
\begin{aligned}
\pi_G &= 0.2\pi_G + 0.05\pi_P \\
\pi_F &= 0.5\pi_G + 0.5\pi_F + 0.4\pi_P \\
\pi_P &= 0.1\pi_G + 0.3\pi_F + 0.55\pi_P \\
1 &= \pi_G + \pi_F + \pi_P
\end{aligned}
$$

Doing Some algebra gives the following results: $\pi_G = \frac{5}{154}, \pi_F = \frac{69}{154}$ and $\pi_P = \frac{80}{154}$.

3 Exercise 9.11

In a continuous review inventory system for a company that is constantly monitored. If the inventory position is lower than $r = 1$. Then $R = 3$ units are ordered. The demand probabilities for 0,1 and 2 are given.

Part a)

State Variable: Let $x(t)$ be the inventory position just before it is checked.

State Space: $\mathcal{I} = \{x(t) \in \{4, 3, 2, 1, 0\}\}$

Transition Probabilities:

$$\mathbf{P_{ij}} = \begin{array}{cc} & \begin{array}{ccccc} 4 & 3 & 2 & 1 & 0 \end{array} \\ & \begin{bmatrix} \frac{3}{8} & \frac{1}{2} & \frac{1}{8} & 0 & 0 \\ 0 & \frac{3}{8} & \frac{1}{2} & \frac{1}{8} & 0 \\ 0 & 0 & \frac{3}{8} & \frac{1}{2} & \frac{1}{8} \\ \frac{3}{8} & \frac{1}{2} & \frac{1}{8} & 0 & 0 \\ 0 & \frac{3}{8} & \frac{1}{2} & \frac{1}{8} & 0 \end{bmatrix} \begin{array}{c} 4 \\ 3 \\ 2 \\ 1 \\ 0 \end{array} \end{array}$$

Steady States:

$$\begin{aligned}
\pi_4 &= \frac{3}{8}\pi_4 + \frac{3}{8}\pi_1 \\
\pi_3 &= \frac{1}{2}\pi_3 + \frac{3}{8}\pi_2 + \frac{1}{2}\pi_1 \\
\pi_2 &= \frac{1}{8}\pi_4 + \frac{1}{2}\pi_3 + \frac{3}{8}\pi_2 + \frac{1}{8}\pi_1 \\
\pi_1 &= \frac{1}{8}\pi_3 + \frac{1}{2}\pi_2 \\
\pi_0 &= \frac{1}{8}\pi_2 \\
1 &= \pi_4 + \pi_3 + \pi_2 + \pi_1 + \pi_0
\end{aligned}$$

With some algebra we have that: $\pi_0 = \frac{1}{24}, \pi_1 = \frac{5}{24}, \pi_2 = \frac{8}{24}, \pi_3 = \frac{7}{24}$ and $\pi_4 = \frac{3}{24}$

Part b)
Fixed Costs:

$$\begin{aligned}
\mathbb{E}(\text{fixed costs}) &= 6 \times (\pi_0 + \pi_1) \\
&= \frac{36}{24} \\
&= 1.50
\end{aligned}$$

Inventory Costs:

$$
\begin{aligned}
\mathbb{E}(\text{inventory costs}) &= 1 \times (4\pi_4 + 3\pi_3 + 2\pi_2 + \pi_1) \\
&= \frac{12}{24} + \frac{21}{24} + \frac{16}{24} + \frac{5}{24} \\
&= \frac{54}{24} \\
&= 2.25
\end{aligned}
$$

Hence we have a total expected daily costs of 3.75.

4 Exercise 9.12

We here again have another review inventory system, with inventory position IP. When the IP is lower than s, then we order up to S. Hence, we order $S - IP$ extra units whenever we are below s. The inventory gets reviewed every 2 days. Any order made during the day will be delivered by the next.

Part a)

State Variable: Let $x(t)$ be the inventory position or level just before it gets checked.

