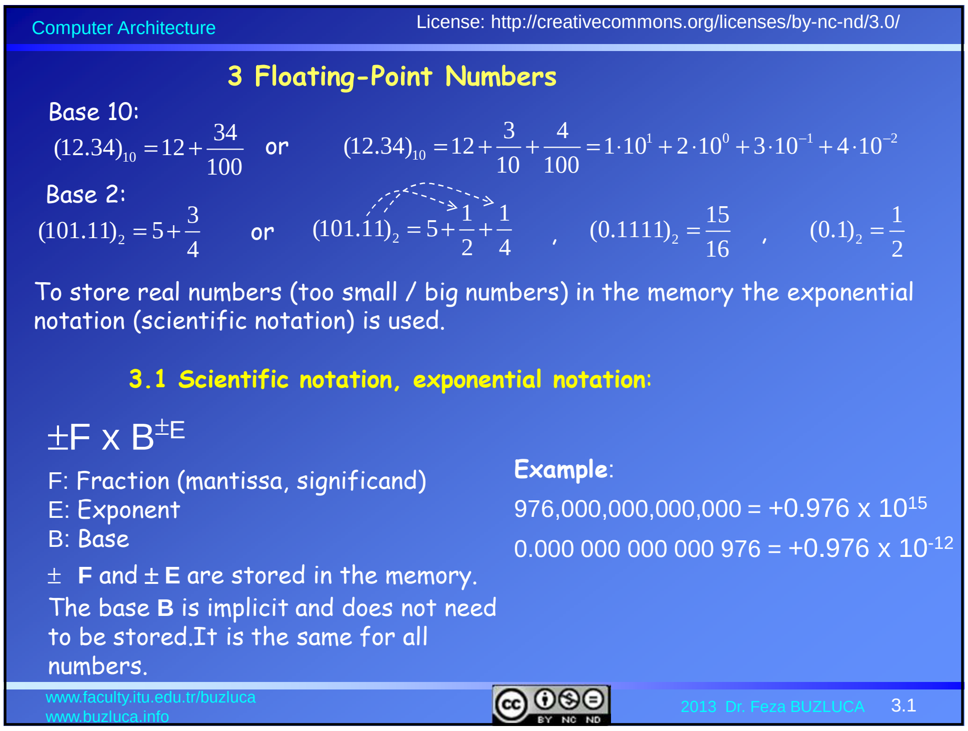


Computer Architecture

3 Floating-Point Numbers

Base 10:

$(12.34)_{10} = 12 + \frac{34}{100}$ or $(12.34)_{10} = 12 + \frac{3}{10} + \frac{4}{100} = 1 \cdot 10^1 + 2 \cdot 10^0 + 3 \cdot 10^{-1} + 4 \cdot 10^{-2}$

Base 2:

$(101.11)_2 = 5 + \frac{3}{4}$ or $(101.11)_2 = 5 + \frac{1}{2} + \frac{1}{4}$, $(0.1111)_2 = \frac{15}{16}$, $(0.1)_2 = \frac{1}{2}$

To store real numbers (too small / big numbers) in the memory the exponential notation (scientific notation) is used.

