Extracting the wisdom of crowds in probabilistic judgments (DRAFT)

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Abstract

Decision makers constantly face the problem of estimating the likelihood of an uncertain event. Leveraging the wisdom of crowds is an increasingly popular solution. Previous findings suggest that statistical aggregation of independent judgments produces remarkably accurate predictions. However in some decision problems individual judgments could be collectively over-optimistic or pessimistic, typically due to shared information available to all individuals. In such cases, probabilistic judgments are likely to err on the same side and the average prediction could exhibit a bias. This paper proposes an algorithm to aggregate probabilistic judgments effectively. Individuals are asked to report a probabilistic prediction and a *meta-prediction* where the latter is an estimate on the average of other individuals' predictions. In a Bayesian setup I show that if average prediction is unbiased, the percentage of predictions and meta-predictions that overshoot the average prediction should be the same. When the two measures differ, an *overshoot surprise* occurs. I present the Surprising Overshoot (SO) algorithm, which uses the information revealed in an overshoot surprise to produce a

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more accurate estimator for the unknown probability. I test the SO algorithm using experimental data from various studies. The SO algorithm consistently outperforms basic aggregates such as unweighted average and median prediction. Furthermore, the SO algorithm compares favorably to alternative aggregation mechanisms when individual predictions are highly dispersed.

1 Introduction

In episode 'Errand of Mercy' of *Star Trek: The Original Series*, Captain Kirk and his first officer Mr. Spock find themselves in a dangerous situation. Captain Kirk plans to infiltrate an enemy camp and asks Mr. Spock the odds of getting out alive. Although being humanoid in appearance, Mr. Spock is from a different planet and his race is known for enhanced abilities in logical and quantitative reasoning. He estimates the odds to be '7824.7 to 1', which Captain Kirk half-jokingly calls 'a pretty close approximation'. In his usual seriousness, Mr. Spock responds by saying 'I endeavor to be accurate'.

The days of starships and deep space exploration might still be far away, but researchers and decision makers today find themselves facing uncertain decision problems, some of which may have stakes as high as Captain Kirk's. Scientists make probabilistic projections on natural phenomena, such as occurrence of a major earthquake or the effects of anthropogenic climate change. Strategists assess the likelihood of important geopolitical events. Investors form judgments on investment opportunities. Economists are interested in predicting the chances of downturns and crashes.

In many such problems, the decision maker needs a well-calibrated probabilistic forecast. Unfortunately, a superior mind such as Mr. Spock's is not available (yet). Individual judgments might be subject to biases such as optimism, overconfidence, being anchored on an initial estimate, focusing too much on easily available information, neglecting an event's base rate and many more (9, 22, 8). Combining multiple judgments to leverage 'the wisdom of crowds' is known to be an effective approach in improving accuracy (21, 11). To illustrate, consider a decision maker who aims to predict the chances of an armed conflict between two hostile neighboring countries in the next few months. She elicits estimates from a panel of 10 experts. Suppose the actual (latent) probability is 60%. Figure 1 below depicts indivdual expert predictions:

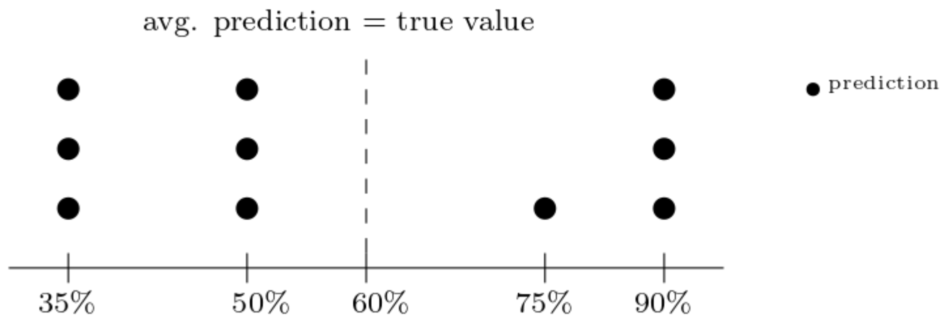


Figure 1: Example expert predictions on the chances of armed conflict

In Figure 1, $35\%, 50\%$ and 90% are reported by three experts each while one experts predicts 75%. Note that none of the experts is perfectly accurate in her prediction. However, the average of all predictions gives 60%. Figure 1 presents an example of the 'wisdom of crowds' effect (21). Individual experts either over or underestimate the likelihood of the event. However, when judgments are aggregated, individual errors cancel out and the average prediction provides a perfectly calibrated estimate.

The use of wisdom of crowds involves choosing an aggregation method that combines individual prediction into a single aggregate prediction (1, 4, 16). In the example in Figure 1, simple averaging provides an accurate prediction. Previous work found simple averaging to be surprisingly effective, typically outperforming more sophisticated aggregation methods and showing robustness across various settings (11, 12, 24, 6). Even though it is difficult to beat simple averaging consistently across many tasks, there are cases where we may expect simple average to be inaccurate. To illustrate, consider the same problem as in Figure 1. Suppose the countries in question have had recurrent conflicts and historically in about 40% of instances, tensions escalate into an armed conflict. So, the experts who study this conflict conflict have some shared information suggesting that the chances of an armed conflict is around 40%. Upon observing the more case-specific evidence, individual experts might reach different conclusions. But their judgments are still affected by the shared information. Figure 2 illustrates one such case:

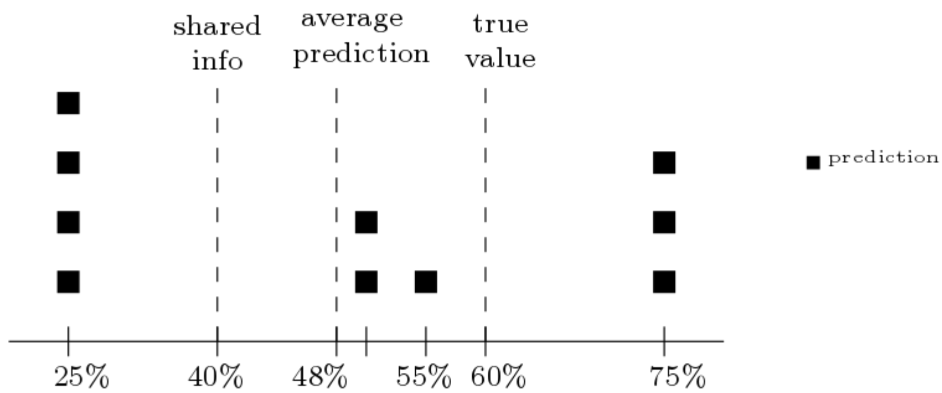


Figure 2: Predictions in the example case where experts have shared information in the form of historical data

In Figure 2, the average prediction underestimates the true value. In this example, the decision maker would be better off using a weighted average where the three 75% predictions are weighted more heavily.

Why does simple averaging fail in Figure 2? Intuitively, simple averaging allows individual forecasting errors to cancel (10), leading to a more accurate prediction. In Figure 2, three experts overshot the true probability even though the shared information underestimates the likelihood. Nevertheless, most experts ended up underestimating the likelihood. When there is shared information that deviates from the true value, experts are more likely to err in the same direction, causing a positive correlation in prediction errors and smaller error-cancellation. Since the shared information in Figure 2 underestimates the chances of the event, most individual predictions in Figure 2 undershoot the true probability. As a result, the simple average of predictions exhibits a bias (3, 18).

In general, the optimal aggregation method in a particular estimation task depends on the information structure and differences in expertise among forecasters (5, 15). For example, in Figure 2, the decision maker could improve the aggregate estimate by weighting the overshooting predictions more heavily compared to the simple unweighted average. In theory, the decision maker in a given task can select and weight judgments such that the errors perfectly cancel out (13, 2). However, optimal weights relate to how experts' prediction errors are correlated and are typically unknown to decision maker. Conventional approach aims to estimate appropriate weights using past data from similar tasks (2, 13). The effectiveness of this approach is limited by the availability and reliability of past data.

