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PAPER

On the Origin of Conservation

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Abstract

We prove the constrained manifold theorem (CMT). Riemannian manifolds of dimension n embedded in R^{n+k} with $k > n^2 - n - 1$ orthonormal normal directions form overdetermined constraint systems. For 2D manifolds in R^5, 12 constraints act on 9 extrinsic curvature components, yielding -3 degrees of freedom. The Gauss equation combined with orthonormality and metric positivity forces a universal lower bound $K_G \geq K_{\min}^2$ on Gaussian curvature, uniquely determined for all C^∞ embeddings. The bound induces an infinite derivative hierarchy $|\nabla^m K| \leq C_m K_{\min}^{2+m/2}$ that acts as ultraviolet regulator. The derivative hierarchy induces Hamiltonian mechanics, with momentum arising from the $m = 1$ structure and the symplectic form from $m = 2$. The curvature bound produces a Hamiltonian constraint, forces dimensional emergence $\mathcal{M}^2 \to \mathcal{M}^3$ and resolves the 10^{123} vacuum energy discrepancy without fine-tuning. With $K_{\min} \sim H_0/c$, the effective cosmological constant $\Lambda_{eff} = (3/2)K_{\min}^2 = 7.96 \times 10^{-53}$ m^{-2} is within a factor of 1.37 agreement with the observed value $\Lambda_{observed} = 1.09 \times 10^{-52}$ m^{-2}. Given hamiltonian mechanics, time functions as a parameter in timeless configuration space, reproducing ADM formalism. The parameter evolution of this system in the fourth embedding dimension yields spacetime. We derive Einstein's field equations as the Gauss-Codazzi compatibility conditions.

We prove the Evolving Manifold pair of Theorems (EMT). One extends spatial embedding to spacetime with characteristic velocity $c = K_{\min}^{1/2}$ and its inverse pair removes observed constant variations to recover curvature perturbations. Applying EMT, we prove that while exact conservation laws hold in the full embedding space; on the observable manifold they acquire corrections bounded by $|dQ/dt| \leq CK_{\min}^{5/2}$. Noether's theorem emerges in the limit $K_{\min} \to 0$. We show that fundamental constants vary with embedding curvature according to $\Delta\alpha/\alpha = -\frac{1}{2}(\Delta K_{\min}/K_{\min})$ and $\Delta c/c = +\frac{1}{2}(\Delta K_{\min}/K_{\min})$. Quasar absorption spectroscopy confirms fine structure variation at 4.7σ; spatial dipole measurements reach 4.2σ.

We show why the fifth embedding dimension is quantized by topological constraints. Simple connectivity of the ambient space forces holonomy around closed loops to integer multiples of 2π. This quantization produces uncertainty relations and the Schrödinger equation with $\hbar \sim K_{\min}^{-1}$, establishing $\ell_{\min} = 1/\sqrt{K_{\min}}$ as the fundamental length scale. We derive quantum field theory and Standard Model gauge groups, with this structure and recover string theory in the full five-dimensional embedding.

Thirteen falsifiable results follow from CMT and EMT. Three are consistent with current observations: fine structure variation (4.7σ, 4.2σ), cosmological constant (factor 1.37), and CMB geometric alignment. Four are near-term predictions and six are specifically testable by LIGO O5 in 2026. Observation of $w < -1$ increasing with redshift, equivalent to $\Lambda < K_{\min}^2$, would falsify this embedding theory.

1 Introduction

1.1 Geometric Motivation

Consider a Riemannian manifold embedded in higher-dimensional Euclidean space. Embedding determines both induced metric and extrinsic curvature—how the manifold bends within the ambient space. We ask what restrictions arise when the normal bundle carries multiple orthonormal normal vector fields.

Orthonormality and perpendicularity conditions constrain extrinsic curvature. Normal vectors must remain orthogonal to each other and to the tangent space at every point. Above a dimensional threshold, constraints exceed degrees of freedom and the system becomes overdetermined—forcing universal curvature bounds that would otherwise remain unconstrained.

1.2 Constrained Manifold Theorem

We prove that n-dimensional Riemannian manifolds embedded in (n+k)-dimensional Euclidean space with k orthonormal normal directions form overdetermined systems when k exceeds a threshold determined by n. For the canonical case—a two-dimensional surface in five-dimensional Euclidean space with three orthonormal normals—twelve constraints act on nine extrinsic curvature components, yielding negative three degrees of freedom.

Gauss's equation relates intrinsic to extrinsic curvature. Combined with orthonormality and metric positivity, it forces a universal lower bound on Gauss curvature, uniquely determined for all smooth embeddings satisfying these conditions.

1.3 Physical Consequences

Curvature bounds propagate through Bianchi identities, which connect curvature to its derivatives. In an overdetermined system these identities become constraints. Higher derivatives cannot be independent but must satisfy compatibility conditions, producing an infinite hierarchy of bounded derivatives controlled by the curvature scale.

Physical quantities constructed from curvature inherit these bounds, constraining time derivatives and yielding conservation laws—bounded rather than exact. Exact Noether conservation is recovered only as minimum curvature vanishes.

Fundamental constants become constrained ranges. Fine structure constant and speed of light exhibit variations tied to minimum curvature, while cosmological constant is bounded by the curvature scale, consistent with observations.

Dimensional interpretation extends this structure. A fourth embedding dimension provides a temporal parameter, with time emerging as the evolution coordinate of a Hamiltonian system. Evolving Manifold Theorems translate between geometry and temporal observations, and general relativity emerges as the consistency condition for spacetime extension.

A fifth dimension introduces quantum structure, as its geometry cannot support classical dynamics. Topological constraints force quantization, derivative bounds generate uncertainty relations, and normal-direction evolution yields wave mechanics with Planck's constant emerging from the curvature scale.

Appendices develop quantum field theory from normal bundle geometry, Standard Model gauge structure from grand unification, and string theory as overdetermined embeddings in the full five-dimensional structure.

1.4 Paper Structure

Section 2 establishes embedded Riemannian geometry. Section 3 proves the Constrained Manifold Theorem and classifies overdetermined configurations. Section 4 derives the derivative hierarchy. Section 5 establishes the Hamiltonian constraint and dimensional emergence. Section 6 treats the cosmological constant. Section 7 proves the Evolving Manifold Theorems. Section 8 derives bounded conservation laws. Section 9 addresses fundamental constant variations and CMB anisotropies. Section 10 develops quantum structure. Section 11 confronts predictions with data. Section 12 discusses implications.

Proofs, derivations, and extended analyses appear in Supplemental Material [**?**, **?**, **?**, **?**, **?**, **?**, **?**, **?**].

2 Overdetermined Embeddings

What geometric conditions force universal physical laws? Consider a Riemannian manifold $\mathcal{M}^n$ embedded in higher-dimensional Euclidean space R^{n+k} via map $X : \mathcal{M}^n \to R^{n+k}$. This embedding determines intrinsic geometry, the metric on $\mathcal{M}^n$, and extrinsic geometry, how $\mathcal{M}^n$ curves within R^{n+k}. Tangent space $T_p\mathcal{M}^n$ represents observable spacetime directions; normal space $\mathcal{N}_p\mathcal{M}^n$ represents extra dimensions perpendicular to our universe at each point.

Requiring k independent orthonormal normal directions imposes strong constraints. When k exceeds a threshold relative to n, Gauss-Codazzi-Ricci equations overdetermine the system, with constraints exceeding geometric degrees of freedom. Overdetermination forces universal bounds on curvature and all its derivatives.

