Simplification (Minimization) of Logical Functions

A logic function has many algebraic expressions. (See canonical forms and simplified expressions)

The purpose of simplification is to choose the most appropriate (least cost) expression from the set of all possible expressions according to a cost criteria.

The cost criteria may change and depend on the application.

For example, the design criteria may require the expression to have minimum number of products (or sums), minimum number of variables in each product, using the same type of gates (such as NAND), using the gates that can be found in stock.

Simplification Related Definitions

Prime Implicant:

Reminder: Each minterm (product) of 1st canonical form corresponds to a single "true" point.

It is possible to combine some of the minterms to products that contain less variables and cover multiple "true" points.

Products that cannot be simplified any further and covers the maximum number of true points (of the function) are called prime implicants.

A	B	C	F
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	1

$F(A, B, C)= \Sigma m(1,3,5,6,7)$: 1st canonical form

$= A'B'C + A'BC + AB'C + ABC' + ABC$

These products are not prime implicants as they can be simplified into products (that have less variables), which are still covered by the function.

This function was simplified previously and the following expression was obtained for the function:

$F= AB + C$

While the minterms in the canonical form covers only a single "true" point, the AB product covers 2 "true" points and C covers 4 "true" points.

Therefore, a prime implicant is a product that cannot be simplified and covers the maximum number of true points of the function.

- For the given function above, ABC' is not a prime implicant as it can be simplified as AB, which is covered by the function.
- AB is a prime implicant as it cannot be simplified as A and B, which are not covered by the function.

Simplification process of a Boolean function has two steps:

1. Finding the complete **set of all prime implicants.**
2. Selection of the "most appropriate" subset of the prime implicants that covers all the true points of the function.

Finding Prime Implicants:

Boolean algebra can be used to combine minterms to obtain products that have less variables and cover more true points.

It is hard to perform these simplifications manually especially for a large scale function (with many variables). Therefore, computer software can be used.

A practical way (without using the logical expression of the function) is:

- Investigating the true points (output = 1) in the truth table,
- Combinations with one or more constant variables (inputs) are combined. Constant variables are retained and the rest (variables with changing values) are removed.

For example:

Algebraic combining: $F = A'B'+AB' = (A'+A)B' = B'$

These combinations are adjacent. Hamming distance =1

A	B	F
0	0	1
0	1	0
1	0	1
1	1	0

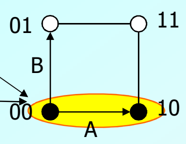
B is constant. For both lines B=0.
Hence B will be retained in the new product.

Value of A is changing.
Hence it will be removed from the new product.

As B=0 the new product will be B' .

Visualization of the process in the Boole cube

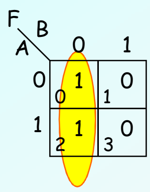
A	B	F
0	0	1
0	1	0
1	0	1
1	1	0



Two points (with dimension 0) is combined to obtain a line (with dimension 1).

This line represents B=0 (B is constant zero and A changes) which means the complement of B namely (B').

Visualization of the process in the Karnaugh diagram



This kind of groupings are performed easier on Karnaugh diagrams.

Neighboring points can be grouped using adjacency property.

In the grouped column above B=0 is fixed and A is changing.

This column represents the complement of B namely (B').

- If more than one variable are fixed, their product is the result of their grouping.

For example:

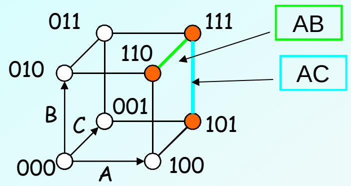
A	B	C	F
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	0
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	1

A=1 , C=1 are constant. B is changing. AC product is formed as the result of this grouping.

Algebraically: AB'C + ABC = AC(B'+B) = AC

A=1 , B=1 are constant. C is changing. AB product is formed as the result of this grouping.

Algebraically: ABC' + ABC = AB(C'+C) = AB



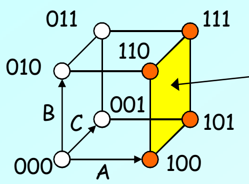
F: A \ BC	00	01	11	10
0	0	0	0	0
1	0	1	1	1

AC

AB

- More than 2 points can also be combined to establish new groups.

