Contour Integration

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April 26, 2021

Contents

1 Introduction

In this handout, we're going to be covering integration in the complex plane. Obviously, some background of calculus will be required to understand what's going on, and some analysis techniques as well. Calculus of complex variables, otherwise known as Complex Analysis, is notably a very beautiful subject, and the results are rather enjoyable.

There are going to be a lot of parallels to vector calculus, as we can think of the complex plane, $\mathbb{C}$, as $\mathbb{R} \times \mathbb{R}$. However, the concept of a differentiable function is *much* stronger in the complex plane than the real plane, which is why many of the results in complex analysis are very clean (unlike real analysis, consider Weirstrauss' monster for example).

2 Review: Functions, Limits, and Derivatives

Before we can talk about integration, we need to define the functions that we're integrating!

Definition 2.1: A **function** $f : \mathbb{C} \to \mathbb{C}$ can be defined in terms of its real and imaginary parts:

$$f(x + iy) = u(x, y) + iv(x, y).$$

Just like real functions, complex functions have a **domain** and **image**.

We can analogously define the limit and derivative of a function in $\mathbb{C}$. Just like in $\mathbb{R}$, limits are unique if they exist. All properties of limits are preserved in the complex plane – linearity, multiplication, and division of limits are all maintained.

Definition 2.2: A function $f : \mathbb{C} \to \mathbb{C}$ is **continuous** at $z_0 \in \mathbb{C}$ if $\lim_{z\to z_0} f(z) = f(z_0)$. Furthermore, if $f = u + iv$ is continuous at $z_0 = x_0 + iy_0$, then u and v are both continuous at (x_0, y_0).

The definition of the derivative is also very similar in $\mathbb{C}$, and differentiability implies continuity. Specifically, we have that

$$f'(z_0) = \lim_{z\to z_0} \frac{f(z) - f(z_0)}{z - z_0}.$$

Linearity of the derivative is preserved in the complex plane. However, as we'll soon see, the derivative in $\mathbb{C}$ is a much stronger result than the derivative in $\mathbb{R}$.

One important result in complex analysis are the **Cauchy Riemann Equations**, which describe a set of necessary conditions for differentiability. However, they are *not* sufficient, but are an efficient way to check a functions differentiability at a point.

Theorem 2.3 (Cauchy Riemann)
Let $f(x + iy) = u(x, y) + iv(x, y)$, and assume that $f'(z)$ exists at $z = x + iy$. Then the first order partial derivatives of u and v exist, and

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

We also have that

$$f'(z) = \frac{\partial u}{\partial x}(x, y) + i\frac{\partial v}{\partial x}(x, y).$$

These are not sufficient for the existence of the derivative at a point, as there are additional continuity conditions that must be satisfied as well (Note that nowhere in the above theorem statement does f have to be continuous).

We now define the holomorphic, or analytic function in $\mathbb{C}$.

> **Definition 2.4:** A function $f : \mathbb{C} \to \mathbb{C}$ is said to be **holomorphic** if it is differentiable at every point of an open set $S \subseteq \mathbb{C}$.

While this may just seem like the traditional "differentiable everywhere" in $\mathbb{R}$, this result is much stronger in $\mathbb{C}$.

Similarly to limits in higher dimensions of $\mathbb{R}$, a limit at a point $z_0 \in \mathbb{C}$ must be the same across all paths one can take to arrive at the point z_0, or else it does not exist. This inherently makes the definition of the derivative much stronger in the complex plane, in particular, this allows holomorphic functions to satisfy the property of **analyticity**. This means that, at every point on its domain, the function is given by a convergent power series. By consequence, all holomorphic functions are infinitely differentiable.

This is in contrast to real functions, as there are infinitely differentiable real functions which are analytic nowhere. When a function is holomorphic on all of $\mathbb{C}$, it is called an **entire** function.

Almost all elementary functions are entire on $\mathbb{C}$, such as polynomials, trigonometric functions, and the exponential function. Others are **meromorphic** – they are holomorphic everywhere except at a set of isolated points (for example, $f(z) = \frac{1}{z}$).

> **Remark.** With all these mentions of "properties of the derivative are preserved in $\mathbb{C}$", you may think that the same holds for he mean value theorem. However, this is *not* the case for a function $w : \mathbb{R} \to \mathbb{C}$ (a complex valued function of a real variable).

With that out of the way, we're ready to talk about complex integration.

✾ 3 Integration: Complex Valued Functions of a Real Variable

We begin with the simplest starting point: integrating complex valued functions $w = u + iv$ of a real variable t. These function exactly the same as real functions, including their definite integrals. We have

$$\int_a^b w(t)\,\mathrm{d}t = \int_a^b u(t)\,\mathrm{d}t + i\int_a^b v(t)\,\mathrm{d}t,$$

provided that these integrals exist.

The fundamental theorem of calculus, and all properties of integrals all hold for these integrals as well. The only difference is that the function is inherently split into 2 different functions, $w(t) = u(t) + iv(t)$.

