Task 2

(1)

zz: $\|a+b\|^2 = \|a\|^2 + 2\langle a,b\rangle + \|b\|^2$ und
$\|a-b\|^2 = \|a\|^2 - 2\langle a,b\rangle + \|b\|^2$

Bew

$$\|a+b\|^2 = \langle a+b, a+b\rangle \overset{\text{Bilinear}}{=} \langle a, a+b\rangle + \langle b, a+b\rangle$$
$$\overset{\text{Bilinear}}{=} \langle a,a\rangle + \langle a,b\rangle + \langle b,a\rangle + \langle b,b\rangle$$
$$\overset{\text{Symmetrie}}{=} \langle a,a\rangle + \langle a,b\rangle + \langle a,b\rangle + \langle b,b\rangle$$
$$= \|a\|^2 + 2\langle a,b\rangle + \|b\|^2$$

$$\|a-b\|^2 = \langle a-b, a-b\rangle \overset{\text{Bilinear}}{=} \langle a, a-b\rangle + \langle -b, a-b\rangle$$
$$\overset{\text{Bilinear}}{=} \langle a,a\rangle + \langle a,-b\rangle + \langle -b,a\rangle + \langle -b,-b\rangle$$
$$\overset{\text{Symmetrie}}{=} \langle a,a\rangle - \langle a,b\rangle - \langle a,b\rangle + \langle b,b\rangle$$
$$= \|a\|^2 - 2\langle a,b\rangle + \|b\|^2$$

$\square$

(2)

zz: $\|c\|^2 = \|a\|^2 + \|b\|^2$ für a, b orthogonal

Bew

$$\|c\|^2 \overset{(c=a-b)}{=} \|a-b\|^2 \overset{(*)}{=} \|a\|^2 - 2\langle a,b\rangle + \|b\|^2$$
$$= \|a\|^2 - \underbrace{2\cdot 0}_{} + \|b\|^2 = \|a\|^2 + \|b\|^2$$

$\square$

da a, b orthogonal
$\Rightarrow \langle a,b\rangle = 0$

(3)

zz: $b = -\frac{\langle w,x\rangle + \langle w,y\rangle}{2}$ für w, b erfüllt (*)

<u>Bew</u>

$$\langle w,x\rangle + b + \langle w,y\rangle + b = +1 - 1$$

$$\Leftrightarrow \quad \langle w,x\rangle + \langle w,y\rangle + 2b = 0$$

$$\Leftrightarrow \quad 2b = -(\langle w,x\rangle + \langle w,y\rangle)$$

$$\Leftrightarrow \quad b = -\frac{\langle w,x\rangle + \langle w,y\rangle}{2} \qquad \square$$

(4)

zz: für $\tilde{w} = \frac{2(x-y)}{\|x-y\|^2}$, $b \equiv b(\tilde{w})$ gilt (*)

<u>Bew</u>

zeige: $\langle \tilde{w}, x \rangle + b = 1$

$$\begin{aligned} \langle \tilde{w}, x \rangle + b &\overset{(3)}{=} \langle \tilde{w}, x \rangle - \tfrac{1}{2}(\langle \tilde{w}, x \rangle + \langle \tilde{w}, y \rangle) \\ &= \tfrac{1}{2}(2\langle \tilde{w}, x \rangle - \langle \tilde{w}, x \rangle - \langle \tilde{w}, y \rangle) \\ &= \tfrac{1}{2}(\langle \tilde{w}, x \rangle - \langle \tilde{w}, y \rangle) \\ &= \tfrac{1}{2}\left\langle \frac{2(x-y)}{\|x-y\|^2}, x-y \right\rangle \\ &= \frac{1}{\|x-y\|^2}\langle x-y, x-y \rangle \\ &= \frac{\|x-y\|^2}{\|x-y\|^2} = 1 \end{aligned}$$

zeige: $\langle \tilde{w}, y \rangle + b = -1$

$$\begin{aligned} \langle \tilde{w}, y \rangle + b &\overset{(3)}{=} \langle \tilde{w}, y \rangle - \tfrac{1}{2}(\langle \tilde{w}, x \rangle + \langle \tilde{w}, y \rangle) \\ &= \tfrac{1}{2}(2\langle \tilde{w}, y \rangle - \langle \tilde{w}, x \rangle - \langle \tilde{w}, y \rangle) \\ &= \tfrac{1}{2}\langle \tilde{w}, y-x \rangle \\ &= -\tfrac{1}{2}\langle \tilde{w}, x-y \rangle \\ &= -\tfrac{1}{2}\left\langle \frac{2(x-y)}{\|x-y\|^2}, x-y \right\rangle \\ &= -\frac{\|x-y\|^2}{\|x-y\|^2} = -1 \end{aligned}$$

