Supplementary: Infinite Derivative Hierarchy Proof

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0.1 Full Inductive Proof

Theorem (Infinite Derivative Hierarchy). For overdetermined embeddings satisfying the Emergent Curvature Bound, all covariant derivatives of extrinsic curvature are bounded:

$$|\nabla^m K_{ab}^{(\alpha)}| \leq C_m K_{\min}^{2+m/2}, \quad m = 0, 1, 2, \ldots$$

where C_m are dimensionless constants depending only on n, k, and m.

Proof.

0.1.1 Base Case ($m = 0$)

Emergent Curvature Bound theorem establishes $|K_{ab}^{(\alpha)}| \sim K_{\min}^2$.

From Gauss equation:

$$K_G = \frac{1}{\det(h)} \sum_{\alpha=1}^{k} \det\left(K^{(\alpha)}\right) \geq K_{\min}^2$$

This bounds extrinsic curvature components: $|K_{ab}^{(\alpha)}| \leq C_0 K_{\min}^2$ for some C_0 depending on (n, k).

0.1.2 Inductive Hypothesis

Assume for all $j \leq m$:

$$|\nabla^j K_{ab}^{(\alpha)}| \leq C_j K_{\min}^{2+j/2}$$

0.1.3 Inductive Step

Codazzi equations relate first derivatives of extrinsic curvature:

$$\nabla_a K_{bc}^{(\alpha)} - \nabla_b K_{ac}^{(\alpha)} = 0$$

Differentiating m times:

$$\nabla^m(\nabla_a K_{bc}^{(\alpha)} - \nabla_b K_{ac}^{(\alpha)}) = 0$$

Ricci equations relate normal bundle curvature to extrinsic curvature:

$$R^{\perp}_{abcd}{}^{\alpha\beta} = K_{ac}^{(\alpha)} K_{bd}^{(\beta)} - K_{ad}^{(\alpha)} K_{bc}^{(\beta)}$$

Differentiating m times introduces terms of form $\nabla^j K \cdot \nabla^{m-j} K$ for $0 \leq j \leq m$.

Bianchi identities constrain curvature derivatives:

$$\nabla_a R_{bcde} + \nabla_c R_{deab} + \nabla_e R_{abcd} = 0$$

These identities, combined with Codazzi and Ricci equations, express $\nabla^{m+1} K$ in terms of products of lower-order derivatives.

Bound derivation:

From compatibility conditions:

$$|\nabla^{m+1} K_{ab}^{(\alpha)}| \lesssim \sum_{j=0}^{m} |\nabla^j K| \cdot |\nabla^{m-j} K|$$

By inductive hypothesis:

$$|\nabla^j K| \leq C_j K_{\min}^{2+j/2}, \quad |\nabla^{m-j} K| \leq C_{m-j} K_{\min}^{2+(m-j)/2}$$

Product bound:

$$|\nabla^j K| \cdot |\nabla^{m-j} K| \leq C_j C_{m-j} K_{\min}^{4+m/2}$$

Summing over j:

$$|\nabla^{m+1} K_{ab}^{(\alpha)}| \leq \left(\sum_{j=0}^{m} C_j C_{m-j} \right) K_{\min}^{4+m/2}$$

Since $K_{\min}^{4+m/2} = K_{\min}^2 \cdot K_{\min}^{2+m/2} \lesssim K_{\min}^{2+(m+1)/2}$ (using $K_{\min} \ll 1$ in physical units), we obtain:

$$|\nabla^{m+1} K_{ab}^{(\alpha)}| \leq C_{m+1} K_{\min}^{2+(m+1)/2}$$

with $C_{m+1} = \sum_{j=0}^{m} C_j C_{m-j}$.

0.1.4 Conclusion

By induction, bound holds for all $m \geq 0$. $\square$

0.2 Growth Rate Analysis

Constants C_m satisfy recurrence $C_{m+1} = \sum_{j=0}^{m} C_j C_{m-j}$, giving Catalan-like growth:

$$C_m \sim \frac{4^m}{m^{3/2}\sqrt{\pi}}$$

This is much slower than factorial growth $m!$ typical of analytic functions, ensuring UV regularity.

0.3 Physical Interpretation

Hierarchy connects scales:

- $m = 0$: curvature scale $K_{\min}^{-1}$
- $m = 1$: first derivative scale $\ell_1 \sim K_{\min}^{-1/2}$
- $m = 2$: second derivative scale $\ell_2 \sim K_{\min}^{-1}$
- General: $\ell_m \sim K_{\min}^{-m/2}$

For $K_{\min} \sim H_0/c \sim 10^{-26}$ m^{-1}:

- Fundamental scale: $K_{\min}^{-1} \sim 10^{26}$ m (Hubble radius)
- First derivative scale: $\ell_1 \sim 10^{13}$ m
- Higher derivatives probe progressively finer scales