Discrete Mathematics

Predicates and Sets

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Topics

Predicate

Definition
predicate (or open statement): a declarative sentence which

- contains one or more variables, and
- is not a proposition, but
- becomes a proposition when the variables in it are replaced by certain allowable choices

Universe of Discourse

Definition
universe of discourse: $\mathcal{U}$
set of allowable choices

- examples:
  - $\mathbb{Z}$: integers
  - $\mathbb{N}$: natural numbers
  - $\mathbb{Z}^+$: positive integers
  - $\mathbb{Q}$: rational numbers
  - $\mathbb{R}$: real numbers
  - $\mathbb{C}$: complex numbers

Predicate Examples

Example
$\mathcal{U} = \mathbb{N}$
$p(x)$: $x + 2$ is an even integer

$p(5)$: F
$p(8)$: T

$\neg p(x)$: $x + 2$ is not an even integer

Example
$\mathcal{U} = \mathbb{N}$
$q(x, y)$: $x + y$ and $x - 2y$ are even integers

$q(11, 3)$: F, $q(14, 4)$: T

Quantifiers

Definition
existential quantifier:
predicate is true for some values

- symbol: $\exists$
- read: *there exists*

- symbol: $\exists!$
- read: *there exists only one*

Definition
universal quantifier:
predicate is true for all values

- symbol: $\forall$
- read: *for all*

Quantifiers

existential quantifier
$\mathcal{U} = \{x_1, x_2, \ldots, x_n\}$
$\exists x \; p(x) \equiv p(x_1) \vee p(x_2) \vee \cdots \vee p(x_n)$

- $p(x)$ *is true for some* x

universal quantifier
$\mathcal{U} = \{x_1, x_2, \ldots, x_n\}$
$\forall x \; p(x) \equiv p(x_1) \wedge p(x_2) \wedge \cdots \wedge p(x_n)$

- $p(x)$ *is true for all* x

Quantifier Examples

Example
$\mathcal{U} = \mathbb{R}$

- $p(x) : x \geq 0$
- $q(x) : x^2 \geq 0$
- $r(x) : (x-4)(x+1) = 0$
- $s(x) : x^2 - 3 > 0$

are the following expressions true?

- $\exists x \; [p(x) \wedge r(x)]$
- $\forall x \; [p(x) \rightarrow q(x)]$
- $\forall x \; [q(x) \rightarrow s(x)]$
- $\forall x \; [r(x) \vee s(x)]$
- $\forall x \; [r(x) \rightarrow p(x)]$

Negating Quantifiers

- replace $\forall$ with $\exists$, and $\exists$ with $\forall$
- negate the predicate

$$
\begin{aligned}
\neg\exists x \; p(x) &\Leftrightarrow \forall x \; \neg p(x) \\
\neg\exists x \; \neg p(x) &\Leftrightarrow \forall x \; p(x) \\
\neg\forall x \; p(x) &\Leftrightarrow \exists x \; \neg p(x) \\
\neg\forall x \; \neg p(x) &\Leftrightarrow \exists x \; p(x)
\end{aligned}
$$

Negating Quantifiers

Theorem
$\neg\exists x \; p(x) \Leftrightarrow \forall x \; \neg p(x)$

Proof.

$$
\begin{aligned}
\neg\exists x \; p(x) &\equiv \neg[p(x_1) \vee p(x_2) \vee \cdots \vee p(x_n)] \\
&\Leftrightarrow \neg p(x_1) \wedge \neg p(x_2) \wedge \cdots \wedge \neg p(x_n) \\
&\equiv \forall x \; \neg p(x)
\end{aligned}
$$

$\square$

Predicate Equivalences

Theorem
$\exists x \; [p(x) \vee q(x)] \Leftrightarrow \exists x \; p(x) \vee \exists x \; q(x)$

Theorem
$\forall x \; [p(x) \wedge q(x)] \Leftrightarrow \forall x \; p(x) \wedge \forall x \; q(x)$

Predicate Implications

Theorem
$\forall x\ p(x) \Rightarrow \exists x\ p(x)$

Theorem
$\exists x\ [p(x) \wedge q(x)] \Rightarrow \exists x\ p(x) \wedge \exists x\ q(x)$

Theorem
$\forall x\ p(x) \vee \forall x\ q(x) \Rightarrow \forall x\ [p(x) \vee q(x)]$

Multiple Quantifiers

- $\exists x \exists y\ p(x, y)$
- $\forall x \exists y\ p(x, y)$
- $\exists x \forall y\ p(x, y)$
- $\forall x \forall y\ p(x, y)$

Multiple Quantifier Examples

Example
$\mathcal{U} = \mathbb{Z}$
$p(x, y) : x + y = 17$

- $\forall x \exists y\ p(x, y)$:
  for every x there exists a y such that $x + y = 17$
- $\exists y \forall x\ p(x, y)$:
  there exists a y so that for all x, $x + y = 17$
- what if $\mathcal{U} = \mathbb{N}$?

