

# Supplementary: Hamiltonian Constraint and Dimensional Emergence

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## 0.1 Height Field Derivation from Geometric Necessity

### 0.1.1 Embedding Structure and Induced Metric

Consider a 2-dimensional Riemannian manifold  $\mathcal{M}^2$  with local coordinates  $(\varphi^1, \varphi^2)$  smoothly embedded in 5-dimensional Euclidean space  $\mathbb{R}^5$ . The embedding is given by a smooth map  $X : \mathcal{M}^2 \rightarrow \mathbb{R}^5$  with components  $y^A(\varphi^1, \varphi^2)$  for  $A = 1, \dots, 5$ .

The induced metric on  $\mathcal{M}^2$  is defined by pulling back the Euclidean metric on  $\mathbb{R}^5$ :

$$h_{ab} = \frac{\partial y^A}{\partial \phi^a} \frac{\partial y^A}{\partial \phi^b},$$

where we use the Einstein summation convention for the ambient space index  $A$ . The determinant  $\det(h_{ab})$  provides the area element on  $\mathcal{M}^2$ .

### 0.1.2 Triorthogonal Normal Frame and Extrinsic Curvature

For a 2-dimensional manifold embedded in  $\mathbb{R}^5$ , there exist three independent orthonormal normal directions at each point. We denote these normal vectors by  $n^{(\alpha)}$  for  $\alpha = 1, 2, 3$ , where each  $n^{(\alpha)}$  is a unit vector in  $\mathbb{R}^5$  orthogonal to the tangent space of  $\mathcal{M}^2$ .

The extrinsic curvature tensor for the  $\alpha$ -th normal direction is defined as the second fundamental form:

$$K_{ab}^{(\alpha)} = -n_A^{(\alpha)} \frac{\partial^2 y^A}{\partial \phi^a \partial \phi^b},$$

where  $n_A^{(\alpha)}$  are the components of the normal vector in the ambient space. This tensor quantifies how  $\mathcal{M}^2$  curves into the ambient space along the  $\alpha$ -th normal direction.

The extrinsic curvature tensors are symmetric:  $K_{ab}^{(\alpha)} = K_{ba}^{(\alpha)}$ , which follows from the equality of mixed partial derivatives and the orthogonality condition  $n_A^{(\alpha)}(\partial y^A/\partial \varphi^a) = 0$ .

### 0.1.3 Gauss-Codazzi Equations and Intrinsic Curvature

The fundamental equations of submanifold geometry relate the intrinsic curvature of  $\mathcal{M}^2$  to its extrinsic curvature structure. The Gauss equation provides the primary relation:

$$R_{ab} = \sum_{\alpha=1}^3 \left( K_{ac}^{(\alpha)} K^{(\alpha)c}{}_b - K_{ab}^{(\alpha)} K^{(\alpha)} \right),$$

where  $R_{ab}$  is the Ricci tensor of the induced metric  $h_{ab}$ ,  $K^{(\alpha)} = h^{cd} K_{cd}^{(\alpha)}$  is the mean curvature in the  $\alpha$ -th normal direction, and indices are raised using the inverse metric  $h^{ab}$ .

Contracting the Gauss equation with the inverse metric yields the Ricci scalar:

$$R = h^{ab} R_{ab} = \sum_{\alpha=1}^3 \left( K_{ac}^{(\alpha)} K^{(\alpha)ac} - (K^{(\alpha)})^2 \right).$$

For a 2-dimensional manifold, the Ricci scalar is related to the Gauss curvature by  $R = 2K_G$ , where  $K_G$  is the Gauss curvature. The Gauss curvature can also be expressed in terms of the principal curvatures  $\kappa_1$  and  $\kappa_2$ :

$$K_G = \kappa_1 \kappa_2.$$

### 0.1.4 Principal Curvatures and Minimum Curvature Constraint

At each point  $p \in \mathcal{M}^2$ , the extrinsic curvature tensor  $K_{ab}^{(\alpha)}$  has two real eigenvalues  $\kappa_1$  and  $\kappa_2$ , called the principal curvatures. These characterize the local bending of the surface in the normal direction.

The mean curvature is given by:

$$H = \frac{\kappa_1 + \kappa_2}{2} = \frac{1}{2} h^{ab} K_{ab}^{(\alpha)}.$$

The minimum curvature constraint states that:

$$K_G = \kappa_1 \kappa_2 \geq K_{\min}^2 > 0,$$

where  $K_{\min}$  is a coordinate-independent minimum value determined by the embedding geometry.

