OR - HAND INS 2

C.Arican i6081025

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1 Exercise 5.16

Suppose we have a connected graph $G = (V, E)$, that is, there exists a path between every pair of vertices, and that for every edge in the graph, $e = \{u, v\}$ s.t. $(u, v) \in V$ has its own respective cost/weight, c_e . A Minimum Spanning Tree is a spanning tree, $T = (V, E')$ with minimum total edge weight. that is T minimizes $\sum_{e \in E'} c_e$, over all possible spanning trees of G.

Generally, a cut of a graph, is the removal of minimal amount of edges such that the graph becomes disjointed and we have two separate connected components.

Take a look at the following graph, its Minimum Spanning Tree (MST) and a cut:

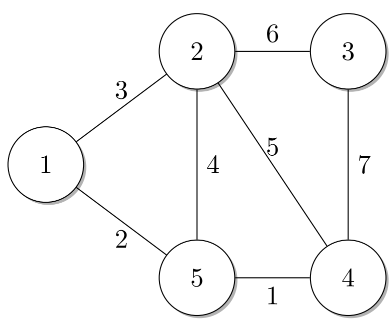

Figure 1: Original Graph

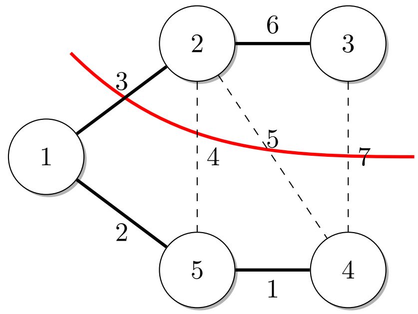

Figure 2: MST with a cut

Theorem - Cut Optimality Condition: Given a connected graph $G = (V, E)$ with edge costs $c_e(e \in E)$. A spanning tree T is a MST if and only if the following cut condition holds:

$$\forall e \in T : \forall f \in \delta_T(e) : c_e \leq c_f$$

In other words, there is a cut that disjoints the graph in a way that there is an edge that also cuts the tree. That cut may be arbitrary, then for every 'crossing edge' from the tree, is smaller or equal to all other elements of the cut i.e. the tree edge is smaller or equal to all other non-tree edges. An example is given above. The red line represents a cut such that the graph and the tree becomes disjoint into 2 components, where we see that c_{12} is smaller than c_{24}, c_{25}, c_{34} which are all elements of the cut, $\delta_T(e)$. Note that the cost is 12.

Proof of correctness:

We prove this optimally condition via a contradiction. Suppose that we have a connected graph $G = (V, E)$, and T is a MST and $e \in T$, with edge cost c_e. But now suppose that there exists a true MST, T' such that $T' = (T \cup \{c_f\}) \backslash \{c_e\}$ and assuming that $c_f \geq c_e$. Then:

$$\begin{aligned} T' &\leq T \\ T + c_f - c_e &\leq T \\ c_f &\leq c_e \end{aligned}$$

But this is a contradiction as we assumed that $c_f \geq c_e$.

The question to be answered is to create a cut optimality condition for a Maximum Spanning Tree. Then as we are maximizing, it is logical to consider the cut optimality condition to be:

$$T \text{ is a Maximum Spanning Tree} \iff \forall e \in T : \forall f \in \delta_T(e) : c_e \geq c_f$$

Lets look at an example, consider the original graph from above along its maximum spanning tree and a cut:

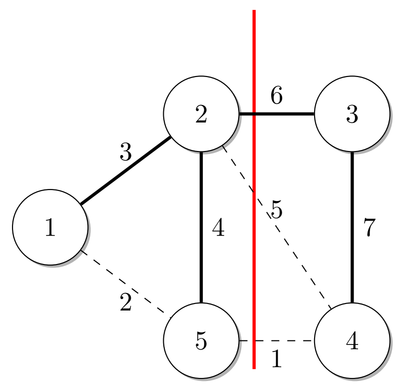

Figure 3: Max.ST with a cut

In other words the edge that is part of the cut, such that it disjoints the tree, is larger or equal to any other edge that is part of the cut. By the example we see that $c_{2,3}$ is larger than c_{24}, c_{45}, which are all part of the cut. Note that the total cost is 20. Then similarly as before, we prove its correctness by contradiction:

Suppose that T is a maximum spanning tree with $e \in T$. Now assume that there exists a true maximum spanning tree T' with $f \in T'$ and that $c_f \leq c_e$. Then $T' = (T \cup \{f\}) \backslash \{e\}$. So:

$$\begin{aligned} T' &\geq T \\ T + c_f - c_e &\geq T \\ c_f &\geq c_e \end{aligned}$$

Which is a contradiction. Hence we have proven the cut optimality condition for a maximum spanning tree.

2 Exercise 6.19

Preliminary For a directed graph G = (V, A), we use l(v) to denote the length of the chosen path between the starting node **s** and node **v**. c_{uv} represents the length between node **u** and node **v**, pred(v) denotes the predecessor node of node **v**.

l(s)=0, pred(s)=0, S = $\emptyset$
$\forall v \in V \backslash \{s\}$, l(v) = ∞

while S $\neq V, do$
    u= arg min { $l(v); u \in V \backslash S\}$ }
    $S = S \cup$ { u }
    for all V: (u,v) $\in A$ do
        if l(v) $>$ max { $l(u)$, c_{uv} }
            l(v)=max { $l(u)$, c_{uv} }
            pred(v)=u