IOC 5275: Brain Computer Interface **(Due: 05/04/2021)**

Lab3: Event Related Potential Analysis

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Submission Policy

Read all the instructions below carefully before you start working on the assignment, and before you make a submission. For this assignment, please hand in the following your report (pdf) and code (.ipynb or .m file).

- **PLAGIARISM IS STRICTLY PROHIBITED. (0 point for Plagiarism)**
- For mathematical problem(s), please show your work step by step and clarify statement of theorem you use (if any). Answering without mathematical derivations will get 0 point.
- Submission deadline: **2021.05.04 09:00:00 AM**.
- **Late submission penalty formula:**

$$\textit{original score} \times (0.7)^{\#(\textit{days late})}$$

File Format

- Each group submits 1 report (.pdf and .tex file) and 1 code (.ipynb or .m).
- **Report** must contains observations, results and explanations. Please name your .pdf and .tex file as **5275_Lab3_GroupNum.pdf** and **5275_Lab3_GroupNum.tex**,respectively.
- Paper submission is not allowed. **Please use our LaTeX template to complete your report**.
- **Code** file must contains comments to explain your code. Please name your code file as **5275_Lab3_GroupNum.ipynb/.m**
- Implementation will be graded by completeness, algorithm correctness, model description, and discussion.
- **Illegal format penalty:** −5 points for violating each rule of file format.

Prerequest

To finish programming problem, you could choose Matlab or Python base on your programming preference.

Matlab 2020a+

- NYCU installation page
- NCTU installation tutorial
- EEGLab official installation page (v2020.0+ is recommended)

Python 3.7+

- MNE official installation page (0.20.7+ is recommended)

1 Mathematical problem

1.1 Denoising the ERP signals–Kalman Filtering

1.1.1 Framework of Kalman filtering

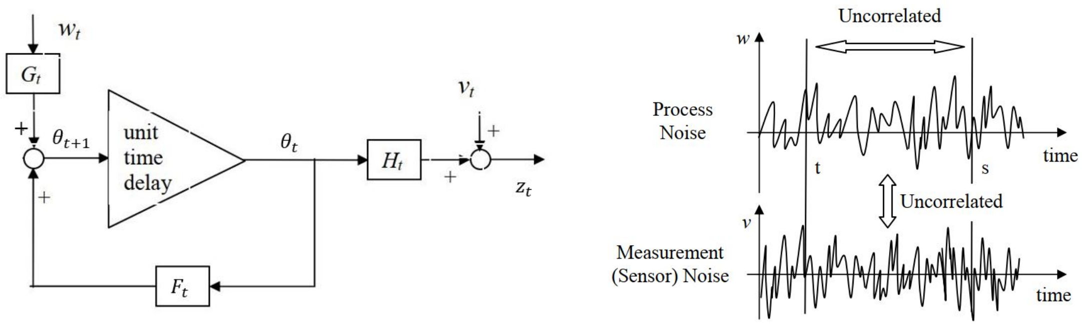


Figure: (Left) State space representation of linear time varying system (ERP) with process noise and measurement noise. (Right) Noise characteristics.

Model Formalization

Let $\theta_t \in \mathbb{R}^{k\times 1}$ denotes the input signal (state) and $z_t \in \mathbb{R}^{M\times 1}$ denotes the output signal at time t. According to *S. D. Georgiadis et al.*, both θ_t and z_t are defined as vector-valued processes.We could construct a state-space model for the linear dynamic systems (ERP) as below:

$$\theta_{t+1} = F_t\theta_t + G_t w_t,\ z_t = H_t\theta_t + v_t \tag{1.1}$$

with initial condition θ_0, Here we define $w_t \in \mathbb{R}^{k\times 1}$ as the process noise with zero mean, $v_t \in \mathbb{R}^{M\times 1}$ as the measurement noise with zero mean, $F_t \in \mathbb{R}^{k\times k}$ as the transition matrix, and $H_t \in \mathbb{R}^{M\times k}$, $G_t \in \mathbb{R}^{k\times k}$.

Hypothesis of ERP system

- F_t, G_t, and H_t are known sequences of matrices
- (θ_0, w_t, v_t) is a sequence of mutually uncorrelated random vectors with finite variance
- $E[w_t] = \mathbf{0}_{k\times 1},\ E[v_t] = \mathbf{0}_{M\times 1}\ \forall t$
- The covariances C_w, C_v, and C_{wv} are known sequences of matrices.

Given the following conditions:

$$C_v(t,s) = E[v_t \cdot v_s^T] = \begin{cases} \mathbf{0}_{M\times M}, & \text{where } t \neq s \\ R_{t,M\times M}, & \text{where} t = s \end{cases} \tag{1.2}$$

$$C_w(t,s) = E[w_t \cdot w_s^T] = \begin{cases} \mathbf{0}_{k\times k}, & \text{where } t \neq s \\ Q_{t,k\times k}, & \text{where} t = s \end{cases} \tag{1.3}$$

$$C_{wv}(t,s) = E[w_t \cdot v_s^T] = \mathbf{0}_{k\times M},\ \forall t,\ \forall s \tag{1.4}$$

To obtain an optimal value of θ_t based on measurements z_t, we minimize the mean square error:

$$J_t = E[(\hat{\theta}_t - \theta_t)^T(\hat{\theta}_t - \theta_t)] \tag{1.5}$$

with the constraints (1.1) to (1.4). We call $\hat{\theta}_{t|t-1}$ as expected state transition based on model, $\hat{z}_t$ as expected output.

