1) Binary Modulation and Demodulation

Part A)

In this part, 0.1 million random bits (0's and 1's) are generated and they are mapped into the following signals;

$$s_1(t) = \begin{cases} -A\,, & t \in [0, T/2) \\ A\,, & t \in [T/2, T) \\ 0\,, & \text{otherwise} \end{cases} \quad , \quad s_2(t) = \begin{cases} 2A\,, & t \in [0, T/2) \\ -2A\,, & t \in [T/2, T) \\ 0\,, & \text{otherwise} \end{cases}$$

Figure 1: Signals for mapping the

Where $A = 1$ and T= 1 milisecond. Bit "0" is mapped into the $s_1(t)$ and bit "1" is mapped into the $s_2(t)$. Randomly generated bits' first 5 symbols are plotted in *Figure 1.*

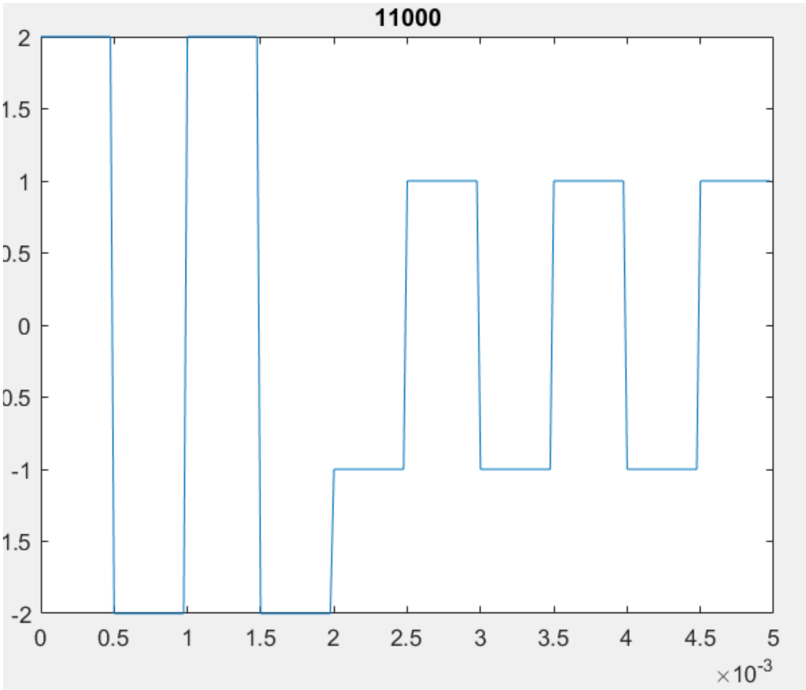

Figure 2: Plot of the first 5 symbols in the generated bits

Plot is consistent with the given signals since it generates 2 for [0 0.5] miliseconds and -2 for [0.5 1] miliseconds for the first symbol.

Part B)

In this part, we will add zero-mean Gaussian noise with variance σ^2 to the generated signal in the previous part. The following equation is used for that purpose;

$$s_{noise} = s + \mu_{noise} + \sigma * \mathrm{N}(0{,}1) \;\; (eq.1)$$

$$\mu: mean, \sigma: standart\ deviation\ , \mathrm{N}: \text{normal distribution}$$

In order to choose appropriate variances, we should know the relation between SNR and variances.

$$SNR = \frac{E_{avg}}{\sigma^2} \quad (eq.2)[1]$$

SNR is inversely proportional with variance and E_{avg} is the average signal power which is;

$$E_{avg} = \frac{1}{2} * \left(\int_0^T {s_1}^2(t)dt + \int_0^T {s_2}^2(t)dt \right) = \frac{1}{400} \quad (eq.3)$$

For high SNR I picked $\sigma^2 = 10^{-4}$ and $SNR = 25$ & $SNR = 13.98\ dB$, for medium SNR $\sigma^2 = 10^{-2}$ and $SNR = 0.25$ & $SNR = -6.02\ dB$ and for low SNR $\sigma^2 = 1$ and $SNR = 0.0025$ & $SNR = -26.02\ dB$. These results give the following waveforms for the same first 5 symbols.