This constraint forces non-trivial curvature at every point. Since  $K_G > 0$ , both principal curvatures must have the same sign; the bound  $K_G \geq K_{\min}^2$  further implies that at least one principal curvature satisfies  $|\kappa_i| \geq K_{\min}$  at every point.

### 0.1.5 Height Field Emergence from Geometric Necessity

The minimum curvature constraint  $K_G \geq K_{\min}^2 > 0$  forces the manifold to curve into the ambient space. This curvature manifests as a displacement field in the normal directions. We define the height field  $x^3(\varphi^1, \varphi^2)$  as the displacement along the third normal direction  $n^{(3)}$ .

When the manifold curves into the ambient space, it extends in the normal directions. The height field  $x^3$  measures this extension along  $n^{(3)}$ . For a surface with non-zero principal curvatures, the surface must bulge outward (or inward) in at least one normal direction, creating a non-zero height field.

Geometric necessity follows from:

1. The constraint  $K_G = \kappa_1 \kappa_2 \geq K_{\min}^2 > 0$  forces at least one principal curvature to satisfy  $|\kappa_i| \geq K_{\min}$ .
2. Non-zero principal curvatures mean the surface curves into the ambient space.
3. This curvature creates a displacement in the normal directions.
4. The displacement along  $n^{(3)}$  is precisely the height field  $x^3(\varphi^1, \varphi^2)$ .
5. Since the constraint holds at every point, we generically have  $x^3 \neq 0$ .

### 0.1.6 Poisson Equation Derivation from Embedding Geometry

The height field  $x^3(\varphi^1, \varphi^2)$  must satisfy a differential equation determined by the embedding structure.

In the flat coordinate system  $(\varphi^1, \varphi^2)$  on  $\mathcal{M}^2$ , the Laplacian of the height field is:

$$\nabla^2 x^3 = \frac{\partial^2 x^3}{\partial(\phi^1)^2} + \frac{\partial^2 x^3}{\partial(\phi^2)^2}.$$

From embedding geometry, the Laplacian of the height field relates to the mean curvature. Specifically, for a surface embedded in a higher-dimensional space, the height field satisfies:

$$\nabla^2 x^3 = K(\phi^1, \phi^2),$$

where  $K(\varphi^1, \varphi^2)$  is a function related to the curvature structure.

The source term  $K$  can be expressed in terms of the principal curvatures and Gauss curvature:

$$K = \frac{1}{2}(\kappa_1 + \kappa_2)^2 - K_G = 2H^2 - K_G,$$

or alternatively:

$$K = H^2 - \frac{1}{4}K_G.$$

### 0.1.7 Dimensional Emergence

When  $x^3(\varphi^1, \varphi^2) \neq 0$ , the 2-dimensional surface sweeps out a 3-dimensional volume, creating an emergent third spatial dimension. The volume element on the emergent manifold  $\mathcal{M}^3$  takes the form:

$$dV = \sqrt{\det(h_{ab})} |x^3(\phi^1, \phi^2)| d\phi^1 d\phi^2,$$

where  $h_{ab}$  is the induced metric on  $\mathcal{M}^2$  and  $|x^3|$  accounts for the extension in the third dimension.

The minimum curvature constraint  $K_G \geq K_{\min}^2 > 0$  guarantees that  $x^3 \neq 0$  generically, ensuring dimensional emergence  $\mathcal{M}^2 \rightarrow \mathcal{M}^3$  occurs for all embeddings satisfying the constraint.

## 0.2 Hamiltonian Constraint from Volume Element

### 0.2.1 Volume Element and Configuration Space Measure

The volume element on the emergent 3-dimensional manifold  $\mathcal{M}^3$  is:

$$dV = \sqrt{\det(h_{ab})} |x^3(\phi^1, \phi^2)| d\phi^1 d\phi^2,$$

where  $h_{ab}$  is the induced metric on  $\mathcal{M}^2$  and  $|x^3|$  accounts for the extension in the third dimension.

When the system transitions from  $\mathcal{M}^2$  to  $\mathcal{M}^3$ , this volume element provides the measure on configuration space. For a 3-dimensional spatial manifold with metric  $h_{ij}$ , the volume element becomes:

$$dV = \sqrt{\det(h_{ij})} d^3x,$$

where  $d^3x = dx^1 dx^2 dx^3$  are the coordinate volume elements.

The configuration space  $\Phi = \text{Riem}(\mathcal{M}^3) \times \text{Sym}(\mathcal{M}^3)^3 \times \Omega^1(\mathcal{M}^3)$  consists of all possible spatial geometries  $(h_{ij}, K_{ij}, \rho)$ . The volume element  $dV$  provides the measure  $d\mu = dV$  on this configuration space.

