Periodic Complex Exponential and Sinusoidal Signals

$$x(t) = e^{j\omega_0 t} \qquad (1.21)$$

$$e^{j\omega_0 t} = e^{j\omega_0 (t+T)} = e^{j\omega_0 t} \cdot \underbrace{e^{j\omega_0 T}}_{1} \qquad (1.33)$$

which implies that ωT_0 is a multiple of 2π, i.e.,

$$\omega T_0 = 2\pi k, \quad k = 0, \pm 1, \pm 2 \ldots \qquad (1.34)$$

Thus,

$$\omega_0 = \frac{2\pi}{T_0} \qquad (1.35)$$

$$T_0 = \frac{2\pi}{|\omega_0|} \qquad (1.24)$$

Periodicity Properties of Disc-Time Comp. Exp.

$$e^{j(\omega_0 + 2\pi)n} = \underbrace{e^{j2\pi n}}_{\text{always } 1} e^{j\omega_0 n} = e^{j\omega_0 n}$$

* Therefore, in considering disc-time comp. exp., we need only consider a frequency interval of length 2π in which to choose ω_0

* as ω_0 increases from 0, signals that oscillate more and more rapidly until $\omega_0 = \pi$. After $\omega_0 = \pi$, decrease the rate of oscillation until $\omega_0 = 2\pi$

* In order for the signal $e^{j\omega_0 n}$ to be periodic with period $N > 0$,

$$e^{j\omega_0 (n+N)} = e^{j\omega_0 n} \qquad (1.53)$$

$$e^{j\omega_0 N} = 1 \quad (1.54)$$

$$\omega_0 N = 2\pi m \quad (1.55)$$

$$\frac{\omega_0}{2\pi} = \frac{m}{N} \quad (1.56)$$

$\Downarrow$

If rational number the signal is periodic

The fundamental freq. of periodic signal $e^{j\omega_0 n}$ is

$$\frac{2\pi}{N} = \frac{\omega_0}{m} \quad (1.57)$$

The fundamental period of periodic signal $e^{j\omega_0 n}$ is

$$N = m\left(\frac{2\pi}{\omega_0}\right) \quad (1.58)$$

$e^{j\omega_0 t}$	$e^{j\omega_0 n}$
* Distinct signals for distinct values of ω_0	* Identical signals for values of ω_0 seperated by multiples of 2π
* Periodic for any choice of ω_0	* Periodic only if $\omega_0 = 2\pi m/N$ for some integers $N > 0$ and m
* Fundemental freq. ω_0	* Fundemental freq* ω_0/m
* Fundemental period $\omega_0 = 0$: undefined $\omega_0 \neq 0$: $\frac{2\pi}{\omega_0}$	* Fundemental period $\omega_0 = 0$: undefined $\omega_0 \neq 0$: $m\left(\frac{2\pi}{\omega_0}\right)$

* Assumes that m and N do not have any factors in common

Periodicity Ex 1(1.25): Determine whether or not each of the following cs-time signals is periodic. If the signal is periodic, determine its fundamental period.

a) $x(t) = 3\cos\left(4t + \frac{\pi}{3}\right)$

If $x(t)$ is periodic

$$x(t) = x(t+T)$$

$$x(t) \stackrel{?}{=} x(t+T) \Rightarrow x(t+T) = 3\cos\left(4(t+T) + \frac{\pi}{3}\right)$$

$$3\cos\left(4t + \frac{\pi}{3}\right) = 3\cos\left(4(t+T) + \frac{\pi}{3}\right)$$
$$= 3\cos\left(4t + 4T + \frac{\pi}{3}\right)$$

$$4T = 2\pi m \quad , m \in \mathbb{Z}$$

$$T = \frac{2\pi m}{4}$$

Fundamental period is $\underline{T_0 = \frac{2\pi}{4}}$

b) $x(t) = e^{j(\pi t - 1)}$

$$x(t) \stackrel{?}{=} x(t+T)$$

$$e^{j(\pi t - 1)} = e^{j(\pi(t+T) - 1)} = e^{j(\pi t + \pi T - 1)}$$
$$= e^{j(\pi t - 1 + \pi T)} = e^{j(\pi t - 1) + j\pi T} = e^{j(\pi t - 1)} \cdot \frac{e^{j\pi T}}{1}$$

$$e^{j\pi T} = \underbrace{\cos(\pi T)}_{1} + \underbrace{j\sin(\pi T)}_{0}$$, T must be 2 at least, so $T_0 = 2$

c) $x(t) = \left[\cos(2t - \frac{\pi}{3})\right]^2$

From half-angle formula

$$\cos^2(2t - \frac{\pi}{3}) = \frac{\cos(4t - \frac{2\pi}{3}) + 1}{2}$$

does not effect on periodicity

$$\cos(4t - \frac{2\pi}{3}) = \cos(4t + 4T - \frac{2\pi}{3})$$

$$4T = 2\pi m$$

$$T_0 = \frac{2\pi}{4}$$

d) $x(t) = \mathcal{E}v\{\cos(4\pi t)\,u(t)\}$

For $f(t)$,

$$\text{Odd}\{f(t)\} = \frac{1}{2} \cdot \{f(t) - f(-t)\}$$

$$\mathcal{E}v\{f(t)\} = \frac{1}{2}\{f(t) + f(-t)\}$$

$$\mathcal{E}v\{\cos(4\pi t)\,u(t)\} = \frac{1}{2}\{\cos(4\pi t)\,u(t) + \cos(-4\pi t)\,u(-t)\}$$

$$= \frac{1}{2}\{\underbrace{\cos(4\pi t)\,u(t)}_{\substack{t<0 \Rightarrow 0 \\ t=0 \Rightarrow u(0)=\frac{1}{2}}} + \underbrace{\cos(4\pi t)\,u(-t)}_{\substack{t>0 \Rightarrow 0 \\ t=0 \Rightarrow u(0)=\frac{1}{2}}}\}$$

$$= \frac{1}{2}\{\cos(4\pi t)\,u(t)\}$$

$$\cos(4\pi t) = \cos(4\pi t + 4\pi T) \Rightarrow 4\pi T = 2\pi m$$

$$T_0 = \frac{1}{2}.$$

e) $x(t) = \mathcal{E}v\{\sin(4\pi t)\, u(t)\}$

$$= \frac{1}{2}\{\sin(4\pi t)\, u(t) + \sin(-4\pi t)\, u(-t)\}$$

$$= \frac{1}{2}\{\sin(4\pi t)\, u(t) - \sin(4\pi t)\, u(-t)\}$$

$t > 0$

$$x(t) = \frac{1}{2}\{\sin(4\pi t)\}$$

$$\sin(4\pi t) = \sin(4\pi(t+T))$$

$$4\pi T = 2\pi m$$

$$T_0 = \frac{1}{2}$$

$t = 0$ $\quad u(0) = \frac{1}{2}$

$x(t) = 0$

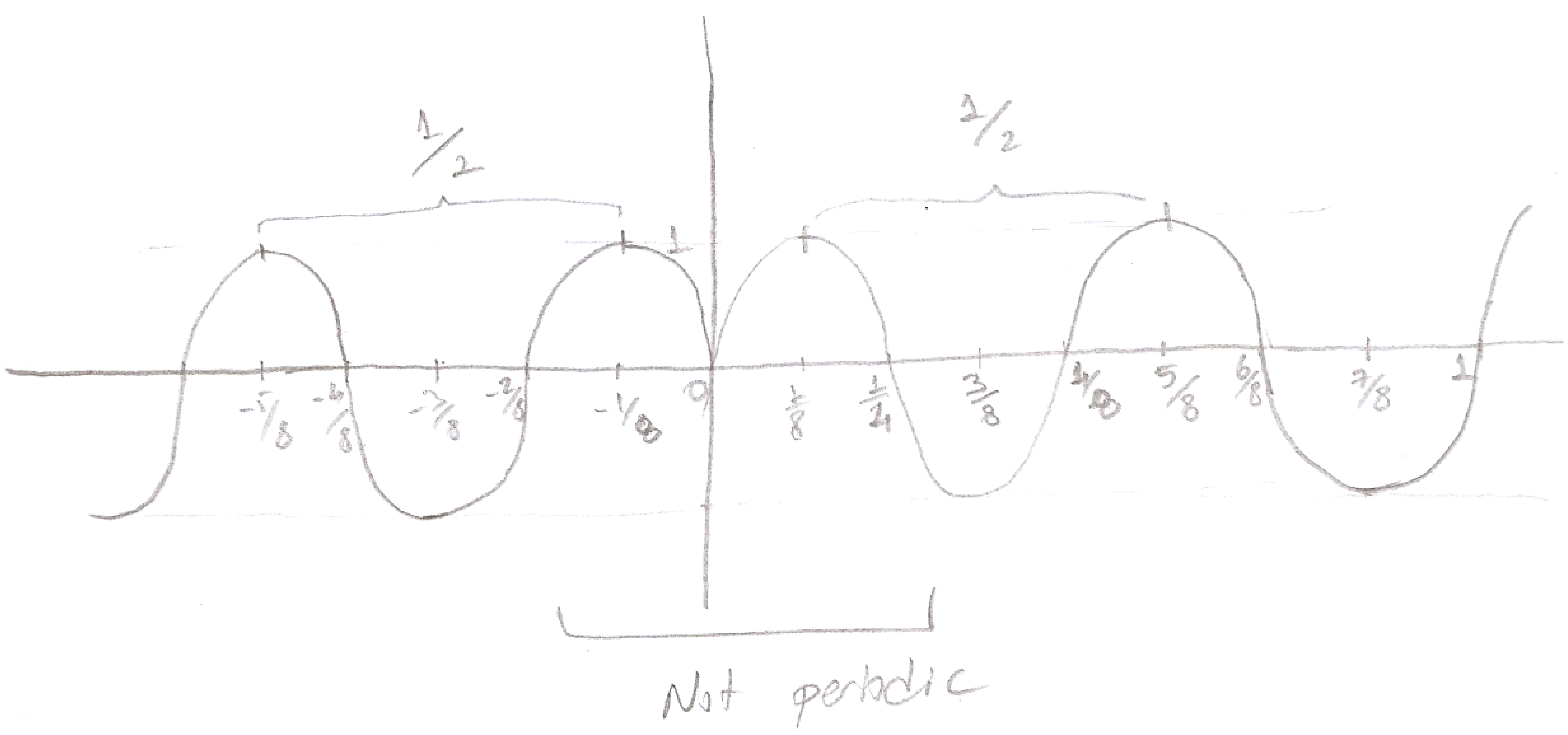
$t < 0$

$$x(t) = \frac{1}{2}\{-\sin(4\pi t)\}$$

$$\sin(4\pi t) = \sin(4\pi(t+T))$$

$$4\pi T = 2\pi m$$

$$T_0 = \frac{1}{2}$$



f) $x(t)=\sum_{n=-\infty}^{\infty} e^{-(2t-n)}\, u(2t-n)$

For any real t, $\lfloor t \rfloor$ represents floor of t (largest integer which is less than or equal to t).

