Minimum Concurrency via Maximum Cycles

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Abstract

An effective algorithmic solution for resource-sharing problems in heavily loaded systems is Scheduling by Edge Reversal, essentially providing some level of concurrency by describing an order of *operation* for nodes in a graph. The resulting concurrency is a hard metric to optimize, since the decision problems associated with obtaining its *extrema* have been proved to be NP-complete. In this paper, we propose a novel approach involving maximum cycles for attaining minimum concurrency, which is desired for problems associated with graph decontamination. We prove the correctness of our algorithm and implement our techniques, showing that reasonably large instances may be solved to proven optimality under acceptable CPU times.

Keywords: Minimum concurrency, Maximum Cycles, Scheduling by edge reversal

1 Introduction

Resource-sharing problems arise naturally in many scenarios, where graph algorithms are often employed to provide a distributed, asynchronous scheduling solution. By representing each process as a node, we may define that each node is connected by an edge if and only if they share a resource. Specifically, in neighborhood-constrained systems, a process is only allowed to *operate* if and only if all of its neighbors are *idle*, meaning that all of its required resources must be available at the time of *operation*. As a consequence, multiple processes requiring the same resource form a clique, a complete sub-graph in which only one node is allowed to *operate* at a time. A connected undirected graph representing resource dependencies among processes, as illustrated in figure 1(a), will be referred to as a *resource graph* throughout this paper.

Under a heavy load assumption, where nodes are constantly demanding access to their required resources, an effective scheduling algorithm to ensure fairness and prevent starvation is *Scheduling by Edge Reversal (SER)*. Introduced in 1989 by Barbosa and Gafni [1], *SER* has inspired many distributed resource-sharing applications ranging from asynchronous digital circuits [2] to the control of traffic lights present in road junctions [3].

The execution of SER may be summarized as follows: by taking a directed acyclic graph *(DAG)* such as the one in figure 1(b) as input, *SER* simultaneously *operates* all sinks, meaning that all nodes with no outgoing edges are allowed to utilize their corresponding resources to perform a given task. Once every sink is done *operating*, the orientation of their incoming edges is reverted, effectively allowing other nodes to become *sinks* themselves. This process is repeated indefinitely, as each new iteration will generate a new *DAG*, allowing different nodes to utilize resources and *operate*. Eventually, orientations will start repeating themselves, leading to the existence of periods. In fact, as has been observed by Barbosa and Gafni, all nodes operate the same number of times within a given period. Figure 1(c) illustrates this procedure.

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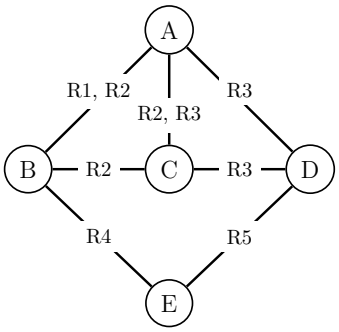
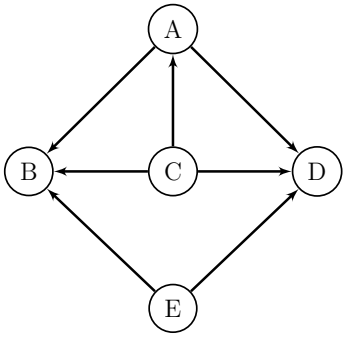
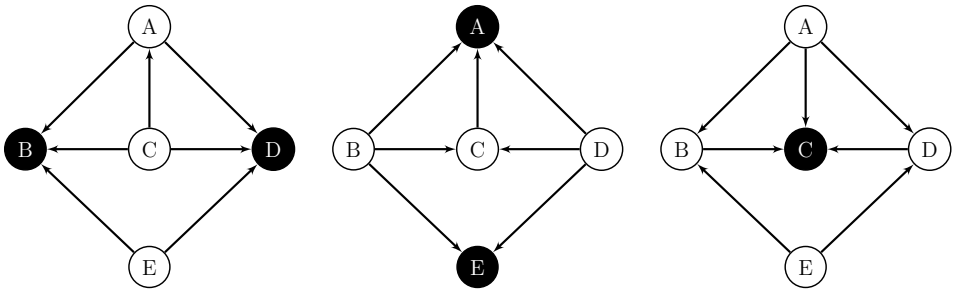


(a) A *resource graph*. Nodes sharing resources are connected by an edge.