2.1 Embedded Manifolds

Let $\mathcal{M}^n$ be a C^∞ n-dimensional Riemannian manifold with local coordinates $(\phi^1, \ldots, \phi^n)$, and let $X : \mathcal{M}^n \to R^{n+k}$ be a C^∞ embedding. Tangent vectors follow from partial derivatives:

$$e_a = \frac{\partial X}{\partial \phi^a}, \quad a = 1, \ldots, n. \tag{1}$$

Induced metric is the Euclidean inner product of tangent vectors:

$$h_{ab} = e_a \cdot e_b. \tag{2}$$

Manifold is Riemannian when $\det(h) > 0$.

Metric h_{ab} determines spacetime distances and causal structure. In $(3+1)$-dimensional spacetime, h_{ab} reduces to Minkowski or Robertson-Walker metric depending on cosmological context. Embedding X specifies how this metric is realized as a surface in higher-dimensional space.

2.2 Normal Bundle Structure

Normal space $\mathcal{N}_p\mathcal{M}^n$ at each point $p \in \mathcal{M}^n$ is the orthogonal complement of $T_p\mathcal{M}^n$ in R^{n+k}. Consider k orthonormal normal vector fields $n^{(\alpha)}$ satisfying: $\mathrm{n}^{(\alpha)} \cdot n^{(\beta)} = \delta^{\alpha\beta}$,

$\mathrm{n}^{(\alpha)} \cdot e_a = 0$. These conditions impose $k(k-1)/2 + nk$ constraints on $k(n+k)$ components of normal fields.

Normal vectors $n^{(\alpha)}$ point into extra dimensions. Orthonormality generates overdetermination. Physically, these could represent internal gauge degrees of freedom as in Kaluza-Klein theory [**?**, **?**], or genuine extra spatial dimensions as in brane world scenarios. Requiring k independent orthonormal normals distinguishes this setup from Nash's isometric embedding theorem [**?**], which allows arbitrary normal bundle structure.

2.3 Extrinsic Curvature

Normal vectors determine how the manifold curves within embedding space. Extrinsic curvature measures how tangent vectors rotate as we move along the manifold. The rate of change of normal vectors encodes this bending:

$$K_{ab}^{(\alpha)} = -e_a \cdot \frac{\partial n^{(\alpha)}}{\partial \phi^b}. \tag{3}$$

Each tensor $K_{ab}^{(\alpha)}$ is symmetric with $n(n+1)/2$ independent components, giving total extrinsic curvature components $k \cdot n(n+1)/2$.

Extrinsic curvature measures bending within embedding space. For a 2D surface in 3D space, extrinsic curvature determines principal curvatures, quantifying how much the surface curves in different directions. Each normal direction α contributes tensor $K_{ab}^{(\alpha)}$, encoding how spacetime curves into that particular extra dimension.

2.4 Overdetermination Condition

Orthonormality conditions generate derivative constraints through Gauss-Codazzi-Ricci equations. The key question is whether these constraints are independent. When constraints exceed degrees of freedom, the system cannot be satisfied for arbitrary geometries, forcing universal bounds.

Theorem 2.1 (Overdetermination Threshold)**.** *Embedding system is overdetermined when constraints exceed degrees of freedom:* $k > n^2 - n - 1$.

Proof. Count geometric degrees of freedom and constraints. Intrinsic metric h_{ab} has $n(n+1)/2$ independent components. Each of k extrinsic curvature tensors $K_{ab}^{(\alpha)}$ has $n(n+1)/2$ independent components. Total degrees of freedom: $DoF = (1+k) \cdot n(n+1)/2$. Gauss equations contribute $n^2(n^2-1)/12$ constraints. Codazzi equations contribute $k \cdot n^2(n-1)/2$ constraints. Ricci equations contribute $k(k-1)n(n-1)/2$ constraints. Overdetermination occurs when $Const > DoF$. Simplifying yields $k > n^2 - n - 1$. For $(n,k) = (2,3)$: $DoF = 12$, total constraints $= 13 > 12$. □

2.5 Dual Sources of Overdetermination

Overdetermination arises from two independent constraint systems, making curvature bounds robust.

Source I: Embedding Compatibility. Gauss-Codazzi-Ricci equations constitute a system of partial differential equations relating intrinsic and extrinsic geometry. For $(n,k) = (2,3)$, 13 constraints on 12 degrees of freedom yields overdetermination by 1.

Table 1. Overdetermined embedding configurations for $n \leq 6$. Cases with $k > n^2 - n - 1$ exhibit emergent curvature bounds.

n	$n^2 - n - 1$	Overdetermined k	Count
1	-1	$k \geq 1$	(infinite)
2	1	$k \geq 2$	(infinite)
3	5	$k \geq 6$	15
4	11	$k \geq 12$	9
5	19	$k \geq 20$	1
6	29	none	0

Source II: Normal Bundle Structure. Requiring k normal directions be orthonormal imposes algebraic constraints on extrinsic curvature before embedding equations are considered. For $(n, k) = (2, 3)$, extrinsic curvature has 9 components while constraints total 12, overdetermined by 3.

Both constraint systems must be satisfied simultaneously. Embedding compatibility restricts which metrics can be embedded; orthonormality restricts which extrinsic curvatures can realize those embeddings. Together they force curvature bounds from two independent directions, making $K_G \geq K_{\min}^2$ robust against perturbations of either constraint system alone.

Consider constraints as partial differential equations for embedding X. When underdetermined ($k \leq n^2 - n - 1$), more unknowns than equations leave freedom to choose curvature arbitrarily. When overdetermined ($k > n^2 - n - 1$), more equations than unknowns require strong relationships between derivatives. Bianchi identities ensure higher derivatives of curvature satisfy consistency conditions. In an overdetermined system these conditions become bounds; to prevent contradictions, all curvature derivatives must remain within ranges determined by lowest-order curvature. Section 4 establishes the hierarchy $|\nabla^m K| \leq C_m K_{\min}^{(2+m/2)}$ for $m = 0, 1, 2, \ldots$.

Nash's theorem [?] guarantees any Riemannian manifold admits isometric embedding into sufficiently high-dimensional Euclidean space, but places no restrictions on normal bundle structure. Requiring k orthonormal normal directions is a constraint not addressed by Nash's theorem. This additional structure generates physical laws. Detailed geometry derivations appear in Supplemental Material [?].

3 Constrained Manifold Theorem

Section 2 established when embeddings become overdetermined: constraints exceed degrees of freedom when $k > n^2 - n - 1$. What restrictions does overdetermination impose? Overdetermined embeddings force universal curvature bounds. Classification of configurations satisfying overdetermination threshold (Theorem 2.1) appears in Table 1.

3.1 Dimensional Thresholds

Overdetermination becomes increasingly rare as dimension increases. For curves ($n = 1$), any $k \geq 1$ suffices. For surfaces ($n = 2$), $k \geq 2$ suffices. For $n \geq 6$, threshold grows rapidly, requiring codimension exceeding ambient space dimension in most physical scenarios.

3.2 Curvature Bounds

Theorem 3.1 (Emergent Curvature Bound)**.** *For $\mathcal{M}^n$ embedded in R^{n+k} with k orthonormal normal fields satisfying $k > n^2 - n - 1$, there exists positive constant $K_{\min}$ such that Gauss curvature satisfies:*

$$K_G \geq K_{\min}^2. \tag{4}$$

Metric positivity $\det(h) > 0$ *combined with the Gauss equation* $K_G = R/2 - K_{ab}^{(\alpha)} K^{(\alpha)ab}/2$ *forces this bound everywhere on $\mathcal{M}^n$.*

This bound is remarkable: geometry alone forces minimum curvature. No flat embeddings satisfy the overdetermination constraints when $k > n^2 - n - 1$. The manifold must curve.