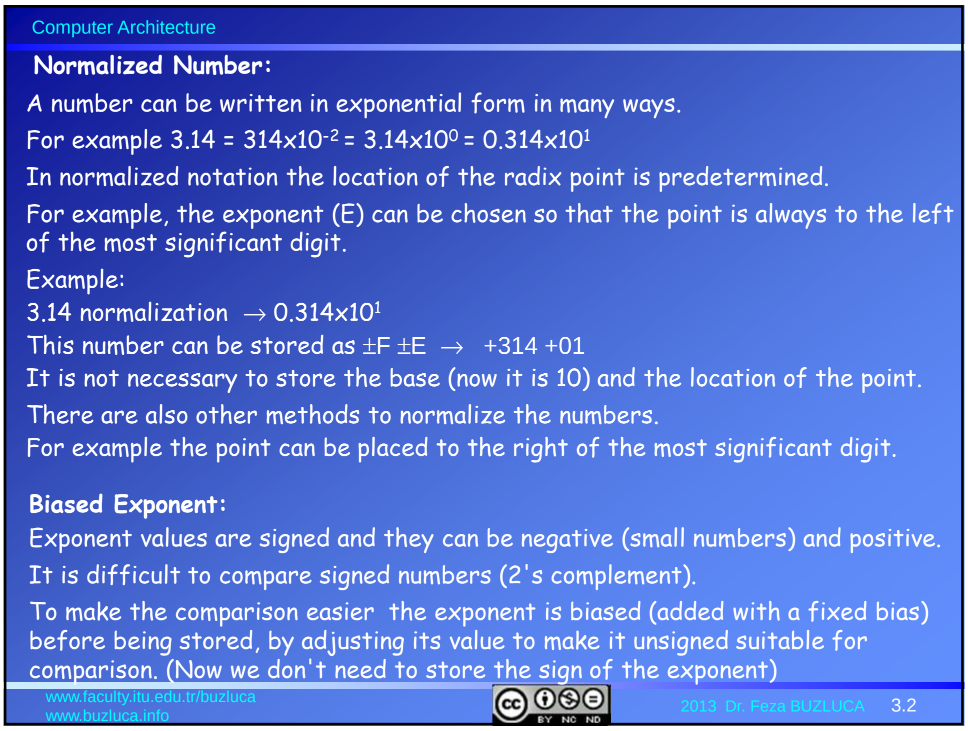
3.1 Scientific notation, exponential notation:

$\pm F \times B^{\pm E}$

F: Fraction (mantissa, significand)
E: Exponent
B: Base

$\pm$ **F** and $\pm$ **E** are stored in the memory.
The base **B** is implicit and does not need to be stored.It is the same for all numbers.

Example:
976,000,000,000,000 = +0.976 x 10^{15}
0.000 000 000 000 976 = +0.976 x 10^{-12}



Computer Architecture

Normalized Number:

A number can be written in exponential form in many ways.

For example 3.14 = 314x10^{-2} = 3.14x10^{0} = 0.314x10^{1}

In normalized notation the location of the radix point is predetermined.

For example, the exponent (E) can be chosen so that the point is always to the left of the most significant digit.

Example:
3.14 normalization $\rightarrow$ 0.314x10^{1}
This number can be stored as $\pm$F $\pm$E $\rightarrow$ +314 +01
It is not necessary to store the base (now it is 10) and the location of the point.

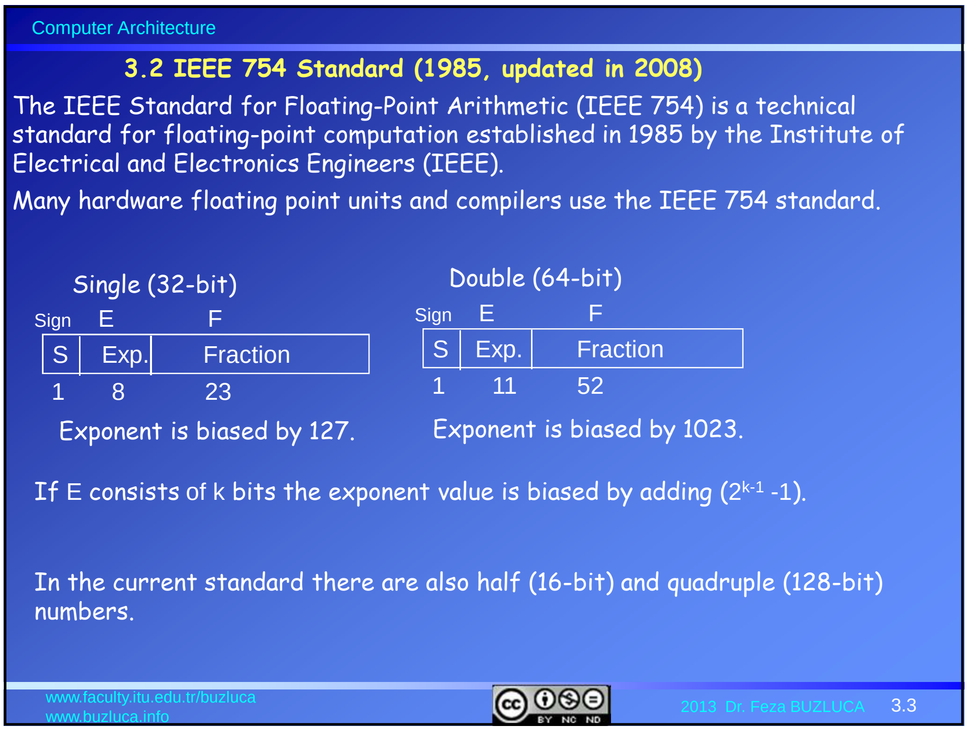
There are also other methods to normalize the numbers.
For example the point can be placed to the right of the most significant digit.