Since $2t-n \geq 0$ for all integers $n \leq 2t$, and since $2t-n<0$ for all integer $n \geq \lfloor 2t \rfloor + 1$,

$$u(2t-n) = \begin{cases} 1 & \text{for all } n \leq \lfloor 2t \rfloor \\ 0 & \text{for all } n \geq \lfloor 2t \rfloor + 1 \end{cases}$$

$x(t)$ given in the problem statement can be rewritten as follows:

$$x(t) = \sum_{n=-\infty}^{\lfloor 2t \rfloor} e^{-(2t-n)} = e^{-2t} \sum_{m=-\lfloor 2t \rfloor}^{\infty} e^{-m}$$

$$= e^{-2t} \frac{e^{\lfloor 2t \rfloor}}{1-e^{-1}} = \frac{e^{-2(t-\lfloor t \rfloor)}}{1-e^{-1}}$$

where $m=-n$. Since $t-\lfloor t \rfloor = (t+1) - \lfloor t+1 \rfloor$ for all t,

we have: $x(t+1) = x(t)$. Therefore, 1 is a period of x.

To see that it is fundamental period of x, if $0<T<1$

the equation $x(t) = x(t+T)$ will not hold for any integer t.

Because, if t is integer and T is a fraction between 0 & 1,

then $t-\lfloor t \rfloor = 0$ whereas $t+T - \lfloor t+T \rfloor = t+T-t = T \neq 0$.

Hence, 1 is the smallest period of x and is therefore the fundamental period.

Periodicity Ex 2: Determine whether or not each of the following discrete-time signals is periodic. If the signal is periodic, determine its fundamental period.

a) $x[n] = \sin\left(\frac{6\pi}{7}n + 1\right)$

$x[n] \stackrel{?}{=} x[n+N]$

$\sin\left(\frac{6\pi}{7}n + 1\right) = \sin\left(\frac{6\pi}{7}(n+N) + 1\right)$

$= \sin\left(\frac{6\pi}{7}n + \frac{6\pi}{7}N + 1\right)$

$\frac{6\pi N}{7} = 2\pi m, \quad m \in \mathbb{Z}$

$N = m\left(\frac{2\pi \cdot 7}{6\pi}\right) = m\left(\frac{7}{3}\right)$

N must be integer in disc-time
so $m=3$
$N = 7$

b) $x[n] = \cos\left(\frac{n}{8} - \pi\right)$

$x[n] \stackrel{?}{=} x[n+N] \Rightarrow \cos\left(\frac{n+N}{8} - \pi\right) = \cos\left(\frac{n}{8} - \pi\right)$

$\frac{N}{8} = 2\pi m \Rightarrow N = 16\pi m \rightarrow$ irrational
so $x[n]$ aperiodic

d) $x[n] = \cos\left(\frac{\pi}{2}n\right)\cos\left(\frac{\pi}{4}n\right)$

Using trigonometric product to sum formula

$$\cos(u).\cos(v) = \frac{1}{2}\left[\cos(u+v) + \cos(u-v)\right]$$

$$\cos\left(\frac{\pi}{2}n\right).\cos\left(\frac{\pi}{4}n\right) = \frac{1}{2}\left[\cos\left(\frac{3\pi}{4}n\right) + \cos\left(\frac{\pi}{4}n\right)\right]$$

Period of $\cos\left(\frac{3\pi}{4}n\right)$

$$\cos\left(\frac{3\pi}{4}n\right) = \cos\left(\frac{3\pi}{4}(n+N)\right)$$

$$= \cos\left(\frac{3\pi}{4}n + \frac{3\pi N}{4}\right)$$

$\frac{3\pi N}{4} = 2\pi m \Rightarrow N = \frac{8}{3}m \Rightarrow$ N must be integer
Least possible value of N is 8

Period of $\cos\left(\frac{\pi}{4}n\right)$

$$\cos\left(\frac{\pi}{4}n\right) = \cos\left(\frac{\pi}{4}(n+N)\right)$$

$$= \cos\left(\frac{\pi}{4}n + \frac{\pi}{4}N\right)$$

$\frac{\pi N}{4} = 2\pi m \Rightarrow N = 8m \Rightarrow N = 8$

e) $x[n] = 2\cos\left(\frac{\pi}{4}n\right) + \sin\left(\frac{\pi}{8}n\right) - 2\cos\left(\frac{\pi}{2}n + \frac{\pi}{6}\right)$

Fundamental period of $\cos\left(\frac{\pi}{4}n\right)$

$$\cos\left(\frac{\pi}{4}n\right) = \cos\left(\frac{\pi}{4}(n+N)\right)$$

$$= \cos\left(\frac{\pi n}{4} + \frac{\pi N}{4}\right) \Rightarrow \frac{\pi N}{4} = 2\pi m$$

$$N = 8$$

Fund. period of $\sin\left(\frac{\pi}{8}n\right)$

$$\cos\left(\frac{\pi}{8}n\right) = \cos\left(\frac{\pi}{8}(n+N)\right)$$

$$= \cos\left(\frac{\pi n}{8} + \frac{\pi N}{8}\right) \Rightarrow \frac{\pi N}{8} = 2\pi m$$

$$N = 16$$

Fund. period of $\cos\left(\frac{\pi}{2}n + \frac{\pi}{6}\right)$

$$\cos\left(\frac{\pi}{2}n + \frac{\pi}{6}\right) = \cos\left(\frac{\pi}{2}n + \frac{\pi N}{2} + \frac{\pi}{6}\right)$$

$$\frac{\pi N}{2} = 2\pi m \Rightarrow N = 4$$

Hence, $x[n]$ is periodic and its fundamental period is the least common multiple of 8, 16, and 4 which is $N = 16$

General properties of systems [x 3 : A system may or may not be

1) Memoryless

2) Time invariant

3) Linear

4) Causal

5) Stable

Determine which of these properties hold and which do not hold for each of the following cts-time systems. Justify your answers. In each example, $y(t)$ denotes the system output and $x(t)$ is the system input.

a) $y(t) = x(t-2) + x(2-t)$ $\quad x(t) \longrightarrow \boxed{\text{system}} \longrightarrow y(t)$

- At the first glance we see that the system has memory
- Non causal → depend on future
- Stable → for bounded input gives bounded output

Is it time-invariant? → New input signal defined as shifted version of $x(t)$

$x_1(t) = x(t-t_0)$ $\quad y_1(t) = x_1(t-2) + x_1(2-t)$

$$= x(t-2-t_0) + \underbrace{x(2-t-t_0)}$$

$y(t) = x(t-2) + x(2-t)$

$$y(t-t_0) = x(t-t_0-2) + \underbrace{x(2-t+t_0)}$$

This parts differ from each other

So the system is time-varying

Is it linear

$x_1(t) \Rightarrow y_1(t) = x_1(t-2) + x_1(2-t)$

$x_2(t) \Rightarrow y_2(t) = x_2(t-2) + x_2(2-t)$

$$x(t) = a_1x_1(t) + a_2x_2(t) \Rightarrow y(t) = \underbrace{a_1x_1(t-2) + \underbrace{a_2x_2(t-2) + a_1x_1(2-t)}_{a_1y_1(t)} + a_2x_2(t-2)}_{a_2y_2(t)}$$

$y(t) = a_1y_1(t) + a_2y_2(t) \Rightarrow$ Linear

b) $y(t) = [\cos(3t)]\, x(t)$ $\quad x(t) \rightarrow \boxed{\text{System}} \rightarrow y(t)$

- Memoryless $\rightarrow$ not depend on past
- Causal $\rightarrow$ not depend on future
- Stable $\rightarrow$ for bounded input gives bounded output

Is it time-invariant?

$x_1(t)$ $\qquad\qquad x_1(t) = x(t-t_0)$

$y(t) = [\cos(3t)]\, x(t) \qquad y_1(t) = [\cos(3t)]\, x_1(t)$

$y(t-t_0) = \underbrace{[\cos(3(t-t_0))]}\, x(t-t_0) \qquad = \underbrace{[\cos(3t)]}\, x(t-t_0)$

Differs from each other. Hence, the system is <u>time-varying</u>

Is it linear?

$x_1(t) \Rightarrow y_1(t) = [\cos(3t)]\, x_1(t)$

$x_2(t) \Rightarrow y_2(t) = [\cos(3t)]\, x_2(t)$

$x(t) = a_1 x_1(t) + a_2 x_2(t) \Rightarrow y(t) = \cos(3t)\,[a_1 x_1(t) + a_2 x_2(t)]$

$$= \underbrace{\cos(3t) \cdot a_1 \cdot x_1(t)}_{a_1 y_1(t)} + \underbrace{\cos(3t)\, a_2 x_2(t)}_{a_2 y_2(t)}$$

$$= \underbrace{a_1 y_1(t) + a_2 y_2(t)}_{\text{linear system}}$$

c) $$y(t) = \int_{-\infty}^{2t} x(\tau)\,d\tau$$

- Has memory → depeds on past and future
- Noncausal
- Diverge

Is it time-invariant?