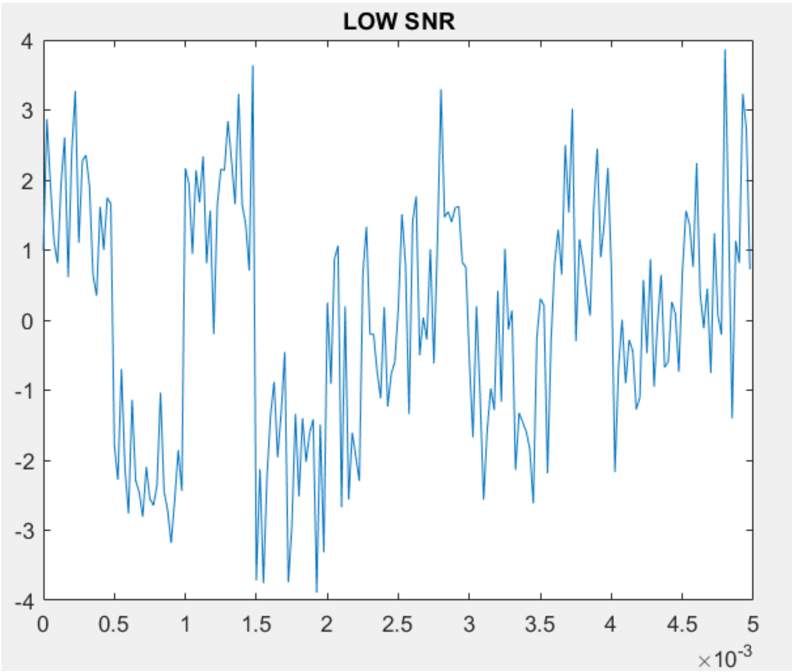

Figure 3: Noisy signal for the first 5 symbols with low SNR

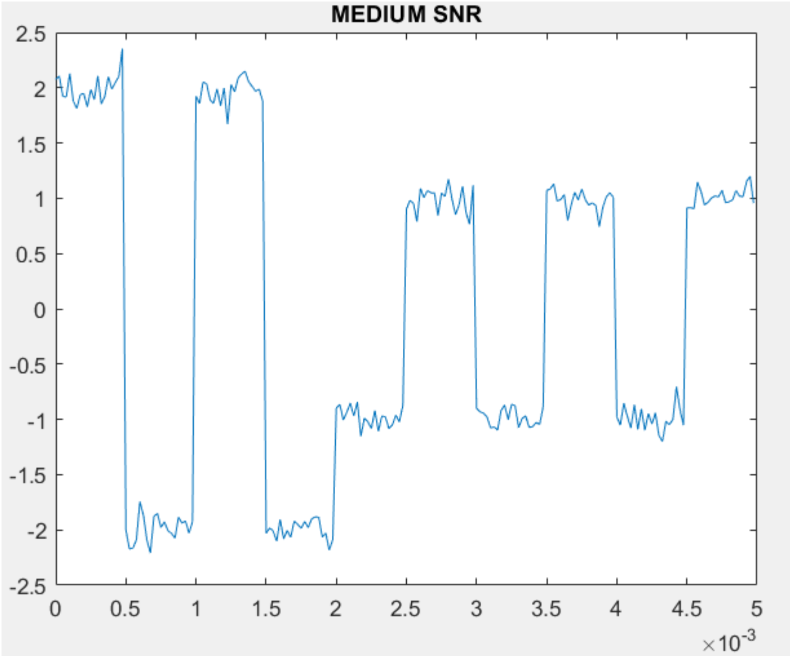

Figure 4: Noisy signal for the first 5 symbols with medium SNR

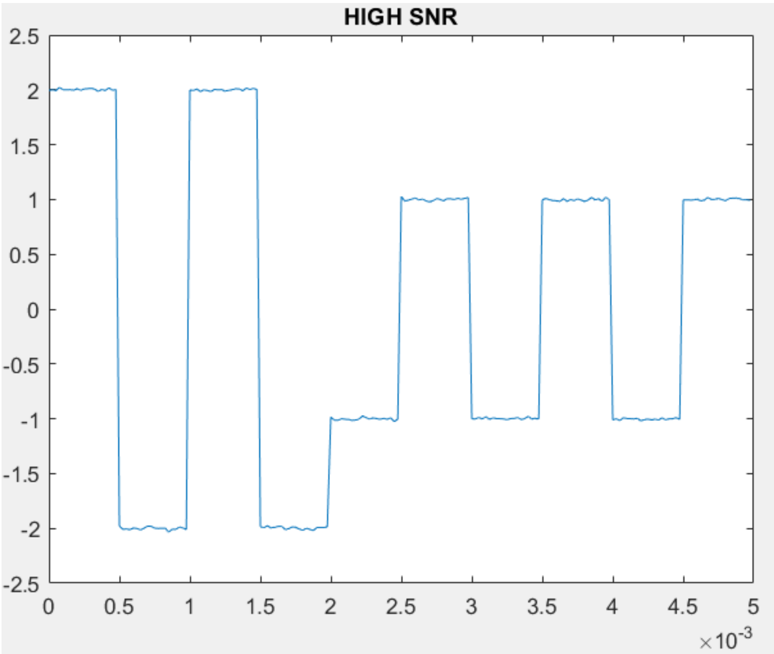

Figure 5: Noisy signal for the first 5 symbols with high SNR

Part C)

In this part, orthonormal basis functions are found for $s_1(t)$ and $s_2(t)$, they are represented in the corresponding signal space and optimal receiver is implemented and found the probability of error. I used Gramm-Schmidt to find orthonormal basis functions.

$$\psi_1(t) = \frac{s_1(t)}{\sqrt{E_{S1}}} = \frac{s_1(t)}{\sqrt{10^{-3}}} \quad (eq.4)$$

Since $s_2(t)$ can be constructed from $s_1(t)$, there is only 1 orthonormal basis function. Thus, signals can be written as;

$$s_1(t) = \sqrt{10^{-3}} * \psi_1(t)$$

$$s_2(t) = -2 * \sqrt{10^{-3}} * \psi_1(t)$$

The signal space representation is the following;

$$s_1 = \left[\sqrt{10^{-3}}\right]$$

$$s_2 = \left[-2 * \sqrt{10^{-3}}\right]$$

For the ML rule, our purpose is to give a message ("0" or "1") according to the received signals' distance to the $s_1(t)$ and $s_2(t)$. If the received signal is closer to the $s_1(t)$ then "0" should be sent. Let r = [r] be the projection of $r(t) = s_i(t) + n(t)$ on $\psi_1(t)$. Then the ML decision rule is;

