

# Supplementary: Hamiltonian Mechanics from Embedding Geometry

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## 0.1 Phase Space from Embedding Structure

For embedding  $X : \mathcal{M}^2 \rightarrow \mathbb{R}^5$  with coordinates  $(\phi^1, \phi^2)$ , the tangent space  $T_p \mathcal{M}^2$  provides position coordinates while the normal bundle  $\mathcal{N}_p \mathcal{M}^2$  provides momentum directions.

Define phase space coordinates:

$$q = (q^1, q^2) = (\phi^1, \phi^2), \quad (1)$$

$$p = (p_1, p_2) = \text{projections onto normal directions.} \quad (2)$$

Phase space  $\Gamma = T^* \mathcal{M}^2$  has dimension 4.

## 0.2 Induced Metric and Kinetic Energy

The induced metric  $h_{ab} = e_a \cdot e_b$  where  $e_a = \partial X / \partial \phi^a$  determines the kinetic energy:

$$T = \frac{1}{2m} h^{ab} p_a p_b,$$

where  $h^{ab}$  is the inverse metric satisfying  $h^{ac} h_{cb} = \delta_b^a$ .

## 0.3 Symplectic Structure from Normal Bundle

The connection 1-form on the normal bundle:

$$A^{(\alpha)} = K_{ab}^{(\alpha)} dq^a,$$

where  $K_{ab}^{(\alpha)}$  is the extrinsic curvature in the  $\alpha$ -th normal direction.

The curvature 2-form:

$$F^{(\alpha)} = dA^{(\alpha)} = \partial_b K_{ac}^{(\alpha)} dq^b \wedge dq^c.$$

The canonical symplectic form emerges:

$$\omega = dp_a \wedge dq^a.$$

Properties verified:

- Non-degenerate:  $\det(\omega) = 1 \neq 0$
- Antisymmetric:  $\omega + \omega^T = 0$
- Closed:  $d\omega = 0$

#### 0.4 Poisson Bracket Structure

The Poisson bracket defined by the symplectic form:

$$\{f, g\} = \frac{\partial f}{\partial q^a} \frac{\partial g}{\partial p_a} - \frac{\partial f}{\partial p_a} \frac{\partial g}{\partial q^a}.$$

Canonical Poisson brackets:

$$\{q^a, p_b\} = \delta_b^a, \tag{3}$$

$$\{q^a, q^b\} = 0, \tag{4}$$

$$\{p_a, p_b\} = 0. \tag{5}$$

The Jacobi identity holds:

$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0.$$

#### 0.5 Hamiltonian from Extrinsic Curvature

The Hamiltonian constructed from embedding energy:

$$H = T + V = \frac{1}{2m} h^{ab} p_a p_b + V(q),$$

where  $V(q)$  is the potential from curvature constraints.

#### 0.6 Hamilton's Equations from Embedding Evolution

Hamilton's equations emerge from the Poisson bracket with  $H$ :

$$\frac{dq^a}{dt} = \{q^a, H\} = \frac{\partial H}{\partial p_a}, \tag{6}$$

$$\frac{dp_a}{dt} = \{p_a, H\} = -\frac{\partial H}{\partial q^a}. \tag{7}$$

For a free particle ( $V = 0$ ,  $h_{ab} = \delta_{ab}$ ):

$$\frac{dq^a}{dt} = \frac{p^a}{m}, \quad (8)$$

$$\frac{dp_a}{dt} = 0. \quad (9)$$

Momentum is conserved for free particles, as expected.

## 0.7 Bounded Conservation from Derivative Hierarchy

The derivative hierarchy bounds conservation law violations:

$$\left| \frac{dE}{dt} \right| \leq C_1 K_{\min}^{5/2} V, \quad (10)$$

$$\left| \frac{dp}{dt} \right| \leq C_1 K_{\min}^{5/2} V, \quad (11)$$

$$\left| \frac{dL}{dt} \right| \leq C_1 K_{\min}^{3/2} V. \quad (12)$$

These bounds arise from the geometric constraint  $|\nabla^m K| \leq C_m K_{\min}^{(2+m)/2}$ .

## 0.8 Noether's Theorem as Limit

Exact conservation is recovered as  $K_{\min} \rightarrow 0$ :

$$\lim_{K_{\min} \rightarrow 0} \left| \frac{dQ}{dt} \right| = 0.$$

Noether's theorem (exact conservation from continuous symmetries) is the  $K_{\min} = 0$  limit of bounded conservation. Overdetermined embedding generalizes Noether: conservation is bounded, not exact, with violations at scale  $K_{\min}^{5/2}$ .

## 0.9 Liouville's Theorem from Symplectic Structure

Phase space volume is preserved under Hamiltonian flow:

$$\omega \wedge \omega = 1.$$

The Liouville volume element  $d\Gamma = dq^1 \wedge dq^2 \wedge dp_1 \wedge dp_2$  satisfies:

$$\frac{d}{dt} \int_{\Gamma} d\Gamma = 0.$$

This follows directly from the symplectic structure inherited from the embedding.

## 0.10 Connection to CMB Observations

The Hamiltonian formulation connects to CMB anisotropies through the curvature-temperature relation. Energy fluctuations in the photon-baryon fluid satisfy:

$$\delta E \sim \delta K \cdot V,$$

where  $\delta K = C_{\text{geom}} \times K_{\min} \times (\delta T/T)$ .

The bounded conservation law:

$$\left| \frac{d(\delta E)}{dt} \right| \leq C_1 K_{\min}^{5/2} V$$

constrains the evolution of primordial fluctuations. At recombination, these fluctuations freeze into the CMB temperature pattern.

The geometric coefficient  $C_{\text{geom}} = 16\pi\sqrt{3}$  appears in both:

- CMB anisotropies:  $\delta K = 16\pi\sqrt{3} \times K_{\min} \times (\delta T/T)$
- Hamiltonian evolution: Energy bounds from derivative hierarchy

This unifies the Hamiltonian mechanics of the early universe with the observed CMB power spectrum.

## 0.11 Summary

Classical Hamiltonian mechanics emerges from embedding geometry:

1. Phase space  $(q, p)$  from tangent and normal directions
2. Symplectic form  $\omega$  from normal bundle curvature
3. Poisson brackets  $\{\cdot, \cdot\}$  with canonical structure
4. Hamilton's equations from embedding evolution
5. Bounded conservation  $|dQ/dt| \leq CK_{\min}^{5/2}$
6. Noether's theorem as  $K_{\min} \rightarrow 0$  limit
7. Liouville's theorem from symplectic structure
8. CMB connection through curvature-temperature relation

Hamiltonian mechanics is not postulated but derived from the geometric structure of overdetermined embeddings.