4CCM112A Solutions to Assignment 9

[0] Cover any questions from Assignment 8 that could not be covered last week.

Solution: See solutions to Assignment 8.

[1] Consider the surface S parametrised by

$$\mathbf{r}(u,v) = (u^2 \cos v, u^2 \sin v, u)$$

with $(u,v) \in \mathbb{R}^2$. Compute in the form $ax + by + cz = 1$ the equation of the tangent plane to S at the point corresponding to $u = 1, v = 0$.

(*Optional*: Can you deduce what the surface looks like (a sketch)? – and observe why the equation for the tangent plane you have computed appears to be correct?)

Solution: Recall that the Cartesian equation of the tangent plane at $\mathbf{x}_0 \in S$ is determined for $\mathbf{x} = (x,y,z)$ by $(\mathbf{x} - \mathbf{x}_0).\mathbf{n} = 0$ with $\mathbf{n}$ a vector normal to S at $\mathbf{x}_0$ (i.e. normal to its tangent plane!). As S is a parametrised surface we know that $\mathbf{N}(1,0) = \mathbf{r}_u(1,0) \times \mathbf{r}_v(1,0) = (-1,0,2)$ is normal to S at $\mathbf{r}(1,0) = (1,0,1)$. So the tangent plane to S at this point is

$$\begin{pmatrix} x-1 \\ y \\ z-1 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} = 0,$$

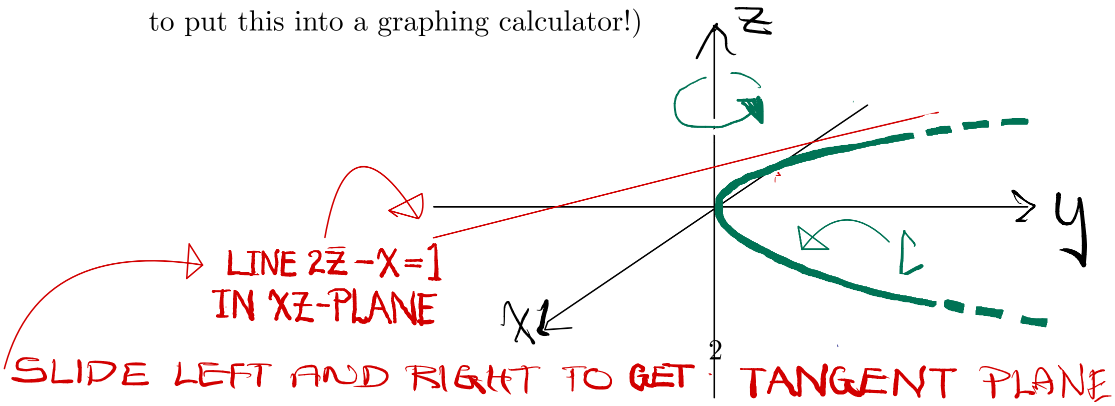
or

$$2z - x = 1.$$

(*Optional part*: The x, y, z entries of $\mathbf{r}(u,v)$ satisfy the Cartesian equation $x^2 + y^2 = z^4$, or, using polar coordinates (r, θ) in (x,y) plane, $r^2 = z^4$, which since $r > 0$ means

$$r = z^2.$$

This is so for any θ (i.e. independent of θ). Thus, the vertical slice down through the surface over the line in the xy-plane emanating from the origin at any fixed angle θ from the x-axis is a parabola 'on its side', i.e. the graph of $z = \pm\sqrt{r}$, shaped like a letter C. The surface is obtained therefore by rotating this C around the z-axis. The tangent plane at $(1,0,1)$ is thus tangent to the C in the xz-plane (C with its 'back' to the z-axis), and must hence be a plane parallel to the y-axis (hence the coefficient of y is zero in the tangent plane equation) and tilted outwards as x and z increase. You might better want to put this into a graphing calculator!)



