5 Interesting Problems from the 2020 AMCs

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§ 1 Introduction

These are five interesting AMC problems from the 2020 cycle and their solutions. I added motivations for what steps are taken to show the thought process involved in solving.

§ 2 Problem 1

Problem 1 (2020 AMC 10B 25/12B 4)

Let $D(n)$ denote the number of ways of writing the positive integer n as a product

$$n = f_1 \cdot f_2 \cdots f_k,$$

where $k \geq 1$, the f_i are integers strictly greater than 1, and the order in which the factors are listed matters (that is, two representations that differ only in the order of the factors are counted as distinct). For example, the number 6 can be written as 6, $2 \cdot 3$, and $3 \cdot 2$, so $D(6) = 3$. What is $D(96)$?

Solution: Knowing the prime factorization of 96 is a good start. It is

$$96 = 2^5 \cdot 3$$

We want to organize these factors into some number of groups where each group has at least one 2 or 3, since we need to split it into groups greater than 1.

Let's drop the 3 for just a second. Consider splitting the 2s into n groups, where each group contains at least one factor of 2. Once we do this, we can either put 3 as a separate factor into any of the gaps (there are $n+1$ gaps, as we are including putting 3 at the beginning or at the end). We can also put 3 into one of the n groups, essentially multiplying the product of the numbers in that group by 3. We can do this in n ways. Thus, there are a total of $2n+1$ ways to incorporate the 3.

Now we can use the Ball and Urn method along with some case work.

Case 1: 1 group There is only one way to split the factors of 2 in one group, and that is by having them all in one group. There are 3 ways to insert the 3, so we have a total of $1 \cdot 3 = \boxed{3}$ for this case.

Case 2: 2 groups

By the Ball and Urn method, we place one factor of 2 in each group, so we have to find the ways to rearrange $5 - 2 =$ three 2's and 1 bar. This is $\binom{4}{1} = 4$. There are 5 ways to insert the 3, giving us a total of $4 \cdot 5 = \boxed{20}$ for this case.

Case 3: 3 groups

Proceeding similarly as the last case, we see that we can distribute the 2's into three groups in $\binom{4}{2} = 6$ ways. There are 7 ways to insert the 3. This gives us a total of $6 \cdot 7 = \boxed{42}$ for this case.

Case 4: 4 groups

Once again using the Ball and Urn method, we see that there are $\binom{4}{1} = 4$ ways to distribute the 2s. There are 9 ways to insert the 3. This gives us a total of $4 \cdot 9 = \boxed{36}$ for this case.

Case 5: 5 groups

There is only one way to do this, have one 2 in each group. There are 11 ways to insert the 3. We have $1 \cdot 11 = \boxed{11}$ for this case.

We can get the answer by summing up the answers for the cases, we get:

$$3 + 20 + 42 + 36 + 11 = \boxed{112}$$

§ 3 Problem 2

Problem 2 (2020 AMC 10A 21/12A 19)

There exists a unique strictly increasing sequence of nonnegative integers $a_1 < a_2 < \ldots < a_k$ such that

$$\frac{2^{289}+1}{2^{17}+1} = 2^{a_1} + 2^{a_2} + \ldots + 2^{a_k}.$$

What is k?

Solution: Let $a = 2^{17}$, we can rewrite the LHS as

$$\frac{a^{17}+1}{a+1}$$

Note that from the geometric sequence formula, this is a geometric sequence with initial term 1, common ratio $-a$, and 17 terms. We get:

$$1 - a + a^2 - a^3 + \cdots - a^{15} + a^{16}$$

We can group the terms as the following, so we can factor out an a. We do this because it seems beneficial given that we are working with powers of 2, as an expression in the form $2^k - 1$ can be split into a sum of powers of 2.

$$1 + (a^2 - a) + (a^4 - a^3) + (a^6 - a^5) + \cdots + (a^{16} - a^{15})$$

$$1 + a(a-1) + a^3(a-1) + a^5(a-1) + \cdots + a^15(a-1)$$

Substituting back $a = 2^{17}$, we get:

$$1 + 2^{17}(2^{17}-1) + 2^{51}(2^{17}-1) + 2^{85}(2^{17}-1) + \cdots + 2^{255}(2^{17}-1)$$

Note that we can write $2^17 - 1 = 2^0 + 2^1 + 2^2 + \cdots + 2^{15} + 2^{16}$. We can distribute the powers of 2, and we can see that there is no overlap between each group where we distribute, since there are 17 terms in the above sequence, and each adjacent power of 2 that gets distributed differs by 34. Thus, our answer is $8 \cdot 17 + 1 = \boxed{137}$.

§ 4 Problem 3

Problem 3 (2020 AMC 12B 22)

What is the maximum value of $\frac{(2^t-3t)t}{4^t}$ for real values of t?

Solution:

We are given a product that we want to maximize. Even more interestingly, the first term of the product in the numerator is $2^t - 3t$, and the second is t. Since we can cancel the linear terms by summing, this motivates the use of AM-GM.

Before we use AM-GM, it's important to note that AM-GM only works for positive reals. We need to rule out negative values of t and $t = 0$. Testing, these always yield values less than or equal to 0. Taking $t = 4$, we get a positive value, so the solution we receive by AM-GM is our answer.

