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GPS single point positioning algorithm

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1 Computation of receiver's position

This paper describes the computation of the position of a GPS receiver based on L1 pseudorange observations, here denoted by P. The algorithm is summarised in Table 1.
The GPS satellites continuously transmit signals on two bands: L1 and L2. Usually, only L1 is used for navigation purposes. There are three data streams modulated on L1: C/A ranging code, P ranging code and navigation message. The receiver correlates the received and generated ranging codes by shifting the generated ones with respect to the received ones. The amount of the shift is a measure of the travel time of the signal from satellite to the receiver. The travel time derived from the code correlation multiplied by the speed of light is called pseudorange. The mathematical definition of the pseudorange is given by the following equation:

$$P_A^s\left(\tilde{t}_A\right) = \left(\tilde{t}_A - \tilde{t}^s\right)c \tag{1}$$

where

$P_A^s\left(\tilde{t}_A\right)$	pseudorange measured at time $\tilde{t}_A$ by receiver A to satellite s
$\tilde{t}_A$	nominal time of the signal reception measured by the clock of receiver A
$\tilde{t}^s$	nominal time of signal transmission measured by the clock of satellite s

It is evident, that the pseudorange is directly affected by the satellite and receiver clock errors. The clock errors are defined by the following equations:

$$\begin{aligned} t_A &= \tilde{t}_A - \delta t_A \\ t^s &= \tilde{t}^s - \delta t^s \end{aligned} \tag{2}$$

where

δt_A	receiver clock error
δt^s	satellite clock error
t_A	system time of the signal reception
t^s	system time of the signal transmission

By introducing (2) into Equation (1), we get:

$$P_A^s(\tilde{t}_A) = \rho_A^s(t_A) + c\,\delta t_A - c\,\delta t^s \tag{3}$$

where $\rho_A^s(t_A)$ is geometric distance between receiver A and satellite s at time t_A (also called topocentric distance):

$$\rho_A^s(t_A) = c\left(t_A - t^s\right) = c\Delta t_A^s \tag{4}$$

Since the system (true) time t_A is unknown, the topocentric distance must be linearised around the known nominal receiver time $\tilde{t}_A$:

$$\rho_A^s(t_A) = \rho_A^s(\tilde{t}_A) - \dot{\rho}_A^s(\tilde{t}_A)\delta t_A \tag{5}$$

The second term in right side of equation (5) is at most about 0.8 m, therefore it can be neglected in case of code single point positioning (other error sources are larger).
We can relate the distance to the coordinates using Pythagoras' Theorem as follows:

$$\rho_A^s(\tilde{t}_A) = \sqrt{\left(X^s - x_A\right)^2 + \left(Y^s - y_A\right)^2 + \left(Z^s - z_A\right)^2} \tag{6}$$

where

X^s, Y^s, Z^s Cartesian coordinates of satellite s in WGS84

x_A, y_A, z_A Cartesian coordinates of receiver A expressed in rotated reference frame, due to the rotation of the Earth during the signal's travel time.

The coordinates are related to WGS84 coordinates by:

$$\begin{aligned} x_A &= X_A - \dot{\Omega}_e Y_A \Delta t^s \\ y_A &= Y_A + \dot{\Omega}_e X_A \Delta t^s \\ z_A &= Z_A \end{aligned} \tag{7}$$

X^A, Y^A, Z^A are WGS84 coordinates of the receiver, which are to be computed and $\dot{\Omega}_e$ is the rotation rate of the Earth. Looking at Equations (3) - (7), we can see that there are four unknowns, namely three receiver coordinates X^A, Y^A, Z^A and receiver clock error δt_A. The satellite coordinates X^s, Y^s, Z^s can be computed using orbit parameters, satellite clock error δt^s can be computed using clock polynomial coefficients transmitted to the user in the navigation message, and the pseudorange $P_A^s\left(\tilde{t}_A\right)$ is measured by the GPS receiver.