State Space: $\mathcal{I} = \{x(t) \in \{6, 5, 4, 3, 2, 1, 0\}\}$

Transition Probabilities:

$$
\mathbf{P_{ij}} = \begin{array}{c} \\ \left[\begin{array}{ccccccc}
6 & 5 & 4 & 3 & 2 & 1 & 0 \\
\end{array}\right. \end{array}
$$

$$
\mathbf{P_{ij}} = \begin{array}{cc}
\begin{array}{ccccccc} 6 & 5 & 4 & 3 & 2 & 1 & 0 \end{array} & \\
\begin{bmatrix}
\frac{9}{64} & \frac{24}{64} & \frac{22}{64} & \frac{8}{64} & \frac{1}{64} & 0 & 0 \\
0 & \frac{9}{64} & \frac{24}{64} & \frac{22}{64} & \frac{8}{64} & \frac{1}{64} & 0 \\
0 & 0 & \frac{9}{64} & \frac{24}{64} & \frac{22}{64} & \frac{8}{64} & \frac{1}{64} \\
\frac{9}{64} & \frac{24}{64} & \frac{22}{64} & \frac{8}{64} & \frac{1}{64} & 0 & 0 \\
\frac{9}{64} & \frac{24}{64} & \frac{22}{64} & \frac{8}{64} & \frac{1}{64} & 0 & 0 \\
\frac{9}{64} & \frac{24}{64} & \frac{22}{64} & \frac{8}{64} & \frac{1}{64} & 0 & 0 \\
\frac{9}{64} & \frac{24}{64} & \frac{22}{64} & \frac{8}{64} & \frac{1}{64} & 0 & 0
\end{bmatrix} & \begin{array}{c} 6 \\ 5 \\ 4 \\ 3 \\ 2 \\ 1 \\ 0 \end{array}
\end{array}
$$

Steady States:

$$
\begin{aligned}
\pi_6 &= \frac{9}{64}\pi_6(\pi_6 + \pi_3 + \pi_2 + \pi_1 + \pi_0) \\
\pi_5 &= \frac{9}{64}\pi_5 + \frac{24}{64}(\pi_6 + \pi_3 + \pi_2 + \pi_1 + \pi_0) \\
\pi_4 &= \frac{24}{64}\pi_5 + \frac{9}{64}\pi_4 + \frac{22}{64}(\pi_6 + \pi_3 + \pi_2 + \pi_1 + \pi_0) \\
\pi_3 &= \frac{22}{64}\pi_5 + \frac{24}{64}\pi_4 + \frac{8}{64}(\pi_6 + \pi_3 + \pi_2 + \pi_1 + \pi_0) \\
\pi_2 &= \frac{8}{64}\pi_5 + \frac{22}{64}\pi_4 + \frac{1}{64}(\pi_6 + \pi_3 + \pi_2 + \pi_1 + \pi_0) \\
\pi_1 &= \frac{1}{64}\pi_5 + \frac{8}{64}\pi_4 \\
\pi_0 &= \frac{1}{64}\pi_4 \\
1 &= \pi_6 + \pi_5 + \pi_4 + \pi_3 + \pi_2 + \pi_1 + \pi_0
\end{aligned}
$$

With some algebra we get the following: $\pi_6 = \frac{27225}{392384}, \pi_5 = \frac{1320}{6131}, \pi_4 = \frac{1786}{6131}, \pi_3 = \frac{12013}{49048}, \pi_2 = \frac{52887}{392384}, \pi_1 = \frac{1951}{49048}$ and $\pi_0 = \frac{893}{196192}$.

Part b)

Fixed Costs:

$$\mathbb{E}(\text{fixed costs}) = 6 \times (\pi_0 + \pi_1 + \pi_2 + \pi_3) \qquad = 2.54$$

Inventory Costs:

$$\mathbb{E}(\text{inventory costs}) = 2 \times (1\pi_1 + 2\pi_2 + 3\pi_3 + 4\pi_4 + 5\pi_5 + 6\pi_6) \qquad = 7.40$$

Hence the total costs is 9.94.

5 Exercise 9.13

We have 5 machines, and 1 service man. From the question we know that the rate at which the machines breakdown is Poisson distributed with $\lambda = \frac{1}{4}$ and the rate of service is also Poisson distributed with $\mu = 1$.