For example: F(A,B,C) = Σ(4,5,6,7)



A=1 is constant. B and C are changed.

This face of the cube (plane) is representing A.

Algebraically: AB'C' + AB'C + ABC' + ABC
= AB'+AB
= A

Using Karnaugh diagram:

F: A \ BC	00	01	11	10
0	0	0	0	0
1	1	1	1	1

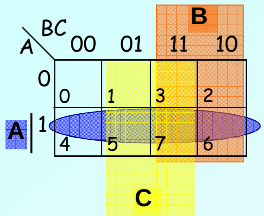
A=1 is constant. B and C are changing.

Finding Prime Implicants Using Karnaugh Maps (Diagrams):

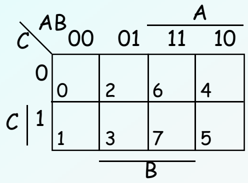
In Karnaugh maps, only a single variable changes between two neighboring cells, the rest remain constant.

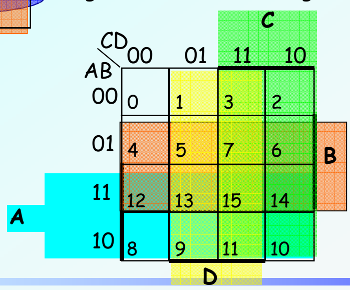
True points (1s) in the neighboring squares (cells) can be clustered into groups that contain 2, 4, 8 … cells.

Below, areas where the variable values stay constant are shown on the Karnaugh maps for functions with 3 and 4 variables.



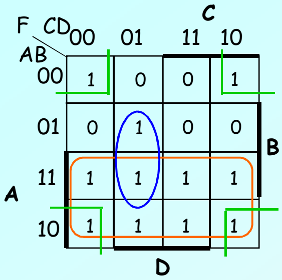
Same diagram can be formed in different ways by changing the order of variables (compare the diagrams on the left & right).





Example: Find the prime implicants of the following function.

$F(A,B,C,D) = \Sigma_1(0,2,5,8,9,10,11,12,13,14,15)$



Prime implicants: A , B'D' , BC'D

Method of finding prime implicants (grouping 1s):

- The groups must be rectangular and must have an area that is a power of two (i.e. 1, 2, 4, 8…).
- "True" points (1s) are placed into the biggest groups possible.
- Two points within bigger group(s) cannot be combined into a smaller subgroup. For example, 2 points within different groups of 4 points cannot be combined to form another 2-point group. It is possible to form a new 4-point group.
- However, if one of the points is not contained by any groups (such as 0101 above), that point can be grouped using another point which is already in another group.

Finding the Set of All Prime Implicants:

In the logic circuit design, the simplification process starts with finding all prime implicants.

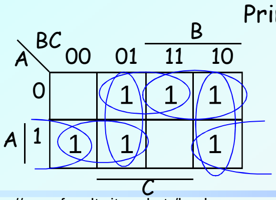
The set that contains all prime implicants is called «set of all prime implicants».

In the second step of the simplification, most appropriate prime implicants are selected from the set of all prime implicants.

The set of prime implicants that covers all true points of a function is called sufficient base. If any of the prime implicants in the sufficient base is removed, some of the true points of the function will not be covered.

Therefore, simplification of a function is the selection of the most appropriate (with minimum cost) sufficient base.

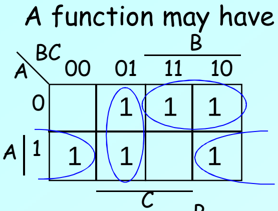
For example: Find the set of all prime implicants of the function.



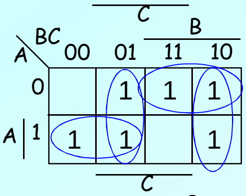
Prime implicants:

BC' , A'B , A'C , AB' , B'C , AC'

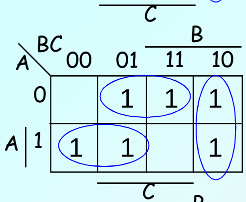
A function may have many sufficient bases.