(b) A given acyclic orientation applied to the *resource graph* from (a).



(c) A schedule (from left to right) taking (b) as the initial orientation. Sinks (in black) are allowed to *operate*, after which they revert their edges so new sinks may be formed. Once node C is done *operating*, the leftmost orientation will be repeated, leading to a period.

Fig. 1. Scheduling by Edge Reversal as a distributed solution for scheduling processes (nodes) in a resource-sharing system.

In order to apply *SER* to any resource graph and obtain a corresponding schedule, an initial acyclic orientation must be generated. This commencing *DAG* will directly impact the overall concurrency obtained through the edge reversal procedure, leading to periods of different lengths and of different orientations. Intuitively, a highly concurrent dynamic will present more nodes *operating* simultaneously while minimizing the amount of steps where each node is *idle*. Although concurrency will be better defined in section 2, it's already inevitable to inquire the complexity of problems such as obtaining the orientations that maximize or minimize this metric. In fact, the decision problems associated with the maximum as well as the minimum concurrency yielded by a given resource graph have been proved to be NP-complete by Barbosa and Gafni [1] and by Arantes Jr [4], respectively.

Contrary to intuition, obtaining the orientations of a resource graph from which *SER* will provide minimum concurrency is advantageous to a number of applications. For instance, in the decontamination of circulant graphs representing link farms in the Web, where links to a target web page are spread throughout other pages to promote visibility [5], one goal is to minimize the number of *web marshals* performing the decontamination procedure. Under a dynamic in which a node becomes healthy once it is no longer exposed to contaminated neighbors, it is shown in Gonçalves et al. [6] that lower concurrency values imply in a reduced number of *web marshals* employed in the dismantling effort. This decontamination dynamic can also be extended to conflagration scenarios, where firefighters must guard a safe node until its neighbors are cleared of fire. In Alves et al. [7], the use of robots to extinguish the flames within an apartment was simulated through the dynamics of *SER* decontamination.

The following is how the remainder of this paper is organized. In section 2, we recall some graph-theoretic definitions associated with Scheduling by Edge Reversal, including a metric for concurrency. Section 3 is our main theoretic contribution, showing that minimum concurrency can be obtained in linear-time given a previously discovered maximum cycle. Finally, in section 4, we implement a two-phase algorithm for finding minimum concurrency of various relevant instances, expressing our concluding remarks in section 5.

2 Graph-Theoretic Background

Initially, as has been defined in Barbosa and Gafni [1], we shall characterize the necessary terminology to define *concurrency* under *SER*. As such, let $G = (V, E)$ be a connected undirected graph where $|E| \geq |V|$ (i.e. G is not a tree). Let κ denote an undirected simple cycle in G, that is, a sequence of $|\kappa|$ vertices $i_0, i_1, ..., i_{|\kappa|-1}, i_0$. If κ is traversed from i_0 to $i_{|\kappa|-1}$, we say that it is traversed in the clockwise direction. Otherwise, we say that it is traversed in the counterclockwise direction. Let K denote the set of all simple cycles of G.

Moreover, an *acyclic orientation* of G is a function of the form $\omega : E \to V$ such that no undirected cycle $i_0, i_1, ..., i_{k-1}, i_0$ exists for which $\omega(i_0, i_1) = i_1$, $\omega(i_1, i_2) = i_2$, ..., $\omega(i_{k-1}, i_0) = i_0$. Let Ω denote the set of all acyclic orientations of G.