$$m = \begin{Bmatrix} 0 \; if \; r < \dfrac{3\sqrt{10^{-3}}}{2} \\ 1 \; if \; r > \dfrac{3\sqrt{10^{-3}}}{2} \end{Bmatrix} \quad (eq. 5)$$

The block diagram of the receiver is the following;

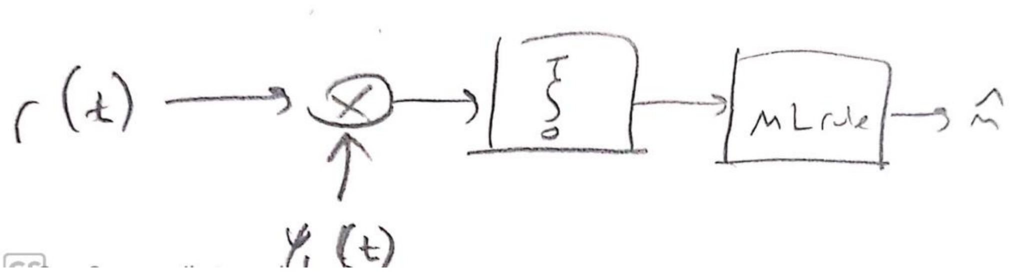

Figure 6: Block diagram of the optimal receiver

Probability of error is the probability of receiving "1" when "0" is sent and it can be calculated as the following equation;

$$P\left(\left(r \geq \frac{3\sqrt{10^{-3}}}{2} \middle| "0" is\ sent\right)\right) = P\left(n \geq \frac{3\sqrt{10^{-3}}}{2}\right) = P\left(\frac{n}{\sigma} \geq \frac{3}{20\sigma\sqrt{10}}\right) = Q\left(\frac{3}{20\sigma\sqrt{10}}\right) \quad (eq. 6)$$

σ is the noise standard deviation.

Part D)

In this part, the signal defined in Part B for medium SNR is processed and received by the optimal receiver and bits are estimated according to the receiver. Then, receiver's estimations are checked and probability of the error is experimentally obtained and it is compared with the theoretical probability of the error which is stated in Part C. Lastly, the effect of the samples per symbol to the SNR is discussed.