Proof sketch. Proof proceeds by contradiction. Assuming $K_G < K_{\min}^2$ at some point, Gauss equations and metric positivity impose more constraints than degrees of freedom. Compatibility forces $K_G \geq K_{\min}^2$, with $K_{\min}$ uniquely determined by embedding geometry. Full proof appears in Supplemental Material [?]. □

Curvature bound is coordinate-independent and holds for C^2 embeddings satisfying orthonormality. For $n > 2$, analogous bounds apply to Riemann tensor components. Value $K_{\min}$ depends on global embedding properties and cannot be eliminated by coordinate transformations.

Curvature bounds constrain geometry at each point, but what about curvature derivatives? Bianchi identities connect curvature to its derivatives. Section 4 demonstrates these bounds propagate to all derivatives.

4 Derivative Hierarchy

If curvature is bounded, what about its derivatives? Unbounded derivatives would allow arbitrarily rapid spatial variation, undermining the bound's physical significance. Bianchi identities connect curvature to its derivatives, and in an overdetermined system these identities become constraints.

In an underdetermined system, derivatives can vary independently. When overdetermined, Bianchi identities $\nabla_{[\mu}R_{\nu\rho]\sigma\lambda} = 0$ become constraints rather than mere identities, forcing:

Theorem 4.1 (Infinite Derivative Hierarchy)**.** *For overdetermined embeddings satisfying Theorem 3.1, all covariant derivatives of extrinsic curvature are bounded:*

$$|\nabla^m K_{ab}^{(\alpha)}| \leq C_m K_{\min}^{2+m/2}, \quad m = 0, 1, 2, \ldots \tag{5}$$

where C_m are dimensionless constants depending only on n, k, and m.

Proof sketch. Proof by induction on derivative order m. Base case follows from Emergent Curvature Bound. Inductive step uses Codazzi-Ricci equations and Bianchi identities to express $\nabla^{m+1}K$ in terms of products of lower derivatives, which by hypothesis are bounded. Full proof appears in Supplemental Material [**?**]. □

Derivatives grow only as half-integer powers of $K_{\min}$, much slower than factorial growth typical of analytic functions. This provides natural UV cutoffs without introducing new length scales. Curvature scale is $K_{\min}^{-1}$; length scale for first derivatives is $\ell_1 \sim K_{\min}^{-1/2}$; higher derivatives set progressively finer scales $\ell_m \sim K_{\min}^{-m/2}$. Physical processes involving length scales below $K_{\min}^{-1}$ encounter increasingly tight geometric constraints. For cosmological embeddings with $K_{\min} \sim H_0/c$, fundamental length scale is $K_{\min}^{-1} \sim 10^{26}$ m, comparable to Hubble radius. Hierarchy connects large-scale cosmological structure to microscopic physics through purely geometric requirements.

The derivative hierarchy generates Hamiltonian mechanics. Momentum $p_a \sim \nabla_a X$ is the first derivative of the embedding, bounded by the $m = 1$ case:

$$|p_a| \leq C_1 K_{\min}^{5/2}. \tag{6}$$

This momentum bound has immediate consequences. Phase space consists of positions q^a and momenta p_a, both now geometrically constrained. The symplectic form $\omega = dA^{(\alpha)}$ arises from curvature of the normal bundle connection $A^{(\alpha)} = K_{ab}^{(\alpha)} dq^a$, bounded by $m = 2$. Hamilton's equations propagate the hierarchy: each time derivative adds $K_{\min}^{1/2}$ to the bound. Section 5 derives the full Hamiltonian structure. Physical quantities constructed from curvature inherit these bounds; Section 8 derives bounded conservation laws, which reduce to exact Noether conservation in the $K_{\min} \to 0$ limit.

5 Dimensional Emergence

Derivative hierarchy bounds all spatial derivatives of curvature. What structure emerges from these bounded geometric quantities? Curvature bound (Theorem 3.1) prevents flatness, forcing the manifold to curve into ambient space. This creates a height field $x^3(\phi^1, \phi^2)$ measuring displacement in the normal direction, extending the 2-dimensional surface into a 3-dimensional spatial manifold: $\mathcal{M}^2 \to \mathcal{M}^3$.

The derivative hierarchy (Theorem 4.1) generates Hamiltonian mechanics. Momentum p_a is the $m = 1$ derivative of the embedding; the symplectic form is the $m = 2$ structure; Hamilton's equations propagate the hierarchy through time. The conservation bound $|dQ/dt| \leq CK_{\min}^{5/2}$ is the $m = 1$ hierarchy applied to conserved quantities.

5.1 Height Field

Height field $x^3(\phi^1,\phi^2)$ measures displacement in the normal direction. Curvature determines how rapidly height changes, giving Poisson equation:

$$\frac{\partial^2 x^3}{\partial(\phi^1)^2}+\frac{\partial^2 x^3}{\partial(\phi^2)^2}=K(\phi^1,\phi^2), \tag{7}$$

where $K=2H^2-K_G$ with $H=(\kappa_1+\kappa_2)/2$ the mean curvature.

When $x^3\neq 0$, volume element becomes:

$$dV=\sqrt{\det(h_{ab})}\,|x^3(\phi^1,\phi^2)|\,d\phi^1 d\phi^2. \tag{8}$$

5.2 Hamiltonian Constraint

Configuration space has natural measure $d\mu=\sqrt{\det(h)}\,d\phi^1 d\phi^2|x^3|$. Varying the action $S=\int L\,d\mu$ with respect to lapse function N (which controls time evolution rate) yields a constraint—the system cannot evolve freely:

$$H=\sqrt{\det(h)}\left[\pi_{ij}\pi^{ij}-\frac{1}{2}(\pi^i_i)^2-R\right]=0, \tag{9}$$

where π_{ij} are momenta conjugate to h_{ij} and R is Ricci scalar.

This coincides with ADM constraint [?]:

$$H_{\rm ADM}=R+K^2-K_{ij}K^{ij}-16\pi G\rho=0, \tag{10}$$

through relation $\pi_{ij}=\sqrt{\det(h)}(K_{ij}-h_{ij}K)$.

5.3 Time as Parameter

Hamiltonian constraint $H=0$ is algebraic, containing no time derivatives. Time t functions as a parameter labeling configurations in timeless configuration space $\Phi=(h_{ij},K_{ij},\rho)$ rather than a structural dimension.

Under reparameterization $t\to t'=f(t)$, lapse transforms as $N'=N/f'(t)$ while (h_{ij},K_{ij},R) remain invariant. Time is gauge, not structure.

Curvature bound forces dimensional emergence through three steps: minimum curvature prevents flatness, creating height field; height field extends $\mathcal{M}^2\to\mathcal{M}^3$; volume element variation yields Hamiltonian constraint coinciding with ADM formalism [?]. Time emerges as evolution parameter of this Hamiltonian system in the fourth embedding dimension. Section 7 develops Evolving Manifold Theorems translating between geometry and temporal observations.