$x(t)$ $$x_1(t) = x(t-t_0)$$

$$y(t) = \int_{-\infty}^{2t} x(\tau)\,d\tau \qquad y_1(t) = \int_{-\infty}^{2t} x(\tau - t_0)\,d\tau$$

$$y_1(t-t_0) = \int_{-\infty}^{2t-2t_0} x(\tau)\,d\tau \neq y_1(t)$$ → time varying

Is it linear?

$$x_1(t) \Rightarrow y_1(t) = \int_{-\infty}^{2t} x_1(\tau)\,d\tau$$

$$x_2(t) \Rightarrow y_2(t) = \int_{-\infty}^{2t} x_2(\tau)\,d\tau$$

$$x(t) = a_1x_1(t) + a_2x_2(t) \Rightarrow y(t) = \int_{-\infty}^{2t} \left(a_1x_1(\tau) + a_2x_2(\tau)\right)d\tau$$

$$y(t) = \underbrace{\int_{-\infty}^{2t} a_1x_1(\tau)\,d\tau}_{a_1y_1(t)} + \underbrace{\int_{-\infty}^{2t} a_2x_2(\tau)\,d\tau}_{a_1y_2(t)}$$

$$y(t) = a_1y_1(t) + a_2y_2(t)$$

So the system is linear

CONVOLUTION

Time - Invariance

Cont. Time :

$$x(t) \rightarrow y(t)$$

Then $\quad x(t-t_0) \rightarrow y(t-t_0)$

Disc - Time :

$$x[n] \rightarrow y[n]$$

$$x[n-n_0] \rightarrow y[n-n_0]$$

Linearity :

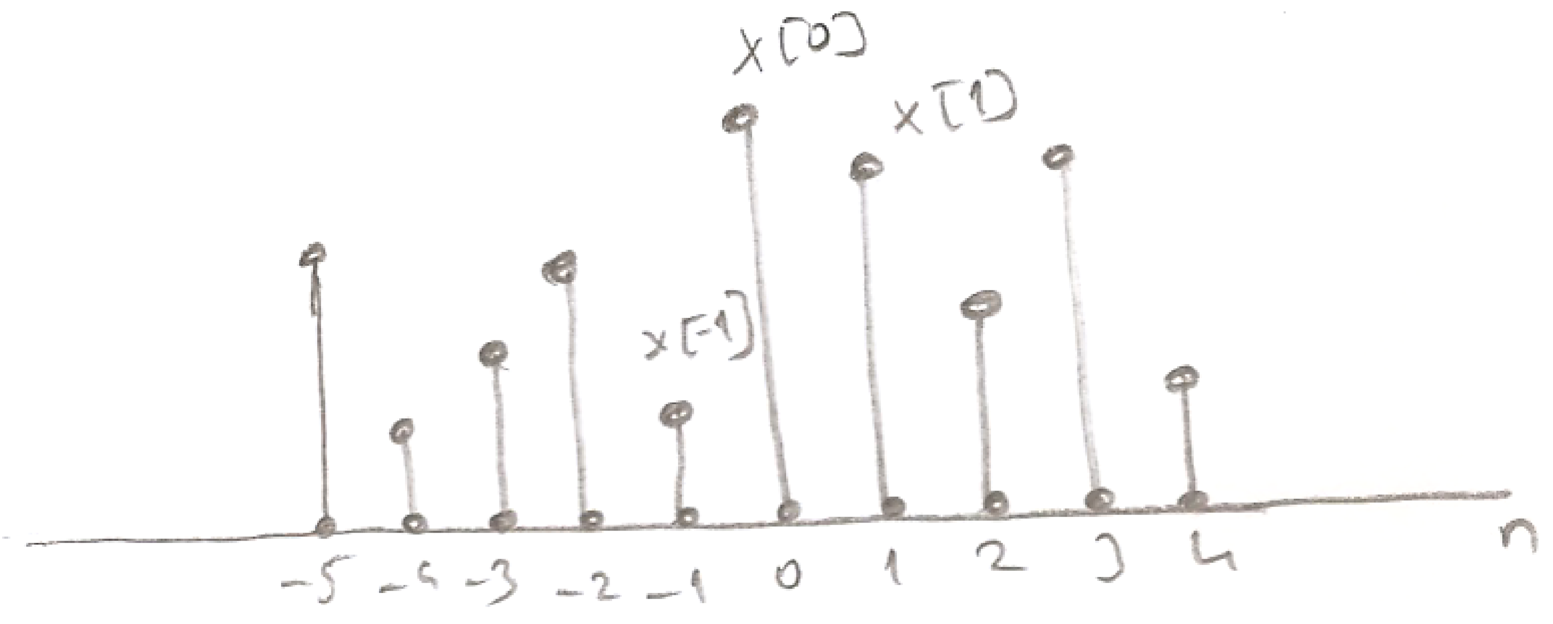
$$\phi_k \longrightarrow \psi_k$$

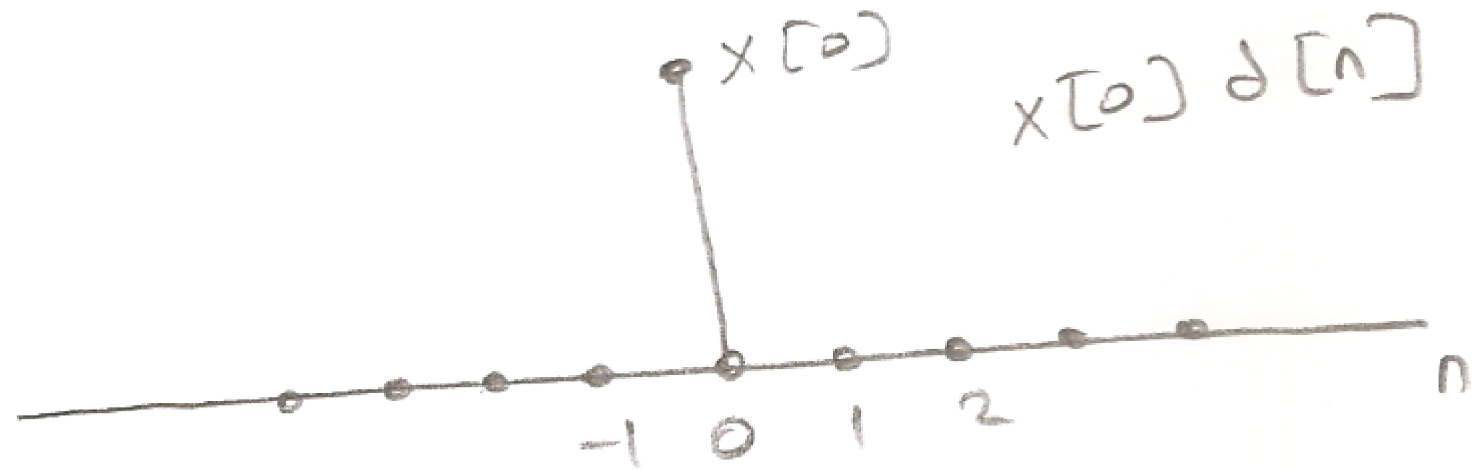
Then

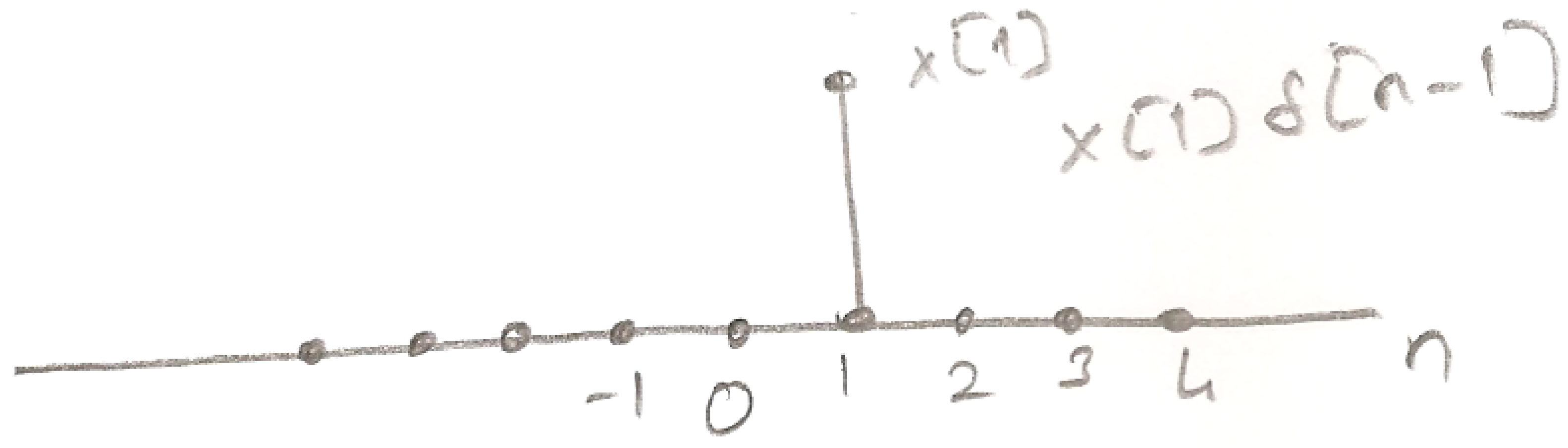
$$a_1\phi_1 + a_2\phi_2 + \dots \longrightarrow a_1\psi_1 + a_2\psi_2 + \dots$$

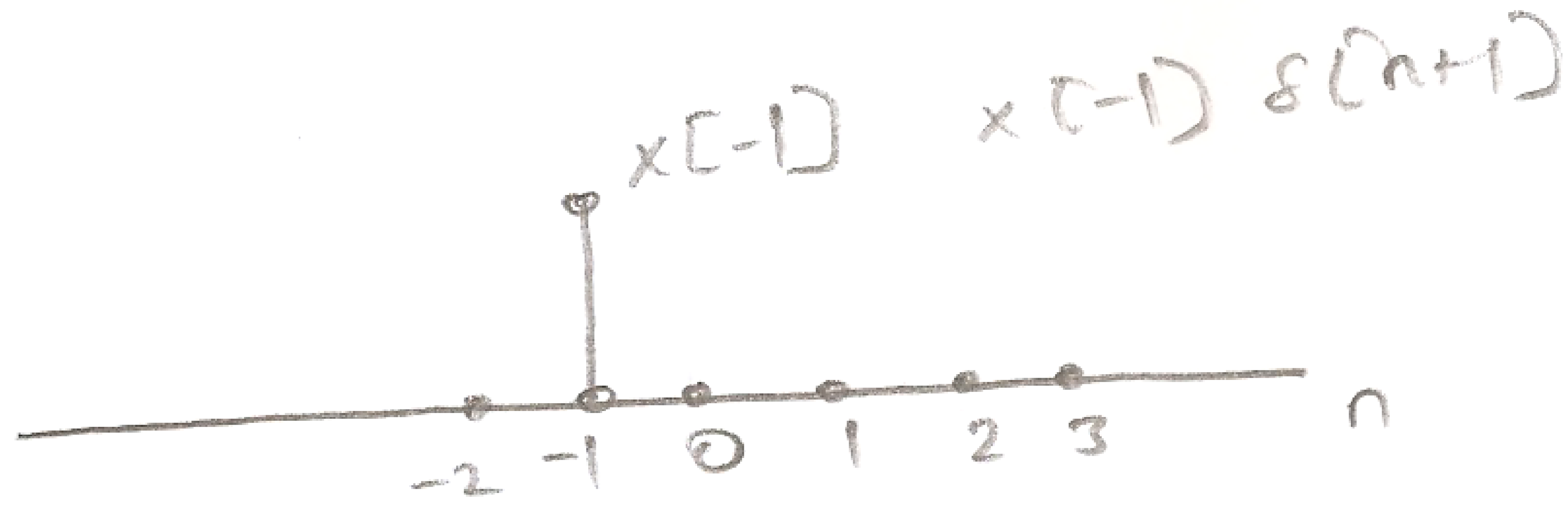
Main Ideas of Convolution !

