

# Supplementary: Embedded Geometry Details

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## 0.1 Dual Sources of Overdetermination

Overdetermination arises from two independent constraint systems, making curvature bounds robust.

### 0.1.1 Source I: Embedding Compatibility

Gauss-Codazzi-Ricci equations constitute a system of partial differential equations relating intrinsic and extrinsic geometry. For  $(n, k) = (2, 3)$ , 13 constraints on 12 degrees of freedom yields overdetermination by 1.

#### **Constraint breakdown:**

- Gauss equations: 1 constraint
- Codazzi equations: 6 constraints
- Ricci equations: 6 constraints
- Total: 13 constraints

#### **Degrees of freedom:**

- Intrinsic metric  $h_{ab}$ : 3 components
- Extrinsic curvature  $K_{ab}^{(\alpha)}$  for 3 normal directions:  $3 \times 3 = 9$  components
- Total: 12 degrees of freedom

### 0.1.2 Source II: Normal Bundle Structure

Requiring  $k$  normal directions be orthonormal imposes algebraic constraints on extrinsic curvature before embedding equations are considered.

#### **Component count:**

- Extrinsic curvature has  $k \cdot n(n+1)/2$  components
- For  $(n, k) = (2, 3)$ :  $3 \times 3 = 9$  components

**Constraint count:**

- Orthonormality  $n^{(\alpha)} \cdot n^{(\beta)} = \delta^{\alpha\beta}$ :  $k(k+1)/2 = 6$  constraints
- Perpendicularity  $n^{(\alpha)} \cdot e_a = 0$ :  $nk = 6$  constraints
- Total: 12 constraints on 9 components

## 0.2 Robustness

Both constraint systems must be satisfied simultaneously. Embedding compatibility restricts which metrics can be embedded; orthonormality restricts which extrinsic curvatures can realize those embeddings. Together they force curvature bounds from two independent directions, making  $K_G \geq K_{\min}^2$  robust against perturbations of either constraint system alone.

## 0.3 Geometric Intuition

Consider constraints as partial differential equations for embedding  $X$ . When underdetermined ( $k \leq n^2 - n - 1$ ), more unknowns than equations leave freedom to choose curvature arbitrarily. When overdetermined ( $k > n^2 - n - 1$ ), more equations than unknowns require strong relationships between derivatives.

Bianchi identities ensure higher derivatives of curvature satisfy consistency conditions. In an overdetermined system these conditions become bounds; to prevent contradictions, all curvature derivatives must remain within ranges determined by lowest-order curvature.

### 0.3.1 Nash's Theorem Comparison

Nash's theorem guarantees any Riemannian manifold admits isometric embedding into sufficiently high-dimensional Euclidean space, but places no restrictions on normal bundle structure. Requiring  $k$  orthonormal normal directions is a constraint not addressed by Nash's theorem. This additional structure generates physical laws.

Key distinction:

- Nash: existence of embedding guaranteed

- This work: embedding with structured normal bundle forces curvature bounds