

# CRMT Team Round

CNCM Administration

October 26th, 2019

1. What is  $(2 - \frac{0}{1})(2 - \frac{1}{2})(2 - \frac{2}{3})(2 - \frac{3}{4})(2 - \frac{4}{5})(2 - \frac{5}{6})(2 - \frac{6}{7})(2 - \frac{7}{8})(2 - \frac{8}{9})$ ?
2. Define  $a_n = a_{n+1} + n$ , with  $a_1 = 1$ . If  $35x = a_x$ , find  $x$ .
3. Define  $f(x)$  as  $\frac{\ln(\frac{1+x}{1-x})}{2}$ . Compute  $\lim_{x \rightarrow \infty} f^{-1}(x)$ .
4. A circle with radius 1 is inscribed inside a right triangle. Find the minimum area of the triangle.
5. Find the last 3 digits of the base 10 number  $5^{2019}$  converted to base 6.
6. Suppose there are 69 lily pads arranged in a circle and a frog is on one of them. Every second, the frog moves to one of the two adjacent lily pads with equal probability. After how many seconds is the frog expected to return to the initial lilypad?
7. Define  $f$  as a fifth degree monic polynomial such that  $f(1) = 0$ ,  $f(2) = 60$ ,  $f(3) = 60$ ,  $f(4) = 60$ ,  $f(5) = 60$ ,  $f(6) = 120$ , and  $f(7) = 480$ . Suppose  $f(8) = a$ , and  $f(9) = b$ . Find  $a + b$ .
8. Define the vertices of an infinite-sided polygon  $P$  inscribed in the unit circle (centered at the origin) as follows:  $V_1 = (0, -1)$ ,  $V_2 = (-1, 0)$ ,  $V_N$  is the midpoint of the smaller arc  $\widehat{V_{N-1}Z}$ , where  $Z$  is  $(0, 1)$ . Measured in units and square units respectively, find the ratio between the perimeter of the polygon and its area.
9. A deck of the first 100 positive integers is randomly shuffled. Find the expected number of draws it takes to get a prime number if there is no replacement.
10. A cube has side length 12. Let  $F$  be a face of the cube and let  $ABCD$  be the square corresponding to face  $F$  on the cube. Pick a point  $P$  such that  $\overline{PA} = \overline{PB} = \overline{PC} = \overline{PD} = \sqrt{136}$  and  $P$  lies outside the cube. Let  $S_1$  be the inscribed sphere of the cube and let  $S_2$  be the inscribed sphere of square pyramid  $PABCD$ . There exists a cone  $C$  such that both  $S_1$  and  $S_2$  are tangent to the lateral surface of the cone and  $S_1$  is tangent to the base of the cone. Compute the volume of cone  $C$ .