$$\begin{aligned} \theta_t &= F_{t-1}\theta_{t-1} + G_{t-1}w_{t-1} \\ \hat{\theta}_{t|t-1} &= E[F_{t-1}\hat{\theta}_{t-1} + G_{t-1}w_{t-1}] = F_{t-1}\hat{\theta}_{t-1} + G_{t-1}E[w_{t-1}] \\ \hat{z}_t &= E[H_t\hat{\theta}_{t|t-1} + v_t] = H_t\hat{\theta}_{t|t-1} + E[v_t] = H_t\hat{\theta}_{t|t-1} \end{aligned} \tag{1.6}$$

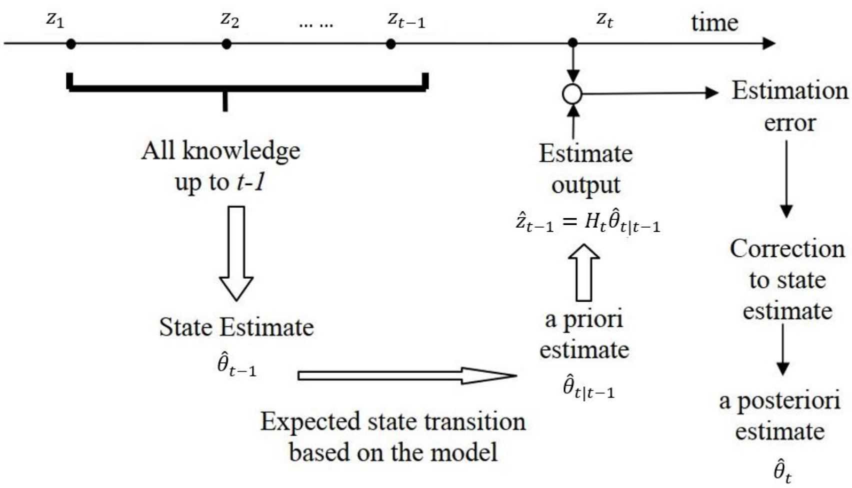


Figure: Outline of the Kalman filter algorithm

1.1.2 Correction of the state estimate

Assimilating the new measurement z_t, we can update the state estimate in proportion to the output estimation error.

$$\hat{\theta}_t = \hat{\theta}_{t|t-1} + K_t(z_t - \hat{z}_t) \tag{1.7}$$

Let $\epsilon_t = \hat{\theta}_{t|t-1} - \theta_t$ be a priori estimation error, then

$$\begin{aligned} e_t \equiv \hat{\theta}_t - \theta_t &= \hat{\theta}_{t|t-1} + K_t\big(z_t - H_t\hat{\theta}_{t|t-1}\big) - \theta_t \\ &= \hat{\theta}_{t|t-1} + K_t\big(H_t\theta_t + v_t - H_t\hat{\theta}_{t|t-1}\big) - \theta_t = \big(\hat{\theta}_{t|t-1} - \theta_t\big) - K_tH_t\big(\hat{\theta}_{t|t-1} - \theta_t\big) + K_tv_t \\ &= \big(I_k - K_tH_t\big)\epsilon_t + K_tv_t \end{aligned} \tag{1.8}$$

Equation (1.7) provides a structure of linear filter in recursive form. Denoting $K_t \in \mathbb{R}^{k\times M}$ as a gain matrix to be optimized so that the mean squared error (expectation of $e_t^Te_t$) of state estimation may be minimized.

Problem 1 (5 points)

Please show that

$$e_t^Te_t = \epsilon_t^T\epsilon_t + \epsilon_t^TH_t^TK_t^TK_tH_t\epsilon_t - 2\epsilon_t^TK_tH_t\epsilon_t + 2\epsilon_t^TK_tv_t - 2v_t^TK_t^TK_tH_t\epsilon_t + v_t^TK_t^TK_tv_t \tag{1.9}$$

Problem 1's answer

First, we calculate and rewrite e_t and e_t^T
$e_t = (I_k - K_tH_t)\epsilon_t + K_tv_t = \epsilon_t - K_tH_t\epsilon_t + K_tv_t$
$e_t^T = \epsilon_t^T(I_k - K_tH_t)^T + v_t^TK_t^T = \epsilon_t^T(I_k - H_t^TK_t^T) + v_t^TK_t^T = \epsilon_t^T - \epsilon_t^TH_t^TK_t^T + v_t^TK_t^T$

$e_t^Te_t = \epsilon_t^T\epsilon_t - \epsilon_t^TK_tH_t\epsilon_t + \epsilon_t^TK_tv_t - \epsilon_t^TH_t^TK_t^T\epsilon_t + \epsilon_t^TH_t^TK_t^TK_tH_t\epsilon_t$
$-\epsilon_t^TH_t^TK_t^TK_tv_t + v_t^TK_t^T\epsilon_t - v_t^TK_t^TK_tH_t\epsilon_t + v_t^TK_t^TK_tv_t$ – eq(1)

We know that $\left(\epsilon_t^TK_tH_t\epsilon_t\right)^T = \epsilon_t^TH_t^TK_t^T\epsilon_t$ and $\left(\epsilon_t^TK_tv_t\right)^T = v_t^TK_t^T\epsilon_t$
and $\left(v_t^TK_t^TK_tH_t\epsilon_t\right)^T = \epsilon_t^TH_t^TK_t^TK_tv_t$ and all of them are const

Hence, we can rewrite eq(1) to
$e_t^Te_t = \epsilon_t^T\epsilon_t + \epsilon_t^TH_t^TK_t^TK_tH_t\epsilon_t - 2\epsilon_t^TK_tH_t\epsilon_t + 2\epsilon_t^TK_tv_t - 2v_t^TK_t^TK_tH_t\epsilon_t + v_t^TK_t^TK_tv_t$

Let us differentiate the scalar function $e_t^T e_t$ with respect to matrix K_t by using the following matrix differentiation rules.