[2] Compute the flux of the vector field $\mathbf{v} = \mathbf{i} + x\mathbf{j} + z\mathbf{k}$ upwards through the hemisphere $x^2 + y^2 + z^2 = 1, z \geq 0$.

Solution: We have to compute the integral

$$\int\int_S \mathbf{v} \cdot \mathbf{n} \; d\sigma$$

with $\mathbf{n}$ the upward unit normal. Since we are on the (upper-half of the) unit sphere we see without any computation that

$$\mathbf{n} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

and hence that

$$\int\int_S \mathbf{v} \cdot \mathbf{n} \; d\sigma = \int\int_S x + xy + z^2 \; d\sigma.$$

In spherical polars $x = \cos\theta\sin\phi, y = \sin\theta\sin\phi, z = \cos\phi$ with $\theta \in [0, 2\pi], \phi \in [0, \pi/2]$, and $|\mathbf{N}(\theta, \phi)| = \sin\phi$ (as seen in lectures and in the course notes), so the above integral is equal to

$$\int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi/2} (\cos\theta\sin\phi + \cos\theta\sin\theta\sin^2\phi + \cos\phi^2)\sin\phi \; d\phi d\theta$$

$$= \int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi/2} (\cos\theta\sin^2\phi \; + \cos\theta\sin\theta\sin^3\phi + \cos^2\phi\sin\phi \; d\phi d\theta.$$

Clearly, the integral of each of the first two summands with respect to θ is zero. This leaves

$$\int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi/2} \cos^2\phi\sin\phi \; d\phi d\theta = \int_{\theta=0}^{2\pi} d\theta \int_{\phi=0}^{\pi/2} \cos^2\phi\sin\phi \; d\phi = -\frac{2\pi}{3}\cos^3\phi\Big|_0^{\pi/2} = \frac{2\pi}{3}.$$

You can, of course, compute the integral from $\int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi/2}\int\int \mathbf{v}(\theta, \phi) \cdot \mathbf{N}(\theta, \phi) \; d\phi d\theta$ and obtain the same answer (and integrand).

[3] Let P be the plane

$$2x + 2y + z = 6$$

and let S be the portion of P which lies inside the cylinder $x^2 + y^2 = 4$. Let C be the curve which is the intersection of of P with the cylinder $x^2 + y^2 = 4$.

(a) Sketch the surface S and indicate the curve C on your sketch.

(b) Compute the area of S.

(c) Consider the vector field $\mathbf{v}(x, y, z) = -2x^2y\mathbf{i} + 2xy^2\mathbf{j} - z^3\mathbf{k}$. Evaluate directly (as a double integral)

$$\int\int_S (\nabla \times \mathbf{v}).\mathbf{n}\, d\sigma,$$

where $\mathbf{n}$ is the upward pointing unit normal to S.