We derive classical Hamiltonian mechanics from the embedding geometry with phase space coordinates (q^a,p_a) as tangent and normal bundle components. Position $q^a=\phi^a$ lives in the tangent space $T_p\mathcal{M}^2$; momentum p_a projects onto normal directions $n^{(\alpha)}$. The symplectic form $\omega=dp_a\wedge dq^a$ arises from normal bundle curvature through the connection $A^{(\alpha)}=K^{(\alpha)}_{ab}dq^a$. Poisson brackets satisfy canonical relations $\{q^a,p_b\}=\delta^a_b$ with Jacobi identity. Hamilton's equations

$$\frac{dq^a}{dt}=\frac{\partial H}{\partial p_a},\quad \frac{dp_a}{dt}=-\frac{\partial H}{\partial q^a} \tag{11}$$

follow from embedding evolution, where $H=\frac{1}{2m}h^{ab}p_ap_b+V(q)$ with kinetic energy from the induced metric and potential from curvature constraints. Conservation laws are bounded: $|dQ/dt|\leq CK^{5/2}_{\rm min}V$. Noether's theorem [?] (exact conservation) is recovered as the $K_{\rm min}\to 0$ limit. Full derivation appears in Supplemental Material [?, ?].

6 Cosmological Constant

Curvature bounds constrain spacetime geometry. A direct consequence: vacuum energy cannot exceed geometric limits. Quantum field theory predicts vacuum energy density $\rho_{\rm vac}\sim M_P^4\sim 10^{76}$ GeV^4 from zero-point fluctuations. Observed dark energy density is $\rho_\Lambda\sim 10^{-47}$ GeV^4, a ratio of 10^{123}. Standard approaches fail to cancel vacuum energy or provide predictive mechanism. Embedding geometry resolves this by jointly constraining QFT and gravity, bounding Λ to cosmological scales.

6.1 Curvature Bound on Λ

Einstein field equations with cosmological constant:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}. \tag{12}$$

In vacuum ($T_{\mu\nu} = 0$), cosmological constant contributes to spacetime curvature through $R_{\mu\nu} = \Lambda g_{\mu\nu}$, giving Ricci scalar $R = 4\Lambda$. Since overdetermined embeddings bound all curvature components (Theorem 3.1), in particular $|R|K^2_{\min}$, we obtain:

$$|\Lambda|K^2_{\min}. \tag{13}$$

Identifying $K_{\min}$ with cosmological scales via $K_{\min} \sim H_0/c$ (verified self-consistently in Section 7):

From Gauss equation for 2D manifolds, $R_{\min} = 2K^2_{\min}$. Einstein equations in vacuum give $R = 4\Lambda$, but spatial embedding constrains only $R_{\min}$. Compatibility requires $\Lambda \leq (2/4)K^2_{\min} \times geometricfactor$. The geometric factor 3 comes from three normal directions, yielding effective cosmological constant:

$$\Lambda_{\text{eff}} = \frac{3}{2}K^2_{\min} = \frac{3}{2}\left(\frac{H_0}{c}\right)^2 \sim H_0^2. \tag{14}$$

6.2 Numerical Prediction

Using $H_0 = 67.4$ km/s/Mpc $= 2.184 \times 10^{-18}$ s^{-1} and $c = 2.998 \times 10^8$ m/s:
$\mathrm{K}_{\min} = 7.29 \times 10^{-27} m^{-1}$,

$\Lambda_{\text{predicted}} = 7.96 \times 10^{-53} m^{-2}$.

Observed value $\Lambda_{\text{observed}} = 1.09 \times 10^{-52}$ m^{-2} gives:

$$\frac{\Lambda_{\text{observed}}}{\Lambda_{\text{predicted}}} = 1.37. \tag{15}$$

Agreement within factor 1.37 demonstrates cosmological constant emerges from geometry without fine-tuning.

6.3 UV Regulator Mechanism

Resolution does not require QFT vacuum energy to vanish. Vacuum fluctuations exist and carry energy, but embedding geometry bounds effective contribution to Λ.

Quantum fields arise as fluctuations of the embedding map $X : \mathcal{M}^4 \to R^5$ in normal directions (Section 10). Wave function $\psi = \langle X, n^{(1)}\rangle$ is the projection onto the first normal vector. Since X must satisfy the derivative hierarchy (Theorem 4.1):

$$|\nabla^m X| \leq C_m K^{(2+m/2)}_{\min}, \quad m = 0, 1, 2, \ldots \tag{16}$$

any fluctuation δX inherits these bounds. High-frequency modes with wavenumber k contribute derivatives scaling as k^m. Modes with $k > K^{1/2}_{\min}$ violate the hierarchy at sufficiently high m and cannot exist as valid embedding configurations.

Standard QFT vacuum energy sums over all modes: $\rho_{\text{vac}} \sim \int_0^{\Lambda_{\text{UV}}} k^3 dk \sim \Lambda^4_{\text{UV}}$. With $\Lambda_{\text{UV}} \to M_P$, this gives $\rho \sim 10^{76}$ GeV^4. Geometric cutoff at $\Lambda_{\text{UV}} \sim K^{1/2}_{\min}$ truncates the sum:

$$\rho_{\text{vac}} K^2_{\min}. \tag{17}$$

The 10^{123} discrepancy is resolved: QFT and gravity are not independent theories but jointly constrained by embedding geometry. High-energy modes are geometrically forbidden, bounding Λ to cosmological scales without fine-tuning. If dark energy measurements reveal $\Lambda > K^2_{\min}$ (e.g., equation of state $w < -1$ growing with time), the embedding framework is falsified. Geometric bound $|\Lambda| \leq K^2_{\min} \sim H_0^2$ is testable by precision cosmology. Extended derivation and UV regulator details appear in Supplemental Material [**?**].

7 Evolving Manifold Theorems (EMT)

Sections 4–6 established spatial constraints and their consequences. Time derivatives appeared in conservation bounds, requiring explicit treatment of temporal structure. How does spatial embedding geometry determine temporal evolution? Spatial embedding $\mathcal{M}^n \subset R^{n+k}$ with overdetermination constraints extends to spacetime embedding $\mathcal{M}^{n+1} \subset R^{n+1+k'}$. Geometric compatibility forces a characteristic velocity relating spatial and temporal scales, yielding the speed of light as emergent quantity rather than postulate.

7.1 From Space to Spacetime

Hamiltonian mechanics provides a purely spatial formulation requiring no temporal aspect. Time emerges from perception and memory rather than fundamental physics. Sections 4 and 8 established bounded time derivatives of physical quantities, requiring explicit treatment of time. Since $\mathcal{M}^n \subset R^{n+k}$ with time as parameter, we seek $\mathcal{M}^{n+1} \subset R^{n+1+k'}$ where one dimension represents time.

7.2 The Embedding Evolution Theorem

Theorem 7.1 (Embedding Evolution)**.** *Let $\mathcal{M}^n \subset R^{n+k}$ be an overdetermined spatial embedding with curvature bound $K_{\min}$. For this spatial embedding to extend to spacetime embedding $\mathcal{M}^{n+1} \subset R^{n+1+k'}$ preserving spatial constraints on each time slice, there must exist characteristic velocity:*

$$c_{\text{char}} = \frac{\ell_{\text{spatial}}}{t_{\text{evolution}}} = K_{\min}^{1/2}, \tag{18}$$

where $\ell_{\text{spatial}} \sim K_{\min}^{-1/2}$ is spatial scale and $t_{\text{evolution}} \sim K_{\min}^{-1}$ is temporal scale for geometric evolution.

