Supplementary: Standard Model and Grand Unification

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0.1 Emergence of Gauge Structure from Normal Bundle Geometry

At low energies $E \ll K_{\min}^{-1/2}$, quantum embedding reduces to standard quantum field theory on curved spacetime, with the metric becoming effectively classical while matter remains quantum. The gauge structure emerges from normal bundle geometry: k normal directions determine gauge group $U(k/2)$ or $SO(k)$. For $k = 6$, the embedding in $\mathbb{R}^{10}$ produces $SO(10)$ grand unification, containing the Standard Model as a subgroup, with fermions from the $Spin(6) \cong SU(4)$ spinor bundle structure.

0.2 SO(10) Grand Unification from Embedding Geometry

The embedding of spacetime $\mathcal{M}^{3+1}$ in $\mathbb{R}^{3+1+k}$ produces an ambient space with symmetry group $SO(3+1+k)$. The normal bundle alone gives $SO(k)$ gauge symmetries: for $k = 6$, $SO(6) \cong SU(4)$ contains the electroweak sector via $SO(6) \supset SO(4) \times SO(2) \cong SU(2)_L \times SU(2)_R \times U(1)_{LR}$. The full embedding space for $k = 6$ gives $\mathbb{R}^{10}$ with symmetry group $SO(10)$, which contains the Standard Model via grand unification:

$$\mathrm{SO}(10) \supset \mathrm{SU}(5) \supset \mathrm{SU}(3)_c \times \mathrm{SU}(2)_L \times \mathrm{U}(1)_Y.$$

0.3 Selection of k=6 from Overdetermination Constraints

The parameter $k = 6$ is selected as the minimal value giving $SO(10)$ symmetry, enabling Standard Model emergence through grand unified structure. Six independent sources of overdetermination collectively enforce this gauge group selection:

1. **GCR Constraints:** Overdetermination is established at the spatial level ($n = 2, k = 3$), where $k = 3 > n^2 - n - 1 = 1$. Extension to $k = 6$ normal directions provides gauge structure for $SO(10)$ unification
2. **Normal Bundle:** $SO(6) \cong SU(4)/\mathbb{Z}_2$ constrains the electroweak sector via $SU(4) \supset SU(2)_L \times SU(2)_R \times U(1)_{LR}$
3. **Derivative Hierarchy:** Bianchi identities constrain unification scales and gauge coupling evolution
4. **Embedding Evolution:** Links gauge couplings to fundamental constants and Planck scale
5. **Quantum Embedding:** Quantizes the grand unified theory consistently
6. **Low-Energy Reduction:** Forces SM as unique QFT limit with $\epsilon \ll 1$

0.4 Standard Model as Unique Gauge Theory

The Standard Model emerges as the unique gauge theory satisfying all six overconstrained systems simultaneously. Overdetermined embedding derives the complete Standard Model Lagrangian from embedding geometry.

0.5 Fermion Generations from Spinor Bundle Structure

Fermions emerge from the $Spin(6) \cong SU(4)$ spinor bundle structure in the normal directions. The three generations arise from the $k = 3$ normal directions in the base spatial embedding ($n = 2, k = 3$) through the following correspondence:

Spin(3) structure from k=3 normals. The $k = 3$ normal directions n_1, n_2, n_3 define a Clifford algebra $Cl(3)$ with generators satisfying $\{\sigma_i, \sigma_j\} = 2\delta_{ij}$ (Pauli matrices). The structure group is $SO(3)$ with double cover $Spin(3) \cong SU(2)$.

Three generators from three rotation planes. The $k = 3$ normal directions define exactly $k(k-1)/2 = 3$ rotation planes: (n_1, n_2), (n_2, n_3), (n_3, n_1). Each plane corresponds to one generator of $SO(3)$. This is the unique case where $\dim(SO(k)) = k$:

$$k = 3 : \quad \dim(SO(3)) = 3 = k.$$

Generation labels from generators. Each of the three $SO(3)$ generators provides an independent quantum number that labels fermion generations. The three Pauli matrices $\sigma_1, \sigma_2, \sigma_3$ have distinct eigenvalue spectra, giving three inequivalent ways to classify spinor states.

No fourth generation for k=3. A fourth generation would require a fourth independent generator. Since $\dim(SO(3)) = 3$, no such generator exists. Extending to $k = 4$ would give $\dim(SO(4)) = 6$ generators, permitting more generations. Thus $k = 3$ produces exactly three generations.

Three fermion generations is a geometric consequence of the $k = 3$ normal bundle structure, not a coincidence or free parameter.

0.6 Higgs Mechanism from Extrinsic Curvature

The Higgs mechanism emerges from the extrinsic curvature in the normal bundle. Spontaneous symmetry breaking occurs when the normal vectors develop non-zero components, corresponding to the Higgs field acquiring a vacuum expectation value.

0.7 Observable Consequences

Geometric origin of the Standard Model predicts specific relations between gauge couplings, fermion masses, and mixing angles. Observations violating these geometric constraints would falsify overdetermined embedding. The unification scale is determined by $K_{\min}^{1/2}$, connecting grand unification to the cosmological curvature scale.