Analysis of Algorithms 1 (Fall 2013) Istanbul Technical University Computer Eng. Dept.

Chapter 7: Quicksort

Course slides from
Susan Bridges @MS State
have been used in
preparation of these slides.

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Purpose

- Learn about quicksort algorithm
- Learn about important subroutine used by quicksort for partitioning
- Analyze randomized quicksort

Outline

- Description of Quicksort and Partition
- Intuitive Discussion of Performance of Quicksort
- Version of Quicksort that Uses Random Sampling
- Analysis of Randomized Quicksort

Why Quicksort?

- Popular algorithm for sorting large input arrays
- Worst-case running time $\Theta(n^2)$ on input array of n numbers
- Slow worst-case running time, but often best practical choice for sorting because remarkably efficient on average
  - expected running time is $\Theta(n \lg n)$, and constant factors hidden in $\Theta(n \lg n)$ notation are quite small
- Advantage of sorting in place
- Works well even in virtual memory environments

Description of Quicksort

- Like merge sort, based on divide-and-conquer paradigm
- **Divide:**
  - Partition A[p..r] into two (possibly empty) subarrays A[p..q-1] and A[q+1 .. r] such each element of A[p..q-1] ≤ A[q] and A[q] ≤ each element of A[q+1..r] (A[q] called pivot)
  - Compute the index q as part of this partitioning procedure
- **Conquer:**
  - Sort the two subarrays by recursive calls to quicksort
- **Combine:**
  - Since the subarrays are sorted in place, no work is needed to combine them: A[p..r] is now sorted

Quicksort Algorithm

```
QUICKSORT(A,p,r)
if p < r
   then q ← PARTITION(A,p,r)
        QUICKSORT(A,p,q-1)
        QUICKSORT(A,q+1,r)
```

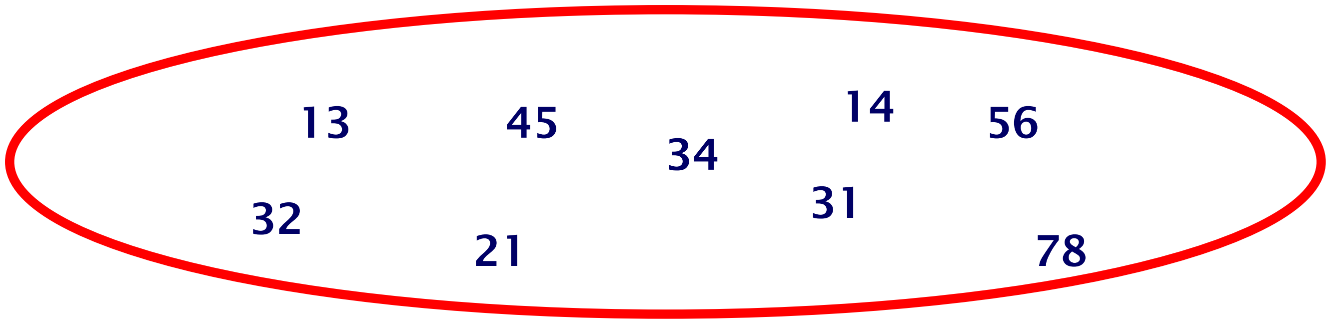
Initial call: QUICKSORT(A,1, length[A])

Merge Sort (REMINDER)