✾ 4 Contour Integrals

Recall that the complex plane $\mathbb{C}$ is *not* an ordered set. We can't say that some complex number $w < z$. We can compare magnitudes, but not the numbers themselves. This makes integration more complicated: we can't integrate over an interval on the complex plane.

Similar to vector calculus (where we also cannot order elements), integration of complex valued functions of a **complex** variable are done over curves in the complex plane.

> **Definition 4.1:** A set of points $z = (x, y)$ in the complex plane is called an **arc** if
>
> $$x = x(t), y = y(t)$$
>
> for $a \leq t \leq b$, where $x(t)$ and $y(t)$ are continuous functions of a real variable t. We write $z = z(t)$ for convienience, where $z(t) = x(t) + iy(t)$.

An arc C is called a **Jordan** or **simple** arc if it is not self intersecting. When C is a simple arc except that $z(a) = z(b)$ for $t \in [a, b]$, we say that C is a **simple closed curve**, or a **Jordan curve**.

For all purposes, the counterclockwise direction is taken as positive.

Example 4.2: Consider the unit circle in $\mathbb{C}$. This can be parameterized by $z = e^{i\theta}$, for $\theta \in [0, 2\pi]$. We can extend this to a circle of radius R centered at z_0:

$$z(\theta) = z_0 + Re^{i\theta},$$

for $\theta \in [0, 2\pi]$.

If we negate the exponent of z, we do *not* get the same curve. The points are the same, but it is traversed in the negative (clockwise) direction. Similarly, multiplying the exponent by a factor of k makes the circle get traversed k times when θ varies from 0 to 2π.

If the component functions x and y of z are differentiable on the interval $t \in [a, b]$, then the arc C is known as a **differentiable arc**. We can integrate the real function

$$|z'(t)| = \sqrt{x'(t)^2 + y'(t)^2}$$

over the interval $[a, b]$, and this produces the **arc length** of C, which is commonly denoted by L.

Just like in vector calculus we can now define the **unit tangent vector**

$$\mathbf{T} = \frac{z'(t)}{|z'(t)|}.$$

This has an angle of inclination (relative to the real axis) of $\arg z'(t)$.

If z' is continuous on the closed interval $[a, b]$, then we call the curve C it parameterizes as a **smooth curve**.

We can now define the contour, the main path of integration in the complex plane.

Definition 4.3: A **contour** is a piecewise smooth arc, or an arc consisting of a finite number of smooth arcs joined end to end. When $z(a) = z(b)$ for $t \in [a, b]$ on a contour C, we say that it is a **simple closed contour**.

One quick result (without proof, as that requires some advanced theory) is the Jordan curve theorem:

Theorem 4.4 (Jordan Curve Theorem)

The Jordan curve theorem states that every Jordan curve divides the plane into an interior region, which is bounded by the curve, and an exterior region. This means that any path connecting the exterior to the interior must intersect the curve at some point.

Contour integrals look very familiar, as they are a special case of line integrals! They behave the same way, and even have the exact same definition. We denote the **contour integral** of f over a contour C by

$$\int_C f(z)\, dz.$$

The following notation denotes an integral over a simple closed contour:

$$\oint_C f(z)\, dz.$$

The circle indicates that the path we integrate over is closed.

Unfortunately, there isn't really a 'nice' geometric or physical interpretation of integrals in the complex plane, unlike in $\mathbb{R}$, where we can consider the integral as the area under a curve.

A few asumptions must be made before we can define the contour integral of a function f over a contour C parameterized by a curve $z(t)$, for $t \in [a, b]$. Namely, we assume that $f(z(t))$ is piecewise continuous on $[a, b]$. We may now define the contour integral:

Definition 4.5: The contour integral of a function f along a contour C, parameterized by a function $z(t)$, for $t \in [a, b]$ is

$$\int_C f(z)\,\mathrm{d}z = \int_a^b f(z(t))z'(t)\,\mathrm{d}t.$$

This definiton should look familiar, as it is the same as the definition for the line integral in $\mathbb{R}^n$.

If a contour C can be expressed as a sum of contours $C_1, C_2, \cdots, C_n$, all joined at their endpoints, then the contour integral of f on C is equivalent to

$$\sum_{k=1}^{n} \int_{C_k} f(z)\,\mathrm{d}z.$$

Let's look at a rather famous example:

Example 4.6: Evaluate the contour integral of $f(z) = \frac{1}{z}$ on the unit circle (traversed positively).

Solution. To parameterize the unit circle, we have $z(\theta) = e^{i\theta}$, from $\theta \in [0, 2\pi]$. Writing out the expression for a contour integral, we have

$$\oint_C f(z)\,\mathrm{d}z = \oint_C \frac{\mathrm{d}z}{z} = \int_0^{2\pi} \frac{1}{e^{i\theta}} ie^{i\theta}\,\mathrm{d}\theta = \int_0^{2\pi} i\,\mathrm{d}\theta,$$

which is simply $2\pi i - 0 = \boxed{2\pi i}$.