Taking into account the tropospheric $T_A^s(\tilde{t}_A)$ and ionospheric $I_A^s(\tilde{t}_A)$ delays as well as Equation (5) neglecting term with δt_A, the observation equation (3) can be written as:

$$P_A^s(\tilde{t}_A) = \rho_A^s(\tilde{t}_A) + c\,\delta t_A - c\,\delta t_{L1}^s + I_A^s(\tilde{t}_A) + T_A^s(\tilde{t}_A) \tag{8}$$

If we insert Equation (7) into (6) and then linearise it using the Taylor's expansion, and neglecting higher order terms, we get:

$$\rho_A^s(\tilde{t}_A) = \rho_{A0}^s(\tilde{t}_A) + \frac{\partial \rho_{A0}^s(\tilde{t}_A)}{\partial X_{A0}}\Delta X + \frac{\partial \rho_{A0}^s(\tilde{t}_A)}{\partial Y_{A0}}\Delta Y + \frac{\partial \rho_{A0}^s(\tilde{t}_A)}{\partial Z_{A0}}\Delta Z \tag{9}$$

$$
\begin{aligned}
\frac{\partial \rho_{A0}^{s}(t_A)}{\partial X_{A0}} &= a_X^s = -\frac{X^s - X_{A0} + Y^s \dot{\Omega}_e \Delta t - X_{A0} \dot{\Omega}_e^2 \Delta t^{s^2}}{\rho_{A0}^s} \\
\frac{\partial \rho_{A0}^{s}(t_A)}{\partial Y_{A0}} &= a_Y^s = -\frac{Y^s - Y_{A0} - X^s \dot{\Omega}_e \Delta t - Y_{A0} \dot{\Omega}_e^2 \Delta t^{s^2}}{\rho_{A0}^s} \\
\frac{\partial \rho_{A0}^{s}(t_k)}{\partial Z_{A0}} &= a_Z^s = -\frac{Z^s - Z_{A0}}{\rho_{A0}^s}
\end{aligned}
\tag{10}
$$

where

$$
\rho_{A0}^s(\tilde{t}_A) = \sqrt{\left(X^s - X_{A0} + \dot{\Omega}_e Y_{A0} \Delta t^s\right)^2 + \left(Y^s - Y_{A0} - \dot{\Omega}_e X_{A0} \Delta t^s\right)^2 + \left(Z^s - Z_{A0}\right)^2} \tag{11}
$$

The terms containing Δt^s in Equations (10) are negligible, so the coefficients can be computed by simplified equations:

$$
\begin{aligned}
\frac{\partial \rho_{A0}^{s}(t_A)}{\partial X_{A0}} &= a_X^s = -\frac{X^s - X_{A0}}{\rho_{A0}^s} \\
\frac{\partial \rho_{A0}^{s}(t_A)}{\partial Y_{A0}} &= a_Y^s = -\frac{Y^s - Y_{A0}}{\rho_{A0}^s} \\
\frac{\partial \rho_{A0}^{s}(t_k)}{\partial Z_{A0}} &= a_Z^s = -\frac{Z^s - Z_{A0}}{\rho_{A0}^s}
\end{aligned}
\tag{12}
$$

X_{A0}, Y_{A0} and Z_{A0} are approximate receiver coordinates, ΔX, ΔY, ΔZ are unknown corrections to the approximate coordinates and Δt_A^s is signal propagation time that can be computed with sufficient accuracy by:

$$
\Delta t_A^s = \frac{P_A^s(\tilde{t}_A)}{c} \tag{13}
$$

$\dot{\Omega}_e$ is the rotation rate of the Earth, numerically given in Section 5.
X^s, Y^s, Z^s are WGS84 coordinates of satellite s computed at time of signal transmission t^s. The nominal transmission time $\tilde{t}^s$ is computed by:

$$
\tilde{t}^s = \tilde{t}_A - \frac{P_A^s(\tilde{t}_A)}{c} \tag{14}
$$

The system transmission time t^s is computed as:

$$
t^s = \tilde{t}^s - \delta t_{L1}^s \tag{15}
$$

Where δt_{L1}^s is satellite clock correction computed by Equation (24).