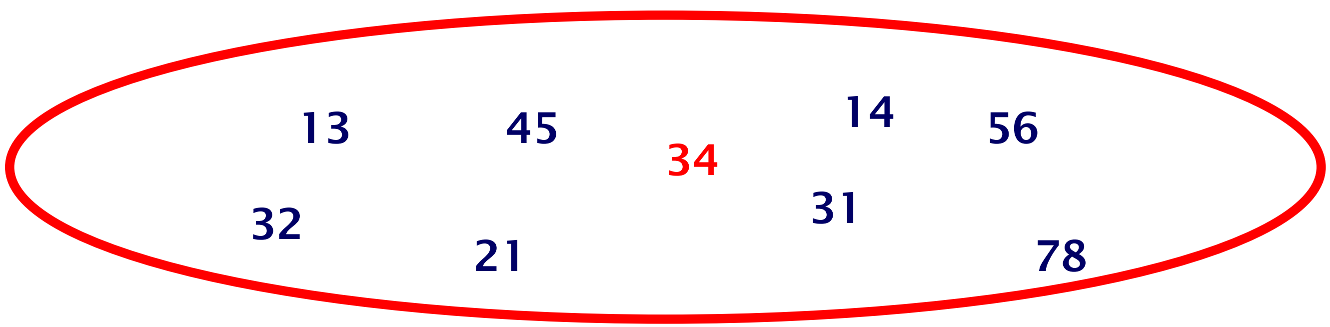
```
Merge-Sort(A, p, r)
   if p < r
   then {q ← ⌊(p+r)/2⌋
              Merge-Sort(A, p, q)
              Merge-Sort(A, q+1, r)
              Merge(A, p, q, r)}
```

- A is the (sub)array *when the procedure is called.*
- p, q, and r are indices numbering elements of the array such that $p \le q \le r$; p is the lowest index and r is the highest index.