Experimental probability of error for medium SNR is 0, in other words, receiver demodulates the signals perfectly. Error probability is calculated by subtracting the correctness probability from 1. Each estimated symbol is compared with the original message and if they match it is added to the correct counter. Then, this counter is divided by the total number of and subtracted from 1 which can be seen in the following equation;

$$P_e = 1 - \frac{correct\ estimates}{total\ symbols} \quad (eq. 7)$$

For theoretical probability of error we can use the previously obtained equation (eq. 6);

$$P_e = Q\left(\frac{3}{20\sigma\sqrt{10}}\right)$$

There are two different SNR's for experiment and for theoretical, from (eq. 2);

$$SNR_{theo} = SNR_{exp} = \frac{E_{theo\_avg}}{{\sigma_{theo}}^2} = \frac{E_{exp\_avg}}{{\sigma_{exp}}^2} \quad (eq. 8)$$

$$\mathrm{\sigma}_{theo}{}^2 = \frac{\mathrm{\sigma}_{exp}{}^2 E_{theo\_avg}}{E_{exp\_avg}} \quad (eq. 9)$$

From Part B, we know that;

$$E_{theo\_avg} = \frac{1}{400}$$

$${\sigma_{exp}}^2 = \frac{1}{100}$$

$$E_{exp\_avg} = \frac{1}{2} * \left(s_{1vector}^2 + s_{2vector}^2\right) (eq. 10)$$

Where, $s_{1vector} = s_1 * {\psi_1}^T$ and s_1, ψ_1 are 1×40 matrices since the received signal is discrete, every sample has an index and no need to take the integral of this term.

$$E_{\exp\_avg} = 100$$

$$\mathrm{\sigma}_{theo}{}^2 = 25 * 10^{-8}$$

Thus,

$$P_e = Q\left(\frac{3}{20\sigma\sqrt{10}}\right) \cong 0$$

Since the experimental and theoretical probability of errors are same (or very close), our method is compatible with theory.

Lastly, number of samples changes the $E_{\exp\_avg}$ but SNR does not change because of (eq. 8). Lessening samples result as lessening $E_{\exp\_avg}$ but it does not affect the SNR.

Part E)

In this part, we are asked to compare the theoretical and experimental probability of errors according to SNR. In order to get experimental probability of error between $[10^{-3}\ 0.5]$, the given variance should be between [9.5 4*10^4]. When we applied the noises between the given interval to the both theoretical and experimental signals, we get the following plot;

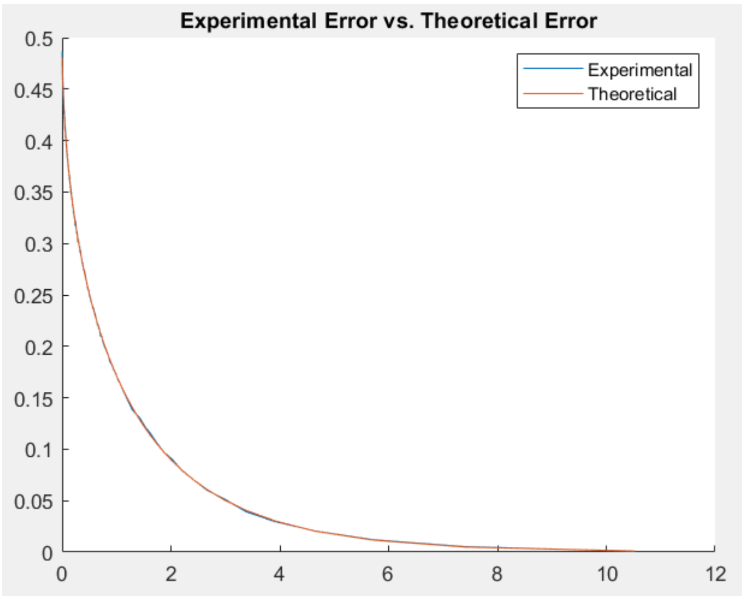

Figure 7: Error probability plot with respect to SNR

It can be seen that theoretical and experimental curves are very close to each other, in other words, our experimental results are compatible with theoretical results.