Biased Exponent:

Exponent values are signed and they can be negative (small numbers) and positive.

It is difficult to compare signed numbers (2's complement).

To make the comparison easier the exponent is biased (added with a fixed bias) before being stored, by adjusting its value to make it unsigned suitable for comparison. (Now we don't need to store the sign of the exponent)



3.2 IEEE 754 Standard (1985, updated in 2008)

The IEEE Standard for Floating-Point Arithmetic (IEEE 754) is a technical standard for floating-point computation established in 1985 by the Institute of Electrical and Electronics Engineers (IEEE).

Many hardware floating point units and compilers use the IEEE 754 standard.

Single (32-bit)

Sign	E	F
S	Exp.	Fraction
1	8	23

Exponent is biased by 127.

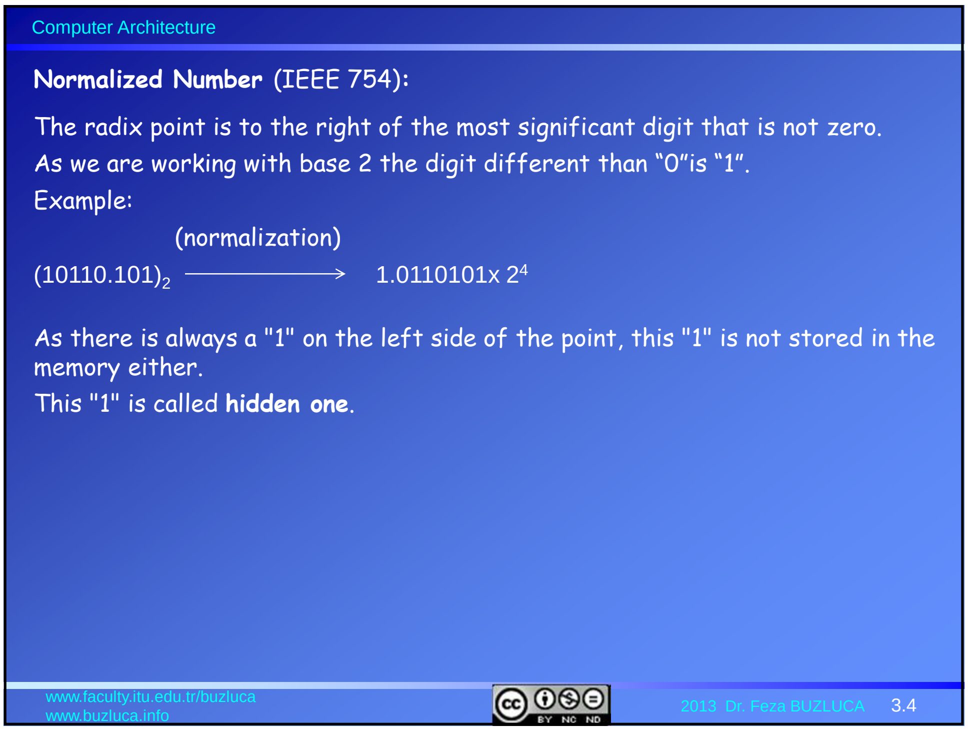
Double (64-bit)

Sign	E	F
S	Exp.	Fraction
1	11	52

Exponent is biased by 1023.

If E consists of k bits the exponent value is biased by adding (2^{k-1} -1).

In the current standard there are also half (16-bit) and quadruple (128-bit) numbers.



Normalized Number (IEEE 754):

The radix point is to the right of the most significant digit that is not zero.

