

PHYS486-HW1: Nuclear Decays

Nicholas Cemenenkoff

April 13, 2018

1 Problem Statement

Consider a radioactive decay problem involving two types of nuclei, A and B, with populations $N_A(t)$ and $N_B(t)$. Suppose that type A nuclei decay to form type B nuclei, which then also decay, according to the differential equations

$$\frac{dN_A}{dt} = -\frac{N_A}{\tau_A}, \quad (1)$$

$$\frac{dN_B}{dt} = \frac{N_A}{\tau_A} - \frac{N_B}{\tau_B}, \quad (2)$$

where τ_A and τ_B are the decay time constants of each type of nucleus. In this model, assume that nuclei of type A are naturally occurring, but nuclei of type B are not: that is, assume type B nuclei only appear as the product of the prior decay of a type A nucleus. Now imagine that we have been asked to determine the age of a meteorite by measuring the amount of type A and type B nuclei within it. Assume we know the time constants of species A and B (i.e. τ_A and τ_B respectively) but that we may only measure the ratio of the two types of nuclei (i.e. $N_B(t)/N_A(t)$) and *not* their individual populations. How well can we measure the age of the meteorite if our measurement of N_B/N_A has an accuracy of $\pm 0.5\%$? Does the precision of an age measurement depend on the age of the object and/or the ratio of the time constants of the two species, $\gamma = \tau_A/\tau_B$? We address these questions in the following sections.

2 Numerical Solution

Define $\gamma \equiv \tau_A/\tau_B$, $T \equiv t/\tau_A$, and $dT \equiv \Delta t/\tau_A$. The following numerical solution employs Euler's method of approximating solutions to differential equations given by

$$N_U(T + \Delta t) \approx N_U(T) + \frac{dN_U}{dt} \Delta t. \quad (3)$$

For N_A , substitute (1) into (3):

$$\begin{aligned} N_A(T + \Delta t) &\approx N_A(T) + \frac{dN_A}{dt} \Delta t \\ &\approx N_A(T) - \frac{N_A(T)}{\tau_A} \Delta t \\ &\approx N_A(T) - N_A(T) \left(\frac{\Delta t}{\tau_a} \right) \\ &\approx N_A(T) - N_A(T) dT. \end{aligned} \quad (4)$$

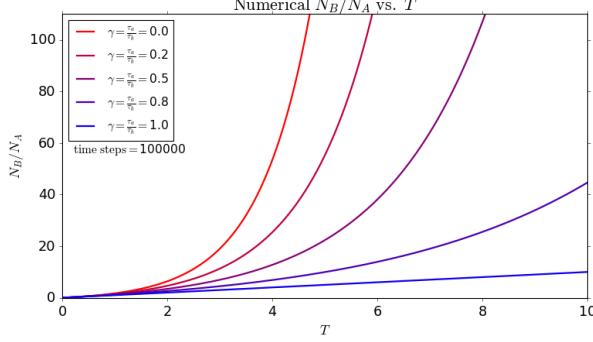


Figure 1: N_B/N_A vs. T for $0 \leq \gamma \leq 1$. Notice that $\frac{d}{dT} [N_B/N_A]$ gets large at a faster rate as $\gamma \rightarrow 0$. Note that $\gamma < 0$ is unphysical because $t, \tau_A, \tau_B > 0$.

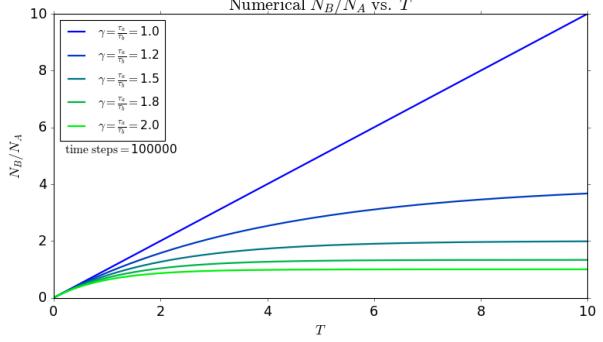


Figure 2: N_B/N_A vs. T for $1 \leq \gamma \leq 2$. Notice that $\frac{d}{dT} [N_B/N_A]$ becomes vanishingly small at a faster rate as $\gamma \rightarrow \infty$. Note that when $\gamma = 1$, $\frac{d}{dT} [N_B/N_A]$ is constant.

