CNCM Generating Functions Handout Part 2

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§1 Partitions with Generating Functions

§1.1 Deriving the Generating Function for Partitions

Definition 1.1

A **partition** of a positive integer n is the number of ways that n can be split into positive integer parts. For example, a partition of 5 is $2+2+1$. Two partitions that have the same parts but are in a different order are considered the same. The partitions $2+2+1$ and $1+2+2$ are considered the same.

We can list out the partitions of the first few integers.

Integer	Partitions	Total
1	1	1
2	1 + 1, 2	2
3	1 + 1 + 1, 1 + 2, 3	3
4	1 + 1 + 1 + 1, 1 + 1 + 2, 1 + 2, 2 + 2, 4	5
5	1 + 1 + 1 + 1 + 1, 1 + 1 + 1 + 2, 1 + 1 + 3, 1 + 4, 1 + 2 + 2, 2 + 3, 5	7

There doesn't seem like there are any patterns to this sequence, especially if you list out the number of partitions of more integers.

There is indeed a generating function that represents the number of partitions of an integer n, and we will derive it here.

We can think of partitions as picking a certain number of 1s, a certain number of 2s, and so on. More formally, we can state this as follows:

$$n = c_1 \cdot 1 + c_2 \cdot 2 + c_3 \cdot 3 + \cdots + c_n \cdot n$$

for non-negative integers $c_1, c_2, \cdot c_n$. There are of course restrictions to the above coefficients, such as $c_n \leq 1$, but I have listed this here just to give you an idea of how we can represent partitions.

Let's think of that equation in terms of generating functions. Let's let $n = x^n$. We can actually translate the above equation into the following equation:

$$x^n = x^{c_1 \cdot 1} \cdot x^{c_2 \cdot 2} \cdot \cdots \cdot x^{c_n \cdot n}$$

Which is true since we add exponents when we multiply variables. We can go a step further. Since we know that $x^{a \cdot b} = \left(x^b\right)^a$, we can write the equation as

$$x^n = \left(x^1\right)^{c_1} \cdot \left(x^2\right)^{c_2} \cdot \cdots \cdot \left(x^n\right)^{c_n}$$

This equation shows that we can pick a multiple of each exponent from $1 \ldots n$. This gives us the following expression:

$$(1 + x + x^2 + x^3 + \dots)(1 + x^2 + x^4 + \dots)(1 + x^3 + x^6 + \dots)\dots$$

To see why this generating function works, we can see that for each group. The first group corresponds to $1 \cdot c_1$ since all the exponents are a multiple of 1, the second group corresponds to $2 \cdot c_2$ since all the exponents are a multiple of 2, and so on. Using the infinite geometric series formula, we can express the generating function as

$$\left(\frac{1}{1-x}\right)\left(\frac{1}{1-x^2}\right)\dots$$

This gives us the following

Definition 1.2

The generating function for partitions is

$$\prod_{k=1}^{\infty} \frac{1}{1-x^k}$$

where the number of partitions for positive integer n corresponds to the coefficient of x^n in the expansion.

§1.2 Generating Functions for Manipulations of Partitions

This reasoning allows us to make some manipulations in regards to partitions. Here are some interesting ones.

Definition 1.3

The generating function for partitions with odd parts is

$$\prod_{k=1}^{\infty} \frac{1}{1-x^{2k-1}}$$

The generating function for partitions with distinct parts is

$$\prod_{k=1}^{\infty} (1+x^k)$$

The generating function for partitions with parts that can be at most m is

$$\prod_{k=1}^{m} \frac{1}{1-x^k}$$

Note that we didn't include the generating functions for partitions with even parts here. That is because the number of partitions of positive integer n (which must be even) into even parts is equal to the number of partitions of $\frac{n}{2}$. (Can you see why?)

Example 1.4 — Prove that the number of partitions of a positive integer n with distinct parts is the same as the number of partitions with odd parts.

The generating function for the number of partitions with distinct parts is

$$\prod_{k=1}^{\infty} (1+x^k)$$

Since we have the option of either picking the integer k to be a part or not to be a part.
The generating function for the number of partitions with odd parts is

$$\prod_{k=1}^{\infty} \frac{1}{1-x^{2k-1}}$$

As you can see, we have simply left out the even parts.

