Section 6 - Indirect Argument

- Method of Proof by Contradiction;
- Method of Proof by Contraposition;
- Examples of Each Method.

Proof by Contradiction

- Instead of the Universal Modus Ponens argument form: $\forall x$, $[P(x) \rightarrow Q(x)$ AND $P(x)] \Rightarrow Q(x)$, a *Proof by Contradiction* (*reductio ad absurdum*) follows the Universal Modus Tollens form: $\forall x$, $[P(x) \rightarrow Q(x)$ AND $\sim Q(x)] \Rightarrow \sim P(x)$.
- We obtain a contradiction when the conclusion of this form is combined with our standard assumption in a direct proof the $P(x)$ holds.
- This differs marginally from the Method of Contraposition which proves directly the validity of the comtrapositive statement.

Method of Proof By Contradiction

- Suppose the statement to be proved is FALSE;
- Show this supposition leads logically to a contradiction (either to the original hypotheses or to some other statement of fact);
- Conclude that the original statement to be proved is TRUE.

Example: No Greatest Integer

Theorem: There is no greatest integer.

Proof: (Contradiction) Suppose there is a greatest integer N. Thus for every integer k, $k \leq N$.

Now, since N is an integer, by closure, $(N+1)$ is an integer. Thus: $N + 1 \leq N$,

hence $1 \leq 0$.*

Therefore, there is no greatest integer. QED

Sums of Rationals and Irrationals

Theorem: The sum of a rational and an irrational is irrational.

Proof: (Contradiction) Let r be rational, s be irrational, and assume $(r + s)$ is rational. Thus there exist $a,b,c,d \in \mathbf{Z}$, with $r = a/b$, $(r + s) = c/d$ and $b,d \neq 0$.

$$\text{Now, } s = (r + s) - r = c/d - a/b$$
$$= (bc - ad)/bd.$$

Since $a,b,c,d \in \mathbf{Z}$ and $b,d \neq 0$, we have $s \in \mathbf{Q}$.*

Therefore $(r + s)$ is irrational. QED

Argument by Contraposition

- Since we know that a statement and its contrapositive are logically equivalent, if we can pose our conjecture in the form of a conditional, we can work, equivalently, with its contrapositive form.
- We call this strategy, simply enough, *Argument by Contraposition*.

Method of Proof by Contraposition

- Express the statement to be proved in the form

  $\forall x$, if $P(x)$ then $Q(x)$.

- Rewrite this as its contrapositive

  $\forall x$, if $\sim Q(x)$ then $\sim P(x)$.

- Prove the contrapositive form directly:
  - *Suppose x is such that Q(x) is FALSE.*
  - *Show that P(x) is FALSE.*

Example of Contraposition

<u>Theorem:</u> Given any integer n, if n^2 is even, then n is even.

(Contrapositive: If n is odd, then n^2 is odd.)

<u>Proof:</u> (Contraposition) Let n be an integer and assume that n is odd. Thus, there is an integer k such that $n = 2k + 1$. Show that n^2 is odd.

$$\text{Now, } n^2 = (2k + 1)^2 = 4k^2 + 4k + 1$$
$$= 2(2k^2 + 2k) + 1.$$

Since k is an integer, $(2k^2 + 2k)$ is an integer.

Therefore n^2 is odd. QED