Loop and cut set Analysis

- *Fundamental theorem of graph theory*
- *Loop Analysis*
- *Two basic facts of loop analysis*
- *Loop analysis of linear time invariant networks*
- *Properties of the loop impedance matrix*
- *Cut set Analysis*
- *Two basic facts of cut-set analysis*
- *Cut-set analysis of linear time invariant networks*
- *Properties of the cut-set admittance matrix*

Loop and cut set Analysis

- Loop and cut set are more flexible than node and mesh analyses and are useful for writing the state equations of the circuit commonly used for circuit analysis with computers.
- The loop matrix **B** and the cutset matrix **Q** will be introduced.

Fundamental Theorem of Graph Theory

A tree of a graph is a connected subgraph that contains all nodes of the graph and it has no loop. Tree is very important for loop and curset analyses. A Tree of a graph is generally not unqiue. Branches that are not in the tree are called links.

Loop and cut set Analysis

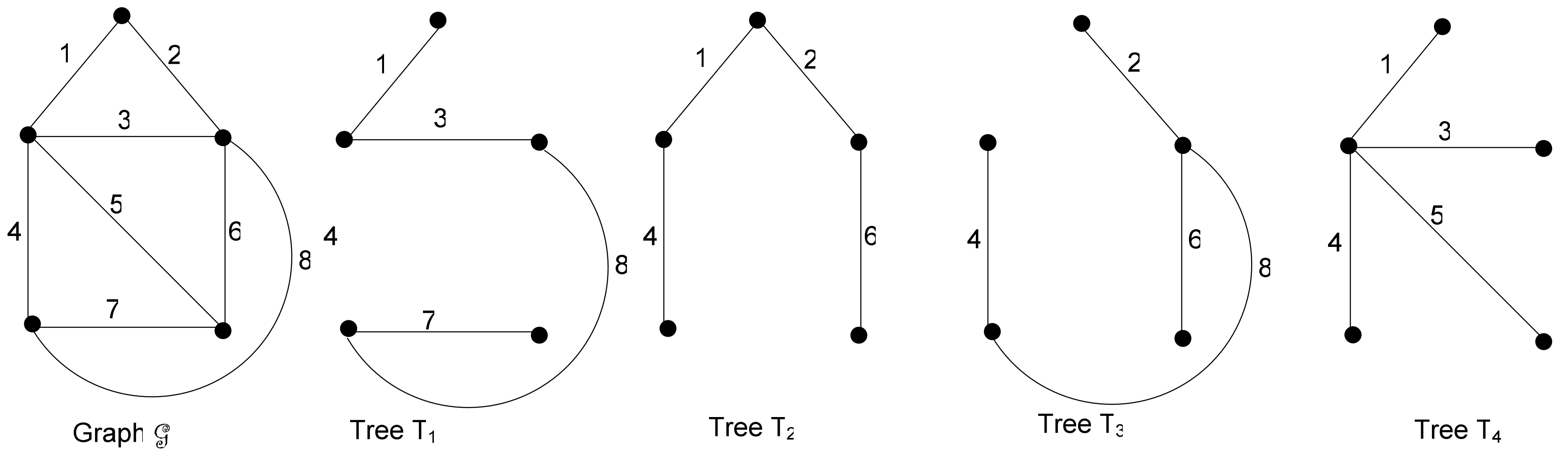


Fig.1 Examples of Tree

Loop and cut set Analysis

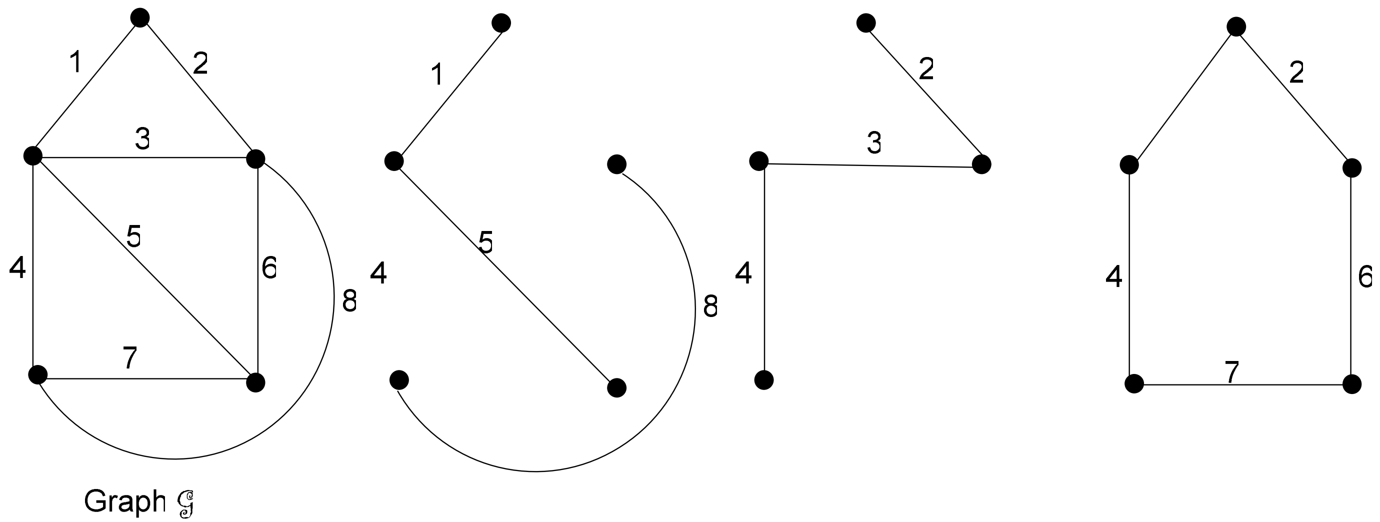


Fig.2 Not a Tree

Loop and cut set Analysis

Properites of loop and cut set

Give a connected graph $\mathcal{G}$ of n_t nodes and b branches and a tree T of $\mathcal{G}$

- There is a unique path along the tree between any two nodes
- There are $n_t - 1$ tree branches $b - n_t + 1$ links.
- Every link of T and the unique tree path between its nodes constitutes a unique loop called "fundamental loop".
- Every Tree branch of T together with some links defines a unique cut set of $\mathcal{G}$. This cut set is called a fundamental cut set.

If $\mathcal{G}$ has n_t nodes, b branches and s separate parts . Let $T_1, T_2, \ldots. T_s$ be trees of each separate part. The set $\{T_1, T_2, \ldots. T_s\}$ is called the forest of $\mathcal{G}$

Loop and cut set Analysis

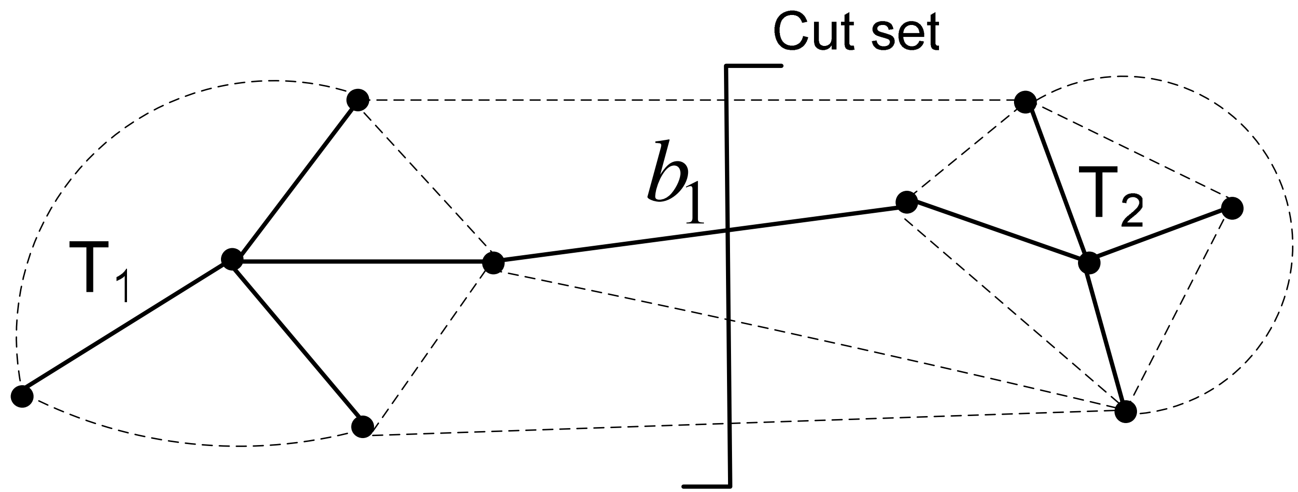
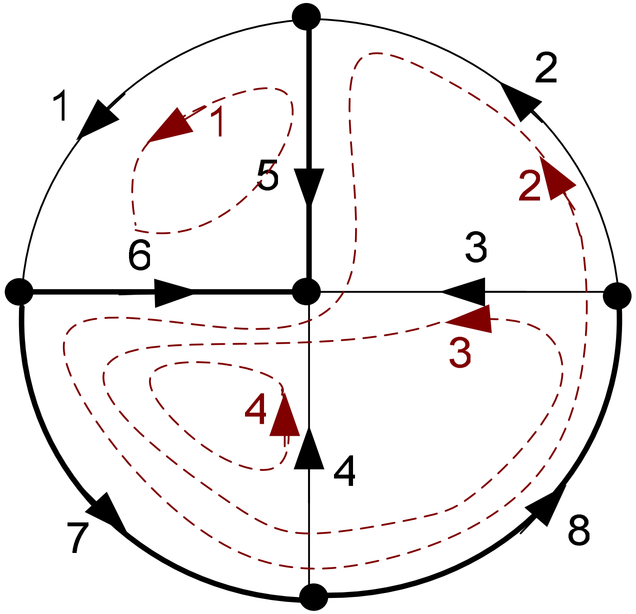