The observation equation (8) can be then rewritten as:

$$P_A^s(\tilde{t}_A) - \rho_{A0}^s(\tilde{t}_A) + c\,\delta t_{L1}^s - I_A^s(\tilde{t}_A) - T_A^s(\tilde{t}_A) = a_X^s \Delta X + a_Y^s \Delta Y + a_Z^s \Delta Z + c\delta t_A \tag{16}$$

The left side of this equation contains the terms that are known (measured and computed), while the right side contains terms with unknown parameters ΔX, ΔY, ΔZ and δt_A. These parameters can be solved if at least 4 measurements are available. The parameters are solved by the least-squares (LSQ) method. This method is most conveniently expressed in matrix notation. Observation equations – system of equations to be solved:

$$\mathbf{L} - \mathbf{v} = \mathbf{AX} \tag{17}$$

where **v** is vector of unknown residuals (random errors of measurements). LSQ solution of (17):

$$\mathbf{X} = (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\mathbf{L} \tag{18}$$

L is the vector with n elements, where n is the number of observed satellites:

$$\mathbf{L} = \begin{bmatrix} P_A^1(\tilde{t}_A) - \rho_{A0}^1(\tilde{t}_A) + c\,\delta t_{L1}^1 - I_A^1(\tilde{t}_A) - T_A^1(\tilde{t}_A) \\ \vdots \\ P_A^n(\tilde{t}_A) - \rho_{A0}^n(\tilde{t}_A) + c\,\delta t_{L1}^n - I_A^n(\tilde{t}_A) - T_A^n(\tilde{t}_A) \end{bmatrix} \tag{19}$$

A is the design matrix containing the coefficients at unknown parameters. Its size is n x 4:

$$\mathbf{A} = \begin{bmatrix} a_X^1 & a_Y^1 & a_Z^1 & 1 \\ \vdots & \vdots & \vdots & \vdots \\ a_X^n & a_Y^n & a_Z^n & 1 \end{bmatrix} \tag{20}$$

X is the vector of the four unknowns:

$$\mathbf{X} = \begin{bmatrix} \Delta X & \Delta Y & \Delta Z & c\delta t_A \end{bmatrix}^T \tag{21}$$

The estimated coordinates of the receiver are finally computed by:

$$\begin{aligned} X_A &= X_{A0} + \Delta X \\ Y_A &= Y_{A0} + \Delta Y \\ Z_A &= Z_{A0} + \Delta Z \end{aligned} \tag{22}$$

Please note that we estimate the receiver clock error multiplied by the velocity of light ($c\delta t_A$), rather than clock error (δt_A), to get better numerical stability of the solution.

Table 1. Algorithm for computation of receiver's position.

1. Compute signal propagation time by (13).
2. Compute signal transmission time $\tilde{t}^s$ by (14).
3. Compute satellite clock correction δt^s_{L1} by (24) and (25), neglect Δt_r.
4. Compute t^s using the correction from the step 3.
5. Compute eccentric anomaly (Table 2)
6. Compute Δt_r by (26) and t^s by (15).
7. Compute satellite coordinates X^s, Y^s, Z^s, for time t^s - Table 2.
8. Compute satellite clock correction δt^s_{L1} by (24) - (27).
9. Compute tropospheric correction $T^s_A(\tilde{t}_A)$.
10. Compute ionospheric correction $I^s_A(\tilde{t}_A)$.
11. Compute approximate distance $\rho^s_{A0}(\tilde{t}_A)$ by (11).
12. Repeat steps 1 – 11 for all measured satellites.
13. Compute elements of vector **L** (19).
14. Compute elements of matrix **A** (20); a^s_X, a^s_Y, a^s_Z by (12).
15. Estimate unknown parameters by (18).
16. Update receiver coordinates by (22).
17. Repeat steps 11 – 16 until the solution has converged. The solution has converged if the following condition is fulfilled: $\left|\left(\mathbf{v}^T\mathbf{v}\right)_i - \left(\mathbf{v}^T\mathbf{v}\right)_{i-1}\right| < \varepsilon$, where ε is a small number and depends on the numerical accuracy, ε = 1e-5 should suffice to preserve mm numerical precision of the computed coordinates; i is iteration number. The vector **v** is computed after step 13 by Equation (17).