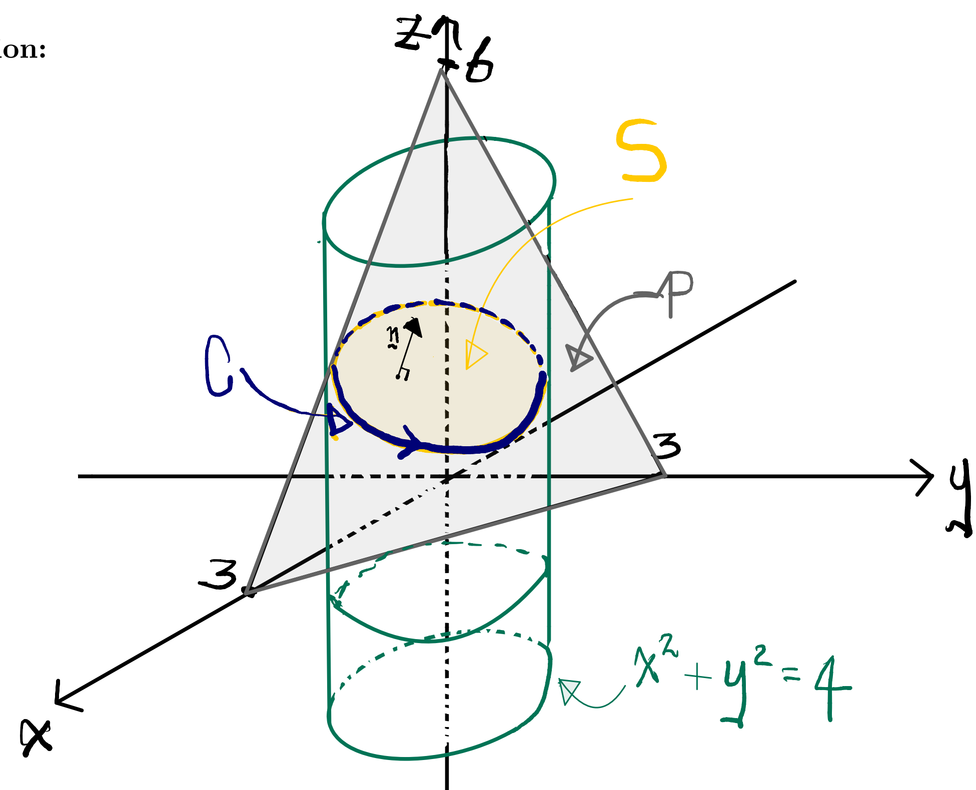
(d) Use Stokes' theorem to evaluate the line integral over C:

$$\int_C -2x^2y\, dx + 2xy^2\, dy - z^3\, dz.$$

(e) Does there exist a function $f : \mathbb{R}^3 \to \mathbb{R}^1$ with $\mathbf{v} = \nabla f$?

Solution:

(a)



(b) The area of S is computed by the surface integral

$$\text{Area}(S) = \int\int_S 1 \; d\sigma.$$

To evaluate this we need a parametrisation of S. Since S is the graph of the function $z = g(x, y) = 6 - 2x - 2y$ over the region $\Omega := \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 4\}$, the natural parametrisation is

$$\mathbf{r} : \Omega \to S \subset \mathbb{R}^3, \quad \mathbf{r}(x, y) = (x, y, 6 - 2x - 2y).$$

Hence,

$$\mathbf{N}(x, y) = (-g_x, -g_y, 1) = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} \tag{0.1}$$

and so $|\mathbf{N}(x, y)| = 3$. Thus,

$$\text{Area}(S) = \int\int_\Omega |\mathbf{N}(x, y)| \; dxdy = 3 \int\int_\Omega 1 \; dxdy = 3\text{Area}(\Omega) = 12\pi.$$

(c) $\nabla \times \mathbf{v} = \begin{pmatrix} 0 \\ 0 \\ 2(x^2 + y^2) \end{pmatrix}$ and so using (0.1) we have

$$\int\int_S (\nabla \times \mathbf{v}).\mathbf{n} \; d\sigma = \int\int_\Omega \begin{pmatrix} 0 \\ 0 \\ 2(x^2 + y^2) \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} \; dxdy = 2 \int\int_\Omega x^2 + y^2 \; dxdy,$$

which evaluated in polar coordinates $2 \int\int_\Omega r^2 \; rdrd\theta = 4\pi \int_{r=0}^2 r^3 \; dr$ gives

$$\int\int_S (\nabla \times \mathbf{v}).\mathbf{n} \; d\sigma = 16\pi.$$

(d) We have

$$\int_C -2x^2y \, dx + 2xy^2 \, dy - z^3 \, dz = \int_C \mathbf{v} \cdot d\mathbf{r} \overset{\text{Stokes}'}{=} \int\int_S (\nabla \times \mathbf{v}).\mathbf{n} \; d\sigma = 16\pi.$$

(e) For any <u>closed</u> curve (loop) C one has by the FTC for curves

$$\int_C \nabla f \cdot d\mathbf{r} = 0.$$

Since $\int_C \mathbf{v} \cdot d\mathbf{r} \neq 0$ from part (d) we conclude that no such f can exist.