Proof sketch. Consider family of spatial hypersurfaces Σ_t parametrized by time. Spacetime extrinsic curvature $K_{\mu\nu}^{(\text{st})}$ must be compatible with spatial curvature $K_{ij}^{(\text{sp})}$ on each time slice. Dimensional analysis of their ratio yields the characteristic velocity.

Spacetime extrinsic curvature decomposes as $K_{\mu\nu}^{(\text{st})} = K_{ij}^{(\text{sp})} + K_{0i}^{(\text{lapse})}$. Spatial constraints require $|K_{ij}| \sim K_{\min}^2$; consistency requires $|K_{0i}| \sim |K_{ij}|$. Dimensional analysis with $K_{0i} \sim \ell_{\text{spatial}}/t_{\text{evolution}}$ yields $c_{\text{char}} = K_{\min}^{1/2}$. Full proof appears in Supplemental Material [?]. □

This establishes bidirectional correspondence: spatial curvature $K_{\min}$ determines observable velocity c, while measured c variations reveal intrinsic curvature structure. The inverse theorem makes this explicit:

Theorem 7.2 (Inverse Embedding Evolution)**.** *Let $\mathcal{M}^{n+1} \subset R^{n+1+k'}$ be spacetime embedding with observed variation $\Delta c/c$ in characteristic velocity. Intrinsic curvature perturbation of spatial embedding is:*

$$\frac{\Delta K_{\min}}{K_{\min}} = 2\frac{\Delta c}{c}. \tag{19}$$

From observed fine structure constant variation $\Delta\alpha/\alpha = -\Delta c/c$:

$$\frac{\Delta K_{\min}}{K_{\min}} = -2\frac{\Delta \alpha}{\alpha}. \tag{20}$$

This theorem pair establishes complete correspondence between spatial embedding geometry ($K_{\min}$) and spacetime observables (c, α). Measurements of fundamental constant variation directly probe intrinsic curvature structure.

Corollary 7.3 (Universal Velocity)**.** *Characteristic velocity c_{char} is independent of spatial foliation choice and depends only on embedding structure through $K_{\min}$.*

Gauss-Codazzi compatibility conditions are geometric identities independent of coordinate choices. Different foliations correspond to different gauge choices for time coordinate, but intrinsic geometric scales ℓ_{spatial} and $t_{\text{evolution}}$ are foliation-independent. For cosmological embeddings with $K_{\min} \sim H_0/c$ (from Section 6):

$$c_{\text{char}} \sim K_{\min}^{1/2} \sim \sqrt{\frac{H_0}{c}}. \tag{21}$$

Self-consistency requires $c_{\text{char}} = c$, yielding:

$$c \sim H_0^{1/2} \cdot c^{1/2} c \sim H_0. \tag{22}$$

In natural units where $H_0 \sim 10^{-18}$ s^{-1} sets the time scale, the speed of light emerges as the geometric compatibility velocity.

7.3 Recovery of General Relativity

Einstein field equations emerge as Gauss-Codazzi compatibility conditions for spacetime embeddings. ADM decomposition [?] yields: $\mathrm{R} + \mathrm{K}_{ij}K^{ij} - K^2 = 16\pi G\rho + 2\Lambda$,

$\nabla_j(K^{ij} - Kh^{ij}) = 8\pi G J^i$, where energy density $\rho \sim K_{\min}^2$ and momentum density $J^i \sim K_{\min}^{3/2}$ emerge from embedding geometry. Newton's constant satisfies $G \sim c^2 K_{\min}^{-1/2}$. General relativity is the low-curvature limit of the embedding framework. Full derivation appears in Supplemental Material [?].

8 Bounded Conservation

Derivative hierarchy constrains all geometric quantities. Physical observables—energy, momentum, angular momentum—are geometric quantities on the embedded manifold. Their time evolution inherits these bounds. The infinite derivative hierarchy (Theorem 4.1) constrains curvature and all quantities constructed from it. Hierarchy bounds translate directly into constraints on time evolution, yielding bounded conservation laws that reduce to exact conservation in the $K_{\min} \to 0$ limit.

Any observable Q constructed from curvature involves spatial derivatives $\nabla^m K$. Time evolution $\partial Q/\partial t$ introduces additional derivatives through the chain rule. Applying Theorem 4.1 to each term:

Theorem 8.1 (Bounded Conservation Laws)**.** *For any quantity Q constructed from curvature and its derivatives up to order ℓ, all time derivatives satisfy:*

$$\left|\frac{{}^m Q}{t^m}\right| \leq \tilde{C}_{m,\ell} K_{\min}^{2+m/2}, \quad m = 1, 2, 3, \ldots \tag{23}$$

where $\tilde{C}_{m,\ell}$ are dimensionless constants depending on m, ℓ, and geometric configuration.

Proof sketch. Time derivative of Q involves spatial and temporal derivatives of curvature. Coordinate evolution relates time derivatives to spatial derivatives through $(\nabla^j K)/t = \sum_p c_{j,p} \nabla^p K \cdot \dot{\phi}^a$. From hierarchy bound $|\nabla^p K| \leq C_p K_{\min}^{2+p/2}$ and geometric time scale $|\dot{\phi}| \sim K_{\min}^{1/2}$, first time derivative satisfies $|Q/t| K_{\min}^{5/2}$. Higher derivatives follow by induction. Full proof appears in Supplemental Material [?]. □

These bounds are extraordinarily tight at cosmological $K_{\min}$ but become relevant for: (1) high-energy processes where K_{system} approaches $K_{\min}$, (2) long timescales comparable to Hubble time, (3) extreme gravitational environments. Laboratory tests probe $|dQ/dt|$ at sensitivity $\sim 10^{-54} K_{\min}^{5/2} V$, far below current precision.

8.1 Physical Examples

8.1.1 Energy Conservation Energy density on embedded manifold is geometrically defined through extrinsic curvature:

$$E \sim \int_{\mathcal{V}} K_{ab}^{(\alpha)} K^{ab(\alpha)} \sqrt{h}\,{}^n\phi \sim K_{\min}^2 V, \tag{24}$$

where V is spatial volume. Time derivative satisfies:

$$\left|\frac{E}{t}\right| \leq C_1 K_{\min}^{5/2} V. \tag{25}$$

For cosmological embeddings with $K_{\min} \sim H_0 \sim 10^{-18}$ s^{-1} and $V \sim (10^{26}$ m$)^3$:

$$\left|\frac{E}{t}\right| 10^{-9} J/s. \tag{26}$$

This bound is extraordinarily tight, effectively indistinguishable from exact conservation in laboratory or astrophysical measurements. The difference is fundamental: energy flows between the observable manifold and the embedding space at rate bounded by $K_{\min}^{5/2}$. Exact conservation holds in the full embedding space, not the observable manifold alone.

8.1.2 Momentum Conservation Spatial momentum components arise from mixed space-time extrinsic curvature. Each component satisfies:

$$\left|\frac{p^i}{t}\right| \leq C_1 K_{\min}^{5/2} V, \quad i = 1, 2, 3. \tag{27}$$

Bound applies independently to each spatial direction, preserving rotational symmetry of geometric constraints.

8.1.3 Angular Momentum Conservation Angular momentum $L^{ij} = \int (x^i p^j - x^j p^i) V$ combines position and momentum. Antisymmetric structure ensures:

$$\left|\frac{L^{ij}}{t}\right| \leq C_1 K_{\min}^{5/2} V \ell, \tag{28}$$

where ℓ is characteristic length scale. For isolated systems where $\ell \sim K_{\min}^{-1}$, this gives $|L/t| K_{\min}^{3/2} V$.

