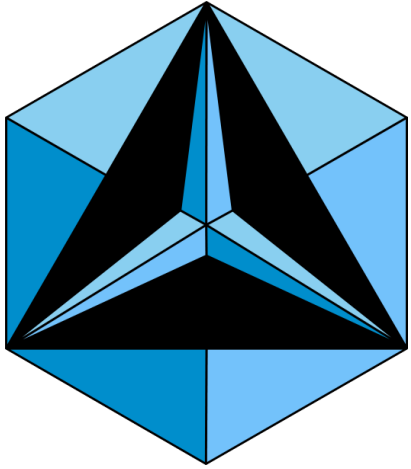


Cyclic National Competitive
Math Group

Member Lecture: Multiplicative Functions

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Technical Details



Definitions

- ▶ An “Arithmetic Function” is a function from the set of positive integers to the set of complex numbers.
- ▶ A “Multiplicative Function” is an arithmetic function f such that for any two positive integers m, n with $\gcd(m, n) = 1$, we have $f(mn) = f(m)f(n)$. The product of two multiplicative functions is also multiplicative.
- ▶ A multiplicative function is “Completely Multiplicative” if for any two positive integers m, n that are not necessarily relatively prime, we have $f(mn) = f(m)f(n)$.
- ▶ When working with multiplicative functions, we can split $f(n) = f(p_1^{e_1})f(p_2^{e_2})f(p_3^{e_3})\dots$ for each prime power dividing n ; this is a key idea in many contest problems that involve multiplicative functions.



Exemplar functions

- ▶ Which functions of the following are multiplicative? And which of the following are completely multiplicative? (assume that the domains are positive integers)

$f(n)=1$ $f(n)=n^7$ sum-of-divisors totient-function number-of-divisors $f(n)=\gcd(n,187)$

C.M.

C.M.

multiplicative

multiplicative

multiplicative

multiplicative



Dirichlet Convolution

- ▶ The Dirichlet Convolution of two arithmetic functions is as follow:

$$(f * g)(n) = \sum_{d|n} f(d) g\left(\frac{n}{d}\right) = \sum_{ab=n} f(a) g(b)$$

- ▶ It is commutative, associative and distributes over addition. Its most important property, for the purpose of this lecture, is that **the Dirichlet convolution of two multiplicative functions is also multiplicative.**
(The proof is left as an exercise :P)



Example of Dirichlet Convolution

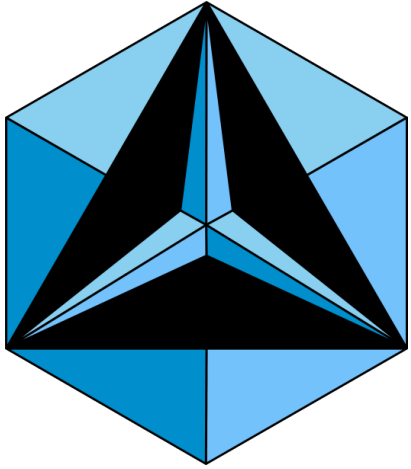
► $\phi * 1 = ?$

This is: $\sum_{d|n} (\phi(d) \cdot 1) = ?$

$$\frac{1}{20}, \frac{2}{20}, \frac{3}{20}, \frac{4}{20}, \frac{5}{20}, \frac{6}{20}, \frac{7}{20}, \frac{8}{20}, \frac{9}{20}, \frac{10}{20}, \frac{11}{20}, \frac{12}{20}, \frac{13}{20}, \frac{14}{20}, \frac{15}{20}, \frac{16}{20}, \frac{17}{20}, \frac{18}{20}, \frac{19}{20}, \frac{20}{20}.$$

$$\frac{1}{20}, \frac{1}{10}, \frac{3}{20}, \frac{1}{5}, \frac{1}{4}, \frac{3}{10}, \frac{7}{20}, \frac{2}{5}, \frac{9}{20}, \frac{1}{2}, \frac{11}{20}, \frac{3}{5}, \frac{13}{20}, \frac{7}{10}, \frac{3}{4}, \frac{4}{5}, \frac{17}{20}, \frac{9}{10}, \frac{19}{20}, \frac{1}{1}.$$

After simplification, we can see that there are $\phi(20)$ fractions left with denominator 20, $\phi(10)$ fractions left with denominator 10, and so on.



Applications on contest problems



2021 AMC 12A #25

Let $d(n)$ denote the number of positive integers that divide n , including 1 and n . For example, $d(1) = 1$, $d(2) = 2$, and $d(12) = 6$. (This function is known as the divisor function.) Let

$$f(n) = \frac{d(n)}{\sqrt[3]{n}}.$$

There is a unique positive integer N such that $f(N) > f(n)$ for all positive integers $n \neq N$. What is the sum of the digits of N ?

(A) 5 (B) 6 (C) 7 (D) 8 (E) 9

d is multiplicative, we can just find individual prime powers e_1, e_2, \dots such that $\frac{e_1+1}{2^{\frac{e_1}{3}}}, \frac{e_2+1}{3^{\frac{e_2}{3}}}, \frac{e_3+1}{5^{\frac{e_3}{3}}}, \dots$ are each maximized and take $N=2^{e_1}3^{e_2}\dots$

After some calculations, we can find that $e_1=3$ and $e_2=2$, which solves the problem as it forces the sum of digits to be a multiple of 9.



2021 AIME I #14

For any positive integer a , $\sigma(a)$ denotes the sum of the positive integer divisors of a . Let n be the least positive integer such that $\sigma(a^n) - 1$ is divisible by 2021 for all positive integers a . Find the sum of the prime factors in the prime factorization of n .

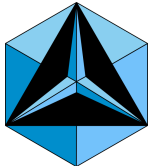
Since the sum of positive divisors is multiplicative, it suffices to compute $\sigma((p^m)^n) = (p^{(m \cdot n + 1)} - 1) / (p^m - 1)$ for each individual prime p . We can divide this into 3 cases.

Case 1: p^m is not equivalent to 0 or 1 mod 43 or 47. By FLT, we have that $p^{(42m+1)}$ is equivalent to p mod 43 and $p^{(46m+1)}$ is equivalent to p mod 47, therefore we would need n to be a multiple of $\text{lcm}(42, 46)$.

Case 2: p^m is equivalent to 1 mod 43 or 47. We can expand the fraction into $1 + p^m + p^{(2m)} + \dots + p^{(m \cdot n)}$; setting each term equivalent to 1 mod 43/47 would give that n is a multiple of 43 and 47.

Case 3: p is 43/47. Then, the fraction is equivalent to $(-1)/(-1) = 1$ mod 43/47.

From here, we can compute the answer as the sum of prime divisors of $\text{lcm}(42, 46) \cdot 43 \cdot 47 = 125$.



2021 HMMT Spring Alg/NT #5

Let n be the product of the first 10 primes, and let

$$S = \sum_{xy|n} \varphi(x) \cdot y,$$

where $\varphi(x)$ denotes the number of positive integers less than or equal to x that are relatively prime to x , and the sum is taken over ordered pairs (x, y) of positive integers for which xy divides n . Compute $\frac{S}{n}$.

Notice that the sum of y over $y|(n/x)$ is equal to $\sigma(n/x)$; the sum can be written as the sum of $\varphi(x)\sigma(n/x)$ over all $x|n$.

We have a Dirichlet convolution, and as φ and σ are both multiplicative, it suffices to compute S/p for each individual p . It is equal to $(p-1+p+1)/p=2$; the answer is therefore $2^{10}=1024$.