Fig.3 Fundamental cut set

Loop Analysis

Consider a connected graph with b branches and n_t nodes. Pick a tree T
There are $n = n_t$-1 tree branches and $\ell = b\text{-}n_t$ links. Number the links first to be 1,2….ℓ and number the tree from ℓ+1 to b . Every link and a unique path of tree branches defines a fundamental loop.

The graph of Fig. 4 illustrates fundamental loop for the chosen Tree



Link 1,2,3,4

Tree 5,6,7,8

Fig.4 Fundamental loop

Loop Analysis

Assign the direction of loop current to the same as the direction of the link the KVL for each fundamental loop are.

$$loop\,1: v_1 - v_5 + v_6 = 0$$

$$loop\,2: v_2 + v_5 - v_6 + v_7 + v_8 = 0$$

$$loop\,3: v_3 - v_6 + v_7 + v_8 = 0$$

$$loop\,4: v_4 - v_6 + v_7 = 0$$

In matrix form

$$\begin{bmatrix} 1 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & -1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \\ v_8 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$1 = b - n$ links

$n = n_t - 1$

tree branches

Loop Analysis

The ℓ linear homogeneous algebraic equations in $v_1, v_2, \ldots v_b$ obtained by applying KVL to each fundamental loop constitute a set of ℓ linearly independent equation

If the reference direction of the loop agrees with that of the link which defines it, the KVL is of the form.

$$\mathbf{Bv} = \mathbf{0}$$

$\mathbf{B}$ is ℓ x b matrix called the fundamental loop matrix

$$b_{ik} = \begin{cases} 1 & \text{If branch } k \text{ is in loop } i \text{ and reference direction agree} \\ -1 & \text{If branch } k \text{ is in loop } i \text{ and reference direction opposite} \\ 0 & \text{If branch } k \text{ is not in loop } i \end{cases}$$

Loop Analysis

The fundamental loop matrix can be partitioned in to

$$\mathbf{B} = [\mathbf{1}_l \;\vdots\; \mathbf{F}]$$

The KCL can be written in the form

$$\mathbf{j} = \mathbf{B}^T \mathbf{i} = \begin{bmatrix} \mathbf{1}_l \\ \cdots\cdots \\ \mathbf{F}^T \end{bmatrix} \mathbf{i}$$

The KCL for Fig.4 is

$$\begin{aligned} j_1 &= i_1 & \qquad j_5 &= -i_1 + i_2 \\ j_2 &= i_2 & \qquad j_6 &= i_1 - i_2 - i_3 - i_4 \\ j_3 &= i_3 & \qquad j_7 &= i_2 + i_3 + i_4 \\ j_4 &= i_4 & \qquad j_8 &= i_2 + i_3 \end{aligned}$$