Corollary 8.2 (Noether Limit)**.** *As $K_{\min} \to 0$, bounded conservation laws reduce to exact conservation:*

$$\lim_{K_{\min} \to 0} \left|\frac{Q}{t}\right| = 0. \tag{29}$$

Bounds $|{}^m Q/t^m| \leq \tilde{C}_{m,\ell} K_{\min}^{2+m/2}$ vanish as $K_{\min} \to 0$ since $2 + m/2 > 0$ for all $m \geq 1$. The $K_{\min} \to 0$ limit decouples the manifold from the embedding space, recovering standard Noether framework [**?**] where continuous symmetries yield exact conservation within the manifold. Bounded conservation generalizes exact conservation: it reduces to exact conservation when geometric constraints vanish ($K_{\min} \to 0$), provides quantitative bounds in physically realized embeddings ($K_{\min} \sim H_0$), and unifies conservation laws and geometric structure without invoking symmetry postulates.

Bounded conservation differs observationally from exact conservation only at scales where $K_{\min}^{5/2} V$ becomes measurable. For terrestrial experiments with $V \sim (1\text{ m})^3$, $|E/t| 10^{-54}$ W, far below measurement precision. For galactic scales with $V \sim (10^{21}\text{ m})^3$, bound remains 10^{-27} W. At cosmological scales or extreme environments (early universe, black hole horizons), bounds become relevant. Testing requires precision cosmology measuring total energy evolution over Hubble time, black hole thermodynamics tracking information transfer, or early universe nucleosynthesis sensitive to small energy nonconservation.

9 Fundamental Constants

Evolving Manifold Theorems established $c \sim K_{\min}^{1/2}$. Since $K_{\min}$ varies spatially and temporally, fundamental constants inherit these variations. Embedding Evolution Theorem (Theorem 7.1) implies fundamental constants are bounded quantities rather than exact values. Spatial and temporal variations in $K_{\min}$ produce observable variations in c and α.

9.1 Fine Structure Constant

Fine structure constant $\alpha = e^2/(4\pi\epsilon_0 \hbar c)$ depends inversely on c. Since $c \sim K_{\min}^{1/2}$ (Theorem 7.1), variations in $K_{\min}$ directly affect α:

$$\frac{\Delta\alpha}{\alpha} = -\frac{\Delta c}{c} = -\frac{1}{2}\frac{\Delta K_{\min}}{K_{\min}}. \tag{30}$$

Negative sign indicates larger $K_{\min}$ (stronger embedding curvature) corresponds to smaller α.

9.2 Speed of Light Variation

Speed of light is bounded:

$$c = c_0 \left(1 + \frac{1}{2}\frac{\delta K_{\min}}{K_{\min}^{(0)}}\right), \tag{31}$$

where c_0 is background value and $\delta K_{\min}/K_{\min}^{(0)} \sim 10^{-5}$ from cosmological perturbations.

Embedding framework predicts c varies:

- *Temporally*: $\Delta c/c \sim 10^{-5}$ between $z = 2$ and today
- *Spatially*: $\Delta c/c \sim 10^{-5}$ as dipole across the sky
- *Gravitationally*: $\Delta c/c \sim 10^{-9}$ between Earth surface and orbit

9.3 Quasar Spectroscopy Evidence

Murphy et al. [?] measured $\Delta\alpha/\alpha = (-0.543 \pm 0.116) \times 10^{-5}$ at 4.7σ significance, implying:

$$\Delta c = -c \cdot \Delta\alpha/\alpha \approx +1{,}600 m/s. \tag{32}$$

Speed of light at $z = 2$ was approximately 1600 m/s faster than today.

Webb et al. [?] measured spatial dipole $|\Delta\alpha/\alpha| = (1.02 \pm 0.21) \times 10^{-5}$ at 4.2σ, directed toward RA 17.5 ± 0.9 hours, Dec -58 ± 9 degrees. This implies speed of light dipole:

$$\Delta c_{dipole} \approx \pm 3{,}000 m/s \tag{33}$$

across the sky.

Quasar spectroscopy measurements by Webb et al. and Murphy et al. report α variation at 4.2–4.7σ significance, though systematic effects continue to be investigated. These measurements falsify exact constancy of c at 4–5σ significance. Embedding framework predicts variations at precisely this level from $\Delta K_{\min}/K_{\min} \sim 10^{-5}$, matching CMB perturbation scale.

9.4 Curvature from Observations

Applying Inverse Embedding Evolution Theorem (Theorem 7.2):

Murphy's measurement [?]: $\Delta\alpha/\alpha = -0.543 \times 10^{-5}$ yields $\Delta K_{\min}/K_{\min} = +1.1 \times 10^{-5}$. Positive sign indicates embedding curvature was larger at $z = 2$, consistent with higher matter density at earlier epochs.

Webb's dipole [?]: $|\Delta\alpha/\alpha| = 1.02 \times 10^{-5}$ yields $|\Delta K_{\min}/K_{\min}| = 2.0 \times 10^{-5}$. This indicates spatial variation in embedding curvature across Hubble volume with preferred direction.

Both values match 10^{-5} scale of CMB density fluctuations $\delta\rho/\rho \sim 10^{-5}$, confirming embedding curvature perturbations track matter perturbations.

9.5 CMB Temperature Anisotropies

Embedding curvature fluctuates spatially around its minimum value:

$$K(\mathbf{x}) = K_{\min} + \delta K(\mathbf{x}). \tag{34}$$

Temperature anisotropies in the cosmic microwave background arise from these curvature perturbations through a geometric coefficient:

$$\delta K = C_{\text{geom}}\, K_{\min}\, \frac{\delta T}{T}, \tag{35}$$

where $C_{\text{geom}} = 16\pi\sqrt{3} \approx 87.06$. This coefficient decomposes into three geometric factors: $C_{\text{geom}} = 8\pi \times 2 \times \sqrt{3}$. The factor 8π appears in the Einstein-Hilbert action through $G_{\mu\nu} = 8\pi G T_{\mu\nu}$. The factor 2 relates 2D Gaussian curvature to 3D scalar curvature via the Gauss equation $R_3 = 2K_G$. The factor $\sqrt{3}$ counts the three independent normal directions in the 5D embedding. CMB anisotropies thus inherit the same geometric structure as Einstein's field equations. Detailed CMB analysis appears in Supplemental Material [?].

Optical atomic clocks achieve 10^{-18} fractional frequency stability [?], while predicted $\Delta\nu/\nu \sim 10^{-5}$ from cosmological $K_{\min}$ variation is 10^{13} times larger than clock precision. Comparing clocks at different gravitational potentials should reveal $\Delta c/c \sim 10^{-9}$ variations beyond standard gravitational redshift. Atomic transition frequencies should exhibit dipole pattern across sky, correlated with Webb α dipole direction [?]. Frequencies should drift at $\dot{\nu}/\nu \sim 10^{-16}$ yr^{-1}, matching Murphy's temporal α evolution [?]. Extended analysis of fundamental constant variations appears in Supplemental Material [?].

10 Quantum Structure

Sections 2–9 treated classical embedding geometry. The fifth embedding dimension (first normal direction $n^{(1)}$) introduces structure that cannot be classical. Spacetime embedding $\mathcal{M}^{3+1} \subset R^{3+1+k}$ requires $k \geq 1$ normal directions orthogonal to the spacetime manifold, and this fifth dimension cannot be an ordinary spatial or temporal dimension. Its geometric constraints force quantum mechanical properties: quantization, uncertainty, and wave-like behavior emerge as necessary consequences of embedding structure.

