SÜREKLİ OLASILIK DAĞILIMLARI:

$$P(a < X < b) = \int_a^b f(x)\ dx.$$

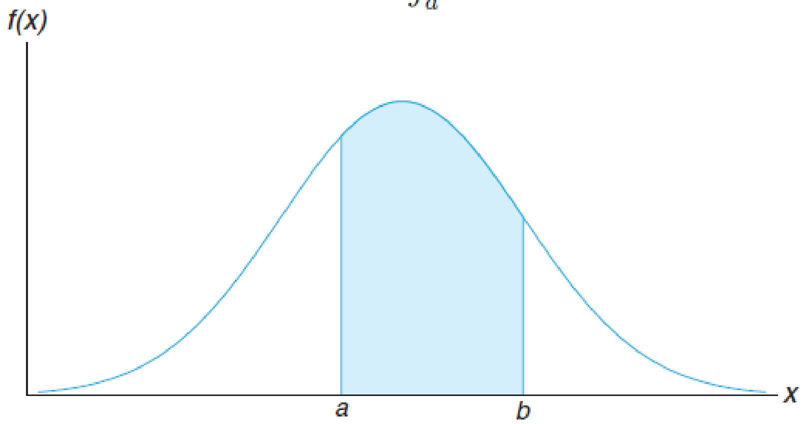

Figure 3.5: $P(a < X < b)$.

The function $f(x)$ is a **probability density function** (pdf) for the continuous random variable X, defined over the set of real numbers, if

1. $f(x) \geq 0$, for all $x \in R$.
2. $\int_{-\infty}^{\infty} f(x)\ dx = 1$.
3. $P(a < X < b) = \int_a^b f(x)\ dx$.

Example 3.11: Suppose that the error in the reaction temperature, in °C, for a controlled laboratory experiment is a continuous random variable X having the probability density function

$$f(x) = \begin{cases} \frac{x^2}{3}, & -1 < x < 2, \\ 0, & \text{elsewhere}. \end{cases}$$

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(a) Verify that $f(x)$ is a density function.

(b) Find $P(0 < X \leq 1)$.

Solution: We use Definition 3.6.

(a) Obviously, $f(x) \geq 0$. To verify condition 2 in Definition 3.6, we have

$$\int_{-\infty}^{\infty} f(x)\ dx = \int_{-1}^{2} \frac{x^2}{3} dx = \frac{x^3}{9} |_{-1}^{2} = \frac{8}{9} + \frac{1}{9} = 1.$$

(b) Using formula 3 in Definition 3.6, we obtain

$$P(0 < X \leq 1) = \int_0^1 \frac{x^2}{3} dx = \left.\frac{x^3}{9}\right|_0^1 = \frac{1}{9}.$$

BİRLEŞİK OLASILIK DAĞILIMLARI:

Definition 3.9: The function $h(x,y)$ is a **joint density function** of the continuous random variables X and Y if

1. $h(x,y) \geq 0$, for all (x,y),
2. $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(x,y)\, dx\, dy = 1$,
3. $P[(X,Y) \in A] = \int\int_A h(x,y)\, dx\, dy$, for any region A in the xy plane.

Example 3.15: A privately owned business operates both a drive-in facility and a walk-in facility. On a randomly selected day, let X and Y, respectively, be the proportions of the time that the drive-in and the walk-in facilities are in use, and suppose that the joint density function of these random variables is

$$h(x,y) = \begin{cases} \frac{2}{5}(2x+3y), & 0 \leq x \leq 1, 0 \leq y \leq 1, \\ 0, & \text{elsewhere.} \end{cases}$$

(a) Verify condition 2 of Definition 3.9.

(b) Find $P[(X,Y) \in A]$, where $A = \{(x,y) \mid 0 < x < \frac{1}{2}, \frac{1}{4} < y < \frac{1}{2}\}$.

***Solution*:** (a) The integration of $h(x,y)$ over the whole region is

$$\begin{aligned}
\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(x,y)\, dx\, dy &= \int_0^1\int_0^1 \frac{2}{5}(2x+3y)\, dx\, dy \\
&= \int_0^1 \left(\frac{2x^2}{5} + \frac{6xy}{5}\right)\Bigg|_{x=0}^{x=1} dy \\
&= \int_0^1 \left(\frac{2}{5} + \frac{6y}{5}\right) dy = \left(\frac{2y}{5} + \frac{3y^2}{5}\right)\Bigg|_0^1 = \frac{2}{5} + \frac{3}{5} = 1.
\end{aligned}$$