Multiple Quantifiers

Example
$\mathcal{U}_x = \{1, 2\} \wedge \mathcal{U}_y = \{A, B\}$

$$\begin{aligned}
\exists x \exists y\ p(x, y) &\equiv [p(1, A) \vee p(1, B)] \vee [p(2, A) \vee p(2, B)] \\
\exists x \forall y\ p(x, y) &\equiv [p(1, A) \wedge p(1, B)] \vee [p(2, A) \wedge p(2, B)] \\
\forall x \exists y\ p(x, y) &\equiv [p(1, A) \vee p(1, B)] \wedge [p(2, A) \vee p(2, B)] \\
\forall x \forall y\ p(x, y) &\equiv [p(1, A) \wedge p(1, B)] \wedge [p(2, A) \wedge p(2, B)]
\end{aligned}$$

References

Required Reading: Grimaldi

- Chapter 2: Fundamentals of Logic
  - 2.4. The Use of Quantifiers

Supplementary Reading: O'Donnell, Hall, Page

- Chapter 7: Predicate Logic

Set

Definition
set: a collection of elements that are

- distinct
- unordered
- non-repeating

Set Representation

- *explicit representation*
  elements are listed within braces: $\{a_1, a_2, \dots, a_n\}$
- *implicit representation*
  elements that validate a predicate: $\{x | x \in G, p(x)\}$
- $\emptyset$: empty set
- let S be a set, and a be an element
  - $a \in S$: a is an element of set S
  - $a \notin S$: a is not an element of set S
- $|S|$: number of elements (cardinality)

Explicit Representation Example

Example
$\{3, 8, 2, 11, 5\}$
$11 \in \{3, 8, 2, 11, 5\}$
$|\{3, 8, 2, 11, 5\}| = 5$

Implicit Representation Examples

Example
$\{x | x \in \mathbb{Z}^+, 20 < x^3 < 100\} \equiv \{3, 4\}$
$\{2x - 1 | x \in \mathbb{Z}^+, 20 < x^3 < 100\} \equiv \{5, 7\}$

Example
$A = \{x | x \in \mathbb{R}, 1 \leq x \leq 5\}$

Example
$E = \{n | n \in \mathbb{N}, \exists k \in \mathbb{N}\ [n = 2k]\}$
$A = \{x | x \in E, 1 \leq x \leq 5\}$

Set Dilemma

- There is a barber who lives in a small town.
  He shaves all those men who don't shave themselves.
  He doesn't shave those men who shave themselves.
  Does the barber shave himself?
- yes $\rightarrow$ but he doesn't shave men who shave themselves $\rightarrow$ no
- no $\rightarrow$ but he shaves all men who don't shave themselves $\rightarrow$ yes

Set Dilemma

- let S be the set of sets that are not an element of themselves
  $S = \{A | A \notin A\}$
  Is S an element of itself?
- yes $\rightarrow$ but the predicate is not valid $\rightarrow$ no
- no $\rightarrow$ but the predicate is valid $\rightarrow$ yes

Subset

Definition
$A \subseteq B \Leftrightarrow \forall x\ [x \in A \rightarrow x \in B]$

- set equality:
  $A = B \Leftrightarrow (A \subseteq B) \wedge (B \subseteq A)$
- proper subset:
  $A \subset B \Leftrightarrow (A \subseteq B) \wedge (A \neq B)$
- $\forall S\ [\emptyset \subseteq S]$

Subset

not a subset

$$\begin{aligned} A \not\subseteq B &\Leftrightarrow \neg\forall x\ [x \in A \rightarrow x \in B] \\ &\Leftrightarrow \exists x\ \neg[x \in A \rightarrow x \in B] \\ &\Leftrightarrow \exists x\ \neg[\neg(x \in A) \vee (x \in B)] \\ &\Leftrightarrow \exists x\ [(x \in A) \wedge \neg(x \in B)] \\ &\Leftrightarrow \exists x\ [(x \in A) \wedge (x \notin B)] \end{aligned}$$