Matrix differentiation rule 1
Let $a \in \mathbb{R}^{k\times 1}$, $b \in \mathbb{R}^{M\times 1}$, and $K \in \mathbb{R}^{k\times M}$ is same as above K_t (We omit the subscript t for brevity).

$$f \equiv [a_1, ..., a_k]\begin{bmatrix} K_{11} & \dots & K_{1M} \\ \vdots & \ddots & \vdots \\ K_{k1} & \dots & K_{kM}\end{bmatrix}\begin{bmatrix} b_1 \\ \vdots \\ b_M\end{bmatrix} = a^T K b \Rightarrow \frac{df}{dK} = \left[\frac{\partial f}{\partial K_{ij}}\right] = [a_i b_j] = ab^T,\ \forall i \in \mathbb{Z}_k, j \in \mathbb{Z}_M \tag{1.10}$$

Matrix differentiation rule 2
Let $b, c \in \mathbb{R}^{M\times 1}$, and $K \in \mathbb{R}^{k\times M}$, and $g = c^T K^T K b$, then

$$\frac{dg}{dK_t} = \left[\frac{\partial g}{\partial K_{im}} \sum_{i=1}^{M}\sum_{j=1}^{k}\sum_{l=1}^{k} K_{il}c_l K_{ij} b_j\right] = \left[\sum_{j=1}^{k} c_m K_{ij} b_j + \sum_{j=1}^{k} K_{il} c_l b_m\right] = Kbc^T + Kcb^T \tag{1.11}$$

Problem 2 (15 points)

Use the matrix differentiation rule 1 and 2 to show that

$$\frac{de_t^T e_t}{dK_t} = 2K_t H_t \epsilon_t \epsilon_t^T H_t^T - 2[K_t H_t \epsilon_t v_t^T + K_t v_t \epsilon_t^T H_t^T] + 2K_t v_t v_t^T + 2[\epsilon_t v_t^T - \epsilon_t \epsilon_t^T H_t^T] \tag{1.12}$$

Problem 2's answer

We get 2 matrix difference rules from previous block:
rule 1: $\frac{d}{dK}(a^T K b) = ab^T$
rule 2: $\frac{d}{dK}(a^T K^T K b) = Kba^T + Kab^T$

$\frac{de_t^T e_t}{dK_t} = \frac{d}{dK_t}(\epsilon_t^T \epsilon_t + \epsilon_t^T H_t^T K_t^T K_t H_t \epsilon_t - 2\epsilon_t^T K_t H_t \epsilon_t + 2\epsilon_t^T K_t v_t - 2v_t^T K_t^T K_t H_t \epsilon_t + v_t^T K_t^T K_t v_t)$
$= \frac{d}{dK_t}(\epsilon_t^T H_t^T K_t^T K_t H_t \epsilon_t) + \frac{d}{dK_t}(-2\epsilon_t^T K_t H_t \epsilon_t + 2\epsilon_t^T K_t v_t - 2v_t^T K_t^T K_t H_t \epsilon_t + v_t^T K_t^T K_t v_t)$
Set $a = b = H_t\epsilon_t$ in red part of above equation, then we can get a new equation by rule2:
$\frac{de_t^T e_t}{dK_t} = 2K_t H_t \epsilon_t \epsilon_t^T H_t^T - 2\frac{d}{dK_t}(\epsilon_t^T K_t H_t \epsilon_t) + \frac{d}{dK_t}(2\epsilon_t^T K_t v_t - 2v_t^T K_t^T K_t H_t \epsilon_t + v_t^T K_t^T K_t v_t)$
Set $a = \epsilon_t$,$b = H_t\epsilon_t$ in red part of above equation, then we can get a new equation by rule1:
$\frac{de_t^T e_t}{dK_t} = 2K_t H_t \epsilon_t \epsilon_t^T H_t^T - 2\epsilon_t \epsilon_t^T H_t^T + 2\frac{d}{dK_t}(\epsilon_t^T K_t v_t) + \frac{d}{dK_t}(-2v_t^T K_t^T K_t H_t \epsilon_t + v_t^T K_t^T K_t v_t)$
Set $a = \epsilon_t$,$b = v_t$ in red part of above equation, then we can get a new equation by rule1:
$\frac{de_t^T e_t}{dK_t} = 2K_t H_t \epsilon_t \epsilon_t^T H_t^T - 2\epsilon_t \epsilon_t^T H_t^T + 2\epsilon_t v_t^T + \frac{d}{dK_t}(-2v_t^T K_t^T K_t H_t \epsilon_t + v_t^T K_t^T K_t v_t)$
$= 2K_t H_t \epsilon_t \epsilon_t^T H_t^T + 2(\epsilon_t v_t^T - \epsilon_t \epsilon_t^T H_t^T) - 2\frac{d}{dK_t}(v_t^T K_t^T K_t H_t \epsilon_t) + \frac{d}{dK_t}(v_t^T K_t^T K_t v_t)$
Set $a = v_t$,$b = H_t\epsilon_t$ in red part of above equation, then we can get a new equation by rule2:
$\frac{de_t^T e_t}{dK_t} = 2K_t H_t \epsilon_t \epsilon_t^T H_t^T + 2(\epsilon_t v_t^T - \epsilon_t \epsilon_t^T H_t^T) - 2(K_t H_t \epsilon_t v_t^T + K_t v_t \epsilon_t^T H_t^T) + \frac{d}{dK_t}(v_t^T K_t^T K_t v_t)$
Set $a = b = v_t\epsilon_t$ in red part of above equation, then we can get a new equation by rule2:
$\frac{de_t^T e_t}{dK_t} = 2K_t H_t \epsilon_t \epsilon_t^T H_t^T - 2(K_t H_t \epsilon_t v_t^T + K_t v_t \epsilon_t^T H_t^T) + 2K_t v_t v_t^T + 2(\epsilon_t v_t^T - \epsilon_t \epsilon_t^T H_t^T)$