This paper develops an algorithm to aggregate judgments on the likelihood of an event. I consider a setup where experts form their judgments by combining shared and private information on the probability of an event. When the shared information differs from the true probability, probabilistic predictions are biased in one side, resulting in a biased average prediction. The algorithm relies on augmented elicitation procedure commonly found in recent work (19, 20, 18, 17, 23): Individuals report a prediction on the probability as well as an estimate on the average of others' predictions, which is referred to as a *meta-prediction*. I establish that when average prediction is unbiased, the percentage of predictions and meta-predictions that overshoot the average prediction should be the same. Whenever the two measures differ an *overshoot surprise* occurs, which indicates a bias in the average prediction. I develop the Surprising Overshoot (SO) algorithm which produces a more accurate estimator. The SO algorithm uses the information in the size and direction of the overshoot surprise. It does not require the use of past data.

The simple example below illustrates the intuition behind an overshoot surprise. Let the cases in Figures 1 and 2 be the 'unbiased' and 'biased' worlds respectively. In the unbiased world the average prediction is accurate while in the biased world the average prediction underestimates the chances of occurrence. Figure 3 below shows the two worlds, which describe two possible scenarios. The empty circles and squares in each figure depict example meta-predictions. An expert's reports (prediction and meta-prediction) are paired in rectangles.

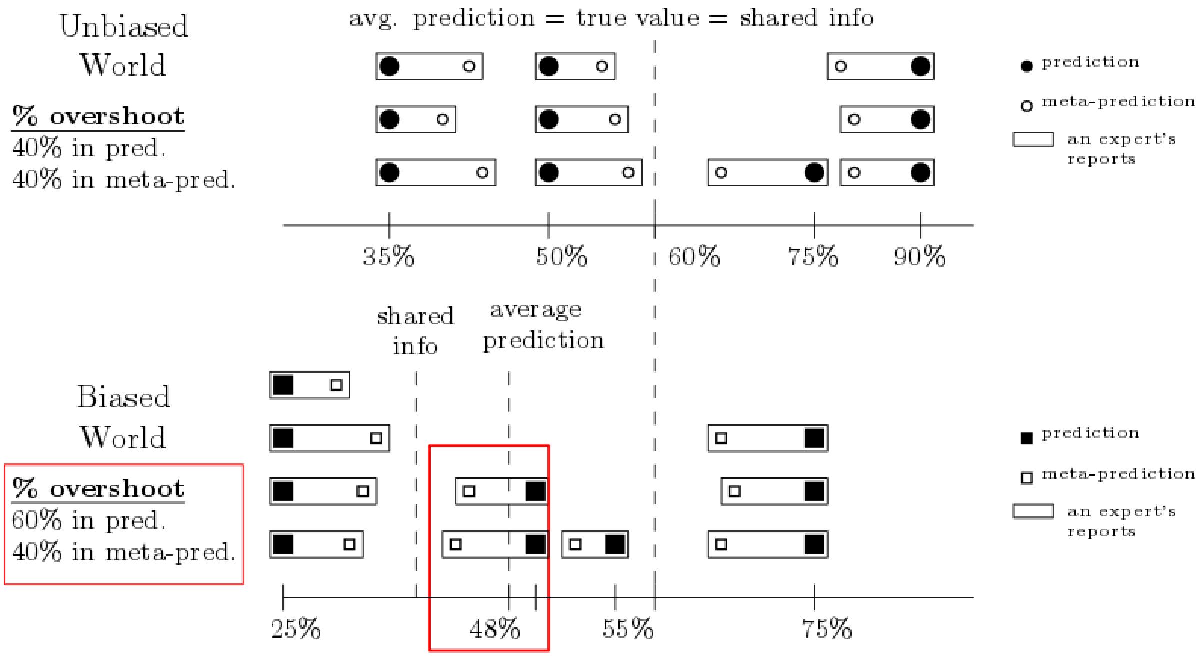


Figure 3: Example cases of unbiased and biased average prediction

In the unbiased world, 4 out of 10 predictions and meta-predictions overshoot the average prediction. Then, we can say the overshoot rate in both reports is 40%. In the biased world, 4 out of 10 predictions overshoot the average prediction, so the overshoot rate in meta-predictions is 40%. However, the overshoot rate in predictions differ. Since 6 out of 10 meta-predictions are higher than the average prediction, the overshoot rate is 60%. The difference is due to the two experts in the biased world, captured in the solid red rectangle. For these experts, the predictions undershoot the average prediction while the meta-predictions overshoot.

Why does a discrepancy between the overshoot rates in the biased world occur? Observe that in both worlds, an expert's meta-prediction is depicted in-between her prediction and the shared information. I will later show that such reporting in meta-predictions is optimal for Bayesian individuals. In the unbiased world, experts' shared information coincides with the average prediction. An expert's prediction and meta-prediction are on the same side of the average prediction. Thus, we do not expect an overshoot surprise. Indeed the overshoot rates in predictions and meta-predictions are both 40%. In contrast the shared information and the true probability differ in the biased world, resulting in a difference between the actual probability of the event and the average prediction. Then, for some experts prediction and meta-prediction may fall on different sides of the average prediction. In Figure 3, two experts undershoot in their meta-predictions even though they overshoot in their predictions, resulting in an overshoot surprise of 20%. The SO algortihm exploits this information to produce an improved estimate of the true probability. We see a negative overshoot surprise when the average prediction is biased downwards (as in the biased world in Figure 3) while a positive overshoot surprise occurs when the average prediction is biased upwards. The SO algorithm infers the direction and size of the bias from the sign and magnitude of the overshoot surprise. In practice, a decision maker typically would not know if the true state of the world is biased or unbiased. The SO algorithm does not distort an unbiased average prediction while removing the bias when it is present.

I test the SO algorithm using experimental data from two sources. Palley and Soll (18) conducted an experimental study where subjects are asked to predict the number of heads in 100 flips of a biased coin. The SO algorithm shares the same Bayesian framework as Palley and Soll (18) and the experiment implements shared and private signals as sample flips from the biased coin. This study allows us to test the SO algorithm in a controlled setup. The second source is Wilkening et al. (23), who conducted two experimental studies. The first experiment replicates the earlier study by Prelec et al. (20) which asked subjects true/false questions about the capital cities of U.S. states. However, unlike Prelec et al. (20) they also ask subjects to report probabilistic predictions and meta-predictions, which allows an implementation of the SO algorithm. In the second experiment, Wilkening et al. (23) generate 500 basic science statements and ask subjects to report probabilistic predictions and meta-predictions on the likelihood of a given statement being true. I use the data from these two experiments to investigate if the SO algorithm produces an aggregate probabilistic prediction closer to the correct answer. Results suggest that the SO algorithm outperforms simple benchmarks such as unweighted averaging and median prediction. I also compare the SO algorithm to alternative solutions for aggregating probabilistic judgments, which elicit similar information from individuals (18, 14, 17, 23). The SO algorithm never underperforms and compares favorably to alternative aggregation mechanisms in prediction tasks where individual predictions are more dispersed. Experimental evidence suggests that the SO algorithm is especially effective in extracting the wisdom of crowds from disagreeing probabilistic judgments.

This paper contributes to the growing literature of judgment aggregation mechanisms that utilize meta-beliefs to improve prediction accuracy. Prelec et al. (20)'s Surprisingly Popular (SP) algorithm uses meta-beliefs to identify the true answer to a multiple choice question even when the majority opinion is inaccurate. In a similar setup, Wilkening et al. (23) develops the Surprisingly Confident (SC) algorithm, which uses meta-beliefs to determine weights that leverage more informed judgments. The SP and SC algorithms aim to identify the correct answer to a binary or multiple-choice question while the SO algorithm produces an estimate of the latent probability of an uncertain event.

Recent work developed aggregation algorithms for probabilistic judgments as well. As mentioned before, the SO algorithm shares the same setup as Palley and Soll (18), in which information commonly available to individuals causes the shared-information bias in the average prediction. Palley and Soll (18) develop the pivoting method which uses meta-predictions to recover and recombine shared and private information in an optimal manner. Palley and Satopää (17) consider a similar setup and propose knowledge-weighting method. Knowledge-weighting constructs a weighted crowd based on the accuracy of combined meta-prediction. The resulting weights are expected to minimize the error in combined prediction as well. The meta-probability weighting algorithm (14) considers a slightly different framework. The absolute difference between an individual's prediction and meta-prediction is considered as an indicator of her expertise. In testing the performance of the SO algorithm, pivoting, knowledge-weighting and meta-probability weighting are considered as benchmarks. As mentioned above the SO algorithm performs especially well when individual judgments are more dispersed. In practice, such problems are likely to be the most challenging ones where expert judgments disagree substantially and even the best experts do not have access to absolute truth.

