CNCM Online Round 3

CNCM Administration

November 7, 2020

3 Problems

Problem 1. Harry, who is incredibly intellectual, needs to eat carrots C_1, C_2, C_3 and solve *Daily Challenge* problems D_1, D_2, D_3. However, he insists that carrot C_i must be eaten only after solving *Daily Challenge* problem D_i. In how many satisfactory orders can he complete all six actions?

Problem 2. Consider rectangle $ABCD$ with $AB = 6$ and $BC = 8$. Pick points E, F, G, H such that the angles $\angle AEB, \angle BFC, \angle CGD, \angle AHD$ are all right. What is the largest possible area of quadrilateral $EFGH$?

Problem 3. Let $a_1 = 1$ and $a_{n+1} = a_n \cdot p_n$ for $n \geq 1$ where p_n is the nth prime number, starting with $p_1 = 2$. Let $\tau(x)$ be equal to the number of divisors of x. Find the remainder when

$$\sum_{n=1}^{2020} \sum_{d|a_n} \tau(d)$$

is divided by 91 for positive integers d. Recall that $d|a_n$ denotes that d divides a_n.

Problem 4. Hari is obsessed with cubics. He comes up with a cubic with leading coefficient 1, rational coefficients and real roots $0 < a < b < c < 1$. He knows the following three facts: $P(0) = -\frac{1}{8}$, the roots form a geometric progression in the order a, b, c, and

$$\sum_{n=1}^{\infty} (a^n + b^n + c^n) = \frac{9}{2}$$

The value $a + b + c$ can be expressed as $\frac{m}{n}$, where m, n are relatively prime positive integers. Find $m + n$.

Problem 5. How many positive integers N less than 10^{1000} are such that N has x digits when written in base ten and $\frac{1}{N}$ has x digits after the decimal point when written in base ten? For example, 20 has two digits and $\frac{1}{20}$= 0.05 has two digits after the decimal point, so 20 is a valid N.

Problem 6. Triangle ABC has side lengths $AB = 13, BC = 14$, and $CA = 15$. Let Γ denote the circumcircle of $\triangle ABC$. Let H be the orthocenter of $\triangle ABC$. Let AH intersect Γ at a point D other than A. Let BH intersect AC at F and Γ at G. Suppose DG intersects AC at X. Compute the greatest integer less than or equal to the area of quadrilateral $HDXF$.

Problem 7. A subset of the positive integers S is said to be a *configuration* if $200 \notin S$ and for all nonnegative integers x, $x \in S$ if and only if both $2x \in S$ and $\left\lfloor \frac{x}{2} \right\rfloor \in S$. Let the number of subsets of $\{1, 2, 3, \ldots, 130\}$ that are equal to the intersection of $\{1, 2, 3, \ldots, 130\}$ with some configuration S equal k. Compute the remainder when k is divided by 1810.