We can prove that both of these functions are equal.

$$(1+x)(1+x^2)(1+x^3)\cdots = \left(\frac{1-x^2}{1-x}\right)\left(\frac{1-x^4}{1-x^2}\right)\left(\frac{1-x^6}{1-x^3}\right)\dots$$

The top consists only of factors in the form of $(1-x^{2k})$, all of those factors will cancel with the factors in the form of $(1-x^{2k})$ in the denominator, leaving only factors in the form of $(1-x^{2k-1})$ in the denominator. Thus,

$$\left(\frac{1}{1-x}\right)\left(\frac{1}{1-x^3}\right)\left(\frac{1}{1-x^5}\right)\cdots = (1+x)(1+x^2)(1+x^3)\dots$$

§2 Power Series

§2.1 Taylor Series and Maclaurin Series

We won't delve into the power series too much, but it helps us set up the generating functions for some common functions. These are much more useful in competition the other way around, where we can find a shorthand function of a power series.

Definition 2.1

A **power series** is an infinite sum in the form of the following

$$\sum_{n=0}^{\infty} a_n(x-c)^n$$

with coefficients a_n and center c

There are two types of power series that are well known, and involve a bit of calculus. If you don't know calculus, you can skim these over and skip to the section of well-known power series. However, it would be best for you to learn the intuition behind some of these well-known power series once you do learn calculus.

Definition 2.2

A **Taylor Series** is an infinite sum in the form of the following

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n = f(c) + f^{'}(c)(x-c) + \frac{f^{''}(c)}{2!}(x-c)^2 + \dots$$

with center c and where $f^{(n)}$ is the nth derivative of f

Definition 2.3

A **Maclaurin Series** is an infinite sum in the form of the following

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = f(0) + f^{'}(0)x + \frac{f^{''}(0)}{2!}x^2 + \dots$$

where $f^{(n)}$ is the nth derivative of f

It is important to note that the Maclaurin Series is a Taylor Series with center 0.

This gives us the intuition behind some well-known power series. If you've ever wanted to know where the factorial expansion of e came from, it is from here. That definition of e is a Maclaurin Series of the function e^x evaluated at $x = 1$. Since the derivative of e^x is e^x, $f^{(n)}(0)$ is always 1.

Definition 2.4

e expressed as a Maclaurin Series

$$e = \sum_{n=0}^{\infty} \frac{1}{n!} = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \dots$$

§2.2 Common Generating Functions You Should Know

Anyways, here are some common generating functions. I won't prove them all, but the intuition from the first few should be clear based on the Series that I have covered in this section.

Definition 2.5

$\dfrac{1}{(1-x)^n}$ covered in Part 1 of this handout

$$\left(\frac{1}{1-x}\right)^n = \sum_{k=0}^{\infty} \binom{n+k-1}{k} x^k$$

e^x as a Maclaurin Series

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$$

$\cos(x)$ as a Maclaurin Series

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$$

$\sin(x)$ as a Maclaurin Series

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n}$$

$\ln(1+x)$ as a Maclaurin Series

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n$$

The Binomial Theorem expansion

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$$

Generating function for the Fibonacci numbers

$$\frac{x}{1-x-x^2} = \sum_{n=0}^{\infty} F_n x^n$$

Generating function for the Catalan numbers

$$\frac{1-\sqrt{1-4x}}{2x} = \sum_{n=0}^{\infty} C_n x^n = \sum_{n=0}^{\infty} \frac{1}{n+1} \binom{2n}{n} x^n$$

A combinatorial identity generating function similar to the one for the Catalan numbers

$$\frac{1}{\sqrt{1-4x}} = \sum_{n=0}^{\infty} \binom{2n}{n} x^n$$

§2.3 Derivatives and Integrals of Power Series

Remember the definition of the power series.

Definition 2.6

A **power series** is an infinite sum in the form of the following

$$\sum_{k=0}^{\infty} a_n(x-c)^n$$

with coefficients a_n and center c

Taking the derivative and the integral of a power series is very useful to know. Sometimes common power series will show up in competition math, but they will be presented after an integral or derivative is taken to make the problem harder. As a general rule of thumb, if you see a common power series with an extra variable in the numerator, you should consider taking a derivative. If you see a common power series with an extra variable in the denominator, you should consider taking an integral.