- Decompose input signal into a linear combination of basic signals
- Choose basic signals so that response easy to compute
- Decompose a signal a linear combination of delayed impulses. That leads to representation for LTI systems which is referred to convolution.









$$x[n] = x[0]\,\delta[n] + x[1]\,\delta[n-1] + x[-1]\,\delta[n+1] + \cdots$$

$$= \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k]$$

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k]$$

If we are talking about linear system, to response to that linear combination is linear combination of the responses.

If we denote the response to the delayed impulse as

$$\underbrace{\delta[n-k]}_{\text{impulse}} \longrightarrow \boxed{\text{LTI system}} \longrightarrow \underbrace{h_k[n]}_{\text{response of system}}$$

Then response to general input

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k]$$

is

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\,h_k[n]$$

The system is time invariant, then the response to impulse at time k is as exactly the same as the response to an impulse at time 0, shifted over time k.

Hence,

$$h_k[n] = h_0[n-k]$$

And it is generally useful to, rather than carrying around $h_0[n]$, just simply define $h_0[n]$ as $h[n]$, which is the unit impulse response of the system.

So the consequence, then, is for a linear time-invariant system, the output can be expressed as this sum

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\,h[n-k]$$

Calculating and plotting convolution Ex 1;

$x[n] = \delta[n] + 2\delta[n-1] - \delta[n-3]$ and

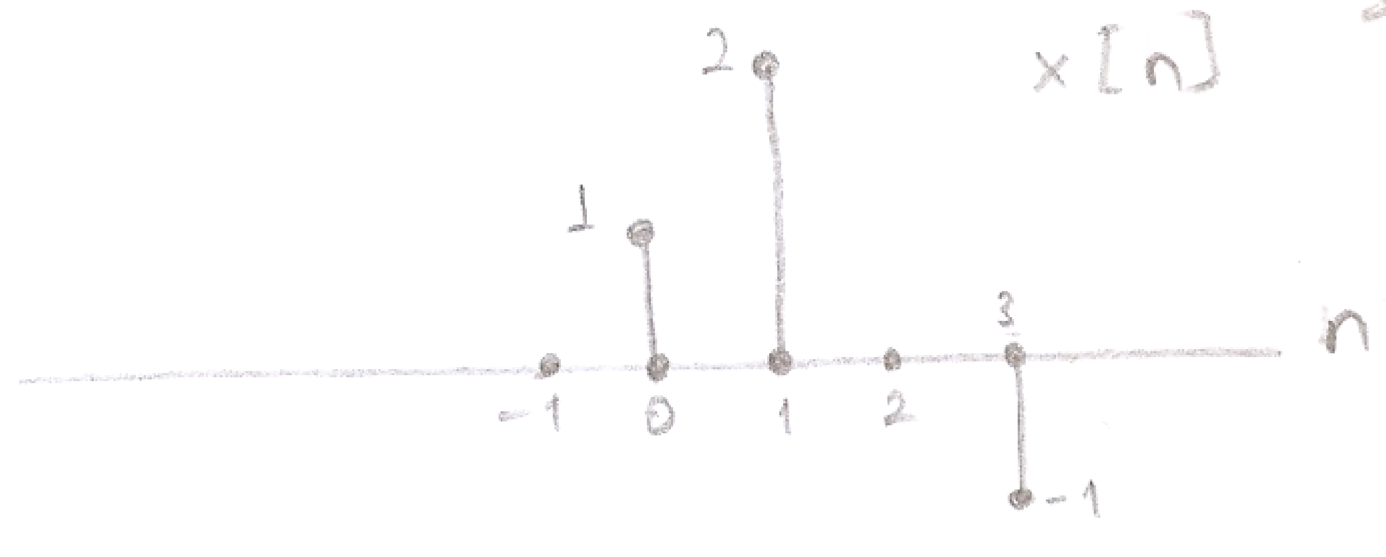
$h[n] = 2\delta[n+1] + 2\delta[n-1]$

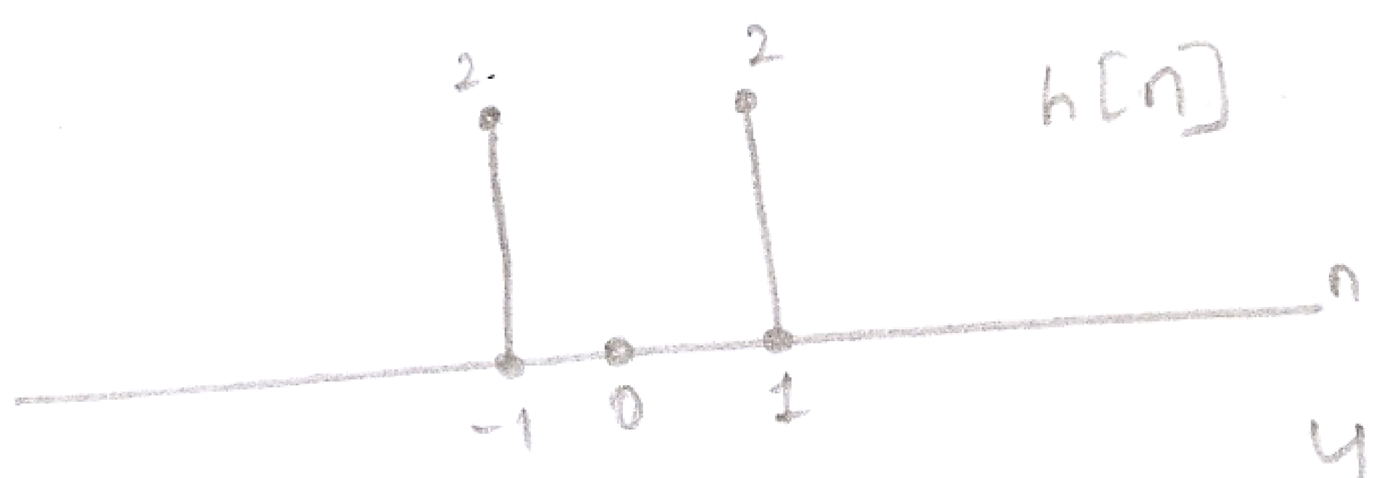
Compute and plot each of the following convolutions:

a) $y_1[n] = x[n] * h[n]$

$$y_1[n] = \sum_{k=-\infty}^{\infty} x[k] \underbrace{h[n-k]}$$

This means that shift impulse response signal from $-\infty$ to ∞ and multiply signal corresponding value of $x[n]$





While shifting $h[n]$, only for three value of n $x[n]$ different from 0. These are 0, 1, and 3