2) QAM Modulation and Demodulation

Part A)

Similar as in the first part, 200 thousand random bits are generated and they are mapped into 16-QAM signals which are described as follows;

$$s_i(t) = s_{i,1}g(t)\cos(2\pi f_c t) - s_{i,2}g(t)\,\mathrm{csin}(2\pi f_c t) \quad (eq. 11)$$

$$t\epsilon[0, T_s], i = 1, \ldots, 16, T_s = 0.2, f_c = 100\ Hz$$

$g(t)$ is a rectangular pulse over $[0, T_s)$ with an amplitude of $\sqrt{\frac{2}{T_s}}$ and the parameters are;

$$s_{1,1} = -3d\ ,\ s_{1,2} = 3d\ ,\ s_{2,1} = -3d\ ,\ s_{2,2} = d\ ,\ s_{3,1} = -3d\ ,\ s_{3,2} = -d\ ,\ s_{4,1} = -3d\ ,\ s_{4,2} = -3d$$
$$s_{5,1} = -d\ ,\ s_{5,2} = 3d\ ,\ s_{6,1} = -d\ ,\ s_{6,2} = d\ ,\ s_{7,1} = -d\ ,\ s_{7,2} = -d\ ,\ s_{8,1} = -d\ ,\ s_{8,2} = -3d$$
$$s_{9,1} = d\ ,\ s_{9,2} = 3d\ ,\ s_{10,1} = d\ ,\ s_{10,2} = d\ ,\ s_{11,1} = d\ ,\ s_{11,2} = -d\ ,\ s_{12,1} = d\ ,\ s_{12,2} = -3d$$
$$s_{13,1} = 3d\ ,\ s_{13,2} = 3d\ ,\ s_{14,1} = 3d\ ,\ s_{14,2} = d\ ,\ s_{15,1} = 3d\ ,\ s_{15,2} = -d\ ,\ s_{16,1} = 3d\ ,\ s_{16,2} = -3d$$

Figure 8: Parameters of the QAM signals

180 samples are used to represent the transmitted signal for each symbol and the following is the plot of the first 5 symbols;

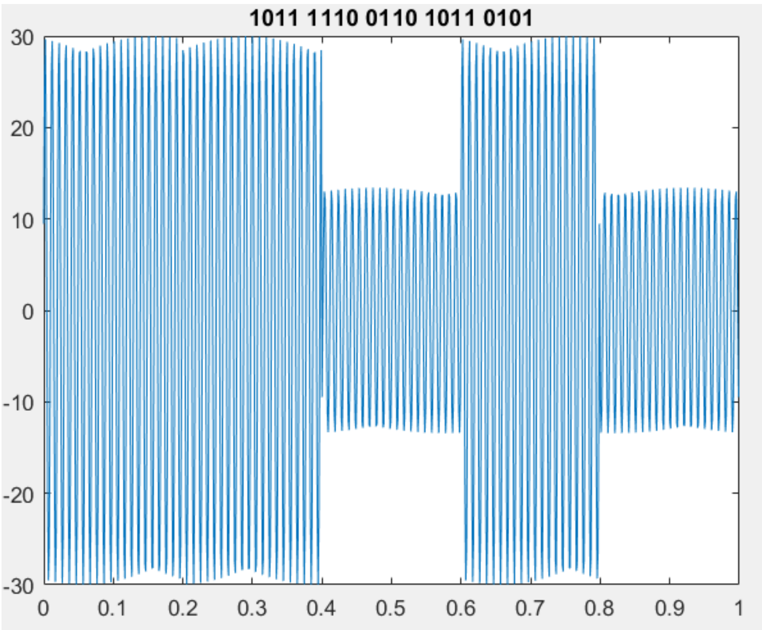

Figure 8: Plot of the first 5 symbols

Part B)

In this part, zero-mean Gaussian noise is added to the signal constructed in the first part. Gaussian noise variance is picked as $\sigma^2 = 5$. After the Gaussian noise received signal is;