F(A,B,C)= A'B + B'C + AC'

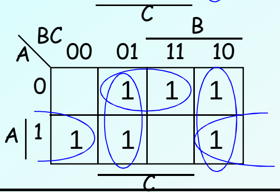


F(A,B,C)= A'B + BC' + B'C + AB'



F(A,B,C)= BC' + A'C + AB'

If any of the prime implicants is removed from the sufficient base, some of the true points will not be covered.



F(A,B,C)= BC' + A'C + B'C + AC'

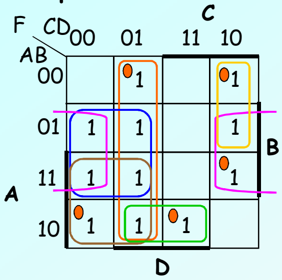
Distinguished Point and Essential prime implicant:

Sometimes some of the true points of a function is covered only with a single prime implicant. These are called **distinguished point**.

The prime implicant that covers a distinguished point is called essential prime implicant.

Inclusion of essential prime implicants into the sufficient base is necessary. Otherwise, it is not possible to cover all true points of a function.

Example:



Set of all prime implicants:

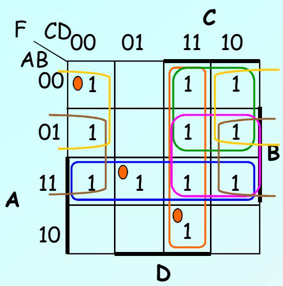
C'D , BC' , AC' , BD' ,A'CD' , AB'D

Distinguished points	Essential prime implicants
0001	C'D
0010	A'CD'
1000	AC'
1110	BD'
1011	AB'D

Here essential prime implicants cover all true points of the function.
This is a specific case.

F= C'D + A'CD' + AC' + BD' + AB'D

Example: Finding the set of all prime implicants, distinguished points and essential prime implicants.



Set of all prime implicants:

CD , AB , A'C , BC , A'D' , BD'

Distinguished points	Essential prime implicants
0000	A'D'
1101	AB
1011	CD

Simplification: Selection of the Most Appropriate Prime Implicants

Reminder: Simplification process has two steps:

1. Finding the set of all prime implicants
2. Selection of a subset of prime implicants with minimum cost that covers the function.

Prime implicant charts are used to select the sufficient base with the minimum cost.

Prime Implicant Chart:

- Simple symbols are assigned to each prime implicant such as A, B, C, ...
- Using a given cost criteria, the cost of each prime implicant is calculated.

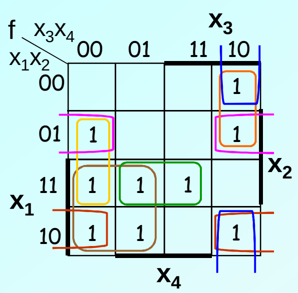
Prime implicant chart is similar to a matrix:

- The symbols of prime implicants are placed to the rows of the chart.
- Number corresponding to the true points of the function are placed to the columns of the chart.
- The costs of the particular prime implicants are placed in the last column.
- If a prime implicant covers a true point their intersection is marked with an 'X'.

Example: Find the set of all prime implicants and form the prime implicant chart for the following function.

$f(x_1, x_2, x_3, x_4)=\Sigma m(2, 4, 6, 8, 9, 10, 12, 13, 15)$

The cost criteria is: 2 units for each variable and 1 unit for each complement sign.



Set of all prime implicants:

	$x_1 x_3'$	$x_2 x_3' x_4'$	$x_1' x_2 x_4'$	$x_1 x_2 x_4$	$x_1' x_3 x_4'$	$x_2' x_3 x_4'$	$x_1 x_2' x_4'$
Symbols:	A	B	C	D	E	F	G
Cost:	5	8	8	6	8	8	8
Covered points:	8,9,12,13	4,12	4, 6	13, 15	2, 6	2, 10	8, 10

Example (cont'd):

Set of all prime implicants:

	$x_1 x_3'$	$x_2 x_3' x_4'$	$x_1' x_2 x_4'$	$x_1 x_2 x_4$	$x_1' x_3 x_4'$	$x_2' x_3 x_4'$	$x_1 x_2' x_4'$
Symbols:	A	B	C	D	E	F	G
Costs:	5	8	8	6	8	8	8
Covered points:	8,9,12,13	4,12	4, 6	13, 15	2, 6	2, 10	8, 10

