Section 3: Mathematical Induction II

Prove: $2^{2n} - 1$ is divisible by 3, for integers $n > 0$.

Proof: (Induction) <u>Basis:</u> $2^{2(1)} - 1 = 2^2 - 1 = 4 - 1 = 3$, which is clearly divisible by 3.

<u>Induction:</u> Assume for some integer k, $2^{2k} - 1$ is divisible by 3.

Now, $2^{2(k+1)} - 1 = 2^{(2k+2)} - 1 = 2^{2k}2^2 - 1$
$= 2^{2k}(4) - 1 = 2^{2k}(3 + 1) - 1 = 3(2^{2k}) + 2^{2k} - 1$
$= 3(2^{2k}) + (2^{2k} - 1)$.

Since each term in parentheses is divisible by 3, we have therefore that $2^{2(k+1)} - 1$ is also. QED

Proving An Inequality

Prove: $2n + 1 < 2^n$, for all integers $n \geq 3$.

Proof: (Induction) <u>Basis:</u> LHS = 2(3) + 1 = 7, and RHS = 2^3 = 8, so clearly $2n + 1 < 2^n$ for $n = 3$.

<u>Induction:</u> Assume for some integer k, $2k + 1 < 2^k$. Show $2(k + 1) + 1 < 2^{(k+1)}$.

$$\begin{aligned} \text{Now, } 2(k + 1) + 1 &= (2k + 1) + 2 \\ &< 2^k + 2 < 2^k + 2^k = 2^{(k+1)}. \end{aligned}$$

Therefore, $2n + 1 < 2^n$, for all integers $n \geq 3$.. QED

Number of Subsets

Prove: A set with n elements has 2^n subsets.

Proof: (Induction) Basis: Since the empty set has 1 subset (itself), and $2^0 = 1$, then a set with 0 elements has 2^0 subsets.

Induction: Assume every k-element set has 2^k subsets. Show every (k+1)-element set has $2^{(k+1)}$ subsets.

Now let A = $\{a_1, a_2, a_3,..., a_k, b\}$, so that A has ($k$+1) elements. We partition P(A) into two sub-collections where the first contains subsets of A which don't have b in them and the second contains

Number of Subsets (*cont'd.*)

subsets of A which do have b in them. Thus:

First Sub-collection	Second Sub-collection
$\{\}$	$\{b\}$
$\{a_1\}$	$\{a_1, b\}$
$\{a_1, a_2\}$	$\{a_1, a_2, b\}$
$\{a_1, a_2, ..., a_k\}$	$\{a_1, a_2, ..., a_k, b\}$

Clearly, the first collection is made up of all the subsets from the k-element set $\{a_1, a_2, ..., a_k\}$ so it has 2^k entries.

Number of Subsets (*cont'd.*)

Now, by construction, it follows that the second collection must have the same number of entries as the first, so it too must have 2^k entries.

Since the collection of all subsets of A has been partitioned into these two sub-collections, we see that A must have $2^k + 2^k = 2^{(k+1)}$ subsets. QED