The necessary condition for the mean squared error of state estimate with respect to the gain matrix K_t is:

$$\frac{dJ_t}{dK} = \mathbf{0}_{k\times M}$$

Taking expectation of $e_t^T e_t$, differentiating it w.r.t. K_t and setting it to zero yield:

$$E[K_t H_t \epsilon_t \epsilon_t^T H_t^T - K_t H_t \epsilon_t v_t^T - K_t v_t \epsilon_t^T H_t^T + K_t v_t v_t^T + \epsilon_t v_t^T - \epsilon_t \epsilon_t^T H_t^T] = \mathbf{0}_{k\times M} \tag{1.13}$$

Which means that K_t and H_t can be factored out,

$$K_t H_t E[\epsilon_t \epsilon_t^T] H_t^T - K_t H_t E[\epsilon_t v_t^T] - K_t E[v_t \epsilon_t^T] H_t^T + K_t E[v_t v_t^T] + E[\epsilon_t v_t^T] - E[\epsilon_t \epsilon_t^T] H_t^T = \mathbf{0}_{k\times M} \tag{1.14}$$

Since we know that

$$\hat{\theta}_{t|t-1} = E[F_{t-1}\hat{\theta}_{t-1} + G_{t-1}w_{t-1}] = F_{t-1}\hat{\theta}_{t-1} \tag{1.15}$$

and

$$\epsilon_t = \hat{\theta}_{t|t-1} - \theta_t,$$

we can examine $E[\epsilon_t v_t^T] = E[(\hat{\theta}_{t|t-1} - \theta_t)v_t^T] = E[\hat{\theta}_{t|t-1}v_t^T] - E[\theta_t v_t^T] = \mathbf{0}_{k\times M}$

Problem 3 (10 points)

Please show that

$$E[\epsilon_t v_t^T] = \mathbf{0}_{k\times M} \ \&\ E[v_t \epsilon_t^T] = \mathbf{0}_{M\times k} \tag{1.16}$$

Problem 3's answer

$E[\epsilon_t v_t^T] = E[\hat{\theta}_{t|t-1}v_t^T] - E[\theta_t v_t^T]$
$\quad = E[(F_{t-1}\hat{\theta}_{t-1})v_t^T] - E[(F_{t-1}\theta_{t-1} + G_{t-1}w_{t-1})v_t^T]$
$\quad = F_{t-1}E[\hat{\theta}_{t-1}v_t^T] - F_{t-1}E[\theta_{t-1}v_t^T] - G_{t-1}E[w_{t-1}v_t^T]$
In the hypothesis of ERP system, we know that w, v and θ are 3 uncorrelated vectors, and $E[v_t] = \mathbf{0}_{M\times 1}$ $\forall t$
2 vectors are called uncorrelated if $E[xy^T] = E[x]E[y]^T$
So we can rewrite the equation to:
$E[\epsilon_t v_t^T] = F_{t-1}E[\hat{\theta}_{t-1}]E[v_t]^T - F_{t-1}E[\theta_{t-1}]E[v_t]^T - G_{t-1}E[w_{t-1}]E[v_t]^T$
$\quad = F_{t-1}E[\hat{\theta}_{t-1}]\mathbf{0}_{1\times M} - F_{t-1}E[\theta_{t-1}]\mathbf{0}_{1\times M} - G_{t-1}E[w_{t-1}]\mathbf{0}_{1\times M}$
$\quad = \mathbf{0}_{K\times M} - \mathbf{0}_{K\times M} - \mathbf{0}_{K\times M} = \mathbf{0}_{K\times M}$

$E[v_t\epsilon_t^T] = E[v_t(\hat{\theta}_{t|t-1} - \theta_t)^T] = E[v_t(F_{t-1}\hat{\theta}_{t-1} - \theta_t)^T] = E[v_t(\hat{\theta}_{t-1}^T F_{t-1}^T - \theta_t^T)]$
$\quad = E[v_t(\hat{\theta}_{t-1}^T F_{t-1}^T)] - E[v_t\theta_t^T] = E[v_t\hat{\theta}_{t-1}^T]F_{t-1}^T - E[v_t\theta_t^T]$
Because v and θ are uncorrelated vectors and $E[v_t] = \mathbf{0}_{M\times 1}$ $\forall t$, we can rewrite the equation to:
$E[v_t\epsilon_t^T] = E[v_t]E[\hat{\theta}_{t-1}]^T F_{t-1}^T - E[v_t]E[\theta_t]^T$
$\quad = \mathbf{0}_{M\times 1}E[\hat{\theta}_{t-1}]^T F_{t-1}^T - \mathbf{0}_{M\times 1}E[\theta_t]^T = \mathbf{0}_{M\times k} - \mathbf{0}_{M\times k} = \mathbf{0}_{M\times k}$