True points of the function

Prime implicants

	2	4	6	8	9	10	12	13	15	Cost
A				X	X		X	X		5
B		X					X			8
C		X	X							8
D								X	X	6
E	X		X							8
F	X					X				8
G				X		X				8

Simplification of Prime Implicant Chart:

1. Distinguished points are determined. If there is a single X in a column, that is a distinguished point.
   The prime implicant that covers the distinguished point (essential prime implicant) is necessarily selected.
   The row of this essential prime implicant and columns that are covered by this implicant are removed from the chart.
2. If there is an X in the i^{th} row for each X in the j^{th} row, then row i covers row j. In other words, all points covered by row j are also covered by row i.

i	X		X	4
~~j~~			~~X~~	~~5~~

   If row i covers row j AND the cost at row i is smaller or equal to the cost at row j, then row j (covered row) is removed from the chart.
3. If a column covers another column, the covering column (with more X) is removed from the chart.

i	X	~~X~~
j	X	~~X~~
k		~~X~~

These rules are applied successively until all true points are covered with the least cost.

Example: Simplification of prime implicant chart of the following function

$f(x_1, x_2, x_3, x_4)=\Sigma m(2, 4, 6, 8, 9, 10, 12, 13, 15)$

True points of the function

		2	4	6	8	9	10	12	13	15	Cost
√ $x_1 x_3'$	A				X	X		X	X		5
$x_2 x_3' x_4'$	B		X					X			8
$x_1' x_2 x_4'$	C		X	X							8
√ $x_1 x_2 x_4$	D								X	X	6
$x_1' x_3 x_4'$	E	X		X							8
$x_2' x_3 x_4'$	F	X					X				8
$x_1 x_2' x_4'$	G				X		X				8

1. step: In this chart 9 and 15 are the distinguished points.

As A and D are essential prime implicants, their rows and the columns that they cover are removed from the chart.

These products are marked to show their inclusion into the final set.

Digital Circuits

	2	4	6	10	Cost
~~B~~		~~x~~			~~8~~
C		x	x		8
E	x		x		8
F	x			x	8
~~G~~				~~x~~	~~8~~

2. step: In this chart, C covers B. As the cost of C is equal to B, B (as the covered row) is removed from the chart.

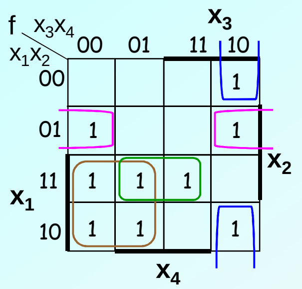
Similarly, F covers G and they have the same cost. So the row of G is removed from the chart. These products (B and G) will not be in the final set.

		2	4	6	10	Cost
√	C		x	x		8
	E	x		x		8
√	F	x			x	8

3. step: In this chart 4 and 10 are distinguished points. Therefore, C and F are selected (and marked). With this selection all true points of the function are covered.

Result: Marked prime implicants for the expression of the function with the least cost.
Selected prime implicants: A + D + C + F
Total cost= 5 + 6 + 8 + 8 = 27
$f(x_1, x_2, x_3, x_4) = x_1 x_3' + x_1 x_2 x_4 + x_1'x_2 x_4' + x_2' x_3 x_4'$

It is possible to see selected prime implicants using the Karnaugh map.



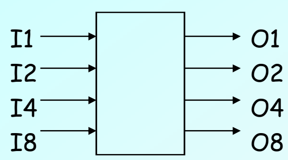
Selected prime implicants should cover all true points and there should be no redundancy.
Selected prime implicants should form a sufficient base. Therefore, removal of any implicant should result in uncovered true point(s).

$x_1 x_3'$ $x_1'x_2 x_4'$ $x_1 x_2 x_4$ $x_2' x_3 x_4'$

Simplification of Incomplete Functions

Reminder: In incomplete functions, the function result is undetermined (we do not care about it) for some input combinations.
These combinations may never appear in the circuit or they are prohibited by the designer.