2 Calculation of satellite coordinates

This section describes the computation of WGS84 coordinates of a GPS satellite for time of signal transmission t^s, computed by Equation (14) and (2). The coordinates are computed by the equations listed in Table 2. The necessary orbital parameters are found in the RINEX navigation file – see chapter 4.
In the course of the coordinates computation, the Kepler's equation for eccentric anomaly must be solved. The following iterative solution is applied:

$$\begin{aligned} E_0 &= M_k \\ E_i &= M_k + e\sin E_{i-1}, \quad i = 1,2,3\ldots \end{aligned} \tag{23}$$

The iteration can be stopped if $\left|E_i - E_{i-1}\right| < \varepsilon$, where ε is a small number. We recommend ε = 1e-13. Usually 4 or 5 iterations suffice.

Table 2. Computation of satellite coordinates from orbital parameters. Source: Interface control document ICD-GPS-200.

$\mu = 3.986005 \text{ x } 10^{14}\ \text{m}^3/\text{s}^2$	Earth' universal gravitational parameter
$\dot{\Omega}_e = 7.2921151467 \text{ x } 10^{-5}$ rad/s	Earth' rotation rate
$A = \left(\sqrt{A}\right)^2$	Semi-major axis
$n_0 = \sqrt{\frac{\mu}{A^3}}$	Computed mean motion
$t_k = t^s - t_{oe}$ $t_k = \begin{cases} t_k - 604800 & \text{if} \quad t_k > 302400 \\ t_k + 604800 & \text{if} \quad t_k < -302400 \\ t_k & \text{otherwise} \end{cases}$	Time from ephemeris reference epoch
$n = n_0 + \Delta n$	Corrected mean motion
$M_k = M_0 + n\, t_k$	Mean anomaly
$E_k = M_k + e \sin E_k$	Kepler's equation for eccentric anomaly. Solution by Eq. (23)
$\sin \nu_k = \frac{\sqrt{1-e^2}\sin E_k}{1 - e\cos E_k}$ $\cos \nu_k = \frac{\cos E_k - e}{1 - e\cos E_k}$	True anomaly. Correct quadrant of ν_k must be determined. We recommend using atan2 function.
$\Phi_k = \nu_k + \omega$	Argument of latitude
$\delta u_k = c_{us}\sin 2\Phi_k + c_{uc}\cos 2\Phi_k$	Argument of latitude correction
$\delta r_k = c_{rs}\sin 2\Phi_k + c_{rc}\cos 2\Phi_k$	Radius correction
$\delta i_k = c_{is}\sin 2\Phi_k + c_{ic}\cos 2\Phi_k$	Inclination correction
$u_k = \Phi_k + \delta u_k$	Corrected argument of latitude
$r_k = A\left(1 - e\cos E_k\right) + \delta r_k$	Corrected radius
$i_k = i_0 + \delta i_k + (\text{IDOT})\, t_k$	Corrected inclination
$x_k^{'} = r_k \cos u_k$ $y_k^{'} = r_k \sin u_k$	Position in orbital plane
$\Omega_k = \Omega_0 + \left(\dot{\Omega} - \dot{\Omega}_e\right) t_k - \dot{\Omega}_e t_{oe}$	Corrected longitude of ascending node
$x_k = x_k^{'}\cos\Omega_k - y_k^{'}\cos i_k \sin\Omega_k$ $y_k = x_k^{'}\sin\Omega_k + y_k^{'}\cos i_k \cos\Omega_k$ $z_k = y_k^{'}\sin i_k$	WGS84 Cartesian coordinates, ($X^s = x_k$, $Y^s = y_k$, $Z^s = z_k$)

3 Satellite clock correction computation

The satellite clock correction δt_{L1}^{s} is needed for receiver position computation (in Equation (19)). For epoch $\tilde{t}^{s}$, it is computed by the following equation:

$$\delta t_{L1}^{s} = \Delta t_{SV} - T_{GD} \tag{24}$$

where

$$\Delta t_{SV} = a_{f0} + a_{f1}\left(\tilde{t}^{s} - t_{oc}\right) + a_{f2}\left(\tilde{t}^{s} - t_{oc}\right)^{2} + \Delta t_{r} \tag{25}$$