Lastly, given an undirected simple cycle κ and an *acyclic orientation* ω, let $n_{cw}(\kappa, \omega)$ be defined as the number of edges oriented clockwise by ω in κ. Similarly, let $n_{ccw}(\kappa, \omega)$ be defined as the number of edges oriented by ω in the counterclockwise direction. Therefore, the *concurrency* of a graph G is defined as a function $\gamma : \Omega \to \mathbb{R}$ such that:

$$\gamma(\omega) = \min_{\kappa \in K} \left\{ \frac{min\left\{n_{cw}(\kappa, \omega), n_{ccw}(\kappa, \omega)\right\}}{|\kappa|} \right\} \tag{1}$$

In other words, given an orientation ω, we check every simple undirected cycle κ of G and calculate the number of edges oriented in the clockwise direction as well as the number of edges oriented in the counterclockwise direction. We take the minimum of these two values and divide the result by the size of the undirected cycle κ. Whichever $\kappa \in K$ returns the smallest value will dictate the system's concurrency.

Finally, we must note that an equivalent result can also be obtained from a dynamic analysis. Let a period of length p be a sequence of distinct acyclic orientations $\alpha_0, ..., \alpha_{p-1}$ induced by the execution of *SER*. Let m be the number of times a node *operates* within a period, which is equal to all nodes. The expression $\gamma(\omega) = m/p$ is equivalent to Equation 1, despite being less significant to this paper.

3 Obtaining Minimum Concurrency

Our main goal in this section is to propose a linear-time algorithm for finding the minimum concurrency yielded by a resource graph G given one of its maximum simple cycles as input, as well as analyzing the correctness and complexity of such algorithm. Initially, we shall derive a different expression for minimizing $\gamma(\omega)$ for all $\omega \in \Omega$, showing how the problem of finding an orientation that yields minimum concurrency is closely related to the problem of obtaining an undirected maximum cycle of G.

Equation 1 is a function of ω. Given that Ω is a finite set, let γ^* denote the minimum value that Equation 1 assumes for every $\omega \in \Omega$. We may then write an expression for minimum concurrency as follows:

$$\gamma^* = \min_{\omega \in \Omega} \left\{ \min_{\kappa \in K} \left\{ \frac{min\left\{n_{cw}(\kappa, \omega), n_{ccw}(\kappa, \omega)\right\}}{|\kappa|} \right\} \right\} \tag{2}$$

However, for every cycle $\kappa \in K$, there will always exist an acyclic orientation $\omega \in \Omega$ such that $n_{cw}(\kappa, \omega) = 1$ and $n_{ccw}(\kappa, \omega) = |\kappa| - 1$. This follows immediately from the fact that a directed cycle would only exist if and only if either n_{cw} or n_{ccw} were equal to zero for a given κ. As such, since the numerator in Equation 2 may assume the value of 1 for every $\kappa \in K$, we may rewrite our expression for γ^* as follows:

$$\gamma^* = \min_{\kappa \in K} \left\{ \frac{1}{|\kappa|} \right\} \tag{3}$$

Equation 3 is essentially the problem of finding a maximum undirected cycle of G, whose minimum concurrency will be equal to the reciprocal for the size of this cycle. However, although we have shown how to calculate γ^*, it still unclear how to obtain ω^*, an orientation for which $\gamma(\omega^*) = \gamma^*$. As such, the remainder of this section is dedicated to describing an algorithm for finding ω^* given a maximum cycle of G, as well as a brief discussion of its correctness and complexity.

Let κ^* be a maximum simple cycle of G, meaning that $|\kappa^*| \geq |\kappa|$ for all $\kappa \in K$. The following theorem holds:

Theorem 3.1 *Given any maximum cycle $\kappa^* \in K$ as input, there exists a linear-time algorithm for finding an orientation $\omega^* \in \Omega$ such that $\gamma(\omega^*)$ is minimum for all $\omega \in \Omega$.*

Proof. Equation 3 states that minimum concurrency will be attained if an orientation ω^* is applied to G under

Algorithm 1: A linear-time algorithm for finding an acyclic orientation that leads to minimum concurrency given a maximum cycle as input.