The rest of this paper is organized as follows: Section 2 introduces the formal framework. Section 3 develops the SO algorithm and illustrates how it produces an aggregate estimate. Section 4 introduces the data sets and benchmarks we consider in testing the SO algorithm empirically. The same section also presents some preliminary evidence on how overshoot surprises relate to biases. Section 5 presents empirical evidence for the effectiveness of the SO algorithm. Section 6 discusses the findings and Section 7 concludes.

2 The Framework

The framework follows the definition of a *linear aggregation problem* in Palley and Soll (18) and Palley and Satopää (17) with a binary outcome. Let $Y \in \{0, 1\}$ be a random variable denoting the outcome of an event. Also let $\theta = P(Y = 1)$ be the latent probability of the outcome 1, representing the occurrence. A decision maker (DM) would like to estimate θ. She elicits judgments from a sample of N agents to develop an estimator. Agents observe a *shared signal* $s \in [0, 1]$ and each agent i receives a *private signal* $t_i \in [0, 1]$. All signals follow the same expectation θ and are conditionally independent given θ. I also assume that the signals are generated from a continuous distribution such that $P(t_i = k) > 0$ for all $k \in [0, 1]$, i.e. any private signal $t_i \in [0, 1]$ has a positive probability.

Each agent i updates her beliefs according to the Bayes' rule. Let $E[\theta|s, t_i]$ denote an agent i's posterior expectation on θ. In a linear aggregation problem, the posterior expectation combines her signals linearly:

$$E[\theta|s, t_i] = (1 - \omega)s + \omega t_i$$

where $\omega \in [0, 1]$ represents weight an agent puts on her private signal relative to the shared signal s. The weight parameter ω is common for all agents. Palley and Soll (18) define such a linear aggregation problem where θ is a latent probability. Agents share the common prior on θ given by a Beta density and each signal is the average of multiple independent realizations of Y. The parameter ω represents the relative informativeness of an agent's private signal t_i compared to the shared signal s.

Suppose a DM who considers the simple average of agents' predictions as an estimator for θ. Let x_i be agent i's reported prediction on θ. Suppose all agents report their best guesses, i.e. $x_i = E[\theta|s, t_i]$. Then the average prediction is given by

$$\bar{x} = \frac{1}{N}\sum_{i=1}^{N} x_i = (1-\omega)s + \omega\frac{1}{N}\sum_{i=1}^{N} t_i$$

Note that $E[\bar{x}|s] = (1-\omega)s + \omega\theta \neq \theta$ if $s \neq \theta$, i.e. the average prediction is not an unbiased estimate of θ unless the shared information is perfectly accurate. Palley and Soll (18) refer to the bias described here as the *shared-information bias*. Section 3 develops the Surprising Overshoot algorithm which constructs an unbiased estimator. Then, I will investigate the effectiveness of the algorithm using experimental data from Palley and Soll (18) and Wilkening et al. (23).

3 The Surprising Overshoot algorithm

This section develops the SO algorithm. For the theoretical analysis below, I assume $N \to \infty$, i.e. the whole population of agents is available. However, the SO algorithm could also operate with a finite sample of agents. Thus, I will later consider finite N in establishing the main result and describing the SO algorithm. The effect of sample size will be one of factors of interest in the empirical analysis.

The decision maker asks each agent i to submit two reports. In the first, she is asked to make a *prediction* on θ, denoted by x_i. In the second, she reports a *meta-prediction* $z_i \in \mathbb{R}$, which is an estimate on the average of other agents' predictions, denoted by $\bar{x}_{-i} =$

$\frac{1}{N-1}\sum\limits_{j\neq i} x_j$. I assume that agents report their best estimates in each report, given by $(x_i, z_i) = (E[\theta|s,t_i], E[\bar{x}_{-i}|s,t_i])$. Such truthful elicitation can be achieved by incentivizing each report by a proper scoring rule (7). Then, we have

$$\begin{aligned}
x_i = E[x_i|s,t_i] &= (1-\omega)s + \omega t_i \\
z_i = E[\bar{x}_{-i}|s,t_i] &= (1-\omega)s + \omega E\left[\frac{1}{N-1}\sum_{j\neq i} t_j \middle| s, t_i\right] \\
&= (1-\omega)s + \omega x_i
\end{aligned}$$

where the expression for z_i comes from $E[t_j|s,t_i] = (1-\omega)s + \omega t_i = x_i$ for all $j \neq i$.

A prediction or meta-prediction is said to *overshoot* the average prediction $\bar{x}$ if it exceeds $\bar{x}$. For any arbitrary agent i, there are two overshoot indicators. For example, if $x_i > \bar{x} > z_i$, agent i's prediction overshoots the average prediction while her meta-prediction does not overshoot. The SO algorithm develops crowd measures based on instances of individual overshoots. Observe that

$$\begin{gathered}
x_i > \bar{x} \\
(1-\omega)s + \omega t_i > (1-\omega)s + \omega\frac{1}{N}\sum_{j=1}^{N} t_j \\
t_i > \frac{1}{N}\sum_{i=1}^{N} t_j
\end{gathered}$$

Recall that θ is the mean of the signal distribution. For $N \to \infty$, we have $\frac{1}{N}\sum_{i=1}^{N} t_i = \theta$. So, agent i's prediction is higher than the average prediction if agent i receives a higher than average signal, that is

$$x_i > \bar{x} \iff t_i > \theta \tag{1}$$

Now consider agent i's meta-prediction z_i. The following holds for z_i:

$$
\begin{aligned}
z_i &> \bar{x} \\
(1-\omega)s + \omega x_i &> (1-\omega)s + \omega \frac{1}{N}\sum_{i=j}^{N} t_j \\
x_i &> \frac{1}{N}\sum_{j=1}^{N} t_j = \theta
\end{aligned}
$$

Then, we get

$$
z_i > \bar{x} \iff x_i > \theta \tag{2}
$$

Equations 1 and 2 suggest a pattern. Recall that, in equation 1, an agent i's prediction x_i is higher than $\bar{x}$ if $t_i > \theta$. However, for her meta-prediction z_i to overshoot $\bar{x}$, we must have $x_i = (1-\omega)s + \omega t_i > \theta$. Since $s < \theta$, there could an agent i such that x_i exceeds the average prediction $\bar{x}$ but still undershoots θ. Thus, we do not necessarily have $z_i > \bar{x}_i$ whenever $x_i > \bar{x}$ is satisfied.

Now consider the following measures computed using predictions and meta-predictions:

$$
\begin{aligned}
p_x &= \frac{1}{N}\sum_{i=1}^{N} \mathbb{1}(x_i > \bar{x}) \\
p_z &= \frac{1}{N}\sum_{i=1}^{N} \mathbb{1}(z_i > \bar{x})
\end{aligned}
$$

The measures p_x and p_z represent the population proportion of predictions and meta-predictions that overshoot the average prediction $\bar{x}$. I refer to p_x and p_z as the *overshoot rate* in predictions and meta-predictions respectively.