Power Set

Definition

power set: $\mathcal{P}(S)$
the set of all subsets of a set, including the empty set and the set itself

- if a set has n elements, its power set has 2^n elements

Example of Power Set

Example

$$\begin{aligned} \mathcal{P}(\{1,2,3\}) = \{ & \\ & \emptyset, \\ & \{1\}, \{2\}, \{3\}, \\ & \{1,2\}, \{1,3\}, \{2,3\}, \\ & \{1,2,3\} \\ \} & \end{aligned}$$

Set Operations

complement
$\overline{A} = \{x | x \notin A\}$

intersection
$A \cap B = \{x | (x \in A) \wedge (x \in B)\}$

- if $A \cap B = \emptyset$ then A and B are disjoint

union
$A \cup B = \{x | (x \in A) \vee (x \in B)\}$

Set Operations

difference
$A - B = \{x | (x \in A) \wedge (x \notin B)\}$

- $A - B = A \cap \overline{B}$
- *symmetric difference*:
  $A \triangle B = \{x | (x \in A \cup B) \wedge (x \notin A \cap B)\}$

Cartesian Product

Definition

Cartesian product:
$A \times B = \{(a, b) | a \in A, b \in B\}$

$A \times B \times C \times \cdots \times N = \{(a, b, \ldots, n) | a \in A, b \in B, \ldots, n \in N\}$

- $|A \times B \times C \times \cdots \times N| = |A| \cdot |B| \cdot |C| \cdots |N|$

Cartesian Product Example

Example

$A = \{a_1 . a_2, a_3, a_4\}$
$B = \{b_1, b_2, b_3\}$

$$\begin{aligned} A \times B = \ \{ & \\ & (a_1, b_1), (a_1, b_2), (a_1, b_3), \\ & (a_2, b_1), (a_2, b_2), (a_2, b_3), \\ & (a_3, b_1), (a_3, b_2), (a_3, b_3), \\ & (a_4, b_1), (a_4, b_2), (a_4, b_3) \\ \} & \end{aligned}$$

Equivalences

Double Complement
$\overline{\overline{A}} = A$

Commutativity
$A \cap B = B \cap A$ $\quad$ $A \cup B = B \cup A$

Associativity
$(A \cap B) \cap C = A \cap (B \cap C)$ $\quad$ $(A \cup B) \cup C = A \cup (B \cup C)$

Idempotence
$A \cap A = A$ $\quad$ $A \cup A = A$

Inverse
$A \cap \overline{A} = \emptyset$ $\quad$ $A \cup \overline{A} = \mathcal{U}$

Equivalences

Identity
$A \cap \mathcal{U} = A$ $\quad$ $A \cup \emptyset = A$

Domination
$A \cap \emptyset = \emptyset$ $\quad$ $A \cup \mathcal{U} = \mathcal{U}$

Distributivity
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ $\quad$ $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Absorption
$A \cap (A \cup B) = A$ $\quad$ $A \cup (A \cap B) = A$

DeMorgan's Laws
$\overline{A \cap B} = \overline{A} \cup \overline{B}$ $\quad$ $\overline{A \cup B} = \overline{A} \cap \overline{B}$

DeMorgan's Laws

Proof.

$$\begin{aligned} \overline{A \cap B} &= \{x | x \notin (A \cap B)\} \\ &= \{x | \neg(x \in (A \cap B))\} \\ &= \{x | \neg((x \in A) \wedge (x \in B))\} \\ &= \{x | \neg(x \in A) \vee \neg(x \in B)\} \\ &= \{x | (x \notin A) \vee (x \notin B)\} \\ &= \{x | (x \in \overline{A}) \vee (x \in \overline{B})\} \\ &= \{x | x \in \overline{A} \cup \overline{B}\} \\ &= \overline{A} \cup \overline{B} \end{aligned}$$

□

Example of Equivalence

Theorem
$A \cap (B - C) = (A \cap B) - (A \cap C)$

Equivalence Example

Proof.