For N_B , substitute (2) into (3):

$$\begin{aligned}
N_B(T + \Delta t) &\approx N_B(T) + \frac{d}{dt} [N_B(T)] \Delta t \\
&\approx N_B + \left(\frac{N_A}{\tau_A} - \frac{N_B}{\tau_B} \right) \Delta t \\
&\approx N_B + N_A \frac{\Delta t}{\tau_A} - N_B \frac{\Delta t}{\tau_B} \cdot \frac{\tau_A}{\tau_A} \\
&\approx N_B + N_A dT - N_B \left(\frac{\tau_A}{\tau_B} \right) \left(\frac{\Delta t}{\tau_A} \right) \\
&\approx N_B(T) + N_A(T) dT - N_B(T) \gamma dT.
\end{aligned} \tag{5}$$

Equation (5) divided by equation (4) constitutes our numerical solution to this problem. Using a for loop, we employ the initial conditions $N_A(0) = 1$ and $N_B(0) = 0$ and a T time step of $10/10^5 = 0.0001$ to construct N_A and N_B arrays, and then divide N_B by N_A element-wise. Notice that by introducing γ , we reduce the number of input parameters and thus the complexity of the analysis.

3 Physical Analysis

The initial conditions not only convey the fact that nuclei B are not naturally occurring (so the initial population of B nuclei is zero), but also that the populations are normalized by the initial number of type A nuclei, $N_A(0)$, thus transforming N_A from an integer into a ratio between 0 and 1.

Figures 1 and 2 show cases when $0 \leq \gamma \leq 1$ and $1 \leq \gamma \leq 2$ respectively. Notice in Figure 1 that $\frac{d}{dT} [N_B/N_A]$ is large for larger values of T and $\gamma \ll 1$. This fact is experimentally desirable because given a measurement of N_B/N_A with an accuracy of 0.5% at a large T -value, the associated error bound for T will be small. This small error bound for T enables us to theoretically measure the age of old objects with high precision. Conversely, Figure 2 shows why $\gamma > 1$ is undesirable: $\frac{d}{dT} [N_B/N_A]$ is nearly zero for higher values of T , meaning the associated error bound for T given a measurement of N_B/N_A with an accuracy of 0.5% will be huge, implying measurements of N_B/N_A at large T -values give wildly imprecise age results.

Note additionally that in the case of small γ , N_A decays much faster than N_B . Because N_B stabilizes far more quickly than N_A , we may compare the two populations without having to account for the possibility of a decrease in N_B as it moves down its decay chain. As long as at least the ratio of the decay constants for the two nuclei, τ_A/τ_B , is known, we are able to make measurements of N_B/N_A to reliably infer the relative age of objects (presumably rocks) containing type A nuclei. This model is a simplified example of a process