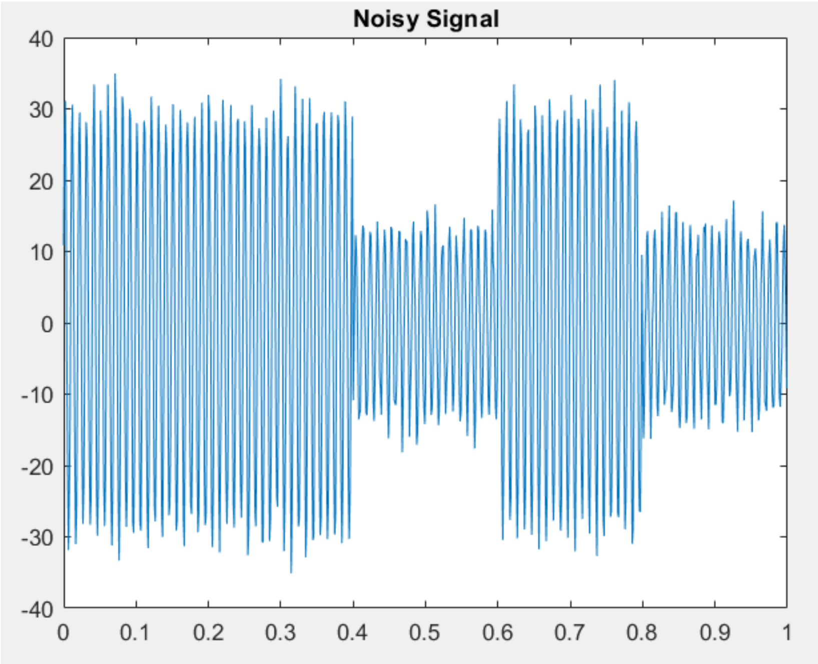

Figure 9: Plot of the first 5 symbols with noise

Then, after finding the basis functions of the signals signal constellation point are found by the equation;

$$s_{1vector} = s_i * \psi_1$$

$$s_{2vector} = s_i * \psi_2$$

Where ψ_1 and ψ_2 are;

$$\psi_1 = \cos(2\pi fct)\,, \psi_2 = \sin(2\pi fct)$$

The correlator outputs are;

$$r_1 = s_{i,noisy} * \psi_1$$

$$r_2 = s_{i,noisy} * \psi_2$$

Noisy signals variance is $\sigma^2 = 5$. First 100 pairs of $[r_1, r_2]$ and the constellation points are in the following figure;

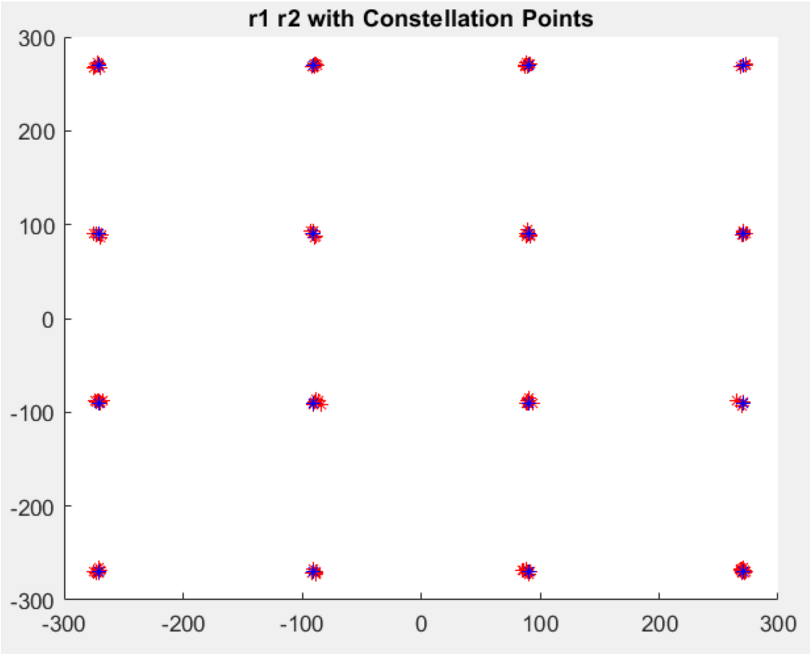

Figure 10: Constellation points for first 100 symbols

Blue points are the signal constellation points and red points are the received or noisy signal constellation points for the first 100 symbols. In a medium SNR received constellation points are closer to the blue points which enable the system to estimate received signals correctly. Variance has direct proportion with red points distance to the blue points. For high variance, received signal constellation points look like they are distributed randomly.