Definition 2.7

A **power series**

$$f(x)=\sum_{n=0}^{\infty} a_n(x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + \dots$$

The **integral of a power series**

$$\int f(x) = \sum_{n=0}^{\infty} \frac{1}{n+1}\cdot a_n(x-c)^{n+1} = C + a_0(x-c) + \frac{a_1}{2}(x-c)^2 + \frac{a_2}{3}(x-c)^2 + \dots$$

The **derivative of a power series**

$$f'(x)=\sum_{n=1}^{\infty} n\cdot a_n(x-c)^{n-1} = a_1 + 2a_2(x-c) + 3a_3(x-c)^2 + \dots$$

Here is an example:

Example 2.8 — Given that the formula for the nth Catalan number is $C_n = \frac{1}{n+1}\binom{2n}{n}$ and

$$\sum_{n=0}^{\infty}\binom{2n}{n}x^n = \frac{1}{\sqrt{1-4x}}$$

Find a closed form for the generating function of the Catalan numbers.

Our goal is to find a closed form for the following:

$$\sum_{n=0}^{\infty} C_n x^n = \sum_{n=0}^{\infty}\frac{1}{n+1}\binom{2n}{n}x^n$$

Note that the integral of the given summation is:

$$\int \sum_{n=0}^{\infty}\binom{2n}{n}x^n = \sum_{n=0}^{\infty}\frac{1}{n+1}\binom{2n}{n}x^{n+1} + C$$

We have:

$$\int \sum_{n=0}^{\infty}\binom{2n}{n}x^n = \int \frac{1}{\sqrt{1-4x}}$$

Which yields

$$\sum_{n=0}^{\infty} \frac{1}{n+1}\binom{2n}{n} x^{n+1} = -\frac{\sqrt{1-4x}}{2} + C$$

Subbing in $x = 0$ for both sides, we have $0 = -\frac{1}{2} + C \implies C = \frac{1}{2}$. The generating function on the LHS is actually shifted to the right by one, which we can fix by dividing by x. This gives us $\boxed{\frac{1-\sqrt{1-4x}}{2x}}$, the same formula presented in the earlier section.

§3 The Snake Oil Method

The Snake Oil Method, made by Herbert Wilf, is a powerful method that is used to solve some summations. The basis of the Snake Oil method is to create a series into a power series, and then create a double summation that will be easier to evaluate afterwards.

Definition 3.1

The **Snake Oil Method** takes a summation

$$S = \sum_{k} \{a_{n,k}\}$$

and converts it to a power series

$$\sum_{n \geq 0} Sx^n = \sum_{n} \sum_{k \geq 0} \{a_{n,k}\} x^n$$

Switching the order of the summation will usually give an easier sum to evaluate

$$\sum_{k} \sum_{n \geq 0} \{a_{n,k}\} x^n$$

If this isn't clear to you, I will cover an example that will hopefully make it more clear.

Example 3.2 — Evaluate the summation

$$\sum_{k=0}^{\infty} \binom{k}{n-k}$$

in terms of n

Our first step is to convert this summation into a power series, this would go.

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \binom{k}{n-k} x^n$$

Next, we will reverse the order of the summations.

$$\sum_{k=0}^{\infty} \sum_{n=k}^{2k} \binom{k}{n-k} x^n$$

We can see that the summation

$$\sum_{n=k}^{2k} \binom{k}{n-k} x^n$$

is very close to a binomial expansion. In fact, if there was an x^{n-k} term instead of an x^n term, it would be the expansion of $(1+x)^k$. We can manipulate the summation to get

$$\sum_{k=0}^{\infty} x^k \sum_{n=k}^{2k} \binom{k}{n-k} x^{n-k}$$

Since we know that the second part of the summation is $(1+x)^k$, we can rewrite the summation as the following

$$\sum_{k=0}^{\infty} x^k \cdot (1+x)^k$$

This is just an infinite geometric series with initial term 1 and common ratio $x(1+x)$. Thus

$$\sum_{k=0}^{\infty} x^k \cdot (1+x)^k = \frac{1}{1-x(1+x)} = \frac{1}{1-x-x^2}$$

The result of this manipulation actually gives a generating function. The coefficient of x^n in the expansion of the generating function will give us the result of the summation in terms of n.

Furthermore, this is actually the generating function for the Fibonacci numbers but divided by x. Thus, the coefficient of x^n will be $\boxed{F_{n+1}}$.