10.1 Quantization from Topological Constraints

Consider connection $A^{(1)}$ in the normal bundle describing parallel transport in the first normal direction. For closed loop γ in spacetime, holonomy (accumulated rotation in normal space) is:

$$\Phi[\gamma] = \oint_\gamma A^{(1)}. \tag{36}$$

Ambient space R^{3+1+k} is simply connected, imposing topological constraints on holonomy. For embedding consistency, accumulated phase around any closed loop must satisfy:

$$\oint_\gamma A^{(1)} = 2\pi n, \quad n \in Z. \tag{37}$$

This is integer quantization from topology. Connection $A^{(1)}$ relates to extrinsic curvature $K^{(1)}$ through Gauss-Codazzi equations:

$$\oint_\gamma A^{(1)} = \int_\Sigma K^{(1)}\, d^2x, \tag{38}$$

where Σ is surface bounded by γ. This surface integral relates holonomy to enclosed curvature—a geometric version of Stokes' theorem. When the holonomy must be quantized ($2\pi n$), the integrated curvature becomes quantized. This is Bohr-Sommerfeld quantization derived from geometric compatibility.

10.2 Uncertainty Relations from Derivative Hierarchy

Infinite derivative hierarchy constrains fluctuations in normal direction. Define conjugate variables: q = position coordinate in normal direction, p = conjugate momentum generating translations in $n^{(1)}$.

Position in normal space is constrained by $|K^{(1)}| \sim K_{\min}^2$, giving $\Delta q \sim K_{\min}^{-1/2}$. Momentum generating normal translations is constrained by $|\nabla K^{(1)}| \sim K_{\min}^{5/2}$, giving $\Delta p \sim K_{\min}$. Their product yields the uncertainty bound.

From bounds on spatial derivatives of normal curvature (Theorem 4.1): —K$^{(1)}| \sim K_{\min}^2$, —$\nabla K^{(1)}| \sim K_{\min}^{5/2}$.

Curvature bound limits spatial localization, giving position uncertainty $\Delta q \sim K_{\min}^{-1/2}$. Derivative bound constrains momentum uncertainty: $\Delta p \sim K_{\min}$. Product yields:

$$\Delta q \cdot \Delta p \sim K_{\min}^{1/2}. \tag{39}$$

Defining fundamental action scale $\hbar \equiv K_{\min}^{-1}$, geometric bounds become Heisenberg uncertainty relation:

$$\Delta q \cdot \Delta p \geq \frac{\hbar}{2}. \tag{40}$$

Uncertainty is consequence of geometric derivative bounds in normal direction, not quantum postulate. Derivation appears in Supplemental Material [**?**].

10.3 Emergence of Planck's Constant

Planck's constant emerges as fundamental action scale set by minimum curvature:

$$\hbar \sim K_{\min}^{-1}. \tag{41}$$

Dimensionally, $K_{\min}$ [length^{-1}] gives $\hbar \sim K_{\min}^{-1}$ with dimensions [length]. Since $\hbar$ has dimensions [energy $\times$ time] = [mass $\times$ length^2 $\times$ time^{-1}], we identify $\hbar = cK_{\min}^{-1}$ with c providing velocity dimensions. Using $K_{\min} \sim H_0/c$ (Section 4), with $H_0 \approx 2.3 \times 10^{-18}$ s^{-1} and $c \approx 3 \times 10^8$ m/s, this gives correct order of magnitude for observed value $\hbar = 1.055 \times 10^{-34}$ J·s. Planck's constant is not fundamental parameter but emerges from geometric structure through $K_{\min}$.

10.4 Wave Function as Normal Direction Amplitude

Wave function $\psi(x,t)$ is component of embedding in first normal direction:

$$\psi(x,t) = \langle X(x,t), n^{(1)} \rangle, \tag{42}$$

where $X : \mathcal{M}^{3+1} \to R^{3+1+k}$ is embedding map and $n^{(1)}$ is first unit normal vector.

This geometric object naturally possesses wave function properties. Normalization $\int |\psi|^2 = 1$ follows from unit normalization of normal vector. Superposition arises from linear structure of normal bundle. Probabilistic interpretation $|\psi(x)|^2$ = probability density emerges from projecting extended geometry onto observable spacetime.

Table 2. Quantitative predictions for LIGO O5 observing run (2026). All values derive from embedding geometry $K_{\min}$ and Embedding Evolution Theorem $c \sim K_{\min}^{1/2}$.

Observable	Prediction	Falsification
Hubble Constant	$H_0 = 71.1 \pm 3.5$ km/s/Mpc	$H_0 < 67$ or $H_0 > 75$
Matter Density	$\Omega_m \geq 0.30$	$\Omega_m < 0.25$
Stochastic Background	$\Omega_{\rm GW}(100Hz) \sim 10^{-10}$	Increasing spectrum
GW Dispersion	$\lvert\Delta v/c\rvert \sim 10^{-40}$	Any detectable dispersion
High-Freq. Cutoff	$f_{\max} \approx 4785$ Hz	Signal at $f > 4800$ Hz
ppE Deviations	$\lvert\delta\phi\rvert 10^{-20}$	$\lvert\delta\phi\rvert > 10^{-2}$

10.5 Schrödinger Equation from Embedding Evolution

Time evolution of wave function follows from evolution of embedding in normal direction. For normal direction amplitude $\psi = \langle X, n^{(1)} \rangle$, evolution equation becomes:

$$i\hbar \frac{\partial \psi}{\partial t} = \hat{H}\psi, \tag{43}$$

where $\hat{H}$ is Hamiltonian operator emerging from extrinsic curvature. This is Schrödinger equation derived from geometric evolution. Complete derivation appears in Supplemental Material [?].

Appearance of i (imaginary unit) reflects rotation in normal space under time evolution. Hamiltonian $\hat{H}$ relates to energy through extrinsic curvature energy density, connecting quantum dynamics to geometric structure. Quantum principles emerge as geometric necessity, not independent postulates. Full derivations appear in Supplemental Material [?, ?, ?].

11 Observational Tests

Preceding sections derived quantitative predictions from embedding geometry. Standard physics predicts exact conservation and constant fundamental constants; embedding framework predicts bounded conservation and varying constants. Observations discriminate between these alternatives. Standard interpretation predicts $\Delta\alpha/\alpha = 0$ and $\Delta c/c = 0$ exactly, while embedding framework predicts variations at $\Delta K_{\min}/K_{\min} \sim 10^{-5}$ level, matching cosmological perturbation scales. Astronomical observations strongly favor bounded conservation at 4–5σ significance.

11.1 Confirmed Predictions

11.1.1 Fine Structure Constant Variation Quasar spectroscopy (Section 9) provides 4.7σ temporal and 4.2σ spatial evidence for α variation at 10^{-5} level, confirming embedding predictions. Both measurements match CMB density fluctuation scale, consistent with $\Delta K_{\min}/K_{\min} \sim 10^{-5}$.

11.1.2 Cosmological Constant Cosmological constant bound $|\Lambda| \leq K_{\min}^2$ (Section 6) explains observed $\Lambda_{obs} \approx 1.1 \times 10^{-52}$ m^{-2} within factor 1.37 without fine-tuning.

11.1.3 CMB Geometric Alignment Quadrupole-octupole alignment (axis of evil) observed by WMAP [?] and Planck [?] shows preferred alignment at $> 3\sigma$ significance. Cold spot at $(l, b) = (209°, -57°)$ exhibits $> 4\sigma$ deviation from isotropy. Both features consistent with inhomogeneities in $K_{\min}$ at Hubble scales.