Hence,

$$y[n] = \sum_{k=-\infty}^{k=\infty} x[k]\,h[n-k]$$

$= x[0]\,h[n-0] + x[1]\,h[n-1] + x[3]\,h[n-3]$

$x[0] = 1$, $x[1] = 2$, $x[3] = -1$

$h[n] = 2\delta[n+1] + 2\delta[n-1]$

$h[n-1] = 2\delta[n] + 2\delta[n-2]$

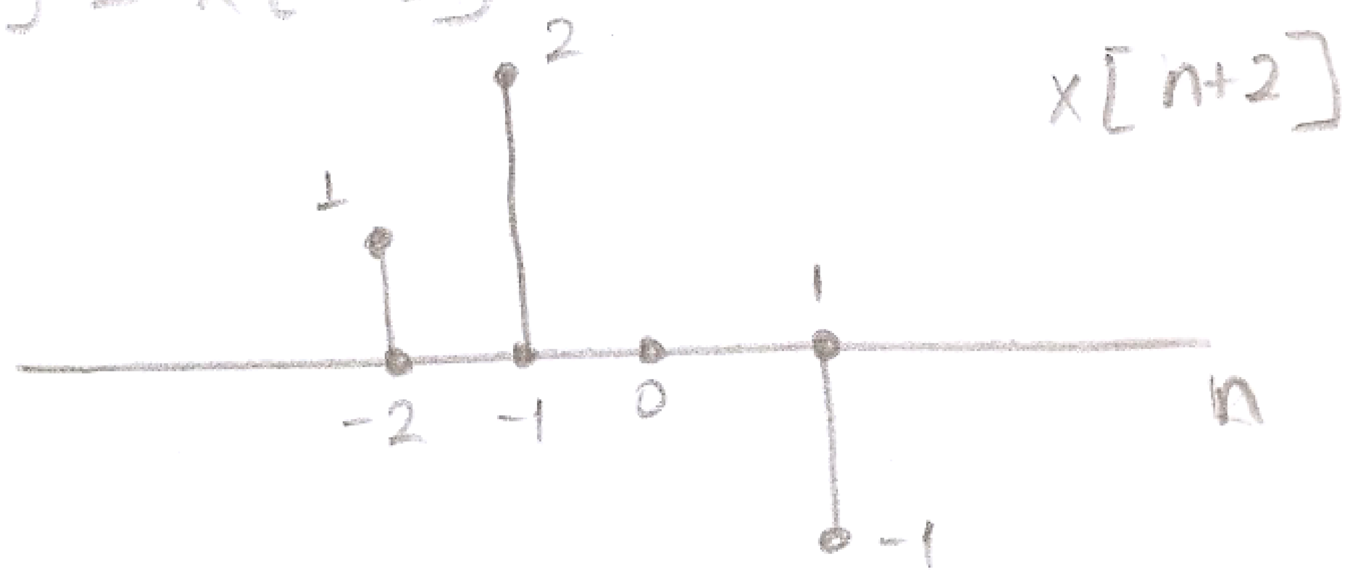
$h[n-3] = 2\delta[n-2] + 2\delta[n-4]$

$\Rightarrow$ Put all this values

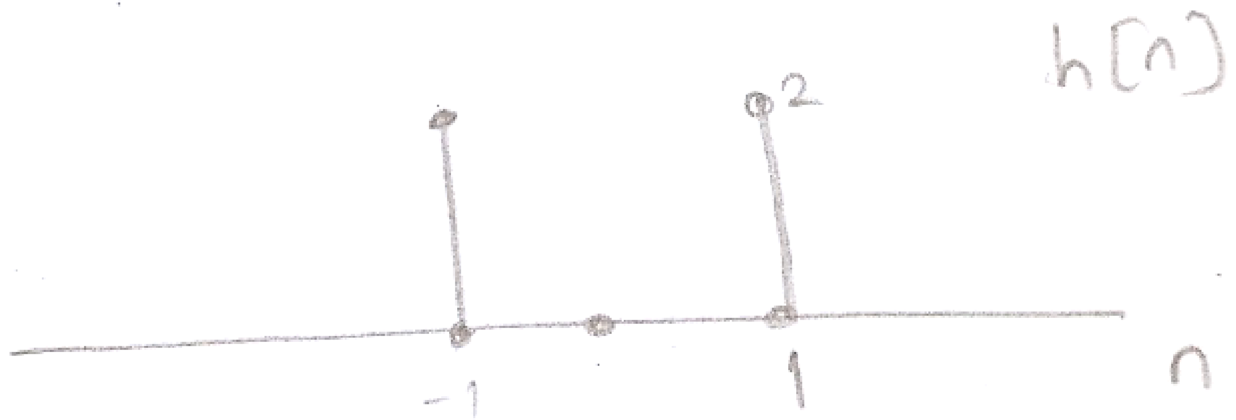
$y[n] = 2\delta[n+1] + 2\delta[n-1] + 2[2\delta[n] + 2\delta[n-2]] - 2\delta[n-2] - 2\delta[n-4]$

$$\boxed{y[n] = 2\delta[n+1] + 4\delta[n] + 2\delta[n-1] + 2\delta[n-2] - 2\delta[n-4]}$$

b) $y_2[n] = x[n+2]\, h[n]$



while shifting $h[n]$, $x[n+2]$ has values different from '0' for only $n=-2,-1,$ and 1.



$$y_2[n] = \sum_{k=-\infty}^{k=\infty} x[k+2]\, h[n-k] = x[0]\, h[n+2] + x[1]\, h[n+1] + x[3]\, h[n-1]$$

$$= 2\delta[n+3] + 2\delta[n+1] + 2\left[2\delta[n+2] + 2\delta[n]\right] - 2\delta[n] - 2\delta[n-2]$$

$$y_2[n] = 2\delta[n+3] + 4\delta[n+2] + 2\delta[n+1] + 2\delta[n] - 2\delta[n-2]$$

$y_2[n] = y_1[n+2]$ → Time invariant system

c) $y_3[n] = x[n] * h[n+2]$ we have showed that the system is time-invariant so

$$y_3[n] = y_1[n+2] = y_2[n]$$

Cts-time compute and sketch convolution Ex 2: Determine and sketch the convolution of the following two signals:

$$x(t) = \begin{cases} t+1, & 0 \le t \le 1 \\ 2-t, & 1 < t \le 2 \\ 0, & \text{elsewhere} \end{cases}$$

$$h(t) = \delta(t+2) + 2\delta(t+1)$$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau = \int_{-\infty}^{\infty} x(t-\tau)\, h(\tau)\, d\tau \quad \text{(commutativity)}$$

$$= \int_{-\infty}^{\infty} x(t-\tau) \underbrace{\Big(\delta(\tau+2) + 2\delta(\tau+1) \Big)}_{\substack{\text{differ from '0' for} \\ \text{two points } -2,\,-1}} d\tau$$

$$= x(t+2) + 2x(t+1)$$

$$x(t+2) = \begin{cases} t+3, & -2 \le t \le -1 \\ -t, & -1 < t \le 0 \\ 0, & \text{elsewhere} \end{cases} \qquad x(t+1) = \begin{cases} t+2, & -1 \le t \le 0 \\ 1-t, & 0 < t \le 1 \\ 0, & \text{elsewhere} \end{cases}$$

$$2x(t+1) = \begin{cases} 2t+4, & -1 \le t \le 0 \\ 2-2t, & 0 < t \le 1 \\ 0, & \text{elsewhere} \end{cases}$$

$$y(t) = x(t+2) + 2x(t+1) = \begin{cases} t+3, & -2 \le t < -1 \\ 3t+7, & t = -1 \\ t+4, & -1 < t \le 0 \\ 2-2t, & 0 < t \le 1 \\ 0, & \text{elsewhere} \end{cases}$$

Disc-time compute convolution Ex 3 (2.3)

Consider an input $x[n]$ and a unit impulse response $h[n]$ given by

$$x[n] = \left(\frac{1}{2}\right)^{n-2} u[n-2]$$

$$h[n] = u[n+2]$$

Determine and plot the output $y[n] = x[n] * h[n]$

Soln:

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$

$$= \sum_{k=-\infty}^{\infty} \left(\frac{1}{2}\right)^{k-2} u[k-2]\, u[n-k+2]$$

$k-2 \geq 0 \Rightarrow k \geq 2$

$n-k+2 \geq 0 \Rightarrow k \leq n+2$

$$= \sum_{k=2}^{n+2} \left(\frac{1}{2}\right)^{k-2} \xrightarrow{k=m+2} = \overbrace{\sum_{m=0}^{n} \left(\frac{1}{2}\right)^{k}}^{\text{geometric series}} = \frac{1-\left(\frac{1}{2}\right)^{n+1}}{1-\frac{1}{2}} = 2\left[1-\left(\frac{1}{2}\right)^{n+1}\right]$$

$$y[n] = 2\left[1-\left(\frac{1}{2}\right)^{n+1}\right] u[n]$$

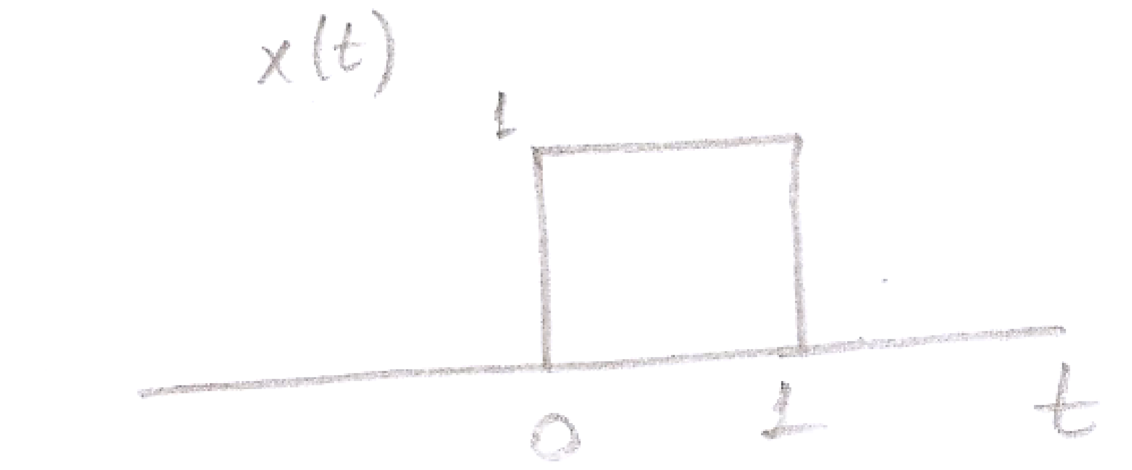
Determine and sketch convolution Ex 4 (2.10)