Loop Analysis

In the matrix form

$$\begin{bmatrix} j_1 \\ j_2 \\ j_3 \\ j_4 \\ j_5 \\ j_6 \\ j_7 \\ j_8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 1 & 0 & 0 \\ 1 & -1 & -1 & -1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \end{bmatrix}$$

Loop analysis of linear time invariant networks

In a resistive circuit, the branch equations are of the form

$$\mathbf{v} = \mathbf{R}\mathbf{j} + \mathbf{v}_s - \mathbf{R}\mathbf{i}_s$$

Premultiply by $\mathbf{B}$ and apply KCL and KVL yields

$$\mathbf{B}\mathbf{R}\mathbf{B}^T = -\mathbf{B}\mathbf{v}_s + \mathbf{B}\mathbf{R}\mathbf{J}_s$$

or

$$\mathbf{Z}_1 \mathbf{i} = \mathbf{e}_s$$

where $\mathbf{Z}_1 \triangleq \mathbf{B}\mathbf{R}\mathbf{B}^T \qquad \mathbf{e}_s = -\mathbf{B}\mathbf{v}_s + \mathbf{B}\mathbf{R}\mathbf{J}_s$

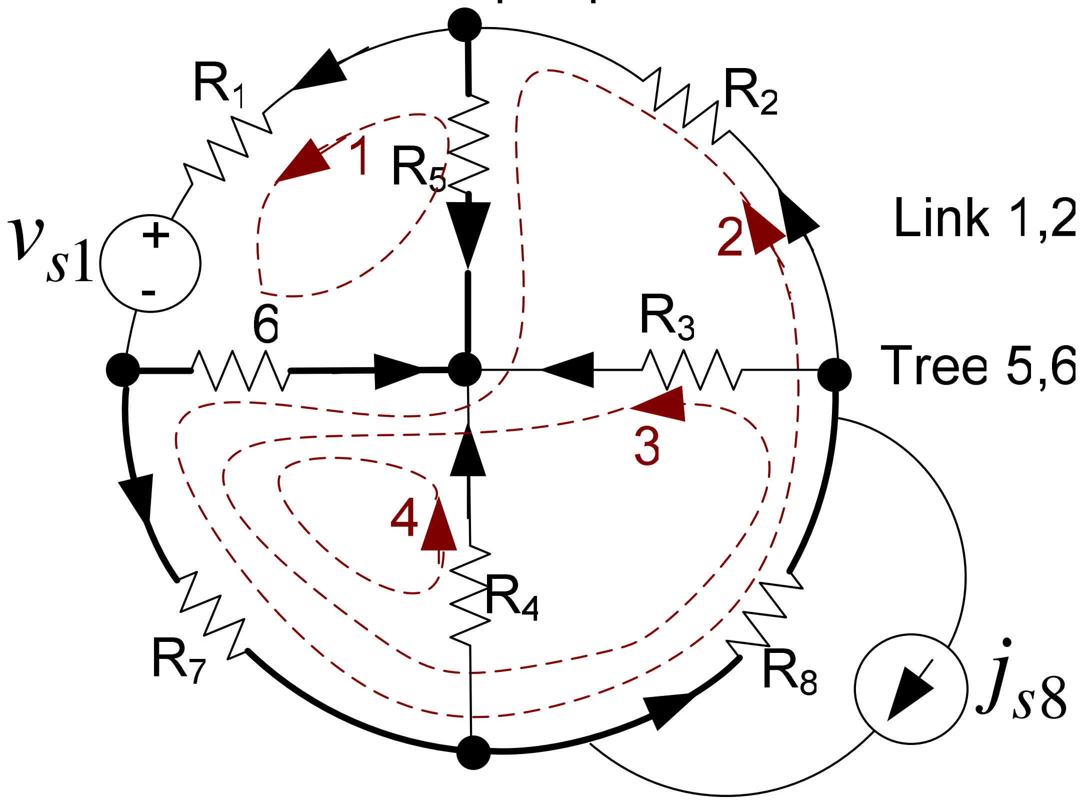
$\mathbf{Z}_1$ is called the loop impedance matrix and

$\mathbf{e}_s$ is the loop voltage source vector

Loop analysis of linear time invariant networks

Example 1

Write the fundamental loop equation for the circuit shown in Fig.5.



