Discrete Mathematics

Principles of Counting

Ayşegül Gençata Yayımlı H. Turgut Uyar

2013

Licence

Topics

Basic Principles

- counting = enumeration
- two basic principles of counting:
  - the rule of sum
  - the rule of product
- decompose more complex problems into smaller ones
- piece together partial solutions to arrive at the final answer

Rule of Sum

The Rule of Sum

- a first task can be performed in m (distinct) ways
- a second task can be performed in n (distinct) ways
- the two tasks cannot be performed simultaneously
- performing either task can be accomplished in any one of $m + n$ ways

Rule of Sum Example

Example

- a college library has 40 textbooks on sociology
- and 50 textbooks on anthropology
- a student can select from $40 + 50 = 90$ textbooks in order to learn more about one or the other subject

Rule of Sum Example

Example

- a computer science instructor has two colleagues
- one colleague has 3 textbooks on "Analysis of Algorithms"
- the other colleague has 5 such textbooks
- n: maximum number of different books that the instructor can borrow
- since both colleagues may own copies of the same book: $5 \leq n \leq 8$

Rule of Product

The Rule of Product

- a procedure can be broken down into first and second stages
- there are m possible outcomes for the first stage
- for each of these outcomes, there are n possible outcomes for the second stage
- the total procedure can be carried out in $m \cdot n$ ways

Rule of Product Example

Example

- the drama club is holding tryouts for a play
- there are 6 men and 8 women auditioning for the leading roles
- the director can cast the leading couple in $6 \cdot 8 = 48$ ways

Rule of Product Examples

Example

- license plates consist of 2 letters, followed by 4 digits
- how many plates?
- if no letter or digit can be repeated: $26 \cdot 25 \cdot 10 \cdot 9 \cdot 8 \cdot 7 = 3,276,000$
- if repetitions are allowed for both letters and digits: $26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 6,760,000$
- if repetitions are allowed for both letters and digits, how many plates consist of only vowels and even digits? $5 \cdot 5 \cdot 5 \cdot 5 \cdot 5 \cdot 5 = 15,625$

Permutation

Definition

permutation: any linear arrangement of n distinct objects

- "arrangement" means that the order is important

Permutation Example

Example

- a class has 10 students: $A, B, C, \ldots, I, J$
- 5 students are to be chosen and seated in a row for a picture:
  - $BCEFI, CEFIB, ABCFG, \ldots$
- how many such linear arrangements are possible?
- the filling of a position is a stage of the counting procedure: $10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 = 30,240$

Permutation Example

Example

$$
\begin{aligned}
10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 &= 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} \\
&= \frac{10!}{5!}
\end{aligned}
$$

Number of Permutations

number of permutations

- n distinct objects
- number of permutations of size r (where $1 \le r \le n$):

$$
\begin{aligned}
P(n, r) &= n \cdot (n-1) \cdot (n-2) \cdots (n-r+1) \\
&= \frac{n!}{(n-r)!}
\end{aligned}
$$

- if repetitions are allowed: n^r

Number of Permutations Example

Example

- what is the number of permutations of the letters in "BALL"?
- the two L's are indistinguishable

A	B	L	L	L	A	B	L
A	L	B	L	L	A	L	B
A	L	L	B	L	B	A	L
B	A	L	L	L	B	L	A
B	L	A	L	L	L	A	B
B	L	L	A	L	L	B	A

- number of permutations: $\frac{4!}{2} = 12$

Number of Permutations Example

Example

- arrangements of all letters in "DATABASES"
- for each arrangement in which the A's are not distinguished, there are $3! = 6$ arrangements with the A's distinguished: $DA_1TA_2BA_3SES$, $DA_1TA_3BA_2SES$, $DA_2TA_1BA_3SES$, $DA_2TA_3BA_1SES$, $DA_3TA_1BA_2SES$, $DA_3TA_2BA_1SES$
- for each of these, there are 2 arrangements where the S's are distinguished: $DA_1TA_2BA_3S_1ES_2$, $DA_1TA_2BA_3S_2ES_1$
- number of arrangements: $\frac{9!}{2! \cdot 3!} = 30,240$

