Burnside's Lemma

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✻ 1 Introduction

In this handout / lecture series, we're going be covering **Burnside's Lemma**. This is a result from group theory, so some of the background material required to understand this may be things you've never seen before! This lemma also has some interesting history associated with it's name – it wasn't even discovered by Burnside!

You can find the lecture video here.

✻ 2 Groups, Actions, and some Theorems

In order to properly define, understand, and prove Burnside's Lemma, we're going to need to look at some group theory first. In particular, we're going to be looking at group actions.
First, let's define what a group actually is.

Definition 2.1 (Groups)**:** A **group** is a pair $(G, +)$ where G is a (nonempty) set, and $+ : G \times G \to G$ is a function on G which satisfies the following:

- **Associatity**: For all $a, b, c \in G$, we have $(a + b) + c = a + (b + c)$.
- **Identity**: There exists a unique element $0 \in G$ such that $0 + a = a + 0 = a$ for all $a \in G$.
- **Inverse**: For all $a \in G$, there exists an inverse element $-a$ such that $a + -a = -a + a = 0$.

In simpler terms, a group is basically a way for us to describe *structures*. Groups are closely associated with *symmetry*, which is what we'll explore below. Now that we've defined a group, we can define symmetry groups, which are the main subject of group actions. In simple terms, a **symmetry group** is a group whose elements are transformations on an object where the object is invariant – the object itself doesn't change. The operation would be composition, $\circ$.
For an easy example, think of rotation: consider the set of operations defined by 'rotating by n°' – it's a symmetry group because the elements themselves are just rotations that can be applied to objects, and they statisfy the group axioms. The operation is again composition, where we can apply rotations in sequence.
A symmetry group can 'act' on a set of objects, transforming them into other objects. We can rotate a square by 90 degrees to make a diamond shape. Formally, we have:

Definition 2.2 (Group Actions)**:** Suppose G is a symmetry group with identity e, and suppose X is a set. A **left group action** φ of G on X is a function $\varphi : G \times X \to X$ that satisfies the following:

- **Identity**: For all $x \in X$, $\varphi(e, x) = x$.
- **Composition**: For all $g, h \in G$ and $x \in X$ we have $\varphi(g \circ h, x) = \varphi(g, \varphi(h, x))$.

Sometimes we can omit the φ and just write $\varphi(g, x)$ as $g \cdot x$ instead.

Now that we've defined this, we can move on to some more concepts that relate to group actions, and by extension Burnside's lemma.
Often when referring to a symmetry group, we understand that the operation is implied as composition, so we can omit the $\circ$ operation and denote the entire group by its set, G. Now, let's define some terms:

Definition 2.3: Suppose G is a group acting on a set X. A **fixed point** of an element $g \in G$ is an $x \in X$ such that $\varphi(g, x) = x$. The **stabilizer** G_x of a point $x \in X$ is the set of elements $g \in G$ such that x is a fixed point of g. The **orbit** of an element $x \in X$ is the set of elements $y \in X$ such that $\varphi(g, x) = y$ for some $g \in G$.

Now that we've defined these, we can move on to a theorem that is used to prove Burnside's lemma. This part is a little complicated, as some of the claims rely on some advanced group theory concepts. Don't worry about not fully understanding the proof of this theorem, unless you know some extra group theory, as it's included mostly for completeness.

Theorem 2.4 (Orbit-Stabilizer Theorem)

Suppose G is a group acting on a set X. Denote the stabilizer of $x \in X$ as G_x, and the orbit of x as O_x. If the orbit of x is finite, then the index $[G : G_x]$ is also finite, and we have $[G : G_x] = |O_x|$. If G is finite, then we have

$$|G_x| \cdot |O_x| = |G|.$$

Proof. Define the map $\varphi : G \to O_x$ such that $\varphi(g) = g \cdot x$. Clearly, φ must be surjective, as x is acted on by all elements of G by definition. As the stabilizer of x by G is a subgroup of G for all $x \in X$,

$$\varphi(g) = \varphi(h) \iff g^{-1}h \in G_x.$$

This implies that $g \equiv h \pmod{G_x}$, so we must have a well defined bijection $G/G_x \to O_x$ given by $gG_x \mapsto \varphi(g)$, so O_x must have the same number of elements as G/G_x, meaning

$$|O_x| = [G : G_x],$$

and the result follows. $\square$

Don't worry too much about understanding the proof above, as it was mostly included for completeness rather than as something essential to understand.
We can define the set of orbits X/G of X acted on by G as the set of orbits over all $x \in X$, not counting repetitions. Then the number of orbits is $|X/G|$.

✾ 3 Burnside's Lemma

Now that we've finally covered all that background theory, we can jump into Burnside's lemma and its proof!