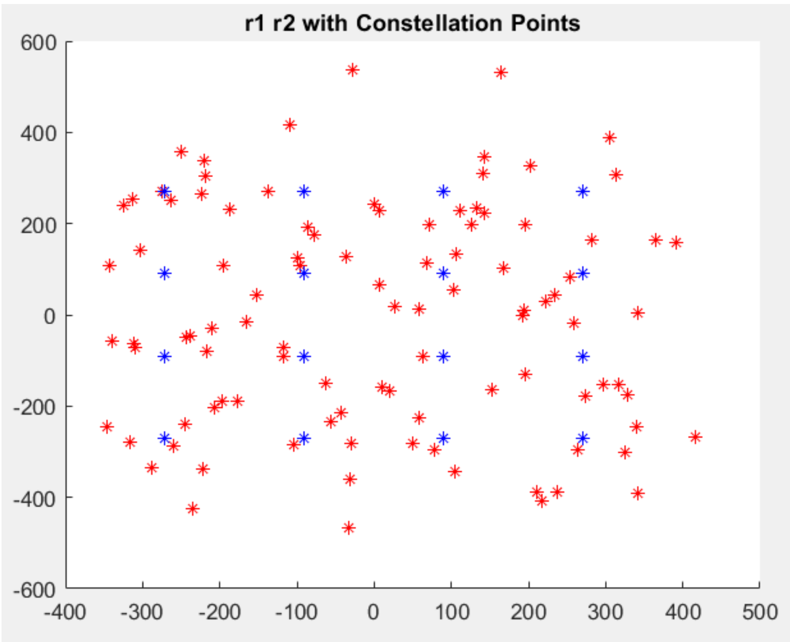

Figure 11: Constellation points for first 100 symbols for $\sigma^2 = 10000$

Part C)

By applying the ML rule to the correlator outputs, experimental probability of both the bit error and the symbol error is 0. These results are calculated according to the equation 7. Receiver perfectly demodulates the symbols and bits. For theoretical probability of error in a M-QAM is [2];

$$P_e = 4\frac{\sqrt{M}-1}{\sqrt{M}}Q\left(\frac{d}{2\sigma}\right) - 4\frac{\left(\sqrt{M}-1\right)^2}{M}Q^2\left(\frac{d}{2\sigma}\right) \ (eq.12)$$

$$= 3Q\left(\frac{3}{2\sigma}\right) - \frac{9}{4}Q^2(\frac{3}{2\sigma})$$

From the equation 9, we know that;

$$\sigma_{theo}{}^2 = \frac{\sigma_{exp}{}^2 E_{theo\_avg}}{E_{exp\_avg}}$$

Where $E_{theo\_avg} = \frac{2d^2(M-1)}{3} = 90$ and $E_{exp\_avg} = 81000$ since;

$$E_{exp\_avg} = \frac{(s_{1,vector}{}^2 + s_{2,vector}{}^2)}{16}$$

Finally, from the equation 9, $\sigma_{theo}{}^2 = 0.044$ and $P_e \cong 0$.

This concludes that theoretical and experimental probability of errors is the same and they are compatible with each other.

In this type of mapping signals symbols' bits are changing sometimes more than 1 bit and it may result demodulating the signal with more than 1 bit. Thus, by mapping according to the gray coding may lessen the probability of bit error.

```
resulted_bits = ["0000" "0001"  "0010" "0011" "0100" "0101" "0110" "0111" "1000" "1001" "1010" "1011" "1100" "1101" "1110" "1111"];
resulted_gray_bits = ["0000" "0001" "0011" "0010" "0100" "0101" "0111" "0110" "1100" "1101" "1111" "1110" "1000" "1001" "1011" "1010"];
```

Figure 12: Bit representation of two types of coding

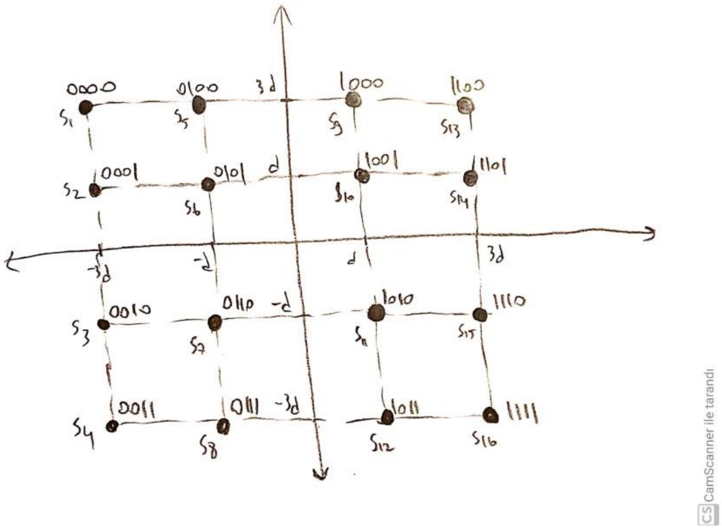

Figure 13: Square constellation plot for gradually increasing encoding (the first type of mapping)