We will use AM-GM on $\dfrac{2^t - 3t}{4^t}$ and $\dfrac{3t}{4^t}$ so we can cancel out the linear terms.

$$\frac{2^t - 3t}{4^t} + \frac{3t}{4^t} \geq 2\sqrt{\frac{(2^t - 3t)(3t)}{4^{2t}}}$$

$$\frac{1}{2^t} \geq 2\sqrt{\frac{(2^t - 3t)(3t)}{4^{2t}}}$$

$$\frac{1}{2^{2t}} \geq 4 \cdot \frac{(2^t - 3t)(3t)}{4^{2t}}$$

$$\frac{4^t}{2^{2t} \cdot 4 \cdot 3} \geq \frac{(2^t - 3t)t}{4^t}$$

$$\frac{1}{12} \geq \frac{(2^t - 3t)t}{4^t}$$

by AM-GM. Thus, our answer is $\boxed{\dfrac{1}{12}}$.

§ 5 Problem 4

Problem 4 (2020 AMC 10A 22)

For how many positive integers $n \leq 1000$ is

$$\left\lfloor \frac{998}{n} \right\rfloor + \left\lfloor \frac{999}{n} \right\rfloor + \left\lfloor \frac{1000}{n} \right\rfloor$$

not divisible by 3? (Recall that $\lfloor x \rfloor$ is the greatest integer less than or equal to x.)

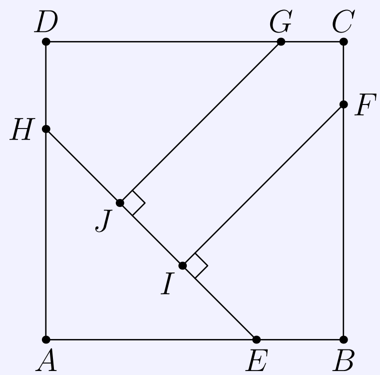
Solution: Note that if the floors are 3 consecutive integers, or all the same, then the number will be divisible by 3. We want to consider such numbers when 2 of the floors give the same result and one is different. This is only possible if $n|999$ or $n|1000$. Before proceeding, note that $n = 1$ does not work, since it yields 3 consecutive integers. 2 works as well, and anything higher that divides either 999 or 1000 will work since the differences between multiples will be too large to encompass 998.

Our answer is simply the number of $d(1000) + d(999) - 2$, where $d(n)$ is the number of divisors of n. $1000 = 2^3 \cdot 5^3$ has 16 divisors, while $999 = 37 \cdot 3^3$ has 8 divisors. Thus, our answer is $16 + 8 - 2 = \boxed{22}$.

§ 6 Problem 5

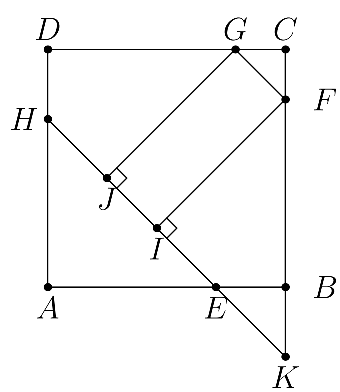
Problem 5 (2020 AMC 10B 21/12B 18)

In square $ABCD$, points E and H lie on $\overline{AB}$ and $\overline{DA}$, respectively, so that $AE = AH$. Points F and G lie on $\overline{BC}$ and $\overline{CD}$, respectively, and points I and J lie on $\overline{EH}$ so that $\overline{FI} \perp \overline{EH}$ and $\overline{GJ} \perp \overline{EH}$. See the figure below. Triangle AEH, quadrilateral $BFIE$, quadrilateral $DHJG$, and pentagon $FCGJI$ each has area 1. What is FI^2?



Solution: We should try to find what lengths would give us FI. Knowing the length JI would be great, because we can find the length of FI using the area of trapezoids. How can we find JI though? We would need to know the length of HJ, and that seems pretty painful to find, so we look for a more clever solution.

Let's look at the angle measures of quadrilateral $FIEB$. The measures are $\angle FIE = 90°, \angle IEB = 135°, \angle EBF = 90°, \angle BFI = 45°$. We can see here that if we were to extend lines IE and FB to meet at a point K, we would form a 45-45-90 triangle with KIF.



(This diagram was made by AoPS user Mudkipswims42, which can be found on the AoPS thread for the problem, I just changed the label for point K to be consistent with this solution)

This seems like an interesting method to use. If we knew the length KF, we could solve the problem, but that also seems tedious to find. We can see that we already know the area of quadrilateral $FIEB$, its just 1. We can also find FI if we knew the area of the triangle. Note that KEB also forms a 45-45-90 triangle. If we knew what EB is, we could solve the problem.

We can see that the side length of the square is 2, as the area of the square is 4. We know that $EB = 2 - AE$. We can find AE since $[AEH] = 1 \, implies AE = \sqrt{2}$. Thus, $EB = 2 - \sqrt{2}$.

We have that $[KEB] = \frac{1}{2}(2 - \sqrt{2})^2 = 3 - 2\sqrt{2}$. Thus, we have $[KIF] = [KEB] + 1 = 4 - 2\sqrt{2}$. Not only can we find FI here, we can find FI^2 directly. Note that $[KIF] = \frac{1}{2}FI^2 = 4 - 2\sqrt{2} \implies FI^2 = \boxed{8 - 4\sqrt{2}}$, and we are done.