Let us define the error covariance of a priori state estimation

$$C_{\hat{\theta}_{t|t-1}} \equiv E[\epsilon_t\epsilon_t^T] = E[(\hat{\theta}_{t|t-1} - \theta_t)(\hat{\theta}_{t|t-1} - \theta_t)^T] \tag{1.17}$$

Problem 4 (5 points)

Please show that

$$K_t = C_{\hat{\theta}_{t|t-1}} H_t^T \left(H_t C_{\hat{\theta}_{t|t-1}} H_t^T + C_v(t,t)\right)^{-1} \tag{1.18}$$

Such K_t is called the **Kalman Gain**.

Problem 4's answer

First, from (1.14) we have
$K_tH_tE[\epsilon_t\epsilon_t^T]H_t^T - K_tH_tE[\epsilon_t v_t^T] - K_tE[v_t\epsilon_t^T]H_t^T + K_tE[v_tv_t^T] + E[\epsilon_t v_t^T] - E[\epsilon_t\epsilon_t^T]H_t^T = \mathbf{0}_{k\times M}$
$\Rightarrow K_tH_tC_{\hat{\theta}_{t|t-1}}H_t^T - K_tH_tE[\epsilon_t v_t^T] - K_tE[v_t\epsilon_t^T]H_t^T + K_tC_v(t,t) + E[\epsilon_t v_t^T] - C_{\hat{\theta}_{t|t-1}}H_t^T = \mathbf{0}_{k\times M}$
$\Rightarrow K_tH_tC_{\hat{\theta}_{t|t-1}}H_t^T - \mathbf{0}_{k\times M} - \mathbf{0}_{k\times M} + K_tC_v(t,t) + \mathbf{0}_{k\times M} - C_{\hat{\theta}_{t|t-1}}H_t^T = \mathbf{0}_{k\times M}$ (1.16)
$\Rightarrow K_t\left(H_tC_{\hat{\theta}_{t|t-1}}H^T + C_v(t,t)\right) - C_{\hat{\theta}_{t|t-1}}H_t^T = \mathbf{0}_{k\times M}$
$\Rightarrow K_t\left(H_tC_{\hat{\theta}_{t|t-1}}H^T + C_v(t,t)\right) = C_{\hat{\theta}_{t|t-1}}H_t^T$
$\Rightarrow K_t = C_{\hat{\theta}_{t|t-1}}H_t^T\left(H_tC_{\hat{\theta}_{t|t-1}}H_t^T + C_v(t,t)\right)^{-1}$

1.1.3 Updating the Error Covariance

The above Kalman gain contains the a priori error covariance $C_{\hat{\theta}_{t|t-1}}$. This must be updated recursively based on each new measurement and the state transition model. Define the a posteriori state estimation error covariance

$$C_{\hat{\theta}_t} = E[e_t e_t^T] = E[(\hat{\theta}_t - \theta_t)(\hat{\theta}_t - \theta_t)^T] \tag{1.19}$$

Problem 5 (5 points)

Please show that

$$C_{\hat{\theta}_t} = (I_k - K_t H_t) C_{\hat{\theta}_{t|t-1}} \tag{1.20}$$

Problem 5's answer

We have $e_t = (I_k - K_t H_t)\epsilon_t + K_t v_t$ and $e_t^T = \epsilon_t^T (I_k - H_t^T K_t^T) + v_t^T K_t^T$

$$\begin{aligned}
C_{\hat{\theta}_t} &= E[e_t e_t^T] = E[(I_k - K_t H_t)\epsilon_t \epsilon_t^T (I_k - H_t^T K_t^T) + (I_k - K_t H_t)\epsilon_t v_t^T K_t^T + K_t v_t \epsilon_t^T (I_k - H_t^T K_t^T) + K_t v_t v_t^T K_t^T] \\
&= (I_k - K_t H_t) E[\epsilon_t \epsilon_t^T](I_k - H_t^T K_t^T) + (I_k - K_t H_t) E[\epsilon_t v_t^T] K_t^T + K_t E[v_t \epsilon_t^T](I_k - H_t^T K_t^T) + K_t E[v_t v_t^T] K_t^T \\
&= (I_k - K_t H_t) C_{\hat{\theta}_{t|t-1}} (I_k - H_t^T K_t^T) + \mathbf{0}_{k\times k} + \mathbf{0}_{k\times k} + K_t C_v(t,t) K_t^T \\
&= (I_k - H_t^T K_t^T - K_t H_t + K_t H_t H_t^T K_t^T) C_{\hat{\theta}_{t|t-1}} + (C_{\hat{\theta}_{t|t-1}} H_t^T - K_t H_t C_{\hat{\theta}_{t|t-1}} H_t^T) K_t^T \\
&= (I_k - H_t^T K_t^T - K_t H_t + K_t H_t H_t^T K_t^T + H_t^T K_t^T - K_t H_t H_t^T K_t^T) C_{\hat{\theta}_{t|t-1}} \\
&= (I_k - K_t H_t) C_{\hat{\theta}_{t|t-1}}
\end{aligned}$$