From equation 2, we can infer that p_z also corresponds the population proportion of predictions such that $x_i > \theta$. Let $f(x)$ denote the population density of predictions given the signal distribution. Figure 4 illustrates two scenarios with an example distribution:

(a) Case $s = \theta = 0.25$ (b) Case $\theta = 0.25 < s = 0.6$

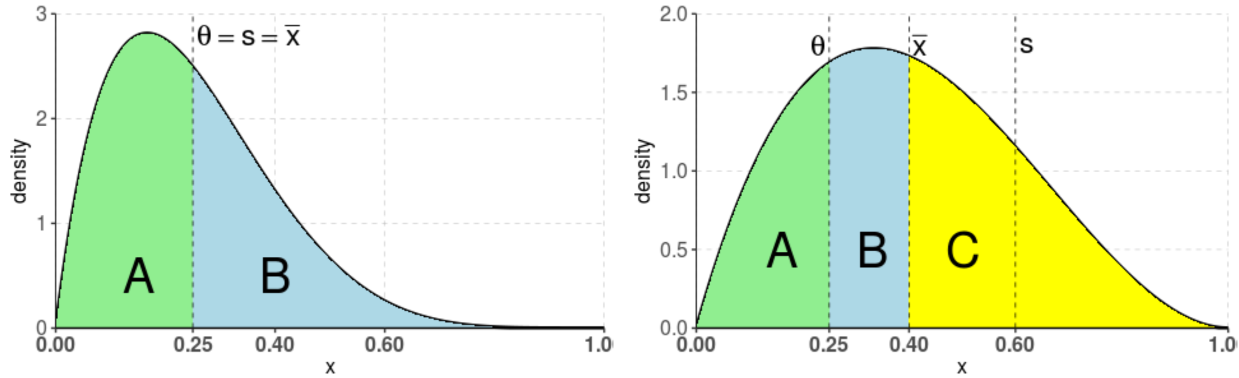


Figure 4: An example theoretical density f of forecasts in two example cases. A, B and C represent the shaded regions.

Recall that p_x measures the predictions with $x_i > \bar{x}$ and p_z captures the predictions satisfying $x_i > \theta$ through the equivalence in equation 2. Figure 4a represents the case where $\bar{x}$ and θ coincides, which follows from $s = \theta$. Then, p_x and p_z both represent the region B and we have $p_x = p_z$. The average prediction $\bar{x}$ reflects θ. In contrast, Figure 4b shows an example density of predictions where the shared signal s differs from θ, resulting in $\bar{x} > \theta$. Observe that in Figure 4b, p_z represents the region $B + C$ while p_x corresponds to the area marked by C only. In this case, we see $\bar{x} > \theta$ due to $s > \theta$. As a result, the percentage of predictions that exceed $\bar{x}$ is lower and we have $p_z > p_z$.

Definition 1 (**Overshoot surprise**). *An overshoot surprise occurs when $p_x \neq p_z$. The overshoot surprise is positive if $p_z > p_x$ and negative if $p_z < p_x$. The size of the overshoot surprise is given by $\Delta p = p_z - p_x$.*

In Figure 4b, we observe a positive overshoot surprise, which indicates an upward bias in $\bar{x}$. A negative overshoot surprise would ensue if $\bar{x}$ is biased downwards instead. In contrast, there is no overshoot surprise in Figure 4a where $\bar{x}$ is unbiased.

Figure 4 suggests that p_z reveals important information on the density of predictions. Consider the cumulative density F of predictions in Figure 4b:

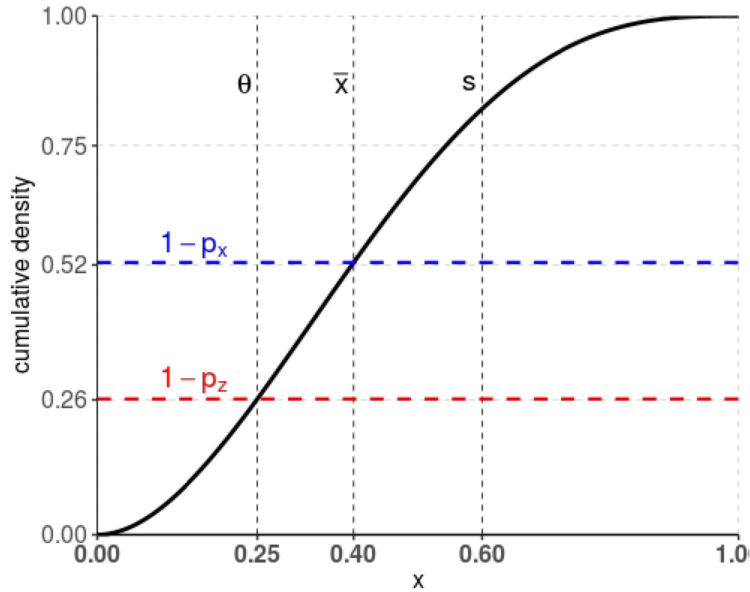


Figure 5: Cumulative density F of predictions for the example in Figure 4b

Since p_z measures the proportion of predictions that overshoot θ, $1-p_z$ gives the quantile of the cumulative density that corresponds θ. We can generalize this finding as follows:

Lemma 1. *Let F be the continuous population density of predictions and $N \to \infty$. Then, $\theta = F^{-1}(1-p_z)$*

Lemma 1 suggests that when the predictions and meta-predictions in the whole population ($N \to \infty$) are known and the population density of predictions is continuous, the exact θ can be recovered using p_z and F. The measure p_z simply reveals the cumulative density of predictions that overshoot θ, implying that $1-p_z$ gives the quantile of F that corresponds to θ.

In practice, a DM can only recruit a finite sample of agents. The population distribution of predictions and the population quantiles are unknown. Let $\hat{F}$ be the empirical cumulative density of the predictions from a random finite sample of agents of size N. Also let $\hat{p}_z$ be the overshoot rate associated with the sample meta-predictions. The definition below introduces the Surprising Overshoot (SO) algorithm:

Definition 2 (The Surprising Overshoot algorithm). *The Surprising Overshoot algo-*

rithm constructs the estimator x^{SO} for θ following the steps below:

1. *Elicit sample predictions and meta-predictions.*
2. *Calculate $\hat{p}_z$ using sample meta-predictions.*
3. *Set $x^{SO} = \hat{F}^{-1}(1 - \hat{p}_z)$.*

The SO algorithm simply locates the $1 - \hat{p}_z$ quantile of the sample predictions. Lemma 1 established that when predictions follow a continuous density, the unknown probability is simply the population quantile $1 - p_z$. The estimator x^{SO} may differ from θ due to sampling errors and/or discontinuities in the population density of predictions. However, we may expect lower prediction errors compared to $\bar{x}$ in tasks where $\bar{x}$ exhibits a bias. When there is a positive (negative) overshoot surprise, x^{S0} is expected to be lower (higher) than $\hat{x}$ as $1-\hat{p}_z$ is expected to be a smaller (larger) quantile than the quantile of $\bar{x}$, given by $1 - \hat{p}_x$. Thus, on average the SO estimator adjusts the aggregate estimate away from the bias. Note that when there is no overshoot surprise, we have $E[\hat{p}_x] = E[\hat{p}_z]$. The SO algorithm is expected to produce $x^{SO} = \bar{x}$ and not distort the average prediction if it is unbiased.

The SO estimator relies on the empirical distribution of predictions as well as agents' meta-predictions. This property has implications about the prediction problems where we may expect the SO algorithm to be more effective. To illustrate, consider the two example empirical densities below. Both figures depict predictions from a sample of 10 agents where the sample average prediction is 0.4 while $\theta = 0.25$. In Figure 6a agents report one of $0.5, 0.3$ or 0.1 as prediction. The distribution of predictions in Figure 6b is more dispersed around the average prediction. Suppose the meta-predictions in each example (not shown on figures) are such that $\hat{p}_z = 0.2$ in both cases. Then the SO estimate is $1 - \hat{p}_z = 0.8$ quantile of the empirical density of predictions. The orange bar in each figure locates the SO estimator.