Suppose that

$$x(t) = \begin{cases} 1, & 0 \le t \le 1 \\ 0, & \text{elsewhere} \end{cases}$$

and $h(t) = x(t/\alpha)$, where $0 < \alpha \le 1$

Determine and sketch $y(t) = x(t) * h(t)$



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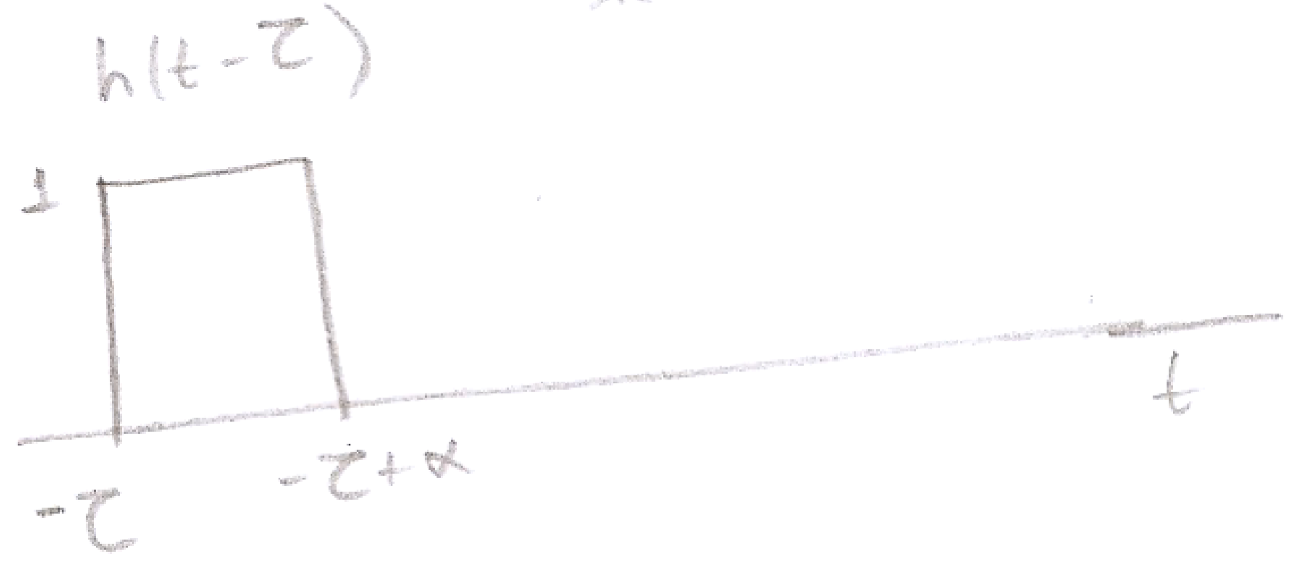
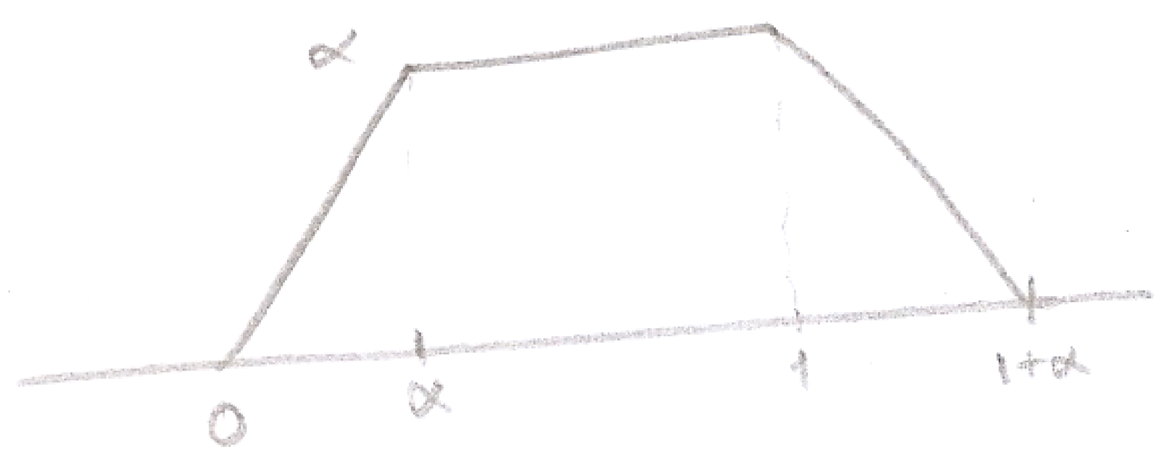




$$y(t) = \begin{cases} t, & 0 \le t \le \alpha \\ \alpha, & \alpha \le t \le 1 \\ 1+\alpha-t, & 1 \le t \le (1+\alpha) \\ 0, & \text{elsewhere} \end{cases}$$

Compute convolution Ex 5: (2.11):

$$x(t) = u(t-3) - u(t-5) \quad \text{and} \quad h(t) = e^{-3t}u(t)$$

Compute $y(t) = x(t) * h(t)$

Sol'n:

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

$$= \int_{-\infty}^{\infty} \underbrace{\left[u(\tau-3) - u(\tau-5)\right]}_{>0} e^{-3(t-\tau)} \underbrace{u(t-\tau)}_{>0}\, d\tau$$

$t - \tau > 0$

$\tau - 3 > 0 \Rightarrow \tau > 3$

$t > \tau$

$\tau - 5 < 0 \Rightarrow \tau < 5$

combine this constraints

case 1:

$3 < t < 5 \Rightarrow$

$$y(t) = \int_{3}^{t} e^{-3t+3\tau}\, d\tau = e^{-3t} \int_{3}^{t} e^{3\tau}\, d\tau$$

$$= e^{-3t} \left[\frac{e^{3\tau}}{3} \Bigg|_{3}^{t} \right]$$

$$= e^{-3t} \left[\frac{e^{3t}}{3} - \frac{e^{9}}{3} \right] = \frac{1 - e^{-3(t-3)}}{3}$$

case 2: $t > 5$

$$y(t) = \int_{3}^{5} e^{-3(t-\tau)}\, d\tau = \int_{3}^{5} e^{-3t} . e^{3\tau}\, d\tau = e^{-3t} \int_{3}^{5} e^{3\tau}\, d\tau$$

$$= e^{-3t} \left[\frac{e^{3\tau}}{3} \Bigg|_{3}^{5} \right] = e^{-3t} \left[\frac{e^{15} - e^{9}}{3} \right]$$

$$y(t) = \begin{cases} \dfrac{1 - e^{-3(t-3)}}{3} , & 3 < t < 5 \\ e^{-3t}\left[\dfrac{e^{15} - e^{9}}{3} \right] , & t > 5 \\ 0 , & \text{elsewhere} \end{cases}$$

Compute convolution Ex 6 (2.21) :

Compute the convolution $y[n] = x[n] * h[n]$ of the following pairs of signals

a) $\left.\begin{aligned} x[n] &= \alpha^n u[n] \\ h[n] &= \beta^n u[n] \end{aligned}\right\} \quad \alpha \neq \beta$

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k] = \sum_{k=-\infty}^{\infty} \alpha^k u[k]\, \beta^{n-k} u[n-k]$$

$\left.\begin{aligned} u[k] &= 1 \text{ for } k \geq 0 \\ u[n-k] &= 1 \text{ for } n \geq k \end{aligned}\right\} \quad n \geq k \geq 0$

$$y[n] = \sum_{k=0}^{n} \alpha^k \beta^{n-k} = \beta^n \sum_{k=0}^{n} \alpha^k . \beta^{-k} = \beta^n \sum_{k=0}^{n} (\alpha/\beta)^k$$

$$= \underbrace{\beta^n \left(\frac{1-\left(\frac{\alpha}{\beta}\right)^{n+1}}{1-\frac{\alpha}{\beta}} \right)}_{\text{from geometric series formula}} = \cancel{\beta^n} \left(\frac{\beta^{n+1} - \alpha^{n+1}}{\cancel{\beta^{n+1}}} \cdot \frac{\cancel{\beta}}{\beta - \alpha} \right)$$

$$= \frac{\beta^{n+1} - \alpha^{n+1}}{\beta - \alpha}$$

If $n < 0$, $y[n] = 0$. Hence,

$$y[n] = \left[\frac{\beta^{n+1} - \alpha^{n+1}}{\beta - \alpha} \right] u[n]$$

b) $x[n] = h[n] = \alpha^n u(n)$

$$y[n] = \sum_{k=-\infty}^{\infty} \alpha^k u[k] \alpha^{n-k} u[n-k]$$

$u[k] = 1$ for $k \geq 0$, so $k \geq 0$

$u[n-k] = 1$ for $n-k \geq 0$, so $n-k \geq 0 \Rightarrow n \geq k$ $\Big\}$ $n \geq k \geq 0$

$$y[n] = \sum_{k=0}^{n} \alpha^k \alpha^{n-k} = \alpha^n \sum_{k=0}^{n} \alpha^0 = \alpha^n (n+1)$$

$y[n] = 0$ for $n < 0$

$y[n] = \alpha^n (n+1) u[n]$

c) $x[n] = \left(-\frac{1}{2}\right)^n u[n-4]$

$h[n] = 4^n u[2-n]$

$$y[n] = \sum_{k=-\infty}^{\infty} x[k] h[n-k] = \sum_{k=-\infty}^{\infty} \left(-\frac{1}{2}\right)^k u[k-4]\, 4^{n-k} u[2-n+k]$$

$u[k-4] = 1$ for $k-4 \geq 0$, so $k \geq 4$

$u[2-n+k] = 1$ for $2-n+k \geq 0$ so $k \geq n-2$

<u>case 1</u>: $n-2 \geq 4 \Rightarrow n \geq 6$

$$y[n] = \sum_{k=n-2}^{\infty} \left(-\frac{1}{2}\right)^k 4^{n-k} = \sum_{k=n-2}^{\infty} \left(-\frac{1}{2}\right)^k \left(\frac{1}{4}\right)^k 4^n$$

$$= \sum_{k=n-2}^{\infty} \left(-\frac{1}{8}\right)^k \cdot 4^n = 4^n \sum_{k=n-2}^{\infty} \left(-\frac{1}{8}\right)^k = 4^n \left(-\frac{1}{8}\right)^{n-2} \left(\frac{1}{1+\frac{1}{8}}\right)$$

$$= (-2)^{n+6} \cdot (9/8)$$

<u>case 2</u>: $n-2<4$

$$y(n) = \sum_{k=4}^{\infty} \left(-\frac{1}{2}\right)^k 4^{n-k} = \sum_{k=4}^{\infty} \left(-\frac{1}{2}\right)^k \cdot 4^n \cdot 4^{-k}$$