Derive (1.21), one cancan compute $C_{\hat{\theta}_{t+1|t}}$ by using the state transition equation (1.1). Consider

$$\epsilon_{t+1} = \hat{\theta}_{t+1|t} - \theta_{t+1|t} = F_t \hat{\theta}_t - (F_t \theta_t + G_t w_t) = F_t e_t - G_t w_t. \tag{1.21}$$

From (1.17)

$$\begin{aligned}
C_{\hat{\theta}_{t+1|t}} &= E[\epsilon_{t+1} \epsilon_{t+1}^T] \\
&= E[(F_t \theta_t + G_t w_t)(F_t \theta_t + G_t w_t)^T] \\
&= F_t E[e_t e_t^T] F_t^T - G_t E[w_t e_t^T] F_t^T - F_t E[e_t w_t^T] G_t^T + G_t E[w_t w_t^T] G_t^T
\end{aligned} \tag{1.22}$$

Problem 6 (5 points)

Please show that

$$C_{\hat{\theta}_{t|t-1}} = F_{t-1} C_{\hat{\theta}_{t-1}} F_{t-1}^T + G_{t-1} C_w(t-1, t-1) G_{t-1}^T \tag{1.23}$$

Problem 6's answer

$$\begin{aligned}
C_{\hat{\theta}_{t|t-1}} &= E[\epsilon_t \epsilon_t^T] = E[(F_{t-1} e_{t-1} - G_{t-1} w_{t-1})(F_{t-1} e_{t-1} - G_{t-1} w_{t-1})^T] \\
&= E[(F_{t-1} e_{t-1} - G_{t-1} w_{t-1})(e_{t-1}^T F_{t-1}^T - w_{t-1}^T G_{t-1}^T)] \\
&= E[F_{t-1} e_{t-1} e_{t-1}^T F_{t-1}^T - F_{t-1} e_{t-1} w_{t-1}^T G_{t-1}^T - G_{t-1} w_{t-1} e_{t-1}^T F_{t-1}^T + G_{t-1} w_{t-1} w_{t-1}^T G_{t-1}^T] \\
&= F_{t-1} E[e_{t-1} e_{t-1}^T] F_{t-1}^T - F_{t-1} E[e_{t-1} w_{t-1}^T] G_{t-1}^T - G_{t-1} E[w_{t-1} e_{t-1}^T] F_{t-1}^T + G_{t-1} E[w_{t-1} w_{t-1}^T] G_{t-1}^T
\end{aligned}$$

We know that w and θ are uncorrelated, and e is the linear transformation of θ

$\Rightarrow w$ and e are uncorrelated to each other.

Furthermore, we know that $E[w] = \mathbf{0}_{k\times 1}$ and 2 uncorrelated vectors have property $E[xy^T] = E[x]E[y]^T$.

Hence we can rewrite the equation to:

$$\begin{aligned}
C_{\hat{\theta}_{t|t-1}} &= F_{t-1} E[e_{t-1} e_{t-1}^T] F_{t-1}^T + G_{t-1} E[w_{t-1} w_{t-1}^T] G_{t-1}^T \\
&= F_{t-1} C_{\hat{\theta}_{t-1}} F_{t-1}^T + G_{t-1} E[w_{t-1} w_{t-1}^T] G_{t-1}^T \quad (1.19) \\
&= F_{t-1} C_{\hat{\theta}_{t-1}} F_{t-1}^T + G_{t-1} C_w(t-1, t-1) G_{t-1}^T
\end{aligned}$$