(a) Low dispersion (b) High dispersion

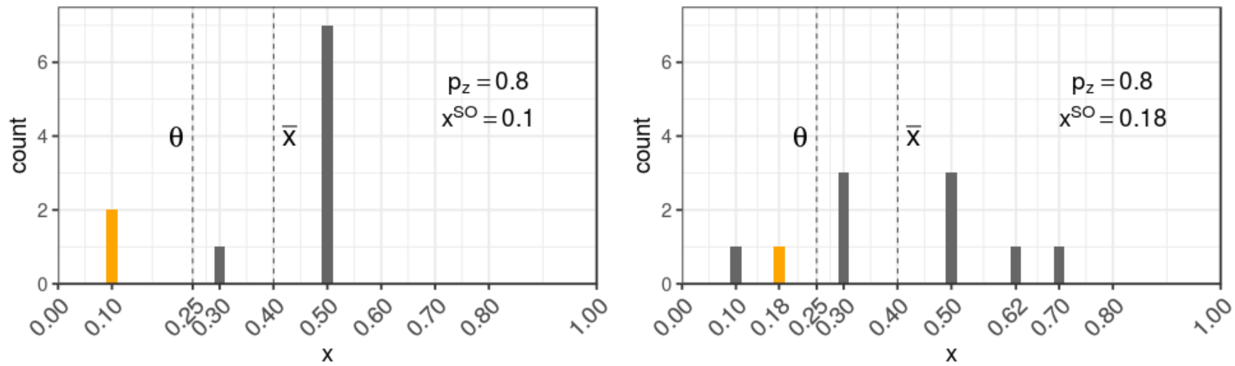


Figure 6: Two examples of empirical density of predictions

The SO estimate is more accurate in the high dispersion case simply because the 0.2 quantile falls closer to θ. The SO algorithm simply picks a reported prediction that corresponds to the sample quantile $1 - \hat{p}_z$. So the set of values x^{SO} can take depens on the empirical density of predictions. Even when $1 - \hat{p}_z$ provides an accurate estimate of the cumulative density at θ, the SO estimate may not be more accurate than $\bar{x}$ simply because $1 - \hat{p}_z$ quantile of the sample predictions is not close to θ. Such cases are less likely when the sample size is higher and/or the empirical density of predictions is more dispersed, as in Figure 6b. Therefore, we may expect the SO algorithm to perform better in larger samples and when the predictions are more dispersed. Intuitively, the latter can be considered as representing prediction tasks where individual judgments disagree, which could occur when the event of interest is highly uncertain and even the best experts lack the absolute truth. The following sections test the SO algorithm using experimental data. In the analyses below, sample size and dispersion of predictions are considered as factors of interest.

4 Testing the SO algorithm

This section outlines the empirical methodology and presents some preliminary evidence on overshot surprises. I use data from various experimental studies to test the SO algorithm. Section 4.1 provides information on the data sets. In testing the SO algorithm, I follow a comparative approach. The analysis will implement various alternative methods as a benchmark and test if the SO algorithm performs significantly better. Section 4.2 introduces the benchmarks. Finally, Section 4.3 provides some preliminary findings on overshoot surprises and how they relate to biases in the simple average.

4.1 Data sets

In testing the SO algorithm, I use data from three experimental studies. The first data set comes from Studies 1 and 2 in Palley and Soll (18). In these studies, Palley and Soll (18) conducted an online experiment where subjects reported their prediction and meta-prediction on the number of heads in 100 flips of a biased two-sided coin. The bias (the actual probability of heads) is unknown to the subjects, who are recruited on Amazon Mechanical Turk. Prior to submitting her report on a coin, each subject observed two independent samples of flips. One sample is common to all subjects and represents the shared signal. The second sample is subject-specific and constitutes a subject's private signal. A subject's best guess on the number of heads in 100 new flips a coin is effectively her best guess on the unknown bias. Thus, the prediction task elicits predictions on a probability unknown to subjects. Studies 1 and 2 Palley and Soll (18) differ in terms of individual incentives. However, they find no significant differences across various schemes. So, I include all applicable data in which subjects reported both a prediction and a meta-prediction, resulting in reports on a total of 120 coins. The Coin Flips data set presents an opportunity to test the SO algorithm in a controlled setup. Since latent probabilities (biases of the coins) are known to the analyst, it is possible to calculate prediction errors directly.

The second source of data is the two experimental studies conducted by Wilkening et al. (23). In their first experiment, Wilkening et al. (23) replicated the experiment initially conducted by Prelec et al. (20). For each U.S. state, subjects are asked if the most populated city is the capital of that state. Prelec et al. (20) required subjects to pick true or false and report the percentage of other subjects who would agree with them. Wilkening et al. (23)

asked subjects to report probabilistic predictions and meta-predictions on the statement (largest city being the capital city), which allows us to implement the SO algorithm. There are 89 subjects in total and each subject answered 50 questions (one per state). In the second experiment, subjects are presented with U.S. grade school level true/false general science statements. Examples include 'Water boils at 100 degrees Celsius at sea level', 'Materials that let electricity pass through them easily are called insulators' and 'Voluntary muscles are controlled by the cerebrum' etc. The experiments elicits judgments on 500 statements in total. Each subject reports her prediction and meta-prediction on the probability of a statement being true for 100 such statements. The number of subjects reporting on a given statement varies between 89 to 95. The State Capital and General Knowledge data sets allow us to test the SO algorithm in relatively more practical settings where only binary outcomes can be observed.

4.2 Benchmarks

The benchmarks in testing the SO algorithm can be categorized in two groups. I will first consider *simple benchmarks*, namely the simple average and median prediction. Simple averaging is an easy and intuitive aggregation method. The median forecast is also popular because it is more robust to outliers. These simple aggregation methods do not require meta-predictions, which makes them easier to implement. However, as shown in Section 2 with simple averaging, these methods may produce biased aggregate judgments. As discussed in Section 1, there exists a growing literature which provide more sophisticated solutions to the aggregation problem utilizing meta-beliefs. I consider three *advanced benchmarks*: Palley and Soll (18)'s pivoting, Palley and Satopää (17)'s knowledge-weighting (KW) and Martinie et al. (14)'s meta-probability weighting (MPW).

The pivoting method first computes simple average of predictions and meta-predictions, $\bar{x}$ and $\bar{z}$ in our notation respectively. Then the mechanism pivots from $\bar{x}$ in different directions. The pivot in the direction of $\bar{z}$ provides an estimate for the shared information (the common prior expectation in our setup) while the opposite direction gives an estimate for the average of private signals. These estimates are combined using Bayesian weights to produce the optimal aggregate estimate. The canonical pivoting method requires the knowledge of Bayesian weight ω to determine the optimal pivot size. Palley and Soll (18) propose *minimal pivoting* (MP) as a simple variant which adjusts $\bar{x}$ by $\bar{x} - \bar{z}$. The adjustment moves the aggregate estimate away from the shared information and alleviates the shared-information bias. MP does not require the knowledge of ω but it only partially corrects for the bias in $\bar{x}$.

The KW mechanism proposes a weighted crowd average as the aggregate prediction. The weights are estimated by minimizing the peer prediction gap, which measures the accuracy of weighted crowds' aggregate meta-prediction in estimating the average prediction. In a similar framework to Section 2, Palley and Satopää (17) show that minimizing the peer prediction gap is a proxy for minimizing the mean squared error of a weighted aggregate prediction. Intuitively, KW builds on the idea that a weighted crowd that is accurate in predicting others could be more accurate in predicting the unknown quantity itself as well. The mechanism's aggregate prediction is simply the weighted average prediction of such a crowd.

The MPW algorithm aims to construct a weighted average of probabilistic predictions. Martinie et al. (14) consider a slightly different Bayesian setup where agents receive a private signal from one of the two signal technologies, one for experts and the other for novices. The absolute difference between an agent's optimal prediction and meta-prediction is higher if the agent's signal is more informative. Based on this result, the MPW algorithm assigns weights proportional to the absolute differences between their prediction and meta-prediction. It is expected that agents with more informative private signals receive higher weights and the weighted average becomes more accurate as a result.

Similar to the advanced benchmarks listed above, the SO algorithm relies on an augmented elicitation procedure which elicits meta-predictions in addition to predictions. In contrast, the mechanisms in simple benchmarks do not require information from meta-predictions. Thus we may expect the SO algorithm to significantly outperform simple benchmarks. The advanced benchmarks have similar information demands to the SO algorithm. Thus, comparisons with the advanced benchmarks may produce different results in different tasks.

4.3 Preliminary evidence on overshoot surprises

Section 3 established a relationship between the size and direction of overshoot surprises and biases. The more p_z differs from p_x, higher the overshoot surprise, suggesting a stronger bias in the crowd average. Presence of an overshoot surprise relates to the performance of SO algorithm as well. We may expect a larger error reduction from using the SO algorithm when $|p_z - p_x|$ is larger as x^{SO} will be further away than the biased $\bar{x}$.

The Coin Flips data set presents an opportunity to investigate whether overshoot surprises correlate with biases. In this experiment, both the shared signal s and the unknown probability θ in each coin are generated by the experimenter. The shared-information bias in $\bar{x}$ is proportional to the absolute difference between s and θ. Furthermore, the bias would be downwards (upwards) if $s < \theta$ ($s > \theta$). Recall that positive (negative) overshoot surprises are associated with upward (downward) biases and we expect no overshoot surprise if $\bar{x}$ is unbiased, which corresponds to the case $s = \theta$. Since the information on s and θ are available, we can test if we observe this pattern. Figure 7 shows the relationship between overshoot surprises and $s - \theta$. Each dot represents an item and the blue lines shows the best linear fit.