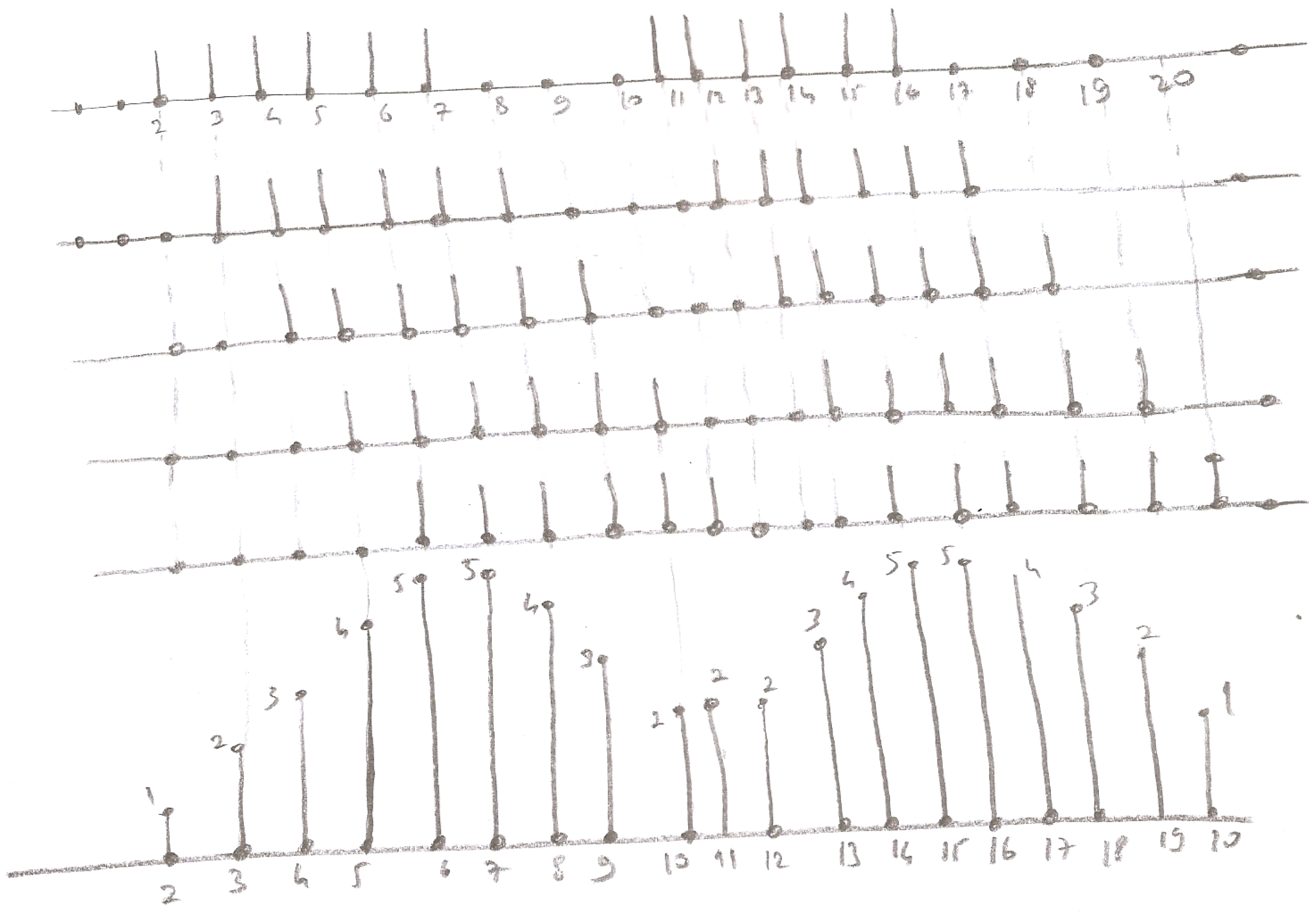
$$= 4^n \sum_{k=4}^{\infty} \left(-\frac{1}{2}\right)\left(\frac{1}{4}\right)^k = 4^n \sum_{k=4}^{\infty} \left(-\frac{1}{8}\right)^k$$

$$= 4^n \left(-\frac{1}{8}\right)^4 \left(\frac{1}{1+\frac{1}{8}}\right) = 4^n (-1/8)^4 \, (8/9)$$

$$y[n] = \begin{cases} (-2)^{-n+6} \cdot (8/9) & , \quad n \geq 6 \\ 4^n \left(-\frac{1}{8}\right)^4 (8/9) & , \quad n < 6 \\ 0 & , \quad \text{otherwise} \end{cases}$$

d)

$$y[n] = x[0]\,h[n] + x[1]\,h[n-1] + x[2]\,h[n-2] + x[3]\,h[n-3] + x[4]\,h[n-4]$$



Cts-time convolution Ex 7 (2.22)

For each of the following pairs of waveforms, use the convolution integral to find the response y(t) of the LTI system with impulse response h(t) to the input x(t).

a) $x(t) = e^{-\alpha t} u(t)$

$h(t) = e^{-\beta t} u(t)$

$\alpha \neq \beta$

$$y(t) = \int_{-\infty}^{\infty} x(\tau) h(t-\tau) d\tau = \int_{-\infty}^{\infty} e^{-\alpha\tau} u(\tau) e^{-\beta(t-\tau)} u(t-\tau) d\tau$$

$$\left.\begin{array}{l} u(\tau) = 1 \text{ for } \tau > 0 \\ u(t-\tau) = 1 \text{ for } t-\tau > 0 \Rightarrow t > \tau \end{array}\right\} 0 < \tau < t$$

$$\int_0^t e^{-\alpha\tau} e^{-\beta(t-\tau)} d\tau = \int_0^t e^{-\alpha t} e^{-\beta t} e^{\beta\tau} d\tau$$

$$= e^{-\beta t} \int_0^t e^{\tau(\beta-\alpha)} d\tau = e^{-\beta t}\left[\left. \frac{e^{\tau(\beta-\alpha)}}{\beta-\alpha} \right|_0^t \right]$$

$$= e^{-\beta t}\left[\frac{e^{t(\beta-\alpha)}}{\beta-\alpha} - \frac{1}{\beta-\alpha} \right] = e^{-\beta t}\left[\frac{e^{t(\beta-\alpha)} - 1}{\beta-\alpha} \right]$$

$$= \frac{e^{-\alpha t} - e^{-\beta t}}{\beta-\alpha}$$

y(t) is zero for $t < 0$

$$y(t) = \left[\frac{e^{-\alpha t} - e^{-\beta t}}{\beta-\alpha} \right] u(t) \quad \alpha \neq \beta$$

b) $x(t) = u(t) - 2u(t-2) + u(t-5)$

$h(t) = e^{2t}u(1-t)$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

$$= \int_{-\infty}^{\infty} \underbrace{\left[u(\tau) - 2u(\tau-2) + u(\tau-5)\right]}_{\substack{\text{differ from zero} \\ \text{if } 0<\tau<5}} e^{2(t-\tau)} u(1-t+\tau)\, d\tau$$

(1) Also for $0<\tau<2$ equal to 1

$2<\tau<5$ equal to -1

$$= \int_0^2 h(t-\tau)\, d\tau - \int_2^5 h(t-\tau)\, d\tau$$

$$= \int_0^2 e^{2(t-\tau)} u(1-t+\tau)\, d\tau - \int_2^5 e^{2(t-\tau)} u(1-t+\tau)\, d\tau$$

(2) $1-t+\tau > 0$

$\tau > t-1$

Combine (1) and (2) constraints and split range of t according to this constraints.

If $t \leqslant 1$, range of t depends on constraint (1)

$$y(t) = \int_0^2 e^{2(t-\tau)}\, d\tau - \int_2^5 e^{2(t-\tau)}\, d\tau$$

$$= \int_0^2 e^{2t} \cdot e^{-2\tau}\, d\tau - \int_2^5 e^{2t} \cdot e^{-2\tau}\, d\tau$$

$$= e^{2t}\left[\int_0^2 e^{-2\tau}d\tau - \int_2^5 e^{-2\tau}d\tau\right]$$

$$= e^{2t}\left[-\frac{e^{-2\tau}}{2}\Big|_0^2 + \frac{e^{-2\tau}}{2}\Big|_2^5\right]$$

$$= e^{2t}\left[-\frac{e^{-4}}{2} + \frac{1}{2} + \frac{e^{-10}}{2} - \frac{e^{-4}}{2}\right]$$

$$= \frac{e^{2t}}{2}\left[+e^{-10} + 1 - 2e^{-4}\right]$$

smaller part of constraint 1 constrained $t-1$

If $1 < t < 3$,

$$y(t) = \int_{t-1}^{2} e^{2(t-\tau)}d\tau - \int_2^5 e^{2(t-\tau)}d\tau$$

$$= \frac{e^{2t}}{2}\left[-e^{-4} + e^{-2t+2} + e^{-10} - e^{-4}\right] = \frac{e^{2t}}{2}\left[e^{-10} - 2e^{-4} + e^{-2t+2}\right]$$

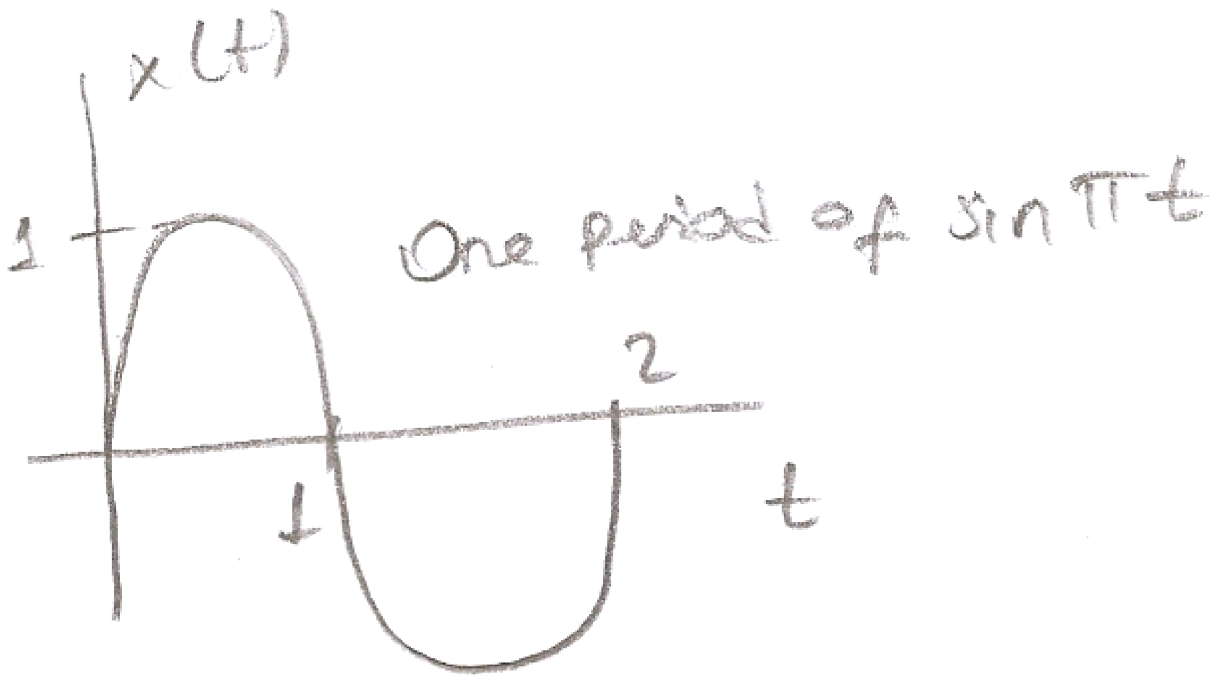
smaller part of constraint 2 constrained $t-1$