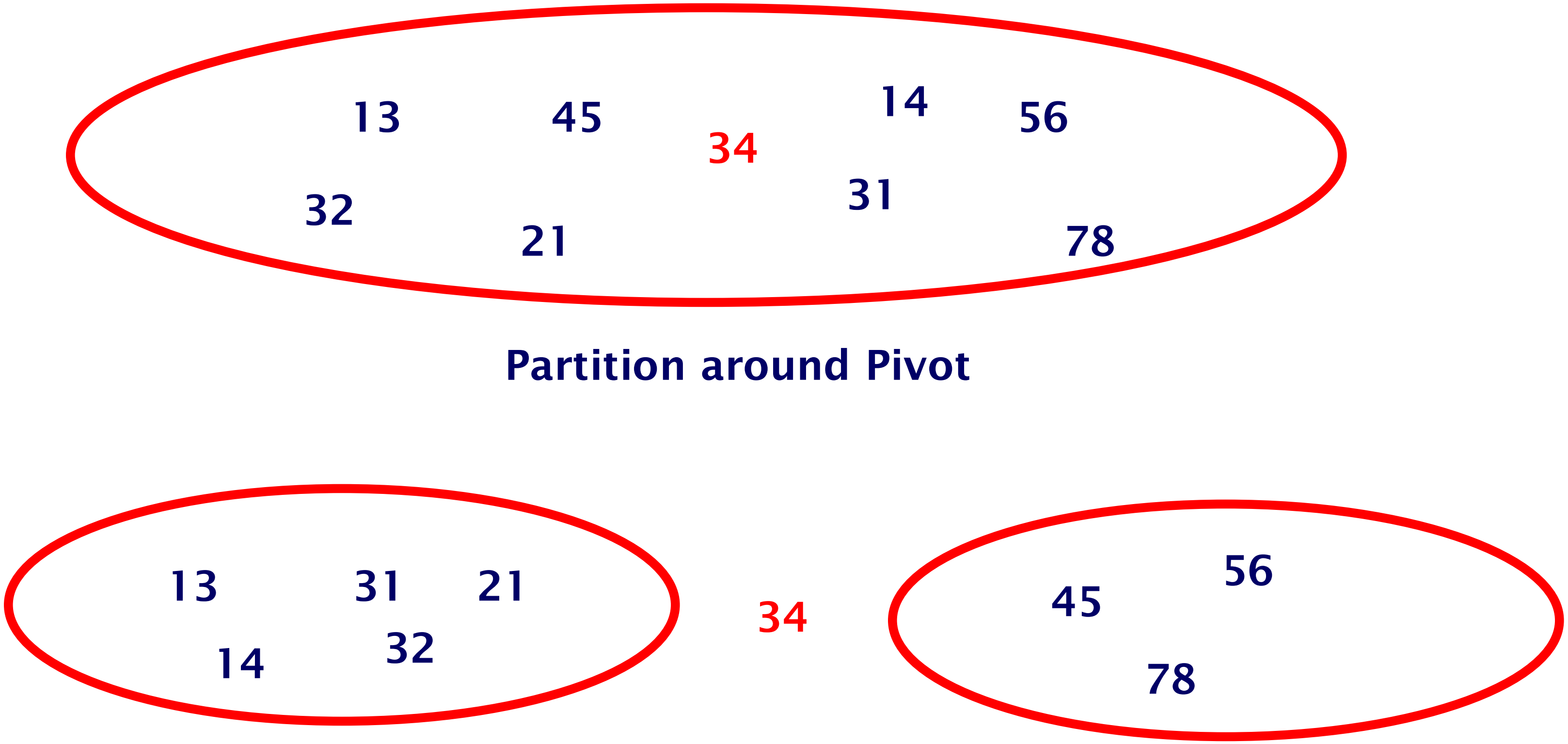
Quicksort Example



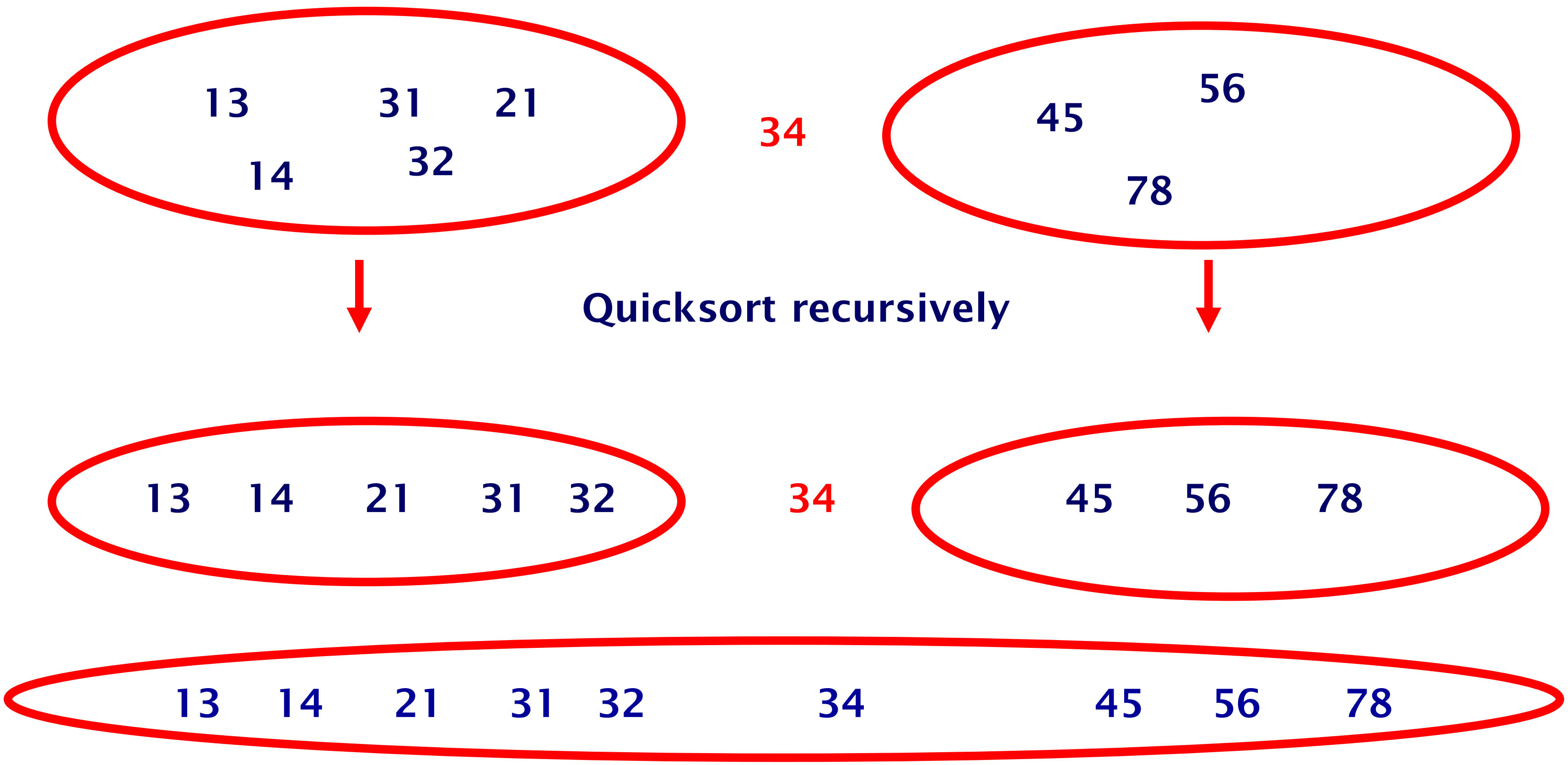
Select Pivot



Quicksort Example



Quicksort Example

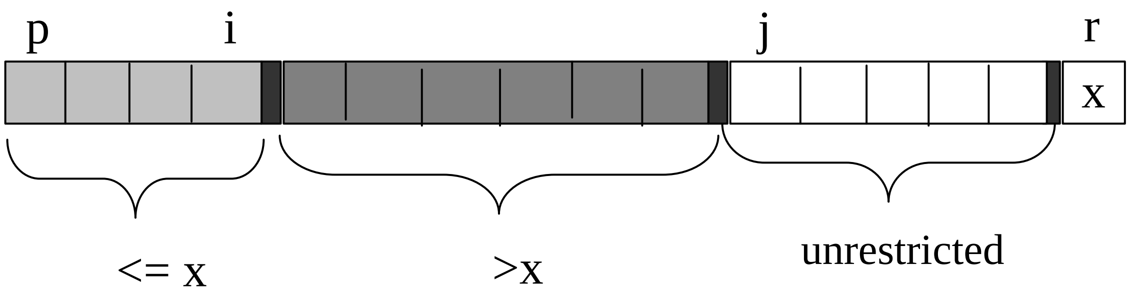


Partition Algorithm

- Rearranges subarray A[p..r] in place

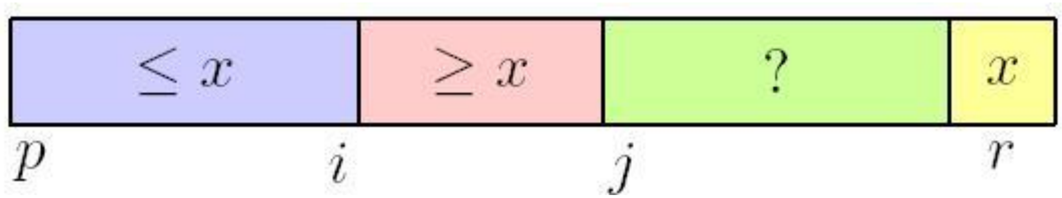
```
PARTITION(A,p,r)
x ← A[r]
 i ← p - 1
 for j ← p to r-1
      do if A[j] ≤ x
         then i ← i + 1
                exchange A[i] ↔ A[j]
 exchange A[i+1] ↔ A[r]
 return i+1
```