$$\begin{aligned} (A \cap B) - (A \cap C) &= (A \cap B) \cap \overline{(A \cap C)} \\ &= (A \cap B) \cap (\overline{A} \cup \overline{C}) \\ &= ((A \cap B) \cap \overline{A}) \cup ((A \cap B) \cap \overline{C}) \\ &= \emptyset \cup ((A \cap B) \cap \overline{C}) \\ &= (A \cap B) \cap \overline{C} \\ &= A \cap (B \cap \overline{C}) \\ &= A \cap (B - C) \end{aligned}$$

□

Principle of Inclusion-Exclusion

- $|A \cup B| = |A| + |B| - |A \cap B|$
- $|A \cup B \cup C| =$
  $|A| + |B| + |C| - (|A \cap B| + |A \cap C| + |B \cap C|) + |A \cap B \cap C|$

Theorem

$$
\begin{aligned}
|A_1 \cup A_2 \cup \cdots \cup A_n| \;=\; & \sum_i |A_i| - \sum_{i,j} |A_i \cap A_j| \\
& + \sum_{i,j,k} |A_i \cap A_j \cap A_k| \\
& \cdots + -1^{n-1} |A_i \cap A_j \cap \cdots \cap A_n|
\end{aligned}
$$

Inclusion-Exclusion Example

Example (sieve of Eratosthenes)

- a method to identify prime numbers

2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
18	19	20	21	22	23	24	25	26	27	28	29	30			
2	3		5		7		9		11		13		15		17
	19		21		23		25		27		29				
2	3		5		7				11		13				17
	19				23		25				29				
2	3		5		7				11		13				17
	19				23						29				

Inclusion-Exclusion Example

Example (sieve of Eratosthenes)

- number of primes between 1 and 100
- numbers that are not divisible by 2, 3, 5 and 7
  - A_2: set of numbers divisible by 2
  - A_3: set of numbers divisible by 3
  - A_5: set of numbers divisible by 5
  - A_7: set of numbers divisible by 7
- $|A_2 \cup A_3 \cup A_5 \cup A_7|$

Inclusion-Exclusion Example

Example (sieve of Eratosthenes)

- $|A_2| = \lfloor 100/2 \rfloor = 50$
- $|A_3| = \lfloor 100/3 \rfloor = 33$
- $|A_5| = \lfloor 100/5 \rfloor = 20$
- $|A_7| = \lfloor 100/7 \rfloor = 14$
- $|A_2 \cap A_3| = \lfloor 100/6 \rfloor = 16$
- $|A_2 \cap A_5| = \lfloor 100/10 \rfloor = 10$
- $|A_2 \cap A_7| = \lfloor 100/14 \rfloor = 7$
- $|A_3 \cap A_5| = \lfloor 100/15 \rfloor = 6$
- $|A_3 \cap A_7| = \lfloor 100/21 \rfloor = 4$
- $|A_5 \cap A_7| = \lfloor 100/35 \rfloor = 2$

Inclusion-Exclusion Example

Example (sieve of Eratosthenes)

- $|A_2 \cap A_3 \cap A_5| = \lfloor 100/30 \rfloor = 3$
- $|A_2 \cap A_3 \cap A_7| = \lfloor 100/42 \rfloor = 2$
- $|A_2 \cap A_5 \cap A_7| = \lfloor 100/70 \rfloor = 1$
- $|A_3 \cap A_5 \cap A_7| = \lfloor 100/105 \rfloor = 0$
- $|A_2 \cap A_3 \cap A_5 \cap A_7| = \lfloor 100/210 \rfloor = 0$

Inclusion-Exclusion Example

Example (sieve of Eratosthenes)

$$
\begin{aligned}
|A_2 \cup A_3 \cup A_5 \cup A_7| \;&=\; (50 + 33 + 20 + 14) \\
&- \; (16 + 10 + 7 + 6 + 4 + 2) \\
&+ \; (3 + 2 + 1 + 0) \\
&- \; (0) \\
&= \; 78
\end{aligned}
$$

- number of primes: $(100 - 78) + 4 - 1 = 25$

References

Required Reading: Grimaldi

- Chapter 3: Set Theory
  - 3.1. Sets and Subsets
  - 3.2. Set Operations and the Laws of Set Theory
- Chapter 8: The Principle of Inclusion and Exclusion
  - 8.1. The Principle of Inclusion and Exclusion

Supplementary Reading: O'Donnell, Hall, Page

- Chapter 8: Set Theory