1.1.4 The Recursive Calculation Procedure for the Discrete Kalman Filter

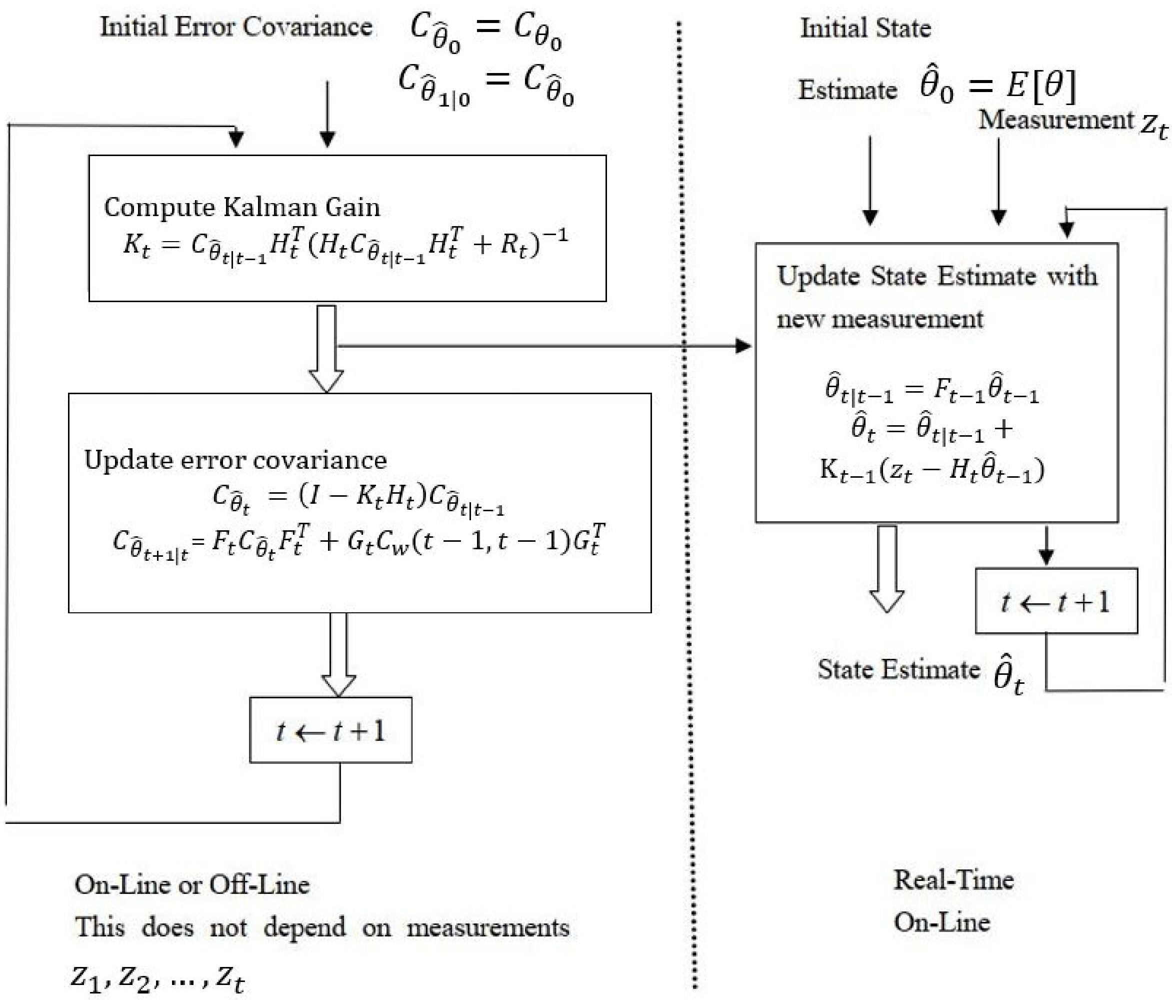


Figure: Solve discrete Kalman Filter recursively

2 Multiple choices

Please give a brief explanation for option(s) you choose. Answering without any description will get 0 point.

Problem 7 (5 points)

Which of the following statements are true regarding baseline correction? Assume that we are using **stimulus-locked epochs**.

(A) To perform baseline correction, the mean voltage is calculated during the prestimulus portion of the epoch, and this value is then subtracted from every point in the prestimulus period.

(B) To perform baseline correction, the mean voltage is calculated during the prestimulus portion of the epoch, and this value is then subtracted from every point in the waveform.

(C) We take the mean of the prestimulus period (rather than just taking the voltage at time zero) so that we can average out random noise during the prestimulus period and obtain a better estimate of the voltage offset.

(D) We take the mean of the prestimulus period (rather than just taking the voltage at time zero) so that we can obtain a better estimate of the noise level.

Problem 8: Single choice (5 points)

Imagine that a researcher conducts a study comparing a patient group with a control group in an oddball paradigm. The researcher conducts a separate patient/control x rare/frequent ANOVA for each time point from 0-800 ms at each electrode site. Each ANOVA yielded 3 p values (main effect of patient/control, main effect of rare/frequent, and interaction). The sampling rate was 250 Hz, so there were 200 time points between 0 and 800 ms. There were 32 electrode sites. If there are no true differences between groups, how many significant p values would you expect the researcher to obtain as a result of noise in the data? [For simplicity, assume that every time point and electrode site is independent of every other time point and electrode site. Assume that $\alpha = 0.05$.]
(A) 1280 (B) 507 (C) 320 (D) 960

Problem 9 (5 points)

Imagine that a researcher conducts a go/no-go experiment in which subjects are supposed to press a button every time they see the word GO (written in green, 80% of trials) and to make no response when they see the word STOP (written in red, 20% of trials). And imagine that they find a larger N1 wave (at 170 ms) for the STOP stimulus than for the GO stimulus. Could this effect be plausibly explained by a physical stimulus confound?

(A) No. It is unlikely that the small physical differences between GO and STOP could explain a difference at 170 ms.

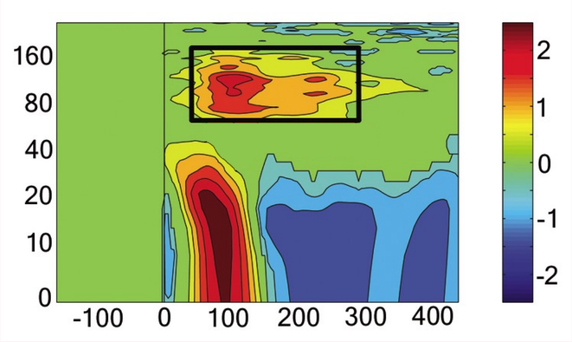
(B) Yes, because subjects may have been looking at a different places on the screen when the words GO and STOP appeared, which would change the position of the stimulus on the retina.

(C) Yes, because the word STOP contains more letters than the word GO and therefore might elicit a larger N1.

(D) Yes, because it is possible that red stimuli elicit a larger N1 than green stimuli.

Problem 10 (5 points)

The time-frequency plot below shows data from the oddballs in a mismatch negativity paradigm. Which of the following statements are true about this plot?



(A) The X axis is time.

(B) The Y axis is a measure of magnitude (typically amplitude or power).

(C) If we wanted to reproduce the original data from a set of wavelets, we would need some high-amplitude wavelets centered at 100 ms with frequencies ranging from 0 to 20 Hz (in addition to wavelets at other frequencies).

(D) The activity shown in the box from 80-280 milliseconds is probably a genuine oscillation.