Input : An undirected graph $G = (V, E)$ and its maximum cycle κ^*
Output: An acyclic orientation $\omega*$ for which $\gamma(\omega^*)$ is minimum
$id = 1$
$v = \kappa^*.getFirstVertex()$
for *i=1 to* $\kappa^*.size()$ **do**
  Assign id to v
  Increment id
  v = $\kappa^*.getClockwiseNeighborOf(v)$
end
while *a vertex* $v \in V$ *with no id exists* **do**
  Assign id to v
  Increment id
end
Create an empty orientation ω^*
foreach *undirected edge* $uv \in E$ **do**
  if $id(v) > id(u)$ **then**
    Orient edge such that $\omega^*(u, v) = v$
  end
  else
    Orient edge such that $\omega^*(u, v) = u$
  end
end
return ω^*

.

the condition that $n_{cw}(\kappa^*, \omega^*) = 1$ and $n_{ccw}(\kappa^*, \omega^*) = |\kappa^*| - 1$, where κ^* is a maximum cycle. Orienting κ^* under the aforementioned conditions can be performed in linear-time by traversing the cycle $\kappa*$ and assigning an increasing identification number $1, ..., |\kappa^*|$ to each visited vertex, resulting in a topological ordering of the cycle. By orienting the corresponding edges towards the vertices with higher identification numbers, only one edge (connecting the vertices with the highest and the lowest identification numbers) will be oriented in the opposing direction from the other $|\kappa^*| - 1$ edges, fulfilling the requirement.

It is now necessary to prove that it is possible to orient the remaining edges of G such that the resulting orientation ω^* is always acyclic. Let $S = V - \kappa^*$ be the set of the remaining vertices of G. Let us assign an increasing identification number $|\kappa^*| + 1, ..., |V|$ to each remaining vertex of S, and then orient all edges of G towards the vertices with higher identification numbers. By contradiction, if the resulting orientation ω^* were cyclic, then there would need to exist at least one directed edge departing from a vertex in S and arriving at a vertex in κ^* as well as at least one directed edge departing from κ^* and arriving at S. However, since all identification numbers assigned to S are strictly greater than the ones assigned to κ^*, such scenario is impossible.

To conclude this proof, we must show that no directed cycle is formed within S. At any point during the identification assignment procedure, let $R \subseteq S$ be the set of vertices not yet assigned a number and $T = S - R$. Let the identification assignment procedure conclude without altering the elements of R or T. A cycle would have been formed if, for a given configuration of R and T, there were at least one directed edge from R to T as well as at least one directed edge from T to R. This is once again impossible, since all vertices in R have an identification number strictly greater than the ones in T. As such, the resulting orientation ω^*, obtained in linear-time, must be acyclic. □

Finally, we structure the proof discussed in Theorem 3.1 as the algorithmic procedure presented in Algorithm 1. Its correctness relies on the aforementioned proof.

4 Experimental Results

In this section, we implement a two-phase algorithm for obtaining acyclic orientations that attain minimum concurrency. The first phase consists of employing a branch-and-cut optimization strategy for identifying a maximum cycle, while the second phase consists of applying Algorithm 1 to obtain an acyclic orientation that results in minimum concurrency. Out of our computational experiments, we show that reasonably large instances relevant to the graph decontamination problem may be solved to proven optimality under acceptable CPU times.

For the first phase of our algorithm, we rely on the Simple Cycle Problem branch-and-cut strategy proposed in Lucena, Cunha and Simonetti [9], which is based on a formulation that decomposes simple cycles into one simple path and an additional edge, connecting the extreme vertices of the path. This algorithm separates Generalized Subtour Elimination Constraints *on the fly* (as they become violated at a linear programming relaxation in the branch-and-cut enumeration tree).