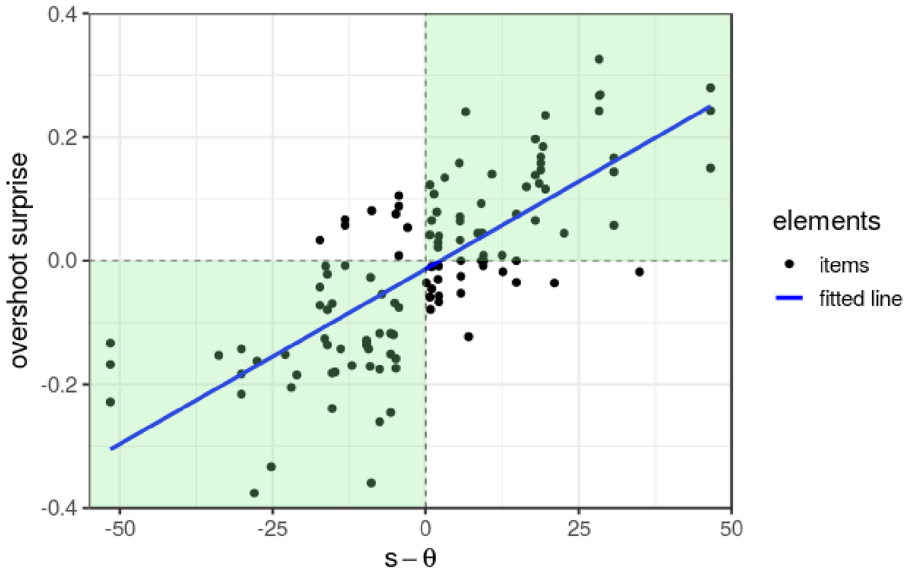


Figure 7: The relation between the shared-information biases and incidences of overshoot surprises across tasks. Shaded areas show the regions where the direction of bias and overshoot surprise is as predicted by the theory.

Figure 7 shows a strong linear association between biases and overshoot surprises. Also observe that most of the points are within the shaded regions. A positive (negative) overshoot surprise is much more likely to occur when we have an upward (downward) bias in $\bar{x}$, which corresponds to $s > \theta$ ($s < \theta$). Furthermore, the magnitudes of overshoot surprises are higher when biases are more substantial. Thus, an overshoot surprise is indeed a strong indicator of the size and direction of a bias in the crowd average. The SO estimator can be thought of as $\bar{x}$ adjusted away from the direction of the bias, identified by the sign of the overshoot surprise. Furthermore, $|x^{SO} - \bar{x}|$ is proportional to the magnitude of the overshoot surprise. Thus, the evidence from overshoot surprises suggests potential error reduction from using the SO algorithm. Section 5 explores whether the SO algorithm improves over various benchmarks.

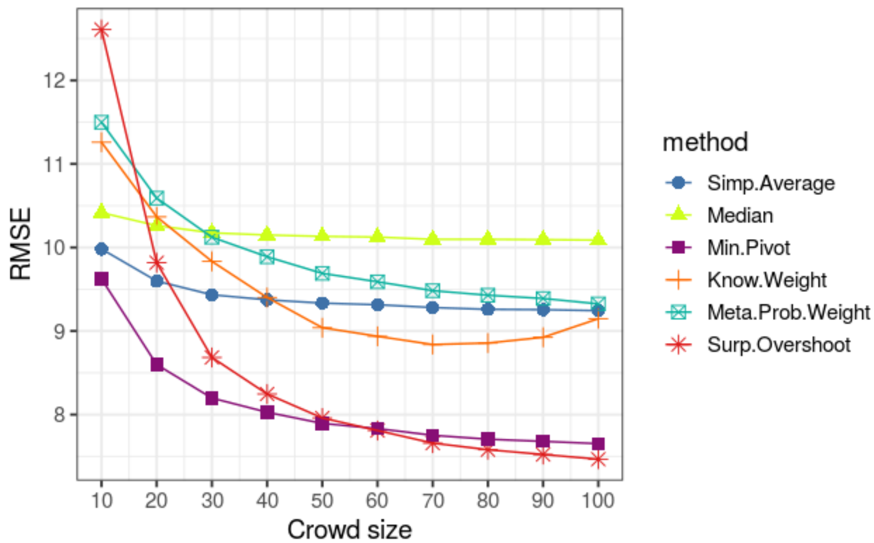
5 Results

5.1 Coin Flips data

In testing the SO estimator with Coin Flips data, I follow a bootstrap approach similar to Palley and Satopää (17). For each item in a given data set, a subset of subjects of size M is randomly selected to construct a bootstrap sample. In each sample (for the associated item), I compute the absolute errors of aggregate predictions from the benchmarks and the SOA. The average of absolute errors across the items gives a measure of the corresponding method's error in that task. I run this procedure 100 times for each crowd size to obtain 100 data points on each method's errors for each crowd size. The observations from bootstrap samples allow us to test for differences in errors between the SO algorithm and a benchmark. I consider two measures for comparison. Firstly, I calculate average RMSE across all iterations for each method. Secondly, I log transform the errors and calculate pairwise differences for each iteration to construct 95% bootstrap confidence intervals for statistical inference. The differences in log-transformed errors can be interpreted as percentage error reduction. The bootstrap approach also allows us to see the effect of crowd size on the relative performance of the SO algorithm.

Figure 8 presents the results of the bootstrap analysis. Figure 8a depicts the average RMSE across iterations. Figure 8b shows the bootstrap confidence intervals for reduction in log absolute error (the SO estimator vs benchmark). The two panels in 8b depict the error reductions compared to simple and advanced benchmarks respectively

(a) Average RMSE (across iterations) vs crowd size



(b) Reduction in log absolute error (averaged across items) in Bootstrap samples

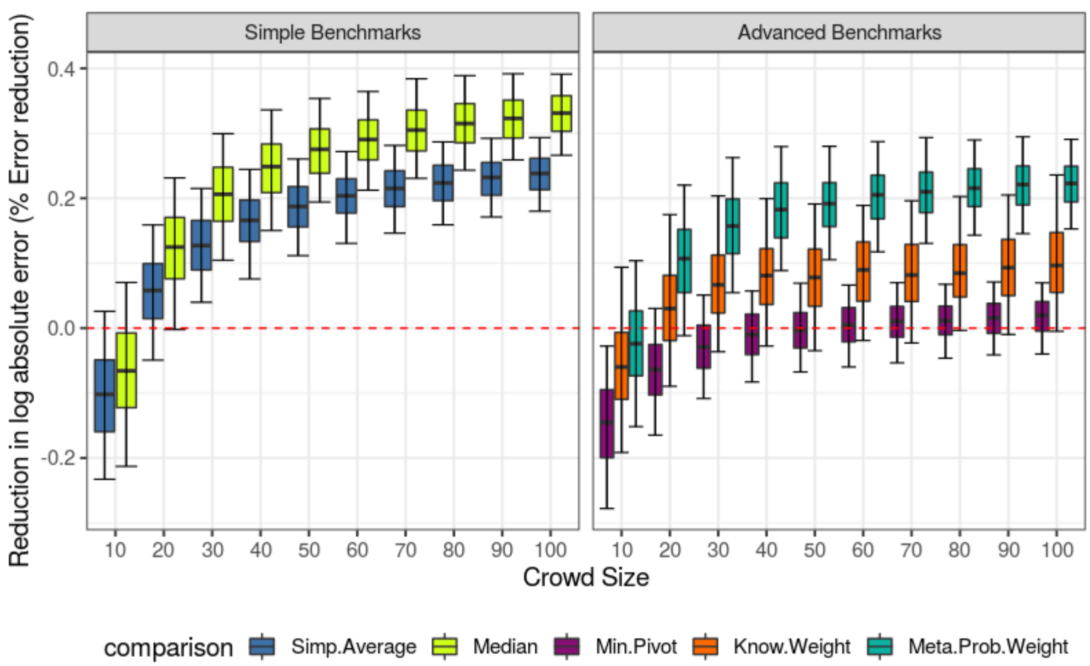


Figure 8: Results of bootstrap analysis on Coin Flips data

Figure 8a shows that the SO algorithm achieves lower errors in sufficiently large samples. Observe that increasing the sample size has a stronger effect on the SO estimator. Almost all aggregation methods benefit from larger samples due to the Wisdom of Crowds effect. For the SO algorithm, benefits of a larger crowd could be twofold. Not only the wisdom of crowds effect become more pronounced, but also a larger sample of predictions is typically more dispersed and allows a more accurate approximation of the population density of predictions.