Regions of Subarray Maintained by PARTITION

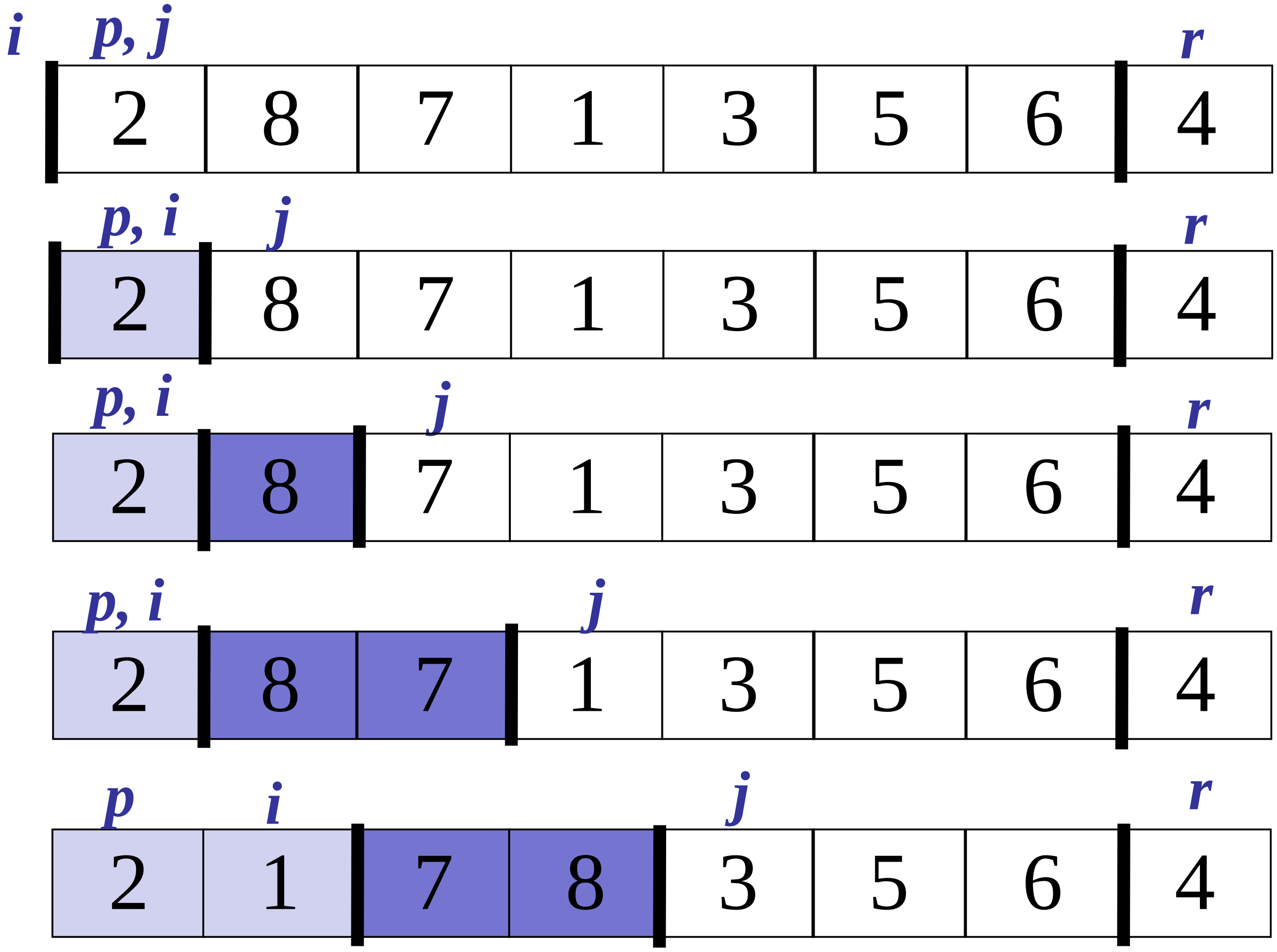


1. Each value in A[p..i] ≤ x
2. Each value in A[i+1..j-1] > x
3. A[r] = x
4. A[j..r-1] can take on any values

Invariant:

