# Coordinate Bashing

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## §1 Useful theorems

**Theorem 1.1** (Sholeace)
In a polygon with coordinates $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$, the area $\mathbf{A}$ is:

$$\begin{aligned}\mathbf{A} &= \frac{1}{2}\left|\left(\sum_{i=1}^{n-1} x_i y_{i+1}\right) + x_n y_1 - \left(\sum_{i=1}^{n-1} x_{i+1} y_i\right) - x_1 y_n\right| \\ &= \frac{1}{2} \cdot |x_1y_2 + x_2y_3 + \cdots + x_{n-1}y_n + x_ny_1 - x_2y_1 - x_3y_2 - \cdots - x_ny_{n-1} - x_1y_n|\end{aligned}$$

**Theorem 1.2** (Point to line)
The distance between the point $(x_0, y_0)$ to the line $Ax + By + C = 0$ can be written as

$$\frac{|Ax_0 + By_0 + C|}{\sqrt{A^2 + B^2}}$$

**Theorem 1.3** (Centroid coordinates)
In a triangle with coordinates $A = (x_a, y_a), B = (x_b, y_b), C = (x_c, y_c)$, the coordinates of the centroid is

$$\left(\frac{x_a + x_b + x_c}{3}, \frac{y_a + y_b + y_c}{3}\right)$$

## §2 Heuristics

- Many people say that coordinate bashing is a "no brain" technique, and is for when you don't know how to synthetic the geometry. This is false. When coordbashing, the most important step is your set up. This means what you define as the origin, and the equations of circles, lines, etc. Having the right setup will save you lots of time.

- Have a plan. Whenever I use coordinates on a problem, I plan out how I would calculate each point, and the method of calculating each point. This helps you get a sense of how long it will take you to solve the problem, and whether its worth it to just move onto another problem, or to try to look for a synthetic solution.

- Know when to apply coordinates! Right angles, lots of lines, problems where you are given triangle side lengths are your friend. Some things that are harder to deal with are incenters, multiple circles, etc.
- Angle conditions are usually pretty hard to deal with, but if they can be reduced to the angle bisector theorem, cyclic quadrilaterals, or tangents, then it is much easier to deal with them.
- When you have a nice central circle, its a good idea to set that circle at the origin, so the equation of the circle is easier to work with.
- For obvious reasons, knowledge of standard geometry theorems is also very helpful. If you can use synthetic techniques to help you simplify the problem, then coordbashing can be much less computationally heavy and much faster.
- When given a triangle with three side lengths, then you can use Heron's to calculate the area, and then calculate the altitude of one of the sides, so you can place the triangle onto the coordinate plane.

## §3 Examples

**Example 3.1** (No Source)

Let $ABC$ be a triangle with $AB = 13$, $BC = 14$, $AC = 15$. Let $H, I, M$ be the orthocenter of $\Delta ABC$, incenter of $\Delta ABC$, and midpoint of $BC$ respectively. What is the area of $\Delta HIM$?

**Walkthrough:**

**(a)** As usual, we want to set $A = (0, a)$, $B = (b, 0)$, $C = (c, 0)$. Find the values of $a, b, c$.

**(b)** For $H$, we can drop an altitude from $B$ to $CA$, and find the intersection of that with the $A$ altitude.

We could calculate the location of $I$ by creating two angle bisectors, and using angle bisector theorem, but there is a better way.

**(c)** Instead, we can calculate the inradius, and note that the $y$-coordinate of the incenter is equal to the inradius.

**(d)** For the $x$-coordinate, we can just use the formulas for distances between the vertices of triangles and the incircle touch points. Alternatively, you could calculate the $A$-angle bisector with angle bisector theorem and intersect that with $y = r$, where $r$ is the inradius.

**(e)** Finish using a good area formula.

**Example 3.2** (2020 AMC 10A Problem 20)

Quadrilateral $ABCD$ satisfies $\angle ABC = \angle ACD = 90^\circ, AC = 20$, and $CD = 30$. Diagonals $\overline{AC}$ and $\overline{BD}$ intersect at point $E$, and $AE = 5$. What is the area of quadrilateral $ABCD$?

**Walkthrough:**

**(a)** We could set the origin to be $C$, as we have $CD$ and $AC$, but there actually is a better point to use. (If you can't figure out where this is, then think about how we know $AC$, and $\angle ABC = 90°$.)

You should have set the origin to be the midpoint of segment $AC$. We are motivated to do this because this allows us to draw a circle $\omega$ centered at the origin that passes through $A$ and $C$, and note that $B$ also passes through the circle.

**(b)** Find the equation to the circle $\omega$, and calculate $B$ by defining it as the second intersection of $\omega$ with $DE$.

**(c)** You should get a quadratic. Elimnate the "wrong" solution.

**(d)** Finish.

Here is an instructional problem for point to line that some of you may be slighlty familiar with.

**Example 3.3** (2021 AIME I Problem 9)

Let $ABCD$ be an isosceles trapezoid with $AD = BC$ and $AB < CD$. Suppose that the distances from $A$ to the lines $BC, CD$, and $BD$ are $15, 18$, and $10$, respectively. Let $K$ be the area of $ABCD$. Find $\sqrt{2} \cdot K$.

**Walkthrough:**

**(a)** Note how we have a bunch of distances from a point to several lines, with the point being on one of 2 parallel lines. What are we motivated to use?

**(b)** Find a nice point that we can set as the origin, so that the coordinates of $A, B, C, D$ are all relatively clean.

What I found to be the cleanest was to set the foot of the altitude from $A$ to $CD$ to be the origin, although some other solutions used different origins.