Part a)

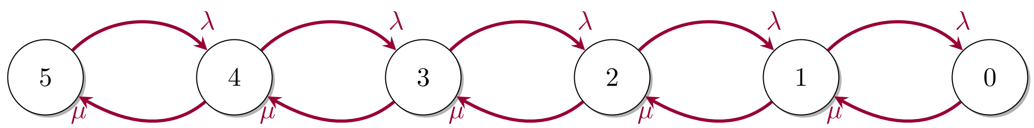

Figure 1: How the system works

State Variable: Let $x(t)$ be the number of machines working at time t.

State Space: $\mathcal{I} = \{x(t) \in \{5, 4, 3, 2, 1, 0\}\}$

Transition Probabilities: what comes in = what comes out:

$$\begin{aligned}
\frac{1}{4}\pi_1 &= \pi_0 \\
\frac{1}{4}\pi_2 + \pi_0 &= \frac{5}{4}\pi_1 \\
\frac{1}{4}\pi_3 + \pi_1 &= \frac{5}{4}\pi_2 \\
\frac{1}{4}\pi_4 + \pi_2 &= \frac{5}{4}\pi_3 \\
\frac{1}{4}\pi_5 + \pi_3 &= \frac{5}{4}\pi_4 \\
\pi_4 &= \frac{1}{4}\pi_5
\end{aligned}$$

subject to $\sum_{i=0}^{5} \pi_i = 1$. Then algebra dictates that the steady states are: $\pi_0 = \frac{1}{1365}, \pi_1 = \frac{4}{1365}, \pi_2 = \frac{16}{1365}, \pi_3 = \frac{64}{1365}, \pi_4 = \frac{256}{1365}$ and $\pi_5 = \frac{1024}{1365}$.

Then π_0 is the average time of machines that are not in use.

Part b)

What fraction of time is each machine is use? - i dont know.

6 Exercise 9.14

7 Exercise 9.15

Part a):

We have a system that handles two types of messages, it has a total memory of M where N_1 and N_2 are already allocated to type of message 1 and 2. Plus a buffer $N_0 = M - N_1 - N_2$. We have to use a continuous Markov Chain, as we have continuous arrival and service rates. Arrival rates are distributed by a Poisson distribution, with a rate of λ_i and the service rates are exponentially distributed with a mean of $\frac{1}{\mu_i}$, hence we can say that the service rates are μ_i, for both type $i = \{1, 2\}$. Then we have the following formulation:

State variables: $(x(t)_1, x(t)_2)$ = a tuple of the number of messages in the system for each type

State space: $\mathbb{I} = \{(x(t)_1, x(t)_2) \mid x(t)_1 + x(t)_2 \leq M, 0 \leq x(t)_1 \leq M - N_2, 0 \leq x(t)_2 \leq M - N_1 \}$

Transition rates:

$$(x(t)_1, x(t)_2) \xrightarrow{\lambda_1} (x(t)_1 + 1, x(t)_2) : \text{if } x(t)_1 \leq M - N_2 \text{ and } x(t)_1 + x(t)_2 \leq M$$

$$(x(t)_1, x(t)_2) \xrightarrow{\lambda_2} (x(t)_1, x(t)_2 + 1) : \text{if } x(t)_2 \leq M - N_1 \text{ and } x(t)_1 + x(t)_2 \leq M$$

For the service rates we have that μ_1 and μ_2 is defined if $x(t)_1, x(t)_2 > 0$. Meaning as long as there are messages in the system, the system will service the messages otherwise it doesnt have to.

part b):

I couldnt solve it, but it is worth looking when to reject a message. That is is for each type of message or both if the buffer is full.

For type 1 message: is rejected when $x(t)_1 = M - N_2$ and $x(t)_2 = N_2$.
For type 2 message: is rejected when $x(t)_2 = M - N_1$ and $x(t)_1 = N_1$.
For both types: it is rejected when $x(t)_1 + x(t)_2 = M$

Then let Y be the number of rejections. Then we want to reduce the number of expected rejects, so we want to minize the following.

$$\mathbb{E}(Y) = \sum_{x(t)_1 < M - N_2} \pi_{(M - N_2, x(t)_2)} \times \lambda_1$$

$+ \sum_{x(t)_2 < M - N_1} \pi_{(x(t)_1, M - N_1)} \times \lambda_2 + \sum_{x(t)_1, x(t)_2 = M} \pi_{(x(t)_1, x(t)_2)} (\lambda_1 + \lambda_2)$

Subject to $N_1, N_2 > 0$