3 Coding problem

Problem 11: Auditory Oddball paradigm (2+2+2+2+2+2+2+2=16 points)

Please use the data: Day1 _ERP.set to answer these questions.
Data Information
This data contains 2 sessions, been down-sampled to $250Hz$, and been band-pass filtered.

Trigger	Event
10	Response
2	High pitch
3	Low pitch
4	End of session

(a) Please guess the range of band-pass filtered and show how you find this range.
(b) Please guess the portion of High Pitch: Low pitch and show how you find it.
(c) Plot the Fz, Cz, and Pz's average ERP for Response respectively.
(d) Plot the Fz, Cz, and Pz's average ERP for High pitch and Low pitch respectively.

For subproblem (e) and (f), please plot topolots for P300.
Suppose that for each channel, P300 occurs during $[300, 400]msec$ when $t = 0$ indicates onset time of High pitch and Low pitch. That is, $P300^{ch} \in \mathbb{R}^{51\times 1}$ $\forall ch \in \mathbb{Z}_{30}$.
(e) Plot topoplot for High pitch and Low pitch respectively. (Use $mean(P300^{ch})$ $\forall ch \in \mathbb{Z}_{30}$)
$\forall ch \in \mathbb{Z}_{30}$, define Min-Max normalization as below

$$\frac{mean(P300^{ch}) - \min\left\{mean(P300^{ch})\middle|ch \in \mathbb{Z}_{30}\right\}}{\max\left\{mean(P300^{ch})\middle|ch \in \mathbb{Z}_{30}\right\} - \min\left\{mean(P300^{ch})\middle|ch \in \mathbb{Z}_{30}\right\}} \tag{3.1}$$

(f) Plot topoplot for High pitch and Low pitch respectively with Min-Max normalization.
If we define signal-to-noise ratio (SNR) for each channel as below:

$$SNR^{ch} = \frac{P300^{ch}}{std(Baseline^{ch})} \tag{3.2}$$

where baseline interval $[-200, 0]msec$, $Baseline^{ch}$ is mean by trial, and $t = 0$ indicates onset time of High pitch and Low pitch.
(g) For each channel, plot SNR for High picth and Low pitch. (Bar plot)
(h) Plot cumulative (by trial) SNR for Fz, Cz, and Pz channel and give a description of your observation.

Problem 12: 5 target SSVEP paradigm (4+8+7=19 points)

Please use the data: Day 2_SSVEP.set to answer these questions.
Data Information

Trigger	Event
11	10 Hz
21	11 Hz
31	12 Hz
41	13 Hz
51	Nan

(a) For Fz and Oz, please plot average ERP for each type of stimuli.
Apply short time Fourier transform (spectrogram in matlab) with the following parameters to answer subproblem (b).

```
% B: SSVEP for certain channel, sfreq:sampling rate
% P is a power spectrum density matrix with size (N_freq, N_time)
[S,F,T,P]=spectrogram(B,sfreq,sfreq/2,sfreq,sfreq);
```

(b) Plot power v.s. frequency for each stimuli at F_z and O_z channel and give description of your observation.
We extract $\{10, 11, 12, 13\}Hz$ **from PSD you get from subproblem (b), and called it as response frequency.**
(c) Plot topolot of response Hz v.s. stimuli Hz with same Min-Max normalization technique. That is, you will plot $4 \times 5 = 20$ topoplots this time.

References

[1] Stefanos D. Georgiadis, Perttu O. Ranta-aho, Mika P. Tarvainen, Pasi A. Karjalainen, *Single-Trial Dynamical Estimation of Event-Related Potentials: A Kalman Filter-Based Approach*, IEEE Transactions on Biomedical Engineering, 52(8), 2005.

[2] Harry Asada, Lecture notes for *Identification, Estimation, and Learning*, Massachusetts Institute of Technology, Department of Mechanical Engineering, 2006.

[3] S. Sanei, J.A. Chambers, *EEG Signal Processing*, Wiley, 2007.

[4] Yuan-Pin Lin, Lecture notes for 3^{rd} *EEG summer workshop*, National Sun Yat-sen University, Institute of Medical Science and Technology, 2020.

[5] Mike X Cohen. *Analyzing neural time series data : theory and practice.* Cambridge, Massachusetts :The MIT Press, 2014.

[6] Donald L. Schomer and Fernando H. Lopes da Silva, *Niedermeyer's Electroencephalography: Basic Principles, Clinical Applications, and Related Fields*,Lippincott William & Wilkins, 2011. **ISBN** 9780781789424.

4 Feedback for Lab 3

This part is not for grading but for understanding learning situation of each student. Please give us your feedback and comments.

4.1 Work Division

For example,

Student ID	Name	Be response for... ...
123456789	Tony	solving Problem 1 and designing preprocessing algorithm in problem 7
987654321	May	solving Problem 3 and designing preprocessing algorithm in problem 7

4.2 Suggestions and Comments

4.2.1 For instructor

4.2.2 For teaching assistant(s)

4.2.2.a For Min-Jiun
4.2.2.b For Eric

Office Hour Information

We'll have limited time to teach EEGLab and MNE on our course; therefore, if you have any question about lab 2, feel free to make an appointment or come to ask me during my office hour.

Day	Time	Office
Tue.	12:20 p.m.-13:10 p.m.	EC120
Thur.	06:30 p.m.-09:30 p.m.	SC207

Note
Actually, my office hour on Thursdays is main for calculus consultation. If there are undergraduate students come to ask calculus problems, I need to teach them first and then to solve your problem during the rest of the office hour on Thursday nights.