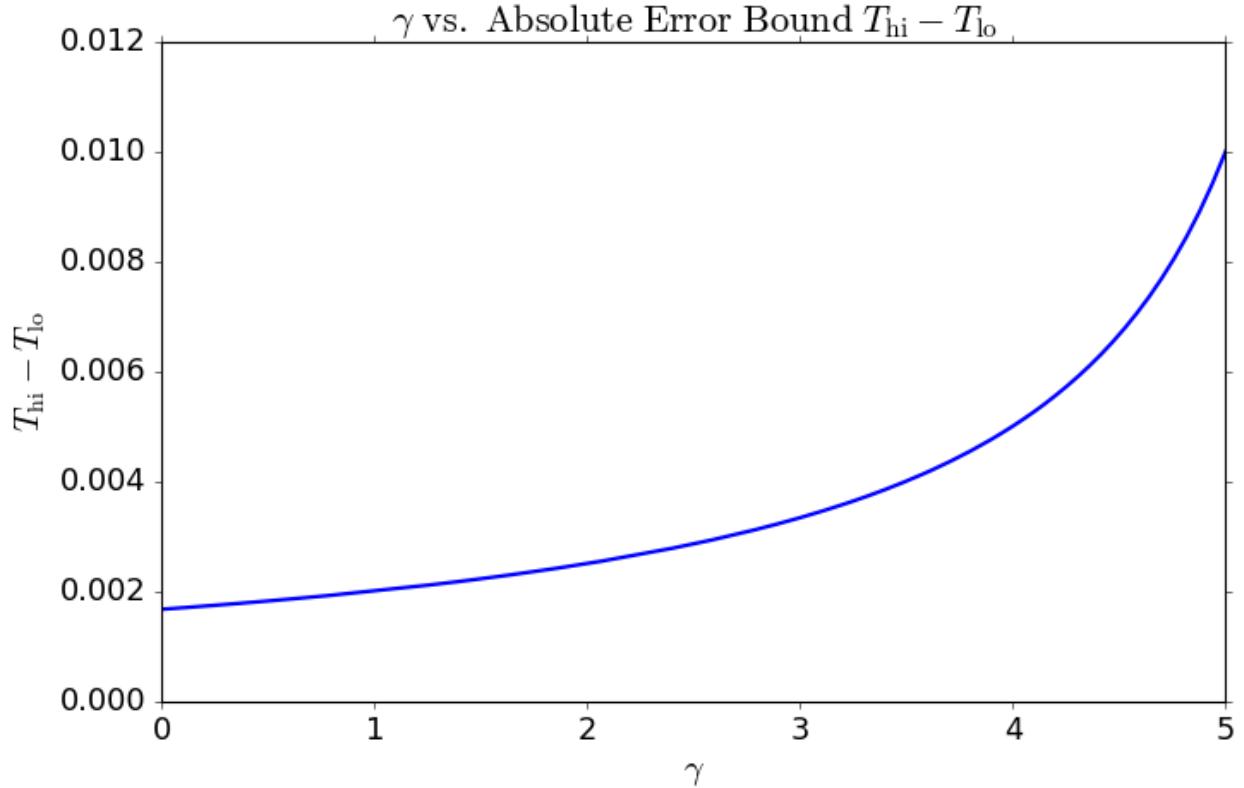


Figure 3: γ vs. $T_{\text{hi}} - T_{\text{lo}}$ for $0 \leq \gamma \leq 5$ and a measured $N_B/N_A = 0.2$ accurate to $\pm 0.5\%$. Notice for $0 < \gamma \ll 1$, we may trust a measurement of $N_B/N_A = 0.2$ accurate to $\pm 0.5\%$ up to $\pm \left(\frac{0.002 \text{ year}}{2} \cdot \frac{365 \text{ days}}{\text{year}} \right) \frac{1}{\tau_A} = \pm 0.365 \frac{\text{days}}{\tau_A}$ (assuming t is measured in years).

used by geologists known as carbon dating. Although this model is crude because it does not account for subsequent steps in the decay chain after type A nuclei decay into type B, as long as γ is small enough, it provides a precise (though possibly physically inaccurate) measurement of the age of a very old object containing type A nuclei.

4 Error Analysis

To what degree may we trust the Euler method for approximating the solution to this set of differential equations? Does the error in T depend on γ ? The first question is addressed by coding the analytical solutions to (1) and (2) to calculate the absolute difference between the analytical and numerical solutions. If the maximum value of this difference exceeds 0.5% of the analytical solution, define the results as inaccurate and remedy the issue by increasing the linear sampling of T . Given accurate enough results, we are then poised to investigate how T depends on γ . I found the associated absolute error bound for T for a measurement of $N_B/N_A = 0.2$ accurate to $\pm 0.5\%$ for 500 γ -values logarithmically sampled from 0 to 5. To recover precise T -values associated with the high and low ends of the N_B/N_A measurement, I ensured T would act as a continuum by using an array of 10 million linearly sampled values between 0 and 10. Figure 3 shows that as long as γ is much less than 1, the measurement of the scaled age T associated with a measurement of $N_B/N_A = 0.2$ accurate to $\pm 0.5\%$ has a precision of $\pm \frac{0.002}{2} \Rightarrow \pm 0.1\%$. We thus conclude that not only is a very small γ -value desirable for theoretical experimental precision, it is desirable for *numerical* precision in that the absolute error bound for T shrinks to a minimum as $\gamma \rightarrow 0$.