Δt_r is the relativistic correction given by:

$$\Delta t_{r} = Fe\sqrt{A}\sin E_{k} \tag{26}$$

where F is a constant:

$$F = -4.442807633e-10\ \mathrm{s/m^{1/2}} \tag{27}$$

e and A are orbit parameters and E_k is eccentric anomaly computed in course of satellite coordinates computation (see Table 2). Please note that $\left(\tilde{t}^{s} - t_{oc}\right)$ is the time difference expressed in seconds.

4 Identification of orbit parameters in RINEX format

RINEX format is defined in [2]. There are several types of RINEX files: observation, meteorological, message and navigation files. Only navigation files are relevant in this report. The name of a GPS navigation file is usually in form *ssssdddf.yyN*
where

ssss	4-character station name designator
ddd	day of the year of first record
f	file sequence number within day
yy	year
N	stands for “navigation”

If *f* = 0, file contains all the existing data of the current day.
Navigation file is a text file consisting of two parts: header and data records. The complete description of header is given in Table 4. Data records follow immediately after the header. One record contains data for one satellite and one reference epoch. Each record consists of 8 lines. The first line contains satellite number (PRN), reference epoch of clock parameters, and clock polynomial parameters. Lines 2 - 8 contain four numbers each. The meaning of the

numbers is described in Table 3. Format of numbers is described by FORTRAN symbolism. The meaning of used symbols:

1X one space
F9.2 floating number expressed by 9 digits (including decimal point and sign), 2 of them are reserved for decimals
A20 20 characters long string
D12.4 double-precision number expressed by 12 digits (including decimal point and sign), 4 of them are reserved for decimals
I6 integer, six digits long
Number before the type specifier (X, F, D, A, I) specifies number of repetitions. For example 4D12.4 specifies 4 double-precision numbers, each of them 12 digits long with four decimals.

5 Constants

The following constants are adopted by GPS exactly with given number of decimals.

Constant	Numerical value	Unit
c	299792458	m/s
π	3.1415926535898	
μ	3.986005×10^{14}	m^3/s^2
$\dot{\Omega}_e$	$7.2921151467 \times 10^{-5}$	rad/s

Table 3. GPS navigation message file - data record description

LINE OF RECORD	DESCRIPTION	FORMAT
1	Satellite PRN number Epoch: t_{oc} - Time of Clock year (2 digits) month day hour minute second SV clock bias a_{f0} SV clock drift a_{f1} SV clock drift rate a_{f2}	I2, 1X, I2 1X, I2, 1X, I2, 1X, I2, 1X, I2, F5.1 D19.12 D19.12 D19.12
2	IODE C_{rs} Δn M_0	3X, D19.12 D19.12 D19.12 D19.12
3	C_{uc} e C_{us} $\sqrt{A}$	3X, D19.12 D19.12 D19.12 D19.12
4	t_{oe} C_{ic} $(OMEGA)_0$, Ω_0 C_{is}	3X, D19.12 D19.12 D19.12 D19.12
5	i_0 C_{rc} ω OMEGADOT, $\dot{\Omega}$	3X, D19.12 D19.12 D19.12 D19.12
6	IDOT Code on L2 Week No. (to go with t_{oe}) L2 P data flag	3X, D19.12 D19.12 D19.12 D19.12
7	SV accuracy SV health T_{GD} IODC	3X, D19.12 D19.12 D19.12 D19.12
8	Transmission time of message Fit interval (Zero if not known) Spare Spare	3X, D19.12 D19.12 D19.12 D19.12

Table 4. GPS navigation message file - header section description, [2]