(b) To calculate the probability, we use

$$\begin{aligned}
P[(X,Y) \in A] &= P\left(0 < X < \frac{1}{2}, \frac{1}{4} < Y < \frac{1}{2}\right) \\
&= \int_{1/4}^{1/2}\int_0^{1/2} \frac{2}{5}(2x+3y)\, dx\, dy \\
&= \int_{1/4}^{1/2} \left(\frac{2x^2}{5} + \frac{6xy}{5}\right)\Bigg|_{x=0}^{x=1/2} dy = \int_{1/4}^{1/2}\left(\frac{1}{10} + \frac{3y}{5}\right) dy \\
&= \left(\frac{y}{10} + \frac{3y^2}{10}\right)\Bigg|_{1/4}^{1/2} \\
&= \frac{1}{10}\left[\left(\frac{1}{2} + \frac{3}{4}\right) - \left(\frac{1}{4} + \frac{3}{16}\right)\right] = \frac{13}{160}.
\end{aligned}$$

Definition 3.10: The **marginal distributions** of X alone and of Y alone are

$$f(x) = \sum_y h(x,y) \quad \text{and} \quad g(y) = \sum_x h(x,y)$$

for the discrete case, and

$$f(x) = \int_{-\infty}^{\infty} h(x,y)\, dy \quad \text{and} \quad g(y) = \int_{-\infty}^{\infty} h(x,y)\, dx$$

for the continuous case.

Example 3.17: Find $f(x)$ and $g(y)$ for the joint density function of Example 3.15.

***Solution*:** By definition,

$$f(x) = \int_{-\infty}^{\infty} h(x,y)\, dy = \int_0^1 \frac{2}{5}(2x+3y)\, dy = \left(\frac{4xy}{5} + \frac{6y^2}{10}\right)\Big|_{y=0}^{y=1} = \frac{4x+3}{5},$$

for $0 \le x \le 1$, and $g(x) = 0$ elsewhere. Similarly,

$$g(y) = \int_{-\infty}^{\infty} f(x,y)\, dx = \int_0^1 \frac{2}{5}(2x+3y)\, dx = \frac{2(1+3y)}{5},$$

for $0 \le y \le 1$, and $g(y) = 0$ elsewhere.

KOŞULLU DAĞILIMLAR:

Definition 3.11: Let X and Y be two random variables, discrete or continuous. The **conditional distribution** of the random variable Y given that $X = x$ is

$$h(y|x) = \frac{h(x,y)}{f(x)}, \quad \text{provided } f(x) > 0.$$

Similarly, the conditional distribution of X given that $Y = y$ is

$$h(x|y) = \frac{h(x,y)}{g(y)}, \quad \text{provided } g(y) > 0.$$

$$P(a < X < b \mid Y = y) = \int_a^b f(x|y)\, dx.$$

Example 3.19: The joint density for the random variables (X,Y), where X is the unit temperature change and Y is the proportion of spectrum shift that a certain atomic particle produces, is

$$h(x,y) = \begin{cases} 10xy^2, & 0 < x < y < 1, \\ 0, & \text{elsewhere.} \end{cases}$$

(a) Find the marginal densities $f(x)$, $g(y)$, and the conditional density $h(y|x)$.

(b) Find the probability that the spectrum shifts more than half of the total observations, given that the temperature is increased by 0.25 unit.

Solution: (a) By definition,

$$f(x) = \int_{-\infty}^{\infty} h(x,y)\, dy = \int_{x}^{1} 10xy^2\, dy$$
$$= \frac{10}{3}xy^3\Big|_{y=x}^{y=1} = \frac{10}{3}x(1-x^3),\ 0 < x < 1,$$
$$g(y) = \int_{-\infty}^{\infty} h(x,y)\, dx = \int_{0}^{y} 10xy^2\, dx = 5x^2y^2\Big|_{x=0}^{x=y} = 5y^4,\ 0 < y < 1.$$

Now

$$h(y|x) = \frac{h(x,y)}{f(x)} = \frac{10xy^2}{\frac{10}{3}x(1-x^3)} = \frac{3y^2}{1-x^3},\ 0 < x < y < 1.$$

(b) Therefore,

$$P\left(Y > \frac{1}{2} \,\Big|\, X = 0.25\right) = \int_{1/2}^{1} h(y \mid x = 0.25)\, dy = \int_{1/2}^{1} \frac{3y^2}{1-0.25^3}\, dy = \frac{8}{9}.$$

BAĞIMSIZLIK:

Definition 3.12: Let X and Y be two random variables, discrete or continuous, with joint probability distribution $h(x,y)$ and marginal distributions $g(x)$ and $h(y)$, respectively. The random variables X and Y are said to be **statistically independent** if and only if

$$h(x,y) = f(x)\, g(y)$$

for all (x,y) within their range.