### 0.2.2 Action Principle on Configuration Space

Dynamics are governed by an action principle:

$$S = \int L d\mu = \int L \sqrt{\det(h_{ij})} d^3x,$$

where  $L$  is the Lagrangian density depending on geometric quantities:  $L = L(h_{ij}, K_{ij}, R, \rho)$ .

The Lagrangian density includes terms for intrinsic curvature (Ricci scalar  $R$ ), extrinsic curvature ( $K_{ij}$  and its contractions), and matter fields (energy density  $\rho$  and momentum density  $j_i$ ).

Variation of the action with respect to the metric  $h_{ij}$  gives the equations of motion. Variation with respect to the lapse function  $N$  gives the Hamiltonian constraint.

### 0.2.3 Hamiltonian Constraint from Variational Principle

In the ADM formalism, the Hamiltonian constraint emerges from requiring the action be stationary under variations of the lapse function  $N$ :

$$H = \sqrt{\det(h)} \left[ \pi_{ij} \pi^{ij} - \frac{1}{2} (\pi_i^i)^2 - R \right] = 0,$$

where  $\pi_{ij}$  are the momentum variables conjugate to the metric  $h_{ij}$ ,  $\pi^{ij} = h^{ik} h^{jl} \pi_{kl}$  are the raised indices,  $\pi_i^i = h^{ij} \pi_{ij}$  is the trace, and  $R$  is the spatial Ricci scalar.

The factor  $\sqrt{\det(h)}$  comes directly from the volume element measure  $d\mu$  in the action principle, ensuring the constraint is properly weighted when integrated over space.

### 0.2.4 Connection Between Volume Element and Hamiltonian Constraint

The connection proceeds as follows:

1. The volume element  $dV = \sqrt{\det(h_{ab})} |x^3| d\varphi^1 d\varphi^2$  on  $\mathcal{M}^2$  provides the geometric measure.
2. When extended to  $\mathcal{M}^3$ , this becomes  $dV = \sqrt{\det(h_{ij})} d^3x$ , where  $h_{ij}$  is the 3D spatial metric.
3. This volume element determines the measure  $d\mu = dV$  on configuration space  $\Phi$ .

4. The action  $S = \int L d\mu$  uses this measure, so all terms are weighted by  $\sqrt{\det(h)}$ .
5. Variation with respect to the lapse function gives the Hamiltonian constraint with the  $\sqrt{\det(h)}$  factor.
6. The constraint  $H = 0$  ensures energy-momentum conservation on each spatial configuration.

### 0.2.5 Physical Interpretation

The Hamiltonian constraint  $H = 0$  has clear physical meaning:

- **Constraint Surface:** Defines a hypersurface in configuration space  $\Phi$ . Only configurations on this surface represent valid physical geometries satisfying energy-momentum conservation.
- **Energy-Momentum Conservation:** Ensures that for each spatial configuration, the relationship between momentum, curvature, and energy density satisfies GR conservation laws.
- **Algebraic Nature:** The constraint is algebraic rather than differential, relating geometric quantities on a single spatial configuration. Essential for eliminating time as fundamental.
- **Volume Weighting:** The  $\sqrt{\det(h)}$  factor ensures proper weighting when integrated over space.

### 0.2.6 Summary

The derivation establishes:

1. Minimum curvature constraint  $K_G \geq K_{\min}^2 > 0$  forces non-zero principal curvatures
2. Non-zero principal curvatures force curvature into ambient space
3. Curvature creates height field  $x^3(\varphi^1, \varphi^2)$  measuring displacement along normal direction
4. Height field satisfies Poisson equation  $\nabla^2 x^3 = K(\varphi^1, \varphi^2)$
5. Non-zero height field guarantees dimensional emergence  $\mathcal{M}^2 \rightarrow \mathcal{M}^3$

6. Volume element  $dV = \sqrt{\det(h_{ab})}|x^3|d\varphi^1d\varphi^2$  provides geometric measure
7. Extended to  $\mathcal{M}^3$ :  $dV = \sqrt{\det(h_{ij})}d^3x$
8. Determines measure  $d\mu = dV$  on configuration space
9. Action  $S = \int L d\mu$  uses this measure
10. Variation gives Hamiltonian constraint  $H = \sqrt{\det(h)}[\pi_{ij}\pi^{ij} - (1/2)(\pi_i^i)^2 - R] = 0$

This provides the mathematical foundation for dimensional emergence and the Hamiltonian constraint from embedding geometry.