**(c)** Use point to line from $A$ to $BC$ and $BD$ to get two equations.

You should have gotten the quadratics

$$\sqrt{324 + (b+d)^2} = \frac{9}{5}b$$
$$\sqrt{324 + d^2} = \frac{6}{5}b$$

or some equivalent expression. This may be slightly different based on how you chose your variables, and the origin.

**(d)** Solve for $b$ and $d$, by squaring and subtracting.

**(e)** Finish.

# §4 Solutions to Examples

## §4.1 No source

We set $A = (0, 12), B = (-5, 0), C = (9, 0)$. Its easy to see that $M = (2, 0)$. Now, basic computation gives us the inradius $r = 4$. If we let $D$ be the $BC$ touchpoint of the incircle, then it isn't hard to see that $BD = 5$, so $I = (1, 5)$. Dropping altitudes gives $H = (0, \frac{15}{4})$, so we can use shoelace to get the answer of $\boxed{\frac{17}{8}}$.

## §4.2 2020 AMC 10A Problem 20

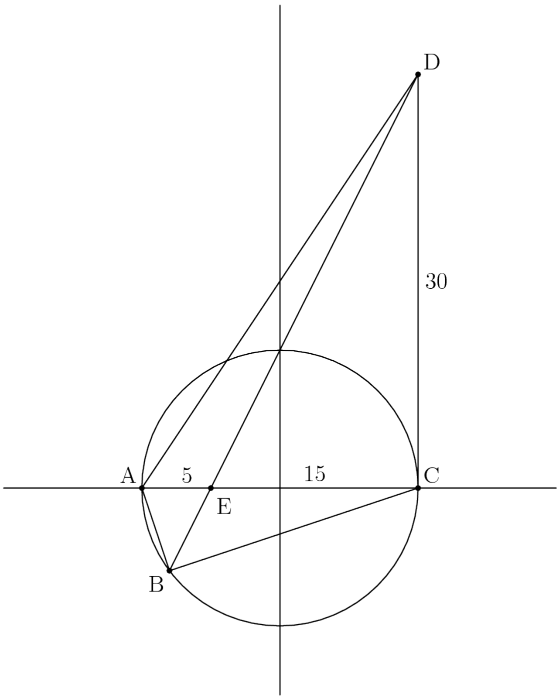


*Diagram from AoPS Wiki.*

We set the origin to be the midpoint of $AC$. Now, construct $(ABC)$, and note that the equation of $(ABC)$ is $x^2 + y^2 = 100$. We have the coordinates are

$$\begin{aligned} A &= (-10, 0) \\ C &= (10, 0) \\ D &= (10, 30) \\ E &= (-5, 0) \end{aligned}$$

It remains to calculate $B$. Now, note that $B$ is simply the intersection of $x^2 + y^2 = 100$ and line $DE$. But line $DE$ can be easily calculated to be $y = -2(x + 5)$ from point slope formula. It remains to solve

$$\begin{aligned} x^2 + (-2(x + 5))^2 &= 100 \\ \implies 5x^2 + 40x &= 0 \\ \implies x &= -8, 0 \end{aligned}$$

Obviously, $x = 0$ is the first intersection of $DE$ with $(ABC)$, so we can toss that out. Then, plugging $x = -4$ back in we get $y = -6$. Finally, we finish with shoelace on $ABCD$, and get an answer of $\boxed{360}$.

## §4.3 2021 AIME I Problem 9

Instead of high IQ similar triangle spam, we use the method of coordinate bashing. We do this by setting the origin to be the foot from $A$ to $CD$, and noting that we can use point to line formula and isosceles trapezoid properties to get the locations of the points $B, C, D$.

We let $X$, $Y$, $Z$ be the feet of the altitudes from $A$ to $CD$, $DB$, and $BC$, respectively. Now, setting $X$ as the origin, we can set

$$\begin{aligned} A &= (0, 18) \\ B &= (b, 18) \\ C &= (b+d, 0) \\ D &= (-d, 0) \\ X &= (0, 0) \end{aligned}$$

We want to use point to line, so we need the equations of $BC$ and $BD$. It's pretty easy to get

$$\begin{aligned} BD : y &= \frac{18}{b+d}(x+d) \\ BC : y &= \frac{18}{-d}(x-b-d) \end{aligned}$$

Now, point to line on $A$ to $BD$ and $BC$ gives

$$\begin{aligned} 10 &= \frac{-18(b+d)+18d}{\sqrt{324+(b+d)^2}} \\ 15 &= \frac{18d+18(-b-d)}{\sqrt{18^2+d^2}} \end{aligned}$$

respectively. Now, rearranging, we get

$$\begin{aligned} \sqrt{324+(b+d)^2} &= \frac{9}{5}b \\ \sqrt{324+d^2} &= \frac{6}{5}b \end{aligned}$$

This is equivalent to

$$\begin{aligned} 324+b^2+2bd+d^2 &= \frac{81}{25}b^2 \\ 324+d^2 &= \frac{36}{25}b^2 \end{aligned}$$

Upon subtracting the two equations, we get $d = \frac{2}{5}b$. Now plugging this back into $324 + d^2 = \frac{36}{25}b^2$, we get

$$\begin{aligned} 324 + \frac{4}{25}b^2 &= 36b^2 \\ \implies 324 &= \frac{32}{25}b^2 \\ \implies b &= \frac{45\sqrt{2}}{4} \\ \implies d &= \frac{9\sqrt{2}}{2} \end{aligned}$$

From here it is not hard to get the answer of $\boxed{567}$.

> **Remark 4.1.** On contest, I calculated the area of the trapezoid to be $\frac{1}{2} \cdot (b+d) \cdot b \cdot 18 \implies 486$. Oops.