BEKLENEN DEĞER:

Definition 4.1: Let X be a random variable with probability distribution $f(x)$. The **mean**, or **expected value**, of X is

$$\mu = E(X) = \int_{-\infty}^{\infty} x f(x)\, dx$$

if X is continuous.

Example 4.3: Let X be the random variable that denotes the life in hours of a certain electronic device. The probability density function is

$$f(x) = \begin{cases} \frac{20{,}000}{x^3}, & x > 100, \\ 0, & \text{elsewhere.} \end{cases}$$

Find the expected life of this type of device.

Solution: Using Definition 4.1, we have

$$\mu = E(X) = \int_{100}^{\infty} x\frac{20,000}{x^3}\, dx = \int_{100}^{\infty} \frac{20,000}{x^2}\, dx = 200.$$

VARYANS:

Definition 4.3: Let X be a random variable with probability distribution $f(x)$ and mean μ. The variance of X is

$$\sigma^2 = E[(X-\mu)^2] = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\, dx, \qquad \text{if } X \text{ is continuous.}$$

The positive square root of the variance, σ, is called the **standard deviation** of X.

Theorem 4.2: The variance of a random variable X is

$$\sigma^2 = E(X^2) - \mu^2.$$

Example 4.10: The weekly demand for a drinking-water product, in thousands of liters, from a local chain of efficiency stores is a continuous random variable X having the probability density

$$f(x) = \begin{cases} 2(x-1), & 1 < x < 2, \\ 0, & \text{elsewhere.} \end{cases}$$

Find the mean and variance of X.

***Solution*:** Calculating $E(X)$ and $E(X^2)$ we have

$$\mu = E(X) = 2\int_1^2 x(x-1)\, dx = \frac{5}{3} \qquad \text{and} \qquad E(X^2) = 2\int_1^2 x^2(x-1)\, dx = \frac{17}{6}.$$

Therefore,

$$\sigma^2 = \frac{17}{6} - \left(\frac{5}{3}\right)^2 = \frac{1}{18}.$$

KOVARYANS ve KORELASYON:

Korelasyon, iki rasgele değişken arasındaki doğrusal ilişkinin yönünü ve gücünü belirtir.

Theorem 4.4: The covariance of two random variables X and Y with means μ_X and μ_Y, respectively, is given by

$$\text{Cov(X,Y)} = E(XY) - \mu_X \mu_Y.$$

Definition 4.5: Let X and Y be random variables with covariance σ_{XY} and standard deviations σ_X and σ_Y, respectively. The correlation coefficient of X and Y is

$$\rho_{XY} = \frac{\sigma_{XY}}{\sigma_X \sigma_Y}.$$

Example 4.14: The fraction X of male runners and the fraction Y of female runners who compete in marathon races are described by the joint density function

$$h(x, y) = \begin{cases} 8xy, & 0 \le y \le x \le 1, \\ 0, & \text{elsewhere.} \end{cases}$$

Find the covariance of X and Y.

***Solution*:** We first compute the marginal density functions. They are

$$f(x) = \begin{cases} 4x^3, & 0 \le x \le 1, \\ 0, & \text{elsewhere,} \end{cases} \quad \text{and} \quad g(y) = \begin{cases} 4y(1-y^2), & 0 \le y \le 1, \\ 0, & \text{elsewhere.} \end{cases}$$

From these marginal density functions, we compute

$$\mu_X = E(X) = \int_0^1 4x^4 \, dx = \frac{4}{5} \text{ and } \mu_Y = \int_0^1 4y^2(1-y^2) \, dy = \frac{8}{15}.$$

From the joint density function given above, we have

$$E(XY) = \int_0^1 \int_y^1 8x^2y^2 \, dx \, dy = \frac{4}{9}.$$

Then

$$Cov(X,Y) = E(XY) - \mu_X \mu_Y = \frac{4}{9} - \left(\frac{4}{5}\right)\left(\frac{8}{15}\right) = \frac{4}{225}.$$

Example 4.16: Find the correlation coefficient of X and Y in Example 4.14.

***Solution*:** Because

$$E(X^2) = \int_0^1 4x^5 \, dx = \frac{2}{3} \text{ and } E(Y^2) = \int_0^1 4y^3(1-y^2) \, dy = 1 - \frac{2}{3} = \frac{1}{3},$$

we conclude that

$$\sigma_X^2 = \frac{2}{3} - \left(\frac{4}{5}\right)^2 = \frac{2}{75} \text{ and } \sigma_Y^2 = \frac{1}{3} - \left(\frac{8}{15}\right)^2 = \frac{11}{225}.$$

Hence,

$$\rho_{XY} = \frac{4/225}{\sqrt{(2/75)(11/225)}} = \frac{4}{\sqrt{66}}.$$