All implementation was done in the C programming language and uses the *XPRESS Mixed Integer Programming* package to solve linear programs and manage the branch-and-cut tree. All other features of the solver, such as pre-processing, primal heuristics and automatic cut generation were kept switched-off. Experiments were conducted on an Intel Core2 Quad Q9650 based machine running at 3.0 GHz with 16 Gbytes of RAM memory and only one thread being used.

In order to assess the viability of our proposal, two sets of instances were selected. For the first set, we generated circulant graphs of the form $Ci_n(1, ..., \lfloor n/2 \rfloor)$, where $n = |V|$, which are equivalent to the complete graph K_n. Subsequently, we removed each edge with a given probability p while ensuring that no additional connected components were formed, allowing us to control the density of our instances and capture less predictable structures. For the second set of graphs, we selected hamiltonian circuit instances present in SITE so as to evaluate the feasibility of our strategy for graphs that are known to be more challenging.

Each instance was executed five times. The results for the first and second set of instances are displayed in Table 1 and Table 2, respectively.

n	$\|E\|$	p	$\|\kappa^*\|$	Min Concurrency	Avg CPU Time (s)
400	300	0.5	245	1/245	0.5
1000	500	0.4	378	1/378	6.7

Table 1
Table1: Minimum concurrency for generated circulant graphs.

Instance Name	n	Min Concurrency	Avg CPU Time (s)
Something	300	1/300	0.5
Something Else	500	1/500	6.7

Table 2
Minimum concurrency for challenging hamiltonian circuit instances.

5 Conclusion

We have presented a novel technique for obtaining minimum concurrency under Scheduling by Edge Reversal, being particularly useful in scenarios derived from graph decontamination problems. By first identifying a maximum cycle of the *resource graph* G, we have demonstrated how to obtain, in linear-time, an orientation that attains minimum concurrency. Despite the decision problem associated with finding maximum cycles still being NP-complete, we have shown through experimental results that recent branch-and-cut techniques are able to solve significantly large instances under acceptable CPU times.

References

[1] Barbosa, V. C., and E. Gafni, *Concurrency in Heavily Loaded Neighborhood-Constrained Systems*, ACM Transactions on Programming Languages and Systems **11(4)** (1989), 562–584.

[2] Cassia, R. F., and V. Alves, and F. Besnard, and F. M. G. França, *Synchronous-To-Asynchronous Conversion of Cryptographic Circuits*, Journal of Circuits, Systems and Computers **18(2)** (2009), 271–282.

[3] Carvalho, D. and F. Protti and M. De Gregorio and F. M. G. França, *A Novel Distributed Scheduling Algorithm for Resource Sharing Under Near-Heavy Load*, Lecture Notes in Computer Science **3544** (2004), 431–442.

[4] Arantes Jr, G. M., "Trilhas, Otimização de Concorrência e Inicialização Probabilística em Sistemas sob Reversão de Arestas", Ph.D. thesis, Universidade Federal do Rio de Janeiro, Rio de Janeiro, 2006.

[5] Luccio, F. and L. Pagli, *Web Marshals Fighting Curly Link Farms*, Lecture Notes in Computer Science **4475** (2007), 240–248.

[6] Gonçalves, V. C. F. and P. Lima, and N. Maculan, and F. M. G. França, *A Distributed Dynamics for WebGraph Decontamination*, Lecture Notes in Computer Science **6415** (2010), 462–472.

[7] Alves, D. S. F., et al., "A Swarm Robotics Approach To Decontamination", Mobile Ad Hoc Robots and Wireless Robotic Systems: Design and Implementation, 1st ed., 107–122.

[8] Moscarini, M., and R. Petreschi, and J. L. Szwarcfiter, *On node searching and starlike graphs*, Congressus Numerantium **131** (1998), 75–84.

[9] Lucena, A., and A. Cunha, and L. Simonetti, *A New Formulation and Computational Results for the Simple Cycle Problem*, Electronic Notes in Discrete Mathematics, **44** (2013), 83–88.