Bases

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§1 Definition

Everyone should be familiar with the base 10 system, even if you don't even know what bases are.

When we count (from 0), we start with our 10 digits $0, 1, 2, 3, 4, 5, 6, 7, 8, 9$, but then when we want to count the next number, we no longer have a single digit representing the number "ten". So instead, we turn the digit to a 0 and add a 1 in front of it, to make it 10. Similarly, we do this carry over each time we want to add a 1 to the number 9.

This is the same in other bases, except we rotate every b for base b.

For example, in base 3, we count like this.

0 1 2 10 11 12 20 21 22 100 ...

Note how there is no '3'. This is because whenever we hit "three", we carry over.

If you are keen, you may notice, what happens in base b if $b > 10$? We only carry over when we hit b, so there must be some symbol for "ten", like how 7 represents "seven". Unfortunately, our number system doesn't really have a symbol, so we just letters. So for example, in base 16 we count as follows.

0, 1, 2, ... 8, 9, A, B, C, D, E, F, 10, 11, ...

Another thing to notice is that in base b, the number 1 followed by k zeroes is equivalent to b^k in base 10. So for example, in base 6, 1000 represents $6^3 = 216$. This means we can represent each base as sums of powers of b. For example, 2020 in base 3 is the same as $2 * 3^3 + 0 * 3^2 + 2 * 3^1 + 0 * 3^0 = 60$. This extends past the decimal point too. Let's formally define it.

Theorem 1.1

Say we have a number x in base b, such that the digits are expressed as follows

$$(d_m)(d_{m-1})(d_{m-2})...(d_2)(d_1)(d_0).(d_{-1})(d_{-2})...(d_{-n+1})(d_{-n})$$

Then in base 10, this can be expressed as

$$\sum_{i=-n}^{m}(d_i * b * i) = d_{-n} * b^{-n} + d_{-n+1} * b^{-n+1} + ... + d_{m-1} * b^{m-1} + d_m * b^m$$

Let's work a simple example.

Problem 1.2 — Convert 23.23 base 12 to a base 10 mixed fraction.

Solution. We use our above theorem.

$$2 * 12^1 + 3 * 12^0 + 2 * 12^{-1} + 3 * 12^{-2}$$
$$24 + 3 + \frac{2}{12} + \frac{3}{144}$$
$$27\frac{3}{16}$$

□

§2 Examples

There's not much theory to bases, other than the definition that was mentioned earlier. Let's take a look at a few contest examples.

> **Problem 2.1** (AMC 10A 2013/9) — In base 10, the number 2013 ends in the digit 3. In base 9, on the other hand, the same number is written as $(2676)_9$ and ends in the digit 6. For how many positive integers b does the base-b-representation of 2013 end in the digit 3?

Solution. First of all, we notice that $b > 3$ because if this were not the case, the units digit could not be 3. So now, we must understand what it means for the right-most digit to be 3. It is easy to see that this must be the case:

$$2013 \equiv 3 \pmod b$$
$$2010 \equiv 0 \pmod b$$
$$b \mid 2010$$

So, b must be a factor of 2010. Factoring 2010, we have

$$2010 = 2^1 \cdot 3^1 \cdot 5^1 \cdot 67^1$$

so 2010 has $2 \cdot 2 \cdot 2 \cdot 2 = 16$ factors. However, we must remember our restriction of $b > 3$. This excludes the following 3 factors of 2010: 1, 2, and 3. Hence, our answer is $16 - 3 = \boxed{13}$ □

> **Problem 2.2** (AIME I 2020/3) — A positive integer N has base-eleven representation $\underline{a}\,\underline{b}\,\underline{c}$ and base-eight representation $\underline{1}\,\underline{b}\,\underline{c}\,\underline{a}$, where a, b, and c represent (not necessarily distinct) digits. Find the least such N expressed in base ten.

Solution. Using our interpretation of numbers in other bases, we rewrite both in base 10 to get the following.

$$121a + 11b + c = 512 + 64b + 8c + a$$
$$120a = 512 + 53b + 7c$$

From here, this is essentially an algebra problem. Since a, b, c have to be non-negative one-digit integers, we see that $120a$ has to be at least 5, or else it is less than 512 and the equation is never true.

Since we want to minimize N, we want to let a be as small as possible. We assume $a = 5$, to see if it produces any solutions.

$$88 = 53b + 7c$$

We can quickly notice that $b = 1$ and $c = 5$ suffices, so our answer is 515 in base 11, which is equal to $5 * 121 + 1 * 11 + 5 * 1 = \boxed{621}$ □

> **Problem 2.3** (AIME I 2001/8) — Call a positive integer N a 7-10 double if the digits of the base-7 representation of N form a base-10 number that is twice N. For example, 51 is a 7-10 double because its base-7 representation is 102. What is the largest 7-10 double?

Solution. Because this is an AIME problems, it is actually valid to just assume that N must have at most 3 digits. But, to be rigorous, we will prove this.

To prove this, let's define the base 7 number $N = \overline{a_n a_{n-1} \cdots a_0}$. We are given the following.

$$2 \cdot 7^n a_n + 2 \cdot 7^{n-1} a_{n-1} + \cdots + 2a_0 = 10^n a_n + 10^{n-1} a_{n-1} + \cdots + a_0$$

We add an subtract like terms. Note that the a_0 and a_1 are positive on the LHS, and all the others terms are positive on the RHS.

$$(2-1)a_0 + (14-10)a_1 = (100-98)a_2 + (1000-686)a_3 + c \cdots + (10^n - 2 \cdot 7^n)a_n$$

$$a_0 + 4a_1 = 2a_2 + 314a_3 + \cdots + (10^n - 2 \cdot 7^n)a_n$$

We know that if N is a valid base 7 number, then all of it's digits must be less than 7. Hence, the maximum of the LHS must be when $a_0 = a_1 = 6$ meaning LHS$= 30$. However, if any of the digits a_k for $k \geq 0$ are positive, then we must have LHS<RHS, meaning they are not equal.

So we now know that N must have 3 or less digits, meaning we can redefine $N = \overline{abc}$ It is here, tempting to just plug in the definitions and try to make some algebraic manipulations, but I'll stomp on that party and just say that we have to do case work on the 100s (or the 49s) digit. We want to maximize, so let's start with $a = 6$.

$$2(6 \cdot 7^2 + 7b + c) = 6 \cdot 10^2 + 10b + c$$

$$588 + 14b + 2c = 600 + 10b + c$$

$$4b + c = 12$$

We see that that there are solutions that work here, so we do not have to check any more possible values of a. Since we want to maximize N, it is priority to maximize b first. We end up with $b = 3$ and $c = 0$, so our value of N is 630. Remember that N is in base 7, so we need to convert this back to base 10, which by our problem statement is simply $630 \cdot \frac{1}{2} = \boxed{315}$

□