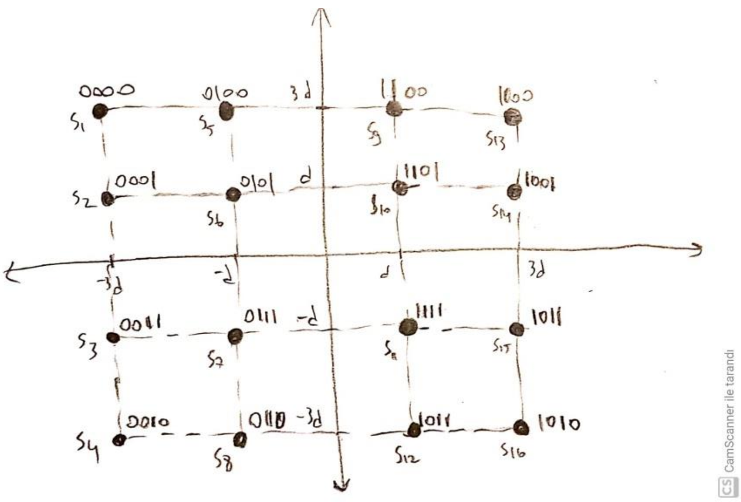

Figure 14: Square constellation plot for gray encoding (the second type of mapping)

For high SNR values probability of error is decreased experimentally for $\sigma^2 = 650$ but they do not have much difference for low SNR;

```
Statistical probability of bit error: 0.000245
Statistical probability of gray bit error: 0.000145
```

Figure 15: Probability of error for two types of coding

Thus, gray coding is more reliable than the first type of coding for high SNR.

Part D)

For the expected probability of error change $\sigma^2\epsilon[12100,670]$. When we applied the noises between the given interval to the both theoretical and experimental signals, we get the following plot;

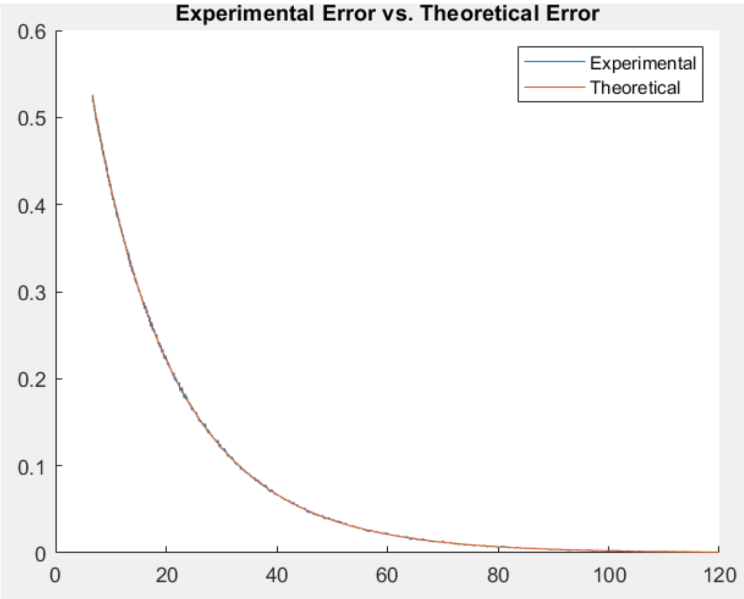

Figure 16: Error probability plot with respect to SNR

It can be seen that theoretical and experimental curves are very close to each other, in other words, our experimental results are compatible with theoretical results.

References

[1] Proakis, John G., and Masoud Salehi. Fundamentals of Communication Systems. 2nd ed., Pearson, 2014, p. 402.

[2] S. Gezici, Telecommunications-1 Lecture Notes Part 3, p. 15.