Generalization

number of arrangements

- n objects
- n_1 indistinguishable objects of $type_1$, n_2 indistinguishable objects of $type_2$, $\ldots$ n_r indistinguishable objects of $type_r$
- $n_1 + n_2 + \ldots + n_r = n$
- number of linear arrangements of these n objects:

$$\frac{n!}{n_1! \cdot n_2! \cdots n_r!}$$

Number of Arrangements Example

Example

- xy-plane from $(2, 1)$ to $(7, 4)$
- staircase path: each step going one unit to the right (R) or one unit upwards (U)
- for example: *RURRURRU*, *URRRUURR*
- how many such paths?
- each path consists of 5 R's and 3 U's
- number of paths: $\frac{8!}{5! \cdot 3!} = 56$

Number of Circular Arrangements Example

Example

- six people are seated around a round table: A, B, C, D, E, F
- how many different circular arrangements?
  - arrangements are considered to be the same when one can be obtained from the other by rotation
  - $ABEFCD, DABEFC, CDABEF, FCDABE, EFCDAB, BEFCDA$
- each circular arrangement (CA) corresponds to 6 linear arrangements (LA)
- $6 \cdot \#CA = \#LA = 6!$
- number of circular arrangements: $\frac{6!}{6} = 120$

Combination Example

Example

- deck of playing cards with 52 cards
- 4 suits: clubs, diamonds, hearts, spades
- 13 ranks in each suit: Ace, 2, 3, ..., 10, Jack, Queen, King
- draw 3 cards in succession, without replacement
- how many possible draws?

$$52 \cdot 51 \cdot 50 = \frac{52!}{49!} = P(52, 3) = 132,600$$

Combination Example

Example

- assume one such draw is: AH (ace of hearts), $9C$ (9 of clubs), KD (king of diamonds)
- if we select all 3 cards at once, the order doesn't matter
- then, the 6 permutations of $(AH, 9C, KD)$ all correspond to just one selection

$$\frac{52!}{3! \cdot 49!} = 22,100$$

Number of Combinations

Combinations

- n distinct objects
- each selection, or combination of r of these objects, with no reference to order, corresponds to $r!$ permutations of size r
- number of combinations of size r (where $0 \leq r \leq n$):

$$C(n, r) = \binom{n}{r} = \frac{P(n, r)}{r!} = \frac{n!}{r! \cdot (n-r)!}$$

Number of Combinations

- number of combinations:

$$C(n, r) = \frac{n!}{r! \cdot (n-r)!}$$

- note that:

$$\begin{aligned} C(n, 0) &= 1 = & C(n, n) \\ C(n, 1) &= n = & C(n, n-1) \end{aligned}$$

Number of Combinations Example

Example

- Lynn and Patti decide to buy a powerball ticket
- to win, one must match five numbers selected from 1 to 49
- and then must also match the powerball, 1 to 42
- how many possible tickets?
- Lynn selects the five numbers from 1 to 49: $C(49, 5)$ ways
- Patti selects the powerball from 1 to 42: $C(42, 1)$ ways
- number of possible tickets: $\binom{49}{5}\binom{42}{1} = 80,089,128$

Number of Combinations Examples

Example

- for a volleyball team, the gym teacher must select nine girls from the junior and senior classes
- 28 junior and 25 senior candidates
- how many different ways?
- if no restrictions: $\binom{53}{9} = 4,431,613,550$
- if two juniors and one senior are the best spikers and must be on the team: $\binom{50}{6} = 15,890,700$
- if there has to be four juniors and five seniors: $\binom{28}{4}\binom{25}{5} = 1,087,836,750$