HEADER LABEL (Columns 61-80)	DESCRIPTION	FORMAT
RINEX VERSION / TYPE	- Format version (2.10) - File type ('N' for Navigation data)	F9.2, 11X, A1, 19X
PGM / RUN BY / DATE	- Name of program creating current file - Name of agency creating current file - Date of file creation	A20, A20, A20
COMMENT	Comment line(s)	A60
ION ALPHA	Ionosphere parameters α_0 - α_3	2X, 4D12.4
ION BETA	Ionosphere parameters β_0 - β_3	2X, 4D12.4
DELTA-UTC: A0,A1,T,W	Almanac parameters to compute time in UTC A0, A1: terms of polynomial T : reference time for UTC data W : UTC reference week number. Continuous number, not mod(1024)!	3X, 2D19.12, 2I9
LEAP SECONDS	Delta time due to leap seconds	I6
END OF HEADER	Last record in the header section.	60X

Satellite nr

Year month day hour min sec

OMEGADOT, $\dot{\Omega}$

```
 2 04  2  2  2  0  0.0 2.677510492500D-04-6.821210263300D-12 0.000000000000D+00
    2.310000000000D+02 1.465625000000D+01 5.271290999000D-09-2.145798944750D+00
    9.164214134220D-07 2.332063857470D-02 5.649402737620D-06 5.153571390150D+03
    9.360000000000D+04 3.650784492490D-07 7.137554984610D-01 7.823109626770D-08
    9.323963691140D-01 2.492500000000D+02-1.713391849830D+00-7.938902115710D-09
    3.907305612110D-10 0.000000000000D+00 1.256000000000D+03 0.000000000000D+00
    2.800000000000D+00 0.000000000000D+00-1.862645149230D-09 2.310000000000D+02
    8.997000000000D+04
```

Table 5. Example of parameter identification in RINEX navigation file

```
     2.10           OBSERVATION DATA    G (GPS)             RINEX VERSION / TYPE
teqc  2002Mar14                         20040202 02:07:05UTCPGM / RUN BY / DATE
_OVO.0                                                      MARKER NAME
108638.0                                                    MARKER NUMBER
SWEPOS              LMV                                     OBSERVER / AGENCY
201                 ASHTECH UZ-12       UFB5                REC # / TYPE / VERS
486                 ASH700936D_M    OSOD                    ANT # / TYPE
  3104219.4530   998383.9820  5463290.5080                  APPROX POSITION XYZ
        0.0710        0.0000        0.0000                  ANTENNA: DELTA H/E/N
     1     1                                                WAVELENGTH FACT L1/2
     7    C1    L1    L2    P1    P2    D1    D2            # / TYPES OF OBSERV
    15.0000                                                 INTERVAL
Forced Modulo Decimation to 15 seconds                      COMMENT
  2004     2     2     1                           GPS         TIME OF FIRST OBS
  2004     4     1    59                           GPS         TIME OF LAST OBS
MSWin2000|IAx86-PII|bcc3                         0|486/DX+     COMMENT
teqc  2002Mar14                                  2 02:07:00UTC
IMoS GFileSrv 3.32  LMV                          2004

 04  2  2  1  0  0.0000000  0 12G24G13G 8G21G29G26G10G17G 2G
  25001257.939 3  -3240782.381 3  -2477390.27041  25001256.6
     -3326.175       -2591.814
  23106629.672 3 -12188598.444 3  -9438884.30442  23106630.26943  23106632.40442
     -3249.560       -2532.122
  20664230.640 4 -27083620.911 4 -21043552.60142  20664230.28544  20664233.24942
      1668.628        1300.232
  24835458.607 3  -1852700.687 3  -1409814.31741  24835459.14243  24835461.18641
      2256.893        1758.620
```

Observation epoch (y m d h m s)

P1 code pseudorange towards satellite 13

Table 6. Example of observation identification in RINEX observation file.

6 References

[1] GPS Interface Control Document ICD-GPS-200C, file ICD-GPS- 200c.pdf

[2] Werner Gurtner: RINEX: The Receiver Independent Exchange Format Version 2.10, Astronomical Institute, University of Berne, January 2002. File rinex.format.

[3] Hofmann-Wellenhof, B., H. Lichtenegger and J. Collins, 2001: GPS, Theory and Practice. Springer Verlag, (Fifth, revised edition).

[4] Leick, A., 1995: GPS satellite surveying. John Wiley and Sons.