Cyclic National Competitive Math Group

Member Lecture: Multiplicative Functions

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Technical Details

Definitions

- An “Arithmetic Function” is a function from the set of positive integers to the set of complex numbers.
- A “Multiplicative Function” is an arithmetic function f such that for any two positive integers m,n with gcd(m,n)=1, we have f(mn)=f(m)f(n). The product of two multiplicative functions is also multiplicative.
- A multiplicative function is “Completely Multiplicative” if for any two positive integers m,n that are not necessarily relatively prime, we have f(mn)=f(m)f(n).
- When working with multiplicative functions, we can split f(n)=f(p_1^e_1)f(p_2^e_2)f(p_3^e_3)... for each prime power dividing n; this is a key idea in many contest problems that involve multiplicative functions.

Examplar functions

- Which functions of the following are multiplicative? And which of the following are completely multiplicative? (assume that the domains are positive integers)

f(n)=1	f(n)=n^7	sum-of-divisors	totient-function	number-of-divisors	f(n)=gcd(n,187)
C.M.	C.M.	multiplicative	multiplicative	multiplicative	multiplicative

Dirichlet Convolution

- The Dirichlet Convolution of two arithmetic functions is as follow:

$$(f * g)(n) = \sum_{d\,|\,n} f(d)\, g\left(\frac{n}{d}\right) = \sum_{ab=n} f(a)\, g(b)$$

- It is commutative, associative and distributes over addition. Its most important property, for the purpose of this lecture, is that **the Dirichlet convolution of two multiplicative functions is also multiplicative**. (The proof is left as an exercise :P)

Example of Dirichlet Convolution

- phi * 1 = ?

This is: $\sum_{d|n}(\phi(d)\cdot 1) = ?$

$$\frac{1}{20}, \frac{2}{20}, \frac{3}{20}, \frac{4}{20}, \frac{5}{20}, \frac{6}{20}, \frac{7}{20}, \frac{8}{20}, \frac{9}{20}, \frac{10}{20}, \frac{11}{20}, \frac{12}{20}, \frac{13}{20}, \frac{14}{20}, \frac{15}{20}, \frac{16}{20}, \frac{17}{20}, \frac{18}{20}, \frac{19}{20}, \frac{20}{20}.$$

$$\frac{1}{20}, \frac{1}{10}, \frac{3}{20}, \frac{1}{5}, \frac{1}{4}, \frac{3}{10}, \frac{7}{20}, \frac{2}{5}, \frac{9}{20}, \frac{1}{2}, \frac{11}{20}, \frac{3}{5}, \frac{13}{20}, \frac{7}{10}, \frac{3}{4}, \frac{4}{5}, \frac{17}{20}, \frac{9}{10}, \frac{19}{20}, \frac{1}{1}$$

After simplification, we can see that there are phi(20) fractions left with denominator 20, phi(10) fractions left with denominator 10, and so on.

Applications on contest problems

2021 AMC 12A #25

Let $d(n)$ denote the number of positive integers that divide n, including 1 and n. For example, $d(1) = 1, d(2) = 2$, and $d(12) = 6$. (This function is known as the divisor function.) Let

$$f(n) = \frac{d(n)}{\sqrt[3]{n}}.$$

There is a unique positive integer N such that $f(N) > f(n)$ for all positive integers $n \neq N$. What is the sum of the digits of N?

$\textbf{(A)}\ 5 \qquad \textbf{(B)}\ 6 \qquad \textbf{(C)}\ 7 \qquad \textbf{(D)}\ 8 \qquad \textbf{(E)}\ 9$

d is multiplicative, we can just find individual prime powers e_1, e_2... such that $\frac{e_1+1}{2^{\frac{e_1}{3}}}, \frac{e_2+1}{3^{\frac{e_2}{3}}}, \frac{e_3+1}{5^{\frac{e_3}{3}}}$... are each maximized and take N=2^(e_1)3^(e_2)...

After some calculations, we can find that e_1=3 and e_2=2, which solves the problem as it forces the sum of digits to be a multiple of 9.

2021 AIME I #14

For any positive integer a, $\sigma(a)$ denotes the sum of the positive integer divisors of a. Let n be the least positive integer such that $\sigma(a^n) - 1$ is divisible by 2021 for all positive integers a. Find the sum of the prime factors in the prime factorization of n.

Sine the sum of positive divisors is multiplicative, it suffices to compute sigma((p^m)^n)=(p^(m*n+1)-1)/(p^m-1) for each individual prime p. We can divide this into 3 cases.

Case 1: p^m is not equivalent to 0 or 1 mod 43 or 47. By FLT, we have that p^(42m+1) is equivalent to p mod 43 and p^(46m+1) is equivalent to p mod 47, therefore we would need n to be a multiple of lcm(42,46).

Case 2: p^m is equivalent to 1 mod 43 or 47. We can expand the fraction into 1+p^m+p^(2m)+...+p^(m*n); setting each term equivalent to 1 mod 43/47 would give that n is a multiple of 43 and 47.

Case 3: p is 43/47. Then, the fraction is equivalent to (-1)/(-1)=1 mod 43/47.

From here, we can compute the answer as the sum of prime divisors of lcm(42,46)*43*47=125.

2021 HMMT Spring Alg/NT #5

Let n be the product of the first 10 primes, and let

$$S = \sum_{xy|n} \varphi(x) \cdot y,$$

where $\varphi(x)$ denotes the number of positive integers less than or equal to x that are relatively prime to x, and the sum is taken over ordered pairs (x, y) of positive integers for which xy divides n. Compute $\frac{S}{n}$.

Notice that the sum of y over y|(n/x) is equal to sigma(n/x); the sum can be written as the sum of phi(x)sigma(n/x) over all x|n.

We have a Dirichlet convolution, and as phi and sigma are both multiplicative, it suffices to compute S/p for each individual p. It is equal to (p-1+p+1)/p=2; the answer is therefore 2^10=1024.