Example: BCD incrementer circuit



I8	I4	I2	I1	O8	O4	O2	O1
0	0	0	0	0	0	0	1
0	0	0	1	0	0	1	0
0	0	1	0	0	0	1	1
0	0	1	1	0	1	0	0
0	1	0	0	0	1	0	1
0	1	0	1	0	1	1	0
0	1	1	0	0	1	1	1
0	1	1	1	1	0	0	0
1	0	0	0	1	0	0	1
1	0	0	1	0	0	0	0
1	0	1	0	X	X	X	X
1	0	1	1	X	X	X	X
1	1	0	0	X	X	X	X
1	1	0	1	X	X	X	X
1	1	1	0	X	X	X	X
1	1	1	1	X	X	X	X

For these input combinations, the output of the circuit (function) is undetermined.
The symbols X or Φ are used to show the undetermined (don't care) outputs.

Selection of Undetermined Values (Φ):

Undetermined values (Φ) can be chosen to be 0 or 1 in order to utilize the least costly expression in the simplification process.

- While searching for the set of all prime implicants, undetermined values are taken as 1 in order to have (larger groups in the Karnaugh diagram and) simpler products.
- While forming the prime implicant chart, undetermined values are taken as 0 as there is no need to cover these points.

Example: Implement the following incomplete function with the least possible cost.

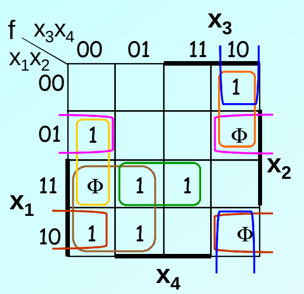
$f(x_1, x_2, x_3, x_4)=\Sigma_m(2, 4, 8, 9, 13, 15) + \Sigma_\Phi(6,10,12)$

Remark: can also be written as

$f(x_1, x_2, x_3, x_4)=\cup_1(2, 4, 8, 9, 13, 15) + \cup_\Phi(6,10,12)$

Cost criteria: 2 units for each variable and 1 unit for each complement.

Finding the prime implicants:



Φ's are assumed to be 1, when we are finding the set of all prime implicants.

Set of all prime implicants:

	$x_1 x_3'$	$x_2 x_3' x_4'$	$x_1' x_2 x_4'$	$x_1 x_2 x_4$	$x_1' x_3 x_4'$	$x_2' x_3 x_4'$	$x_1 x_2' x_4'$
Symbols:	A	B	C	D	E	F	G
Costs:	5	8	8	6	8	8	8
Points covered:	8,9,13	4	4	13,15	2	2	8

Forming the prime implicant chart:

Set of all prime implicants:

	$x_1 x_3'$	$x_2 x_3' x_4'$	$x_1'x_2 x_4'$	$x_1 x_2 x_4$	$x_1'x_3 x_4'$	$x_2' x_3 x_4'$	$x_1 x_2' x_4'$
Symbols:	A	B	C	D	E	F	G
Costs:	5	8	8	6	8	8	8
Points covered:	8,9,13	4	4	13,15	2	2	8

True points of the function

Prime implicants

	2	4	8	9	13	15	Cost
A			X	X	X		5
B		X					8
C		X					8
D					X	X	6
E	X						8
F	X						8
G			X				8

Φ's are assumed to be 0, when we are forming the prime implicant chart

As there is no need to cover the points with undetermined values, these points are not placed into the prime implicant chart.

True points of the function

Prime implicants

		2	4	8	9	13	15	Cost
√	A			X	X	X		5
	B		X					8
	C		X					8
√	D					X	X	6
	E	X						8
	F	X						8
	G			X				8

Step 1: In this chart, points 9 and 15 are distinguished points.
As A and D are the essential prime implicants, they are selected. The rows and columns covered by A and D are removed.

A and D are marked to show that they will be in the final set of prime implicants.

	2	4	Cost
B		x	8
C		x	8
E	x		8
F	x		8

Step 2: B and C are covering the same points and they have the same cost. Therefore, it is not possible to make a choice between B and C. One of B and C can be selected.

Same situation exists for prime implicants E and F.

At the end, the same function can be implemented using any of the following expressions which have the same (lowest) cost.