> **Theorem 3.1** (Burnside's Lemma)
> Suppose G is a finite group acting on the set X. Let X/G be the set of orbits of X. For any $g \in G$, let X^g be the set of points in X which are fixed by g:
> $$X^g = \{x \in X : \varphi(g, x) = x\}.$$
> Then we have
> $$|X/G| = \frac{1}{|G|} \sum_{g \in G} |X^g|.$$

Proof. We start by rewriting the sum in the theorem statement in terms of the elements of X. Note that

$$\sum_{g \in G} |X^g| = \sum_{x \in X} |\{g \in G : \varphi(g, x) = x\}| = \sum_{x \in X} |G_x|.$$

From the Orbit-Stabilizer theorem, we have $|G_x| = \frac{|G|}{|O_x|}$, so we may rewrite the sum as

$$\sum_{x \in X} |G_x| = \sum_{x \in X} \frac{|G|}{|O_x|} = |G| \sum_{x \in X} \frac{1}{|O_x|}.$$

Consider each x's contriution to the sum in a fixed orbit O. Each x contributes $\frac{1}{|O|}$ to the sum, and since there are exactly $|O|$ such x, the contribution from all x is just 1. So the sum evaluates to the number of orbits, which is just $|X/G|$.
We then have

$$\sum_{g \in G} |X^g| = |G| \cdot |X/G|,$$

and the result follows. □

✾ 4 Applying Burnside's Lemma

Now that we've stated and proved Burnside's, let's see how we can apply it to some problems!

> **Example 4.1** (Classic)**:** Compute the number of distinct ways Pranav can color a cube by painting each face of the cube one of 6 given colors. Cubes are distinct if they cannot be obtained from eachother using rotations.

Solution. Let X be the set of all possible colorings of the cube without rotation. ($|X| = 6! = 720$) Define G as the group of rotations of the cube. ($|G| = 24$ as there are 6 ways to fix a top face and 4 ways to fix the front face) Now, it suffices to compute $|X/G|$, which is simple by Burnside's lemma. Since there are no indentical colors (we must color the cube with 6 *distinct* colors), the cube always outputs a distinct coloring under rotation. This means we have no fixed points, $X^g = \emptyset$ for all non-identity g. When g is the identity (in this case, no rotation at all), $X^g = X$, so we have

$$|X/G| = \frac{1}{|G|} \sum_{g \in G} |X^g| = \frac{|X|}{|G|} = \frac{720}{24} = \boxed{30},$$

and we are done.
This was really overkill for this problem, but it's a good way to see how Burnside's works.

Example 4.2 (Classic)**:** Now that he found out how many ways he could color a cube using 6 colors, Pranav is wondering how many ways he could color a cube with n colors, given that he can use each color more than once, and that 2 colorings are considered distinct if one cannot be obtained by a rotation of the other.

Solution. Like before, we let x be the set of all colorings, but this time $|X| = n^6$ since we can use each color more than once, without rotation. Two elements of X belong to the same orbit if and only if one is a rotation of the other. So all we need to do is count $|X/G|$, where G is the same as in the last example, so there are still 24 rotations, except this time we need to describe them since there are fixed points. The rotations are as follows:

- There is one identity rotation that leaves all n^6 elements of X unchanged.
- There are three $180°$ rotations, each of which leaves n^4 of the elements of X unchanged.
- There are six $90°$ face rotation, each of which leave n^3 of the elements of X unchanged.
- There are siz $180°$ edge rotations, each of which leaves n^3 of the elements of X unchanged.
- There are eight $120°$ vertex rotations, each of which leaves n^2 of the elements of X unchanged.

Now that we know the cardinality of X^g for each of the 24 elements of G, we can compute $|X/G|$, giving us the total number of distinct rotations. As before, we have

$$\frac{1}{|G|}\sum_{g\in G}|X^g| = \frac{1}{24}(n^6 + 3n^4 + 6n^3 + 6n^3 + 8n^2).$$

Simplifying, we have $\boxed{\frac{1}{24}\left(n^6 + 3n^4 + 12n^3 + 8n^2\right)}$. ☐

Burnside's lemma can also show up in places that we don't expect it to – take this next example, for instance.

Example 4.3 (AoPS Forums)**:** Prove for all positive integers n and k, that

$$n \mid \sum_{i=0}^{n-1} k^{\gcd(i,n)}.$$

Solution. We can construct a combinatorial argument. Consider a necklace with n beads that we want to color with k different colors. How many distinct ways can this be accomplished, up to rotations? Let X be the set of all colorings, such that $|X| = k^n$, and let G be the group consisting of all rotations with order n. Consider the rotation by i pearls. If there is a rotational symmetry after i turns, the first bead and the $i+1$-th bead must be the same color. We can construct this as a regular polygon with n vertices. Connecting every i-th vertex yields an orbit of length

$$\frac{n}{\gcd(i,n)}.$$

This separates the beads into $\gcd(i,n)$ groups, which would each yield $k^{\gcd(i,n)}$ distinct colorings. By Burnside's lemma, the total number of distinct ways would be

$$\frac{1}{n}\sum_{i=0}^{n-1} k^{\gcd(i,n)},$$

and the result follows. ☐

This above result is actually a look at one of the generalizations of Burnside's lemma, and it's called the Pólya enumeration theorem. In particular, this is the simple, unweighted version of the theorem, and there is a more complex one that involves 'weighting' the colors in one or more ways (which invovles multivariate functions). You can learn more about this theorem here: Pólya Enumeration Theorem.

✾ 5 Sources

I used some sources to make this handout, and I'd like to credit all of them below:

- Brilliant
- Wikipedia