Link 1,2,3,4

Tree 5,6,7,8

Fig. 5

Loop analysis of linear time invariant networks

The branch equations are

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \\ v_8 \end{bmatrix} = \begin{bmatrix} R_1 & & & & & & & \\ & R_2 & & & & 0 & & \\ & & R_3 & & & & & \\ & & & R_4 & & & & \\ & & & & R_5 & & & \\ & 0 & & & & R_6 & & \\ & & & & & & R_7 & \\ & & & & & & & R_8 \end{bmatrix} \begin{bmatrix} j_1 \\ j_2 \\ j_3 \\ j_4 \\ j_5 \\ j_6 \\ j_7 \\ j_8 \end{bmatrix} + \begin{bmatrix} v_{s1} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ R_8 j_{s8} \end{bmatrix}$$

Loop analysis of linear time invariant networks

1

$$=\begin{bmatrix} R_1+R_5+R_6 & -R_5-R_6 & -R_6 & -R_6 \\ -R_5-R_6 & R_2+R_5+R_6+R_7+R_8 & R_6+R_7+R_8 & R_6+R_7 \\ -R_6 & R_6+R_7+R_8 & R_3+R_6+R_7+R_8 & R_6+R_7 \\ -R_6 & R_6+R_7 & R_6+R_7 & R_4+R_6+R_7 \end{bmatrix}$$

And the loop equations are

$$\begin{bmatrix} R_1+R_5+R_6 & -R_5-R_6 & -R_6 & -R_6 \\ -R_5-R_6 & R_2+R_5+R_6+R_7+R_8 & R_6+R_7+R_8 & R_6+R_7 \\ -R_6 & R_6+R_7+R_8 & R_3+R_6+R_7+R_8 & R_6+R_7 \\ -R_6 & R_6+R_7 & R_6+R_7 & R_4+R_6+R_7 \end{bmatrix}\begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \end{bmatrix}=\begin{bmatrix} -v_{s1} \\ -R_8 j_{s8} \\ -R_8 j_{s8} \\ 0 \end{bmatrix}$$

Loop analysis of linear time invariant networks

Properties of the loop impedance matrix

For a RLC networks in sinusoid steady state the loop impedance matrix

$\mathbf{Z}_{1}(j\omega) = \mathbf{B}\mathbf{Z}_{b}(j\omega)\mathbf{B}^{T}$ and has the following properties

- If there is no coupling element the matrix $\mathbf{Z}_{b}(j\omega)$ is diagonal and the loop impedance matrix is symmetric.
- If there is no coupling element the matrix $\mathbf{Z}_{b}(j\omega)$ can be written by inspection

  $Z_{ii}(j\omega)$ is the sum of impedance in the loop i and

  $Z_{ik}(j\omega)$ is the sum or negative sum of impedance of branch k impedance common to loop i the plus sign applied if the branch k direction agree with the loop direction.
- If all current sources are converted to Thevenin voltage sources, then e_{sk} is the sum of voltage sources forcing the current flow in the loop .

Cut set Analysis

- Cut set analysis is a dual of loop analysis
- Every tree branch defines a unique cut set