$f = A + D + B + E = x_1 x_3' + x_1 x_2 x_4 + x_2 x_3' x_4' + x_1' x_3 x_4'$

$f = A + D + B + F = x_1 x_3' + x_1 x_2 x_4 + x_2 x_3' x_4' + x_2' x_3 x_4'$

$f = A + D + C + E = x_1 x_3' + x_1 x_2 x_4 + x_1' x_2 x_4' + x_1' x_3 x_4'$

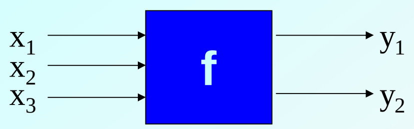
$f = A + D + C + F = x_1 x_3' + x_1 x_2 x_4 + x_1' x_2 x_4' + x_2' x_3 x_4'$

All designs have the same cost (27).

Simplification of General Functions

Remark: General functions have more than one input.

x_1 x_2 x_3	y_1 y_2
0 0 0	1 1
0 0 1	1 Φ
0 1 0	0 0
0 1 1	Φ 0
1 0 0	1 Φ
1 0 1	0 1
1 1 0	0 1
1 1 1	Φ 0



$y_1 = f_1(x_1, x_2, x_3)$

$y_2 = f_2(x_1, x_2, x_3)$

During the simplification of general functions, set of prime implicants for each output is found independently, and prime implicants are selected from these sets.

An important point is to select the common prime implicants of both outputs.

Simplification of general functions is not in the scope of this course.

Finding the Set of All Prime Implicants Using Table (Quine-McCluskey) Method

It is hard to use Karnaugh maps for the function with too many variables as it becomes harder to visualize.

Especially for the functions with 5 or more variables, it is hard to draw and visualize the adjacency of the points.

In table method (Quine-McCluskey), systematic processes are performed successively. Performing these processes manually may be time consuming. However, it is possible to implement this method as a computer software.

Table (Quine-McCluskey) **Method:**

In order to find the set of all prime implicants, true points of the function (minterms) are merged (grouped). Adjacent minterms where a single variable changes are taken into the same group. (See the figure at 4.3)

In the table method, minterms (corresponding to true points) are compared to all other minterms.

If a single variable (input) changes between two minterms, they are merged.

The variable that changes the value is removed, and a new term is obtained.

This process is repeated until no further groups can be formed.

Terms that cannot be grouped are the prime implicants.

Method:

- To ease the grouping; cluster the terms depending on the number of 1s.
- Make comparison between terms that are in the adjacent clusters. Group the combinations where a single variable changes value.
- Variable with changing value will be removed.
- Mark the combinations that are grouped.
- Repeat the grouping on the newly formed combinations until no further groups can be formed.
- Combinations that are not grouped (items that are not signed) form the set of all prime implicants.

Quine-McCluskey method will only let you find the set of all prime implicants. Therefore, you still need to use the prime implicant chart to find a sufficient base with the least cost.

Willard Van Orman Quine (1908-2000), Philosophy, logic
Edward J. McCluskey(1929-) Electric engineer.

Example:

Find the set of all prime implicants of the following function using Quine-McCluskey method.

$f(x_1, x_2, x_3, x_4)= \Sigma_m(0, 1, 2, 8, 10, 11, 14, 15)$

Num.	x_1 x_2 x_3 x_4	
0	0 0 0 0	√
1	0 0 0 1	√
2	0 0 1 0	√
8	1 0 0 0	√
10	1 0 1 0	√
11	1 0 1 1	√
14	1 1 1 0	√
15	1 1 1 1	√

Num.	x_1 x_2 x_3 x_4	
0,1	0 0 0 -	
0,2	0 0 - 0	√
0,8	- 0 0 0	√
2,10	- 0 1 0	√
8,10	1 0 - 0	√
10,11	1 0 1 -	√
10,14	1 - 1 0	√
11,15	1 - 1 1	√
14,15	1 1 1 -	√

Num.	x_1 x_2 x_3 x_4
0,2,8,10	- 0 - 0
0,8,2,10	- 0 - 0
10,11,14,15	1 - 1 -
10,11,14,15	1 - 1 -

No need to rewrite the same items

Set of all prime implicants (Not marked): $x_1' x_2' x_3'$, $x_2' x_4'$, $x_1 x_3$

To find the lowest cost solution (sufficient base), prime implicant chart is used.