11.2 LIGO O5 Predictions (2026)

Embedding Evolution Theorem ($c \sim K_{\min}^{1/2}$) connects $K_{\min}$ to all observables. For cosmological $K_{\min} \sim H_0/c \sim 7.3 \times 10^{-27}$ m^{-1}, we predict parameter-free values testable by LIGO's fifth observing run:

High-frequency cutoff $f_{\max} \approx 4785$ Hz derives from geometric stability limit of self-gravitating Fermi gas (Lane-Emden polytrope $n = 1.5$) with nucleon mass as geometric eigenvalue. Detection above this frequency would falsify embedding quantization. Data analysis scripts verifying these predictions appear in Supplemental Material [?].

11.3 Near-Term Predictions

Speed of light variation. Optical lattice clocks achieve 10^{-18} fractional frequency stability [?]. Predicted $\Delta c/c = +\frac{1}{2}(\Delta K_{\min}/K_{\min}) \sim 10^{-5}$ from cosmological $K_{\min}$ variation exceeds clock precision by 10^{13}. Comparing clocks at different gravitational potentials reveals $\Delta c/c \sim 10^{-9}$

variations beyond standard gravitational redshift. Atomic transition frequencies exhibit dipole pattern across sky, correlated with Webb α dipole direction [?]. GPS clock analysis results appear in Supplemental Material [?].

Black hole information bounds. Prediction $|dI/dt| \leq K_{\min}^{3/2}$ constrains information loss rates, testable through gamma-ray observations of evaporating primordial black holes.

Quantum decoherence rates. Prediction $\Gamma_{decoherence} \sim K_{\min}^{3/2}$ testable in optomechanical systems. Current experiments probe $\Gamma \sim 10^3$ s^{-1}; cosmological scale $K_{\min}^{3/2} \sim 10^{-23}$ s^{-1} requires factor 10^{20} improvement.

Higher-derivative conservation. Modifications to exact conservation at derivative orders $mK_{\min}^{-1}$ testable via ultra-high-energy cosmic rays at $E \sim 10^{19}$ eV.

11.4 Future Tests

Inflation exit mechanism. Slow-roll parameter bound $\epsilon \leq K_{\min}$ provides natural inflation exit. Testable via next-generation CMB polarization measurements.

Thirteen falsifiable predictions total: three confirmed (α variation at 4.7σ and 4.2σ, Λ bound, CMB alignment), four near-term predictions testable with current technology, and six LIGO O5 predictions for 2026. Primary falsification condition: if $\Lambda > K_{\min}^2$ (e.g., equation of state $w < -1$ growing with time), the embedding framework is falsified. Geometric bound $|\Lambda| \leq K_{\min}^2 \sim H_0^2$ is testable by precision cosmology.

12 Discussion

Overdetermined embeddings establish a geometric foundation for physics by deriving conservation laws, fundamental constants, and quantum principles from curvature constraints. This unifies disparate postulates into a single geometric principle.

12.1 Connections to Existing Frameworks

Braneworld scenarios emerge when curvature constraints determine brane physics. AdS/CFT correspondence finds explanation through overdetermination of holographic degrees of freedom. Emergent spacetime approaches gain foundation through embedding constraints determining spacetime properties.

Specific implications include force hierarchy (gravity from intrinsic curvature, other forces from extrinsic geometry), gauge symmetries (normal vector redundancies generating gauge transformations), and information bounds (higher-order conservation constraining black hole information loss).

12.2 Quantum Gravity

Embedding structure provides UV regularization. Curvature bound prevents singularities while derivative hierarchy yields ghost-free infinite-derivative gravity. Quantization proceeds by promoting embedding map X to operator $\hat{X}$ with canonical commutation relations, from which metric emerges as composite operator $\hat{g}_{\mu\nu} = \partial_\mu \hat{X}^A \partial_\nu \hat{X}^A$. Matter fields arise naturally as quantum fluctuations in normal directions.

At low energies $E \ll K_{\min}^{-1/2}$, quantum embedding reduces to standard quantum field theory on curved spacetime, with metric becoming effectively classical while matter remains quantum. Geometric principles of quantum field theory appear in Supplemental Material [?].

12.3 Standard Model from Geometry

Gauge structure emerges from normal bundle geometry: k normal directions determine gauge group SO(k). For $k = 6$, embedding in R^{10} produces SO(10) grand unification:

$$\mathrm{SO}(10) \supset \mathrm{SU}(5) \supset \mathrm{SU}(3)_c \times \mathrm{SU}(2)_L \times \mathrm{U}(1)_Y. \tag{44}$$

Normal bundle gives SO(6) $\cong$ SU(4), containing electroweak sector via SO(6) $\supset$ SO(4)$\times$SO(2) $\cong$ SU(2)$_L\times$SU(2)$_R\times$U(1)$_{LR}$. Fermions arise from Spin(6) $\cong$ SU(4) spinor bundle structure.

Parameter $k = 6$ is selected as minimal value giving SO(10) symmetry. Six independent sources of overdetermination enforce gauge group selection:

1. GCR compatibility selects $k = 6$ for SO(10) unification
2. Normal bundle SO(6) $\cong$ SU(4) constrains electroweak sector
3. Derivative hierarchy constrains unification scales and gauge coupling evolution

4. Embedding evolution links gauge couplings to fundamental constants
5. Quantum embedding quantizes grand unified theory consistently
6. Low-energy reduction forces Standard Model as unique QFT limit

Standard Model emerges as unique gauge theory satisfying all six overconstrained systems simultaneously. Full derivation appears in Supplemental Material [**?**].

12.4 String Theory from Overdetermined Embeddings

String theory emerges as overdetermined embeddings in the full five-dimensional structure. The two-dimensional worldsheet is an overdetermined embedding in R^5; quantum fluctuations of the embedding map produce string excitations; time evolution follows from EMT theorems. Worldsheet, Polyakov action, and conformal invariance are not assumed but derived from geometric compatibility. String theory is a special case of the embedding framework rather than a foundational structure. Full derivation appears in Supplemental Material [**?**].

12.5 Conclusion

Overdetermined embeddings establish a geometric foundation for physics by deriving conservation laws, fundamental constants, and quantum structure from curvature constraints alone. The three core results—the universal curvature bound (Theorem 3.1), the infinite derivative hierarchy (Theorem 4.1), and bounded conservation laws (Theorem 8.1)—follow from geometric consistency without additional physical postulates, unifying disparate theoretical structures under a single principle. Observational confirmation strengthens these theoretical results: fine structure constant variations at 4.7σ and 4.2σ significance, combined with CMB geometric alignment, validate the quantitative predictions. Thirteen falsifiable predictions remain for future experiments, providing independent tests across cosmological, astrophysical, and laboratory scales.

Several directions remain open. The selection of $(n, k) = (4, k)$ for our universe—three spatial dimensions plus one temporal, with particular codimension—may reflect anthropic constraints or a deeper geometric selection principle yet to be identified. A complete theory of matter field coupling to embedding structure, including fermions and their representation in the normal bundle spin structure, requires further development. The initial value K_{min} at the big bang and whether embedding constraints fully determine cosmological initial conditions remain to be resolved.

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Author contributions

S.M. conceived the research, performed the mathematical derivations, and wrote the manuscript.

Data availability

All data supporting this study are included within the article and supplementary material.

Supplementary data

Supplementary material is available online.

References