The results in Figure 8b show that the differences between the SO algorithm and the benchmarks are significant, except for the MP method. Appendix B provides the 95% bootstrap confidence intervals depicted in 8b. The SO algorithm improves the accuracy by 20-40% relative to the simple benchmarks. Compared to the MPW and KW methods, the percentage error reduction is around 20% and 10% respectively.

The Coin Flips study elicits judgments on a controlled prediction task where all private signals are equally informative and the data generation process of signals induces a distribution of predictions. The following section presents evidence from General Knowledge and State Capital data. In those tasks, subjects may differ in knowledge and the distribution of predictions varies across items.

5.2 State Capital and General Knowledge data

I now consider the State Capital and General Knowledge data sets from the experimental studies in Wilkening et al. (23). As discussed in Section 4.1, items in these data sets have a binary truth. The analysis in this section tests if the SO algorithm produces a probabilistic estimate closer to the true answer. I follow a similar approach to Budescu and Chen (2) and Martinie et al. (14) and calculate transformed Brier scores associated with the aggregate estimates of each method in each data set. The transformed Brier score of a method i in a given data set is defined as

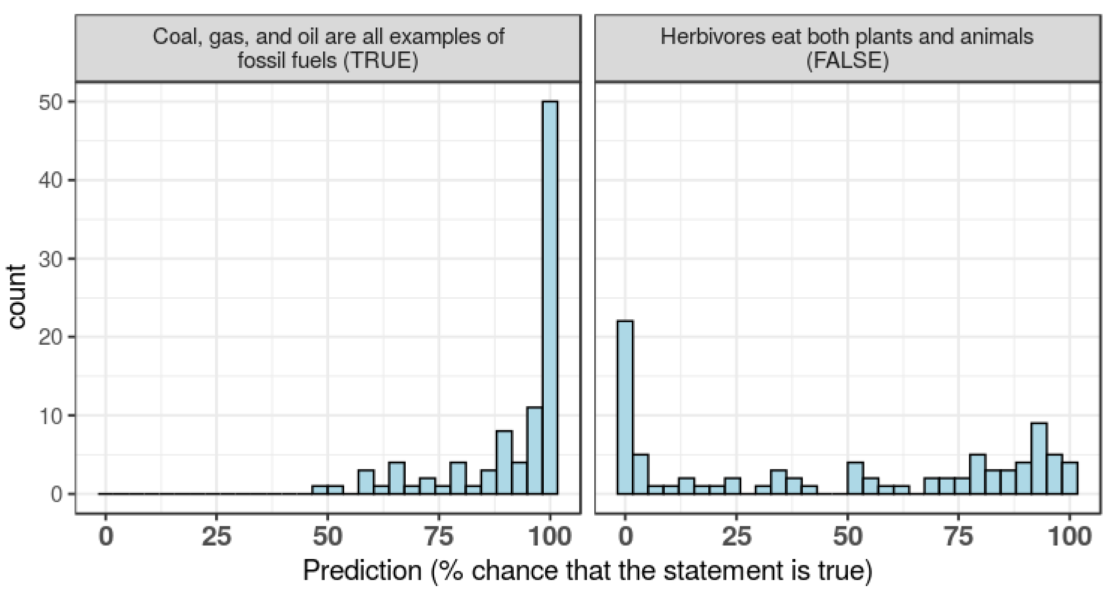
$$S_i = 100 - 100 \sum_{j=1}^{J} \frac{(o_j - x_j^i)^2}{J}$$

where $o_j \in \{0, 1\}$ be the outcome of event j, J is the total number of events in the data set and $x_j^i \in [0, 1]$ is the aggregate probabilistic prediction of method i on event j. The transformed Brier score is strictly proper and assigns a score within $[0, 100]$.

The main question of interest is whether the SO algorithm achieves significantly higher transformed Brier scores compared to the benchmarks. For statistical inference, I will construct 95% confidence intervals for paired differences in Transformed Brier scores using a bootstrap approach. I generate 1000 bootstrap samples for each method in each data set. A bootstrap sample consists of items sampled with replacement and provides a measurement on S_i. Then, I calculate the paired differences between the scores of the SO estimator and the other benchmarks for each bootstrap sample. The 2.5% and 97.5% quantiles of pairwise differences provide a 95% bootstrap confidence interval for testing SO algorithm against a benchmark.

Section 3 discussed that the SO algorithm could be more effective when predictions are more dispersed. In Coin Flips data, the randomness in subjects' private signals result in a variance in predictions. The General Knowledge and State Capital allows us to investigate the extent of dispersion in realistic prediction tasks and how it relates to the performance of SO algorithm. To illustrate, consider the two items from the General Knowledge data in Figure 9 below:

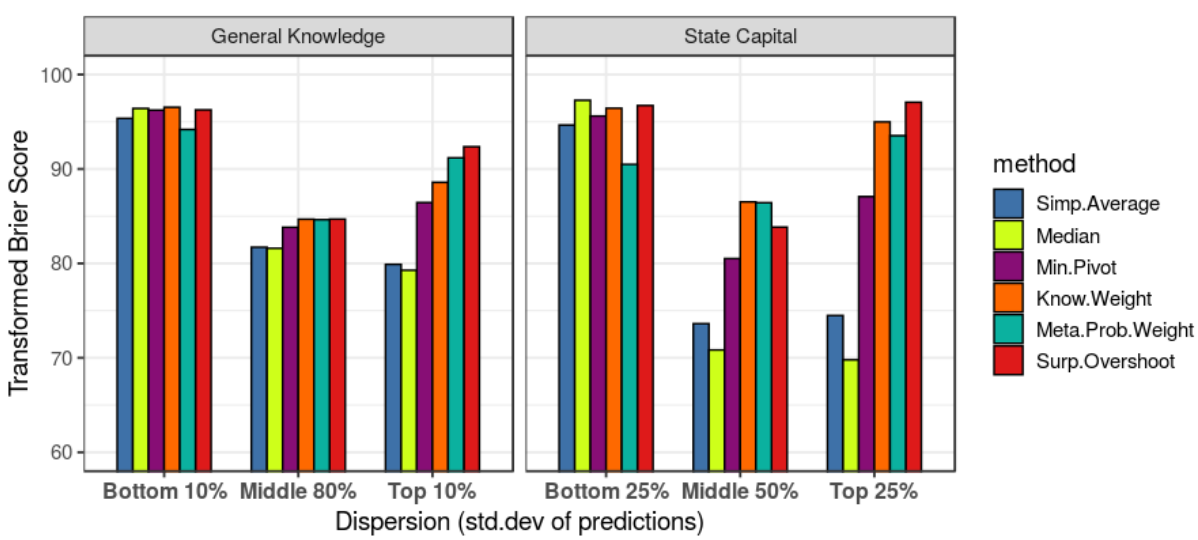
Figure 9: Predictions on two example items from the General Knowledge data



For the item in the left panel, a large number of predictions are at 100% and almost all predictions are 50% or higher. The dispersion of predictions is relatively smaller compared to the item in the right panel, where prediction vary from 0% to 100%. Similar examples can be found in the State Capital data. Given such variety, we can categorize the items in terms of the dispersion of predictions and calculate transformed Brier scores for each category of items. For the main results below, I use standard deviation of predictions as the measure of dispersion in an item. Appendix A replicates the same analysis using kurtosis as the measure and finds qualitatively identical results. In the General Knowledge data, I categorize the items in three groups in terms of the standard deviation of predictions: bottom 10%, middle 80% and top 10%. The bottom and top 10% items represent the low and high dispersion items respectively. The State Capital data includes a lower number of items. In order to have a reasonable number of items in each category, the thresholds are set at 25% and 75%. Thus, the categories in the State capital data are bottom 25%, middle 50% and top 25%. The transformed brier score will be calculated separately for each dispersion category.