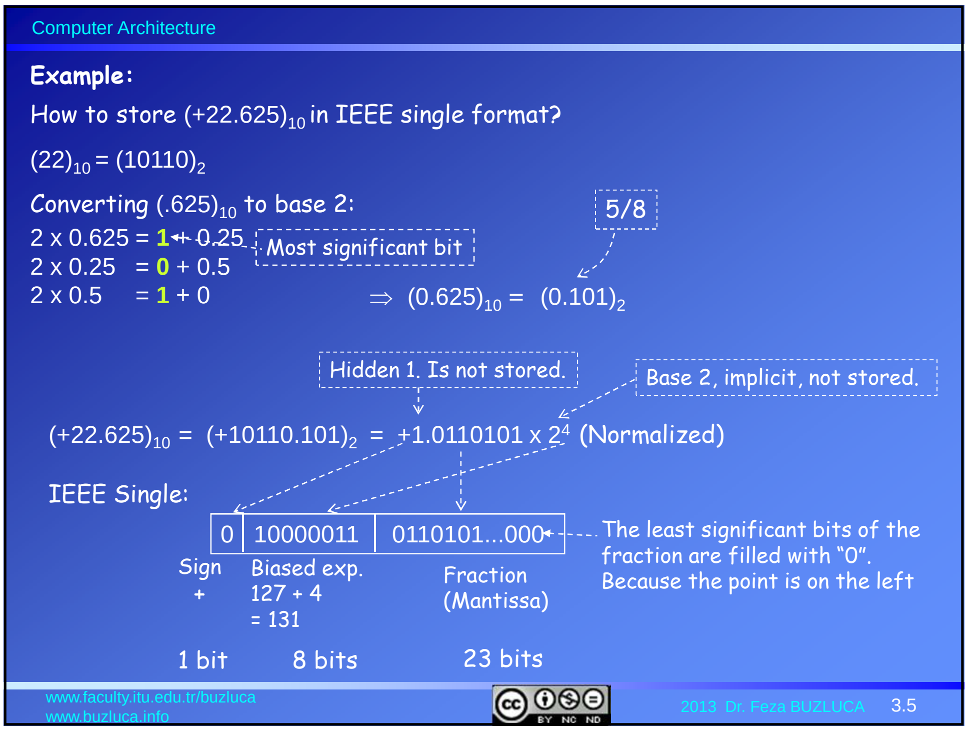
As we are working with base 2 the digit different than “0”is “1”.

Example:

$$(10110.101)_2 \xrightarrow{\text{(normalization)}} 1.0110101\text{x } 2^4$$

As there is always a "1" on the left side of the point, this "1" is not stored in the memory either.

This "1" is called **hidden one**.



Example:

How to store $(+22.625)_{10}$ in IEEE single format?

$(22)_{10} = (10110)_2$

Converting $(.625)_{10}$ to base 2:

2 x 0.625 = **1** + 0.25 ← Most significant bit
2 x 0.25 = **0** + 0.5
2 x 0.5 = **1** + 0

$\Rightarrow (0.625)_{10} = (0.101)_2$ ← 5/8

Hidden 1. Is not stored.

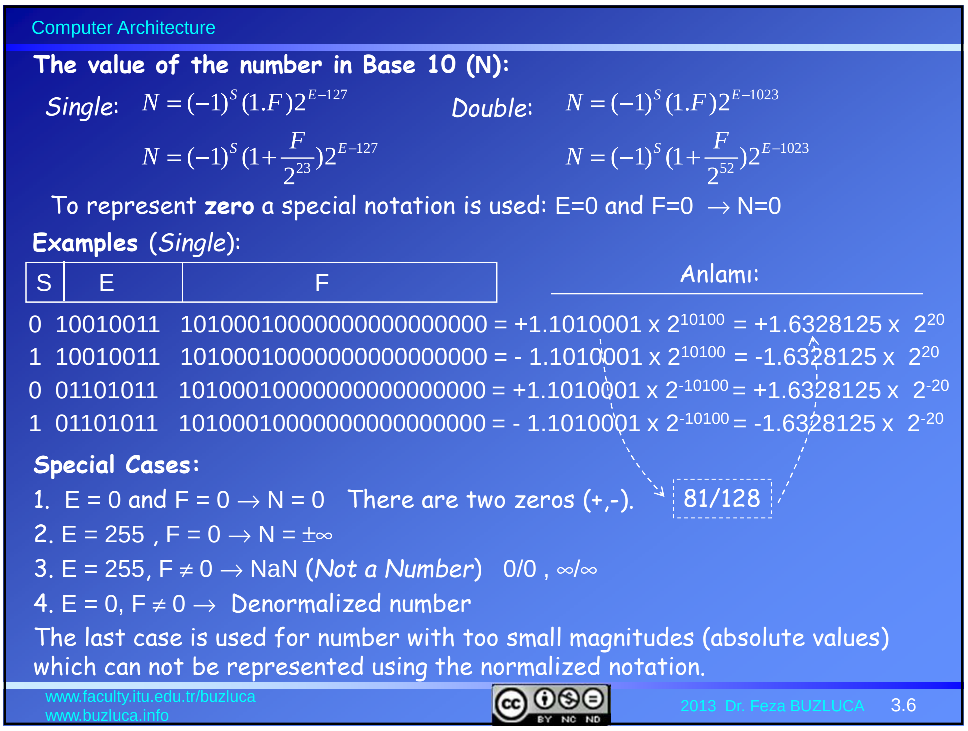
Base 2, implicit, not stored.