$\square$

(5)

zz: $\langle w, x-y\rangle = 2$ für w, b erfüllt (*)

<u>Bew</u>

$$\langle w,x\rangle + b - (\langle w,y\rangle + b) = +1 - (-1)$$

$$\Leftrightarrow \langle w,x\rangle + b - \langle w,y\rangle - b = 2$$

$$\Leftrightarrow \langle w,x\rangle - \langle w,y\rangle = 2$$

$$\Leftrightarrow \langle w, x-y\rangle = 2 \quad \square$$

(6)

zz: $\|w\|^2 = \|\hat{w}\|^2 + \|w-\hat{w}\|^2$ wenn $\langle w, x-y\rangle = 2$

<u>Bew</u>

zeige: $\langle \hat{w}, (w-\hat{w})\rangle = 0$

$$\langle \hat{w}, (w-\hat{w})\rangle = \|\hat{w}\|^2 - \langle w, \hat{w}\rangle = \|\hat{w}\|^2 - \langle w, \frac{2(x-y)}{\|x-y\|^2}\rangle$$

$$= \|\hat{w}\|^2 - \frac{2}{\|x-y\|^2}\langle w, (x-y)\rangle = \|\hat{w}\|^2 - \frac{2}{\|x-y\|^2}\cdot 2$$

$$= \left\|\frac{2(x-y)}{\|x-y\|^2}\right\|^2 - \frac{4}{\|x-y\|^2} = \frac{4\|x-y\|^2}{\|x-y\|^4} - \frac{4}{\|x-y\|^2}$$

$$= \frac{4}{\|x-y\|^2} - \frac{4}{\|x-y\|^2} = 0 \Rightarrow \hat{w} \text{ und } (w-\hat{w}) \text{ orthogonal}$$

Dann ist

$$\|\hat{w}\|^2 + \|w-\hat{w}\|^2 = \|\hat{w}\|^2 + \|w\|^2 - 2\langle w, \hat{w}\rangle + \|\hat{w}\|^2$$

$$\ulcorner \langle \hat{w}, w-\hat{w}\rangle = 0 \Leftrightarrow \langle \hat{w}, w\rangle - \|\hat{w}\|^2 = 0 \Leftrightarrow \langle \hat{w}, w\rangle = \|\hat{w}\|^2 \lrcorner$$

$$= \|\hat{w}\|^2 + \|w\|^2 - 2\|\hat{w}\|^2 + \|\hat{w}\|^2 = \|w\|^2 + 2\|\hat{w}\|^2 - 2\|\hat{w}\|^2 = \|w\|^2$$

$\square$

(7)

zz: $\|w\| > \|\hat{w}\|$ für $w \neq \hat{w}$ und w erfüllt (*)

<u>Bew</u>

$$\begin{aligned} & \|w\| > \|\hat{w}\| \\ \iff\ & \|w\|^2 > \|\hat{w}\|^2 \\ \iff\ & \|\hat{w}\|^2 + \|w-\hat{w}\|^2 > \|\hat{w}\|^2 \quad (6) \\ \iff\ & \underbrace{\|w-\hat{w}\|^2}_{\text{da } w \neq \hat{w} \text{ ist es größer } 0} > 0 \quad \square \end{aligned}$$

(8)

Es ist

$$\hat{w} = \frac{2(x-y)}{\|x-y\|^2} = 2\,\frac{\binom{5}{5}-\binom{2}{1}}{\left\|\binom{5}{5}-\binom{2}{1}\right\|^2} = 2\,\frac{\binom{3}{4}}{\left\|\binom{3}{4}\right\|^2}$$

$$= \frac{2}{9+16}\binom{3}{4} = \frac{2}{25}\binom{3}{4} = \begin{pmatrix} 6/25 \\ 8/25 \end{pmatrix} \overset{\text{Skaliert}}{\Longrightarrow} 25 \cdot \hat{w} = \binom{6}{8}$$

$$b = -\frac{\langle \hat{w}, x\rangle + \langle \hat{w}, y\rangle}{2} = -\frac{1}{2}\left(\frac{6}{25}\cdot 5 + \frac{8}{25}\cdot 5 + \frac{6}{25}\cdot 2 + \frac{8}{25}\cdot 1\right)$$

$$= -\frac{1}{50}(30+40+12+8) = -\frac{90}{50} = -\frac{9}{5}$$