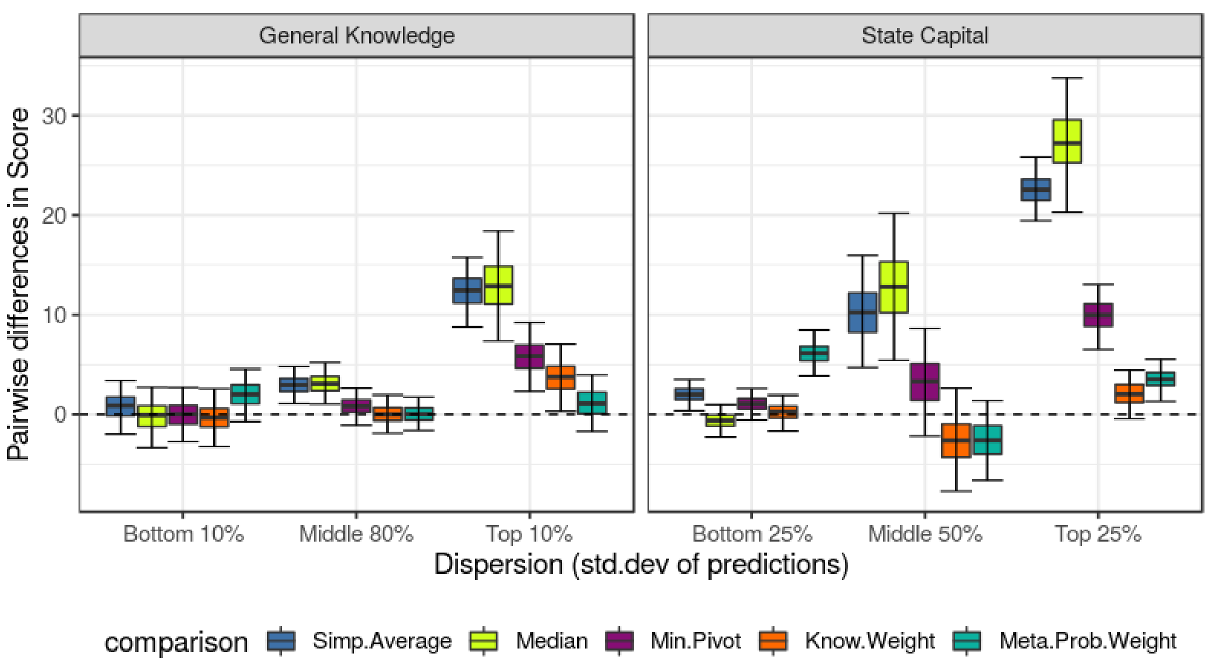
The discussion in Section 3 suggests that the SO algorithm would be relatively more effective in the high-dispersion items. Figure 10 below provides the transformed Brier scores for each method across various levels of dispersion in the General Knowledge and State Capital items.

Figure 10: Transformed Brier Scores in the two data sets



Observe that in both data sets, the transformed Brier scores of the simple aggregation methods (simple average and median) are lower in high dispersion items. In simple aggregation methods we consider, the aggregate prediction is simply a measure of central tendency. Individual predictions are more spread in high-dispersion items, suggesting a higher frequency of inaccurate judgments. As in Figure 9, there could be many predictions putting a high probability on 'True' even though the correct answer is 'False', reducing the accuracy of simple aggregates. Figure 10 demonstrates the the advanced aggregation algorithms improve over simple aggregates in such cases. However, note that the SO algorithm achieves the best transformed Brier score among items with highest dispersion in either data set. So, in such items the SO algorithm is effective relative to other advanced benchmarks as well. Figure 11 below presents 95% bootstrap confidence intervals for pairwise differences in transformed Brier scores. An observation above the 0-line indicates that the SO estimator achieved a higher transformed Brier score than the corresponding benchmark in that particular bootstrap sample.

Figure 11: Bootstrap differences in Transformed Brier Scores



Appendix B provides the Bootstrap confidence intervals depicted in Figure 11. We see that the SO estimator outperforms simple benchmarks except for low-dispersion items. Furthermore, the SO algorithm compares favorably to the advanced benchmarks in high-dispersion items. In the General Knowledge data, differences with the MP and KW algorithms are significant. In the State Capital data, differences with the MP and MPW algorithms are significant at 95% while the difference with the KW algorithm is significant at 90%. The SO algorithm is relatively more effective when individual predictions disagree greatly, resulting in a more dispersed empirical density of predictions.

6 When and why is the SO algorithm effective?

The findings presented in Section 5 not only present SO algorithm as an aggregation mechanism but also provides a 'user's manual' for a DM who intends to use an aggregation algorithm to combine probabilistic judgments. The dispersion in empirical density of predictions is ex-ante observable to the DM. In prediction tasks where individual predictions are clustered at certain values, the simple aggregation methods may perform sufficiently well. When there is high dispersion in predictions, some judgments will end up being less accurate. In such cases, advanced aggregation mechanisms improve over simple aggregates. High dispersion in predictions allows the the SO algorithm to produce a fine-tuned estimate. Evidence from General Knowledge and State Capital data sets suggest that the SO algorithm could be relatively more effective when predictions are more dispersed. A decision maker who observes strong disagreement among individual judgments may consider SO algorithm as a viable solution for extracting the wisdom of crowds in estimating the likelihood of an event.

The SO algorithm differs from the other alternative aggregation algorithms in its use of the empirical density of predictions. Recall that the SO algorithm simply selects a sample quantile that may differ from average prediction is located. For a given level of overshoot surprise, the absolute difference between the SO estimator and the average prediction depends on the dispersion in the empirical density of predictions. Furthermore, the SO algorithm always sets an aggregate estimate that lies within the range of individual predictions. Thus, the SO algorithm's adjustment on the aggregate prediction is informed and restrained by the empirical density. This makes the SO estimator more robust to potential over-adjustments, which could reduce the accuracy even when the aggregate estimate is adjusted in the correct direction (i.e. away from the shared-information bias).

7 Conclusion

Very few decision problems have total absence of uncertainty. Decision makers frequently face the problem of predicting the likelihood of an uncertain event as part of a decision making process. Leveraging the Wisdom of Crowds is shown to be a promising solution. However, the use of collective wisdom is not a trivial problem either, because there are typically no guidelines as to how individual judgments should be aggregated for maximum accuracy. Recent work developed aggregation algorithms that rely on an augmented elicitation procedure (19, 20, 18, 17, 23). These algorithms use individuals' meta-beliefs to aggregate predictions more effectively. This paper proposes a novel algorithm to aggregate probabilistic judgments on the likelihood of an event. The Surprising Overshoot algorithm uses individuals' probabilistic meta-predictions to aggregate their judgments on the chances of an event. However, the SO algorithm also utilizes the information in the empirical density of predictions to determine an aggregate prediction. Experimental evidence suggests that the SO algorithm is very effective when the empirical density of predictions is highly dispersed. Such high dispersion is more likely in larger crowds and more uncertain prediction tasks.

In practice, a decision maker is more likely to need the Wisdom of Crowds when the event in question is highly uncertain. In such cases, even the best experts might disagree greatly. The decision maker could end up collecting deeply conflicting judgments with no straightforward way to combine them. The SO algorithm could be especially powerful in such challenging aggregation problems. The dispersion in predictions that may result from the disagreement among individuals works in the algorithm's favor. Some subjects in General Knowledge and State Capital data sets studied in this paper might know the correct answer in a given item. In many prediction problems, the outcome would not have realized yet and/or the correct answer would remain unknown to all until the realization. We may expect even higher uncertainty and dispersion in predictions in such cases. Future work may test the SO algorithm further in prediction problems which could potentially be more challenging.

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Appendix A

FIGURE 8b WITH RMSE DIFF (INSTEAD OF LOG DIFF) HERE.

FIGURES WITH KURTOSIS (INSTEAD OF STD DEV) HERE

Appendix B

All tables with 90% and 95% bootstrap CIs here