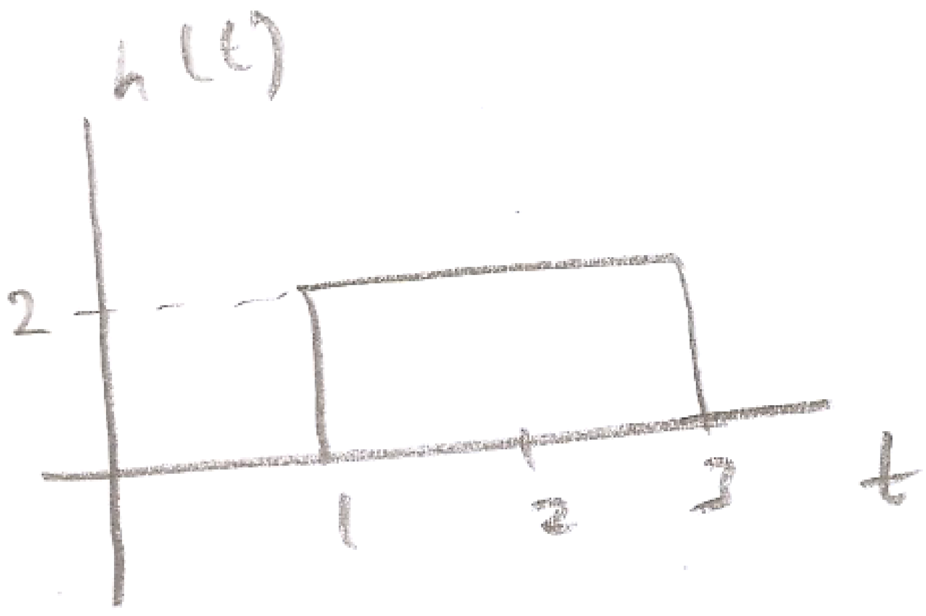
If $3 \le t < 6$,

$$y(t) = -\int_{t-1}^{5} e^{2(t-\tau)}d\tau = \frac{e^{2t}}{2}\left[e^{-10} - e^{-2t+2}\right]$$

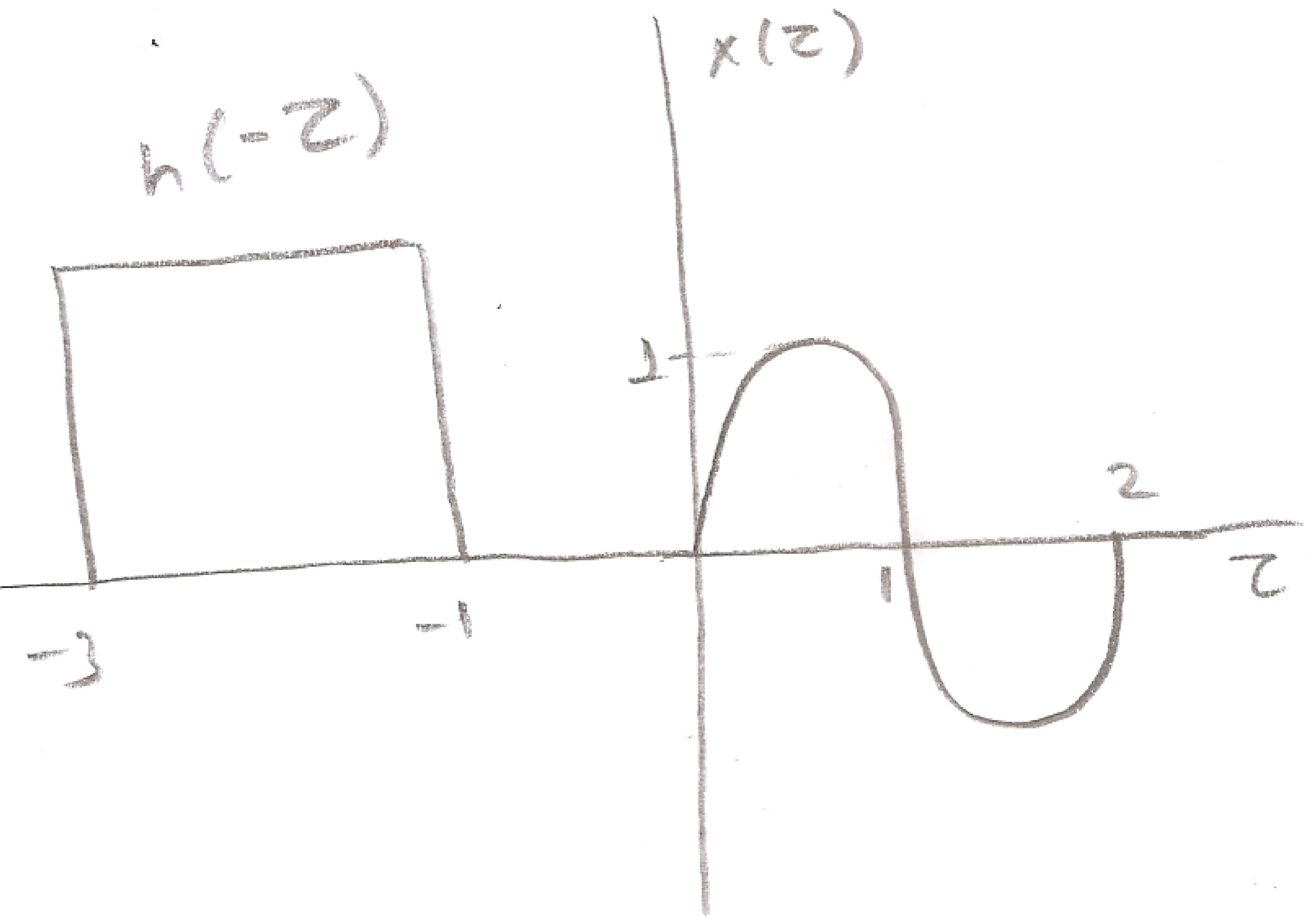
$$y(t) = \begin{cases} \frac{e^{2t}}{2}\left[e^{-10} - 2e^{-4} + 1\right], & t \le 1 \\ \frac{e^{2t}}{2}\left[e^{-10} - 2e^{-4} + e^{-2t+2}\right], & 1 < t < 3 \\ \frac{e^{2t}}{2}\left[e^{-10} - e^{-2t+2}\right], & 3 \le t < 6 \\ 0, & 6 \le t \end{cases}$$

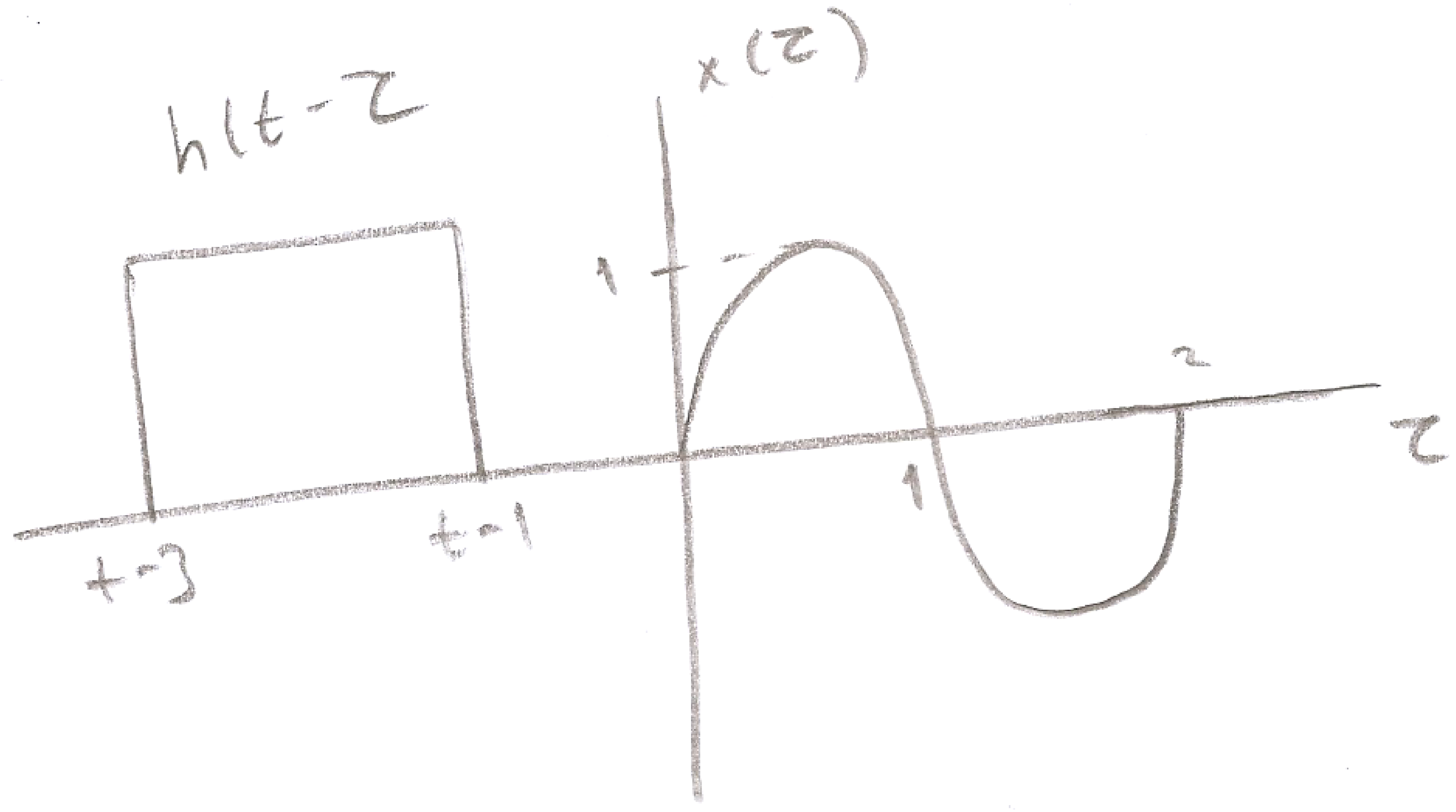
c)





$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$





$$y(t) = \begin{cases} 0 & , t < 1 \\ \int_0^{t-1} 2\sin\pi\tau\, d\tau & , 1 \le t < 3 \\ \int_{t-3}^{2} 2\sin\pi\tau\, d\tau & , 3 \le t < 5 \\ 0 & , 5 \le t \end{cases}$$

$$
y(t) = \begin{cases} 0 & , \quad t < 1 \\ \frac{2}{\pi}\left[1 - \cos(\pi(t-1))\right] & , \quad 1 \le t < 3 \\ \frac{2}{\pi}\left[\cos((t-3)\pi) - 1\right] & , \quad 3 \le t < 5 \\ 0 & , \quad 5 \le t \end{cases}
$$