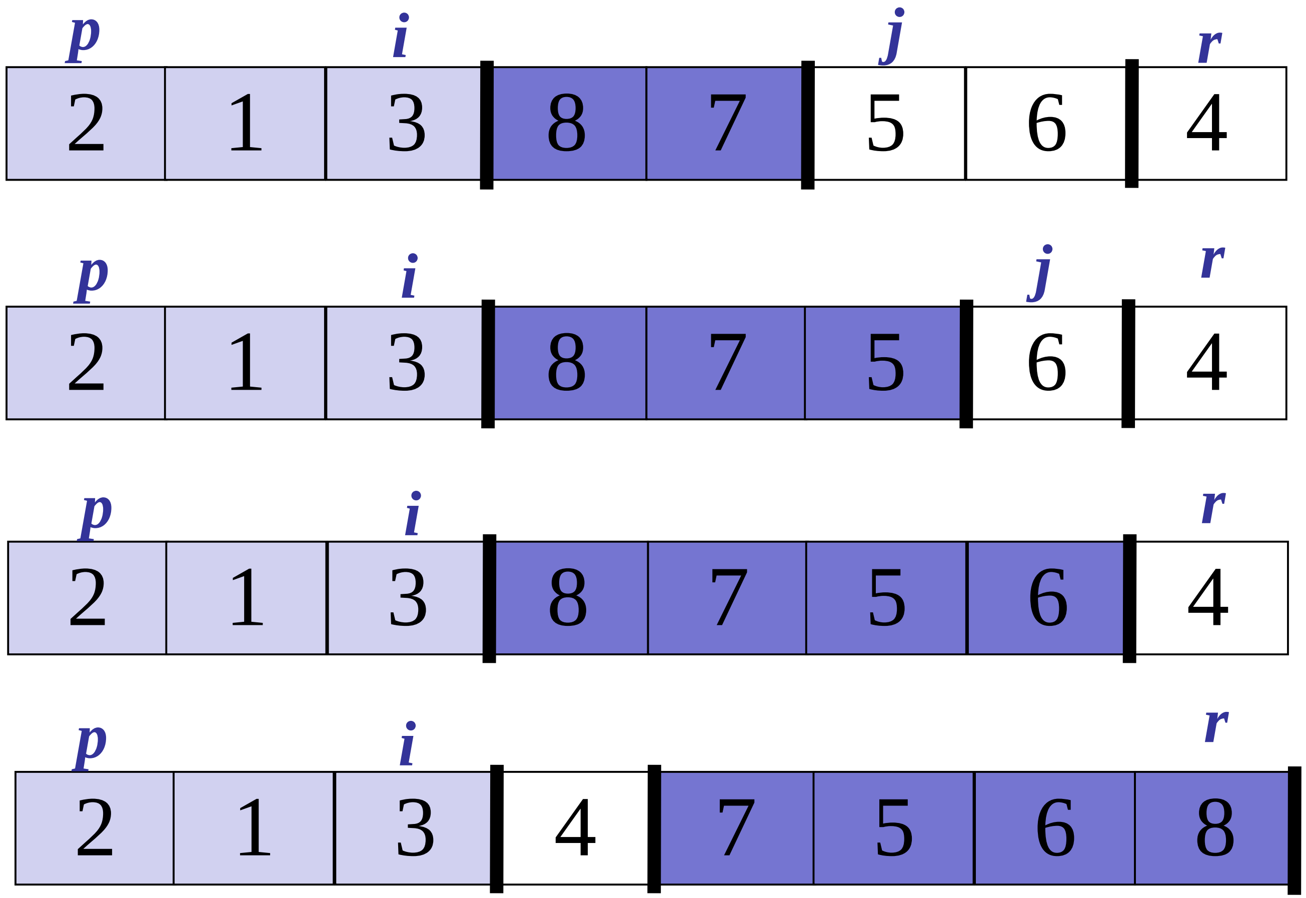


Example of Partition



cont.->

Example of Partition (cont.)



Loop Invariant for Partition

- At beginning of each iteration of loop in lines 3-6, for any array index *k:*
  1. If $p \le k \le i$, then $A[k] \le x$
  2. If $i + 1 \le k \le j-1$, then $A[k] > x$
  3. If $k = r$, then $A[k] = x$

```
PARTITION(A,p,r)
1  x ← A[r]
2  i ← p - 1
3  for j ← p to r-1
4          do if A[j] ≤ x
5             then i ← i + 1
6                    exchange A[i] ↔ A[j]
7    exchange A[i+1] ↔ A[r]
8    return i+1
```

Loop Invariant Correctness

- We need to show that
  - this loop invariant is true prior to first iteration
  - each iteration of loop maintains invariant
  - invariant provides a useful property to show correctness when loop terminates