The fundamental cut set of the circuit of Fig.4 is shown in Fig.6

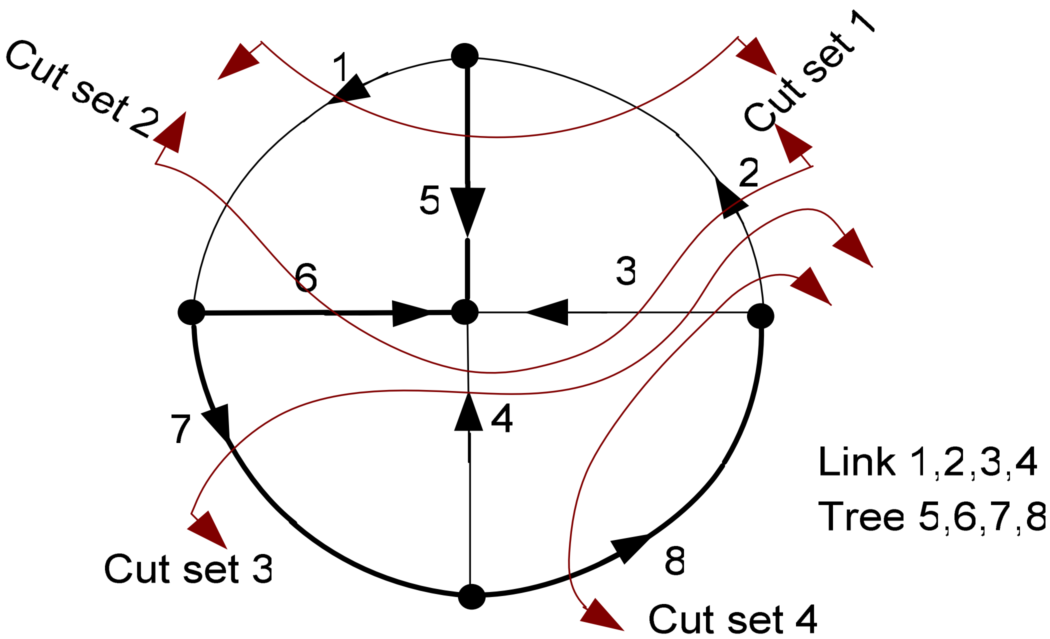


Fig. 6

Cut set Analysis

KCL can be written for each cut set as shown

Cut set 1: $j_1 - j_2 + j_5 = 0$

Cut set 2: $-j_1 + j_2 + j_3 + j_4 + j_5 = 0$

Cut set 3: $-j_2 - j_3 - j_4 + j_7 = 0$

Cut set 4: $-j_2 - j_3 + j_8 = 0$

In matrix form

$$\begin{bmatrix} 1 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & -1 & -1 & 0 & 0 & 1 & 0 \\ 0 & -1 & -1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} j_1 \\ j_2 \\ j_3 \\ j_4 \\ j_5 \\ j_6 \\ j_7 \\ j_8 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Or

$$\mathbf{Qj} = \mathbf{0}$$

Cut set Analysis

- The n linear homogeneous algebraic equations in $j_1, j_2, \dots j_b$ obtained by applying KCL to each fundamental cut set constitute a set of n linearly independent equations.

The fundamental cut set matrix $\mathbf{Q}$ is defined by

$$q_{ik} = \begin{cases} 1 & \text{If branch } k \text{ belongs to cut set } i \text{ and reference direction agree} \\ -1 & \text{If branch } k \text{ belongs to cut set } i \text{ but reference direction opposite} \\ 0 & \text{If branch } k \text{ does not belong to cut set } i \end{cases}$$

The cut set matrix can be partitioned by

$$\mathbf{Q} = [\mathbf{E} \;\vdots\; \mathbf{1}_n] \quad \ell \text{ link} \;\; \text{n cut set}$$

Since the voltage of each branch is a linear combination of tree branch voltages and if tree branch voltages are $e_1, e_2, \dots e_n$ then for Fig.6

Cut set Analysis

KVL

$$v_1 = v_5 - v_6 = e_1 - e_2$$

$$v_2 = -v_5 + v_6 - v_7 - v_8 = -e_1 + e_2 - e_3 - e_4$$

$$v_3 = v_6 - v_7 - v_8 = e_2 - e_3 - e_4$$

$$v_4 = v_6 - v_7 = e_2 - e_3$$

$$v_5 = e_1$$

$$v_6 = e_2$$

$$v_7 = e_3$$

$$v_8 = e_4$$

or

$$\mathbf{v} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & -1 & -1 \\ 0 & 1 & -1 & -1 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{bmatrix}$$

or KVL $\mathbf{v} = \mathbf{Q}^T \mathbf{e}$

Cut-set analysis of linear time invariant networks

In Cut set analysis Kirchhoff's laws are

KCL: $\mathbf{Qj} = \mathbf{0}$ KVL: $\mathbf{v} = \mathbf{Q}^T\mathbf{e}$

And the branch equations

$$\mathbf{j} = \mathbf{Gv} + \mathbf{j}_s - \mathbf{Gv}_s$$

Combine KCL KVL and branch equations to obtain

$$\mathbf{QGQ}^T\mathbf{e} = \mathbf{QGv}_s - \mathbf{Qj}_s$$

or $\mathbf{Y}_q\mathbf{e} = \mathbf{i_s}$

where

$$\mathbf{Y}_q \; @ \; \mathbf{QGQ}^T \qquad \mathbf{i}_s \; @ \; \mathbf{QGv}_s - \mathbf{Qj}_s$$