$(+22.625)_{10} = (+10110.101)_2 = +1.0110101 \times 2^4$ (Normalized)

IEEE Single:

0	10000011	0110101...000
Sign +	Biased exp. 127 + 4 = 131	Fraction (Mantissa)
1 bit	8 bits	23 bits

The least significant bits of the fraction are filled with "0". Because the point is on the left



The value of the number in Base 10 (N):

Single: $N = (-1)^S (1.F) 2^{E-127}$ *Double*: $N = (-1)^S (1.F) 2^{E-1023}$

$N = (-1)^S (1 + \frac{F}{2^{23}}) 2^{E-127}$ $N = (-1)^S (1 + \frac{F}{2^{52}}) 2^{E-1023}$

To represent **zero** a special notation is used: E=0 and F=0 → N=0

Examples (*Single*):

S	E	F	Anlamı:
0	10010011	10100010000000000000000	= +1.1010001 x 2^{10100} = +1.6328125 x 2^{20}
1	10010011	10100010000000000000000	= - 1.1010001 x 2^{10100} = -1.6328125 x 2^{20}
0	01101011	10100010000000000000000	= +1.1010001 x 2^{-10100} = +1.6328125 x 2^{-20}
1	01101011	10100010000000000000000	= - 1.1010001 x 2^{-10100} = -1.6328125 x 2^{-20}

81/128

Special Cases:

1. E = 0 and F = 0 → N = 0 There are two zeros (+,-).
2. E = 255 , F = 0 → N = ±∞
3. E = 255, F ≠ 0 → NaN (*Not a Number*) 0/0 , ∞/∞
4. E = 0, F ≠ 0 → Denormalized number

The last case is used for number with too small magnitudes (absolute values) which can not be represented using the normalized notation.

Denormalized Numbers:

With the normalized notation numbers with too small magnitudes can not be expressed.

The smallest normalized number: S =0, E = 0000 0001 , F = 000.... 0

N= $+(1+0)2^{1-127} = 2^{-126}$

Numbers between 0 and 2^{-126} can not be expressed as normalized numbers.

The value of denormalized numbers (E = 0, F ≠ 0) is calculated as follows:

Single: $N = (-1)^S(\frac{F}{2^{23}})2^{-126}$ *Double*: $N = (-1)^S(\frac{F}{2^{52}})2^{-1022}$

Denormalized smallest number: S =0, E = 0000 0000 , F = 000.... 01

N= $+(1/2^{23})2^{-126} = 2^{-149}$

Numbers between 0 and2^{-149} can not be expressed. (*Underflow*).

Limit values:

The number with the smallest magnitude:

Single: 2^{-149} (Denormalized) , Double: 2^{-1074} (Denormalized)

The number with the largest magnitude :

Single: 0 11111110 11111111111111111111111 = $(2-2^{-23})\text{x}2^{127} \approx 10^{38.53}$ (Normalized)

Double: $(2-2^{-52})\text{x}2^{1023} \approx 10^{308.3}$ (Normalized)