Loop Invariant Correctness: Initialization

- Prior to first iteration of loop, $i = p - 1$, and $j = p$
- There are no values between p and i, and no values between $i + 1$ and $j - 1$, so first two conditions of loop invariant are trivially satisfied
- Assignment in line 1 satisfies third condition

Loop Invariant Correctness: Maintenance

- Two cases to consider depending on outcome of test in line 4
- When A[*j*]>*x*
  - Only action in loop is to increment *j*
  - After *j* is incremented, condition 2 holds for all A[*j*-1] and all other entries remain unchanged
- When A[*j*] ≤ *x*
  - *i* is incremented, A[*i*] and A[*j*] are swapped, and then *j* is incremented
  - Because of the swap, we now have that A[*i*] ≤ *x*, and condition 1 is satisfied
  - Similarly, we also have that A[*j*-1]>*x*, since item that was swapped into A[*j*-1] is, by loop invariant, greater than *x*

Loop Invariant Correctness: Termination

- At termination, $j = r$
- Therefore, every entry in the array is in one of the three sets described by the invariant, and we have partitioned the values in the array into three sets:
  - those less than or equal to x
  - those greater than x
  - singleton set containing x

Partition

- Final two lines move pivot element into its place in middle of array by swapping it with leftmost element greater than x
- Output now satisfies specifications given for the divide step
- Running time on A[p..r] is $\Theta(n)$, where $n = r - p + 1$

```
PARTITION(A,p,r)
x ← A[r]
i ← p - 1
for j ← p to r-1
    do if A[j] ≤ x
        then i ← i + 1
             exchange A[i] ↔ A[j]
exchange A[i+1] ↔ A[r]
return i+1
```

Performance of Quicksort

- Depends on whether partitioning is balanced or unbalanced:
  - **Worst case:** Each time partitioning is done, one subarray contains *n* -1 of *n* elements from previous call and the other is empty
  - **Best case:** Each time partitioning is done, each subarray contains n/2 of elements from previous call

Worst Case

Cost of Partition: $\Theta(n)$

Recurrence for Quicksort:

$$T(n) = T(n-1) + T(0) + \Theta(n)$$
$$= T(n-1) + \Theta(n)$$

Worst Case (cont.)

Solving recurrence by iteration:

$$
\begin{aligned}
T(n) &= \Theta(n) + T(n-1) \\
&= \Theta(n) + \Theta(n-1) + \Theta(n-2) + \ldots + \Theta(1) \\
&= \sum_{k=1}^{n} \Theta(k) \\
&= \Theta\left(\sum_{k=1}^{n} k\right) \\
&= \Theta(n^2)
\end{aligned}
$$

Best Case

Recurrence for Quicksort:

$$T(n) \leq 2T(n/2) + \Theta(n)$$

Solving recurrence by Master Method case 2:

$$T(n) = O(n \lg n)$$

Average Case

- Suppose split is always 9-to-1
- Recurrence:

$$T(n) \le T(9n/10) + T(n/10) + \Theta(n)$$
$$= T(9n/10) + T(n/10) + cn$$

- Solving recurrence by recursion tree:

T(n) = O(n lg n)

Randomized Version of Quicksort

- When an algorithm has average case performance and worst case performance that are very different, we can try to minimize odds of encountering worst case
- For quicksort: Randomly choose pivot element in A[p..r]

Randomized PARTITION

```
RANDOMIZED-PARTITION (A, p, r)
    i ← RANDOM (p, r)
    exchange A[r] ↔ A[i]
    return PARTITION (A, p, r)
```

Randomized Quicksort

```
RANDOMIZED-QUICKSORT (A, p, r)
    if p < r
      then q ← RANDOMIZED-PARTITION (A, p, r)
           RANDOMIZED-QUICKSORT (A, p, q-1)
           RANDOMIZED-QUICKSORT (A, q+1, r)
```

Summary

- Description of Quicksort
- Intuitive Discussion of Performance of Quicksort
- Version of Quicksort that Uses Random Sampling
- Analysis of Randomized Quicksort

Example of Partitioning