Binomial Theorem

Theorem

If x and y are variables and n is a positive integer, then

$$
\begin{aligned}
(x+y)^n &= \binom{n}{0}x^0y^n + \binom{n}{1}x^1y^{n-1} + \binom{n}{2}x^2y^{n-2} + \cdots \\
&\quad + \binom{n}{n-1}x^{n-1}y^1 + \binom{n}{n}x^ny^0 \\
&= \sum_{k=0}^{n}\binom{n}{k}x^ky^{n-k}
\end{aligned}
$$

- $\binom{n}{k}$ is a binomial coefficient

Binomial Theorem Examples

Example

- in the expansion of $(x+y)^7$, the coefficient of x^5y^2: $\binom{7}{5} = \binom{7}{2} = 21$

Example

- in the expansion of $(2a-3b)^7$, the coefficient of a^5b^2:
- $x = 2a, y = -3b$

$$\binom{7}{5}x^5y^2 = \binom{7}{5}(2a)^5(-3b)^2 = \binom{7}{5}(2)^5(-3)^2a^5b^2 = 6048a^5b^2$$

Multinomial Theorem

Theorem

For positive integers n, t, the coefficient of $x_1^{n_1}x_2^{n_2}x_3^{n_3}\cdots x_t^{n_t}$ *in the expansion of* $(x_1+x_2+x_3+\cdots+x_t)^n$ *is*

$$\frac{n!}{n_1!\cdot n_2!\cdot n_3!\cdots n_t!}$$

where each n_i *is an integer with* $0 \le n_i \le n$*, for all* $1 \le i \le t$*, and* $n_1 + n_2 + n_3 + \ldots + n_t = n$

Multinomial Theorem Examples

Example

- in the expansion of $(x+y+z)^7$, the coefficient of $x^2y^2z^3$:

$$\binom{7}{2,2,3} = \frac{7!}{2!\cdot 2!\cdot 3!} = 210$$

- the coefficient of xyz^5:

$$\binom{7}{1,1,5} = \frac{7!}{1!\cdot 1!\cdot 5!} = 42$$

Combinations with Repetition Example

Example

- 7 students visit a restaurant
- each of them orders one of the following: cheeseburger (c), hot dog (h), taco (t), fish sandwich (f)
- how many different purchases are possible?

Combinations with Repetition Example

Example

c c h h t t f	x x | x x | x x | x
c c c c h t f	x x x x | x | x | x
c c c c c c f	x x x x x x | | | x
h t t f f f f	| x | x x | x x x x
t t t t t t t	| | x x x x x x x |
f f f f f f f	| | | x x x x x x x

- enumerate all arrangements of 10 symbols consisting of seven x's and three |'s
- number of different purchases: $\frac{10!}{7! \cdot 3!} = \binom{10}{7} = 120$

Number of Combinations with Repetition

Number of Combinations with Repetition

- select, with repetition, r of n distinct objects
- considering all arrangements of r x's and $n - 1$ |'s

$$\frac{(n+r-1)!}{r! \cdot (n-1)!} = \binom{n+r-1}{r}$$

Number of Combinations with Repetition Example

Example

- distribute 7 bananas and 6 oranges among 4 children
- each child receives at least one banana
- how many ways?
- step 1: give each child a banana
- step 2: distribute 3 bananas to 4 children

1 1 1 0	b | b | b |
1 0 2 0	b | | b b |
0 0 1 2	| | b | b b
0 0 0 3	| | | b b b

- $C(6, 3) = 20$ ways

Number of Combinations with Repetition Example

Example

- step 3: distribute 6 oranges to 4 children

1 2 2 1	o | o o | o o | o
1 2 0 3	o | o o | | o o o
0 3 3 0	| o o o | o o o |
0 0 0 6	| | | o o o o o o

- $C(9, 6) = 84$ ways
- step 4: by the rule of product: $20 \cdot 84 = 1,680$ ways

References

Required Reading: Grimaldi

- Chapter 1: Fundamental Principles of Counting
  - 1.1. The Rules of Sum and Product
  - 1.2. Permutations
  - 1.3. Combinations
  - 1.4. Combinations with Repetition