$\mathbf{Y}_q$ is the cut set admittance matrix and $\mathbf{i}_s$ is the current source vector

Cut-set analysis of linear time invariant networks

Properties of cut set matrix

For RLC circuit with sinusoid sources in steady state the properties of the Cut set admittance matrix $\mathbf{Y}_q$ are

$$\mathbf{Y}_q(j\omega) = \mathbf{Q}\mathbf{Y}_b(j\omega)\mathbf{Q}^T$$

- If the network has no coupling element the branch admittance is diagonal and the cut set admittance matrix $\mathbf{Y}_q$ is symmetric
- If there are no coupling $\mathbf{Y}_q$ can be written by inspection

$Y_{ii}(j\omega)$ is the sum of admittance in the cut set i and

$Y_{ik}(j\omega)$ is the sum or negative sum of branch admittance common to cut set i and cut set k the plus sign applied if the branch i and branch k has the same direction.

- If all voltage sources are converted to Norton sources, then i_{sk} is the algebraic sum of those currents in opposite to the direction of the cut set.

Cut-set analysis of linear time invariant networks

Example2

Write the cut set equation of Fig. 7 by inspection.

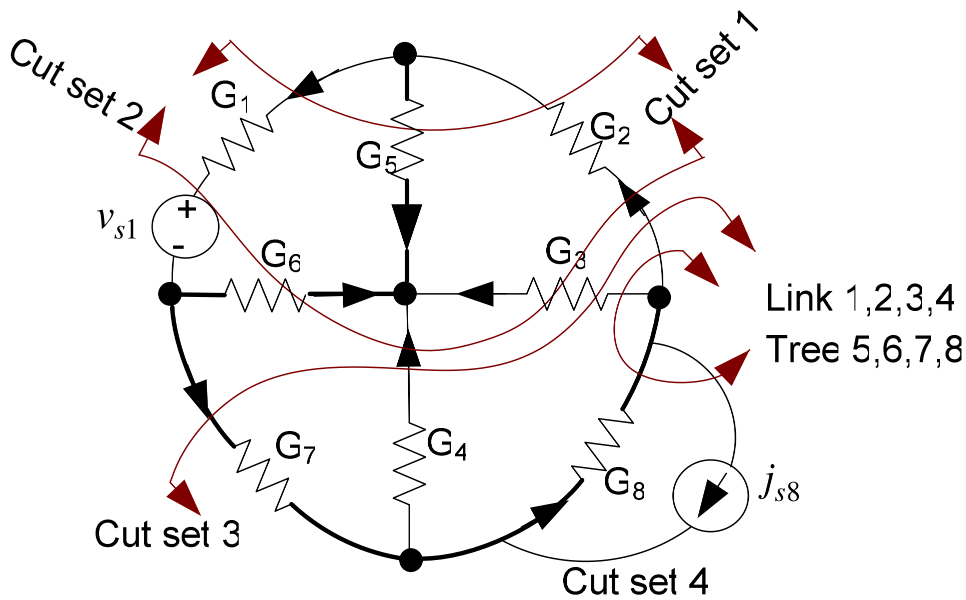


Fig. 7

Cut-set analysis of linear time invariant networks

$$\begin{bmatrix} G_1+G_2+G_5 & -G_1-G_2 & G_2 & G_2 \\ -G_1-G_2 & G_1+G_2+G_3+G_4+G_6 & -G_2-G_3-G_4 & -G_2-G_3 \\ G_2 & -G_2-G_3-G_4 & G_2+G_3+G_4+G_7 & G_2+G_3 \\ G_2 & -G_2-G_3 & G_2+G_3 & G_2+G_3+G_8 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{bmatrix} = \begin{bmatrix} G_1 v_{s1} \\ -G_1 v_{s1} \\ 0 \\ j_{s8} \end{bmatrix}$$

Comments on loop and cut set analysis

Loop and cut set analysis are more general than node and mesh analysis since Tree can be selected in many ways . With certain Tree the loop analysis Becomes the mesh analysis and Cut set analysis becomes the node analysis.

Relation between $\mathbf{B}$ and $\mathbf{Q}$

$$\mathbf{B}\mathbf{Q}^T = \mathbf{0} \quad \text{and} \quad \mathbf{Q}\mathbf{B}^T = \mathbf{0}$$