We used $\oint$ because the unit circle is a closed contour. Here's another example, from a recent exam of mine:

Example 4.7: Evaluate the contour integral of $f(z) = z^2$ on $C = C_1 + C_2$, where C_1 is the upper half of the unit circle, and C_2 is the path from -1 to 1.

Solution. We should split this up into two integrals, $\int_{C_1} f(z)\,\mathrm{d}z$, and $\int_{C_2} f(z)\,\mathrm{d}z$.

1. To parameterize C_1, we can use $z(\theta) = e^{i\theta}$, where $\theta \in [0, \pi]$. Writing out our integral, we have

$$\int_{C_1} f(z)\,\mathrm{d}z = \int_0^{\pi} ie^{i\theta}e^{2i\theta}\,\mathrm{d}\theta = i\int_0^{\pi} e^{3i\theta}\,\mathrm{d}\theta,$$

which is simply

$$i\left(-\frac{1}{3i} - \frac{1}{3i}\right) = -\frac{2}{3i}i = \boxed{-\frac{2}{3}.}$$

2. To parameterize C_2, we can use $z(t) = t$ for $t \in [-1, 1]$. Writing out the integral, we have

$$\int_{C_2} f(z)\,\mathrm{d}z = \int_{-1}^{1} t^2 \cdot 1\,\mathrm{d}t = \int_{-1}^{1} t^2\,\mathrm{d}t,$$

which is just $\boxed{\frac{2}{3}.}$

Summing up our integrals, we have

$$\int_C f(z)\,\mathrm{d}z = \int_{C_1} f(z)\,\mathrm{d}z + \int_{C_2} f(z)\,\mathrm{d}z = -\frac{2}{3} + \frac{2}{3} = \boxed{0},$$

and we're done.

While it may seem coincidental that the answer happened to be zero, this is not the case. Keep this example in the back of your head, because we'll return to it later.

Consider the following statement:

Lemma 4.8

If $w(t)$ is a piecewise continuous complex valued function defined on an interval $t \in [a, b]$, then we have

$$\left|\int_a^b w(t)\,\mathrm{d}t\right| \le \int_a^b |w(t)|\,\mathrm{d}t.$$

This statement should look familiar, as it is alike to a similar result that holds in $\mathbb{R}$. With this, we can state the following theorem.

Theorem 4.9 (Maximum Modulus)

Let C denote a contour of length L, and suppose that a function $f(z)$ is piecewise continuous on C. If M is a nonngegative constant such that

$$|f(z)| \le M$$

for all points z on C for which $f(z)$ is defined, then

$$\left|\int_C f(z)\,\mathrm{d}z\right| \le ML.$$

Proof. Let $z = z(t)$, for $t \in [a, b]$, such that $z(t)$ parameterizes C. From the above lemma we have

$$\left|\int_C f(z)\,\mathrm{d}z\right| = \left|\int_a^b f(z(t))z'(t)\,\mathrm{d}t\right| \le \int_a^b |f(z(t))z'(t)|\,\mathrm{d}t.$$

Obviously we have $|f(z(t))z'(t)| \le M|z'(t)|$. Rewriting the integral, we have

$$\left|\int_C f(z)\,\mathrm{d}z\right| \le M\int_a^b |z'(t)|\,\mathrm{d}t.$$

But this is just M times the expression for the arclength of C, which is just L! And thus we are done. □

✾ 5 The Cauchy-Goursat Theorem

To begin, we need to use a result from vector calculus, **Green's Theorem**:

Theorem 5.1 (Green's Theorem)

Suppose C is a positively oriented piecewise smooth simple closed curve in a plane, and let D be a region bounded by C. If P and Q are functions of (x, y), defined on an open region containing D, and have continuous partial derivatives there, then we have

$$\oint_C (P\,\mathrm{d}x + Q\,\mathrm{d}y) = \iint \left(\frac{\partial P}{\partial x} - \frac{\partial Q}{\partial y}\right) \mathrm{d}x\,\mathrm{d}y.$$

But suppose we are dealing with a complex valued function $f(z) = u(x, y) + iv(x, y)$. With Green's theorem, we can write

$$\int_C f(z)\,\mathrm{d}z = \iint_R (-v_x - u_y)\,\mathrm{d}A + i\iint_R (u_x - v_y)\,\mathrm{d}A.$$

But suppose f is analytic, and thus the Cauchy Riemann equations must hold, meaning

$$u_x = v_y, u_y = -v_x.$$

This makes the above two integrals evalutate to 0, and thus

$$\int_C f(z)\,\mathrm{d}z = 0.$$

For the above to work, f must be analytic and f' must be continuous. However, f' need not be continuous, it just has to exist! We won't prove this here, but we can now state a generalized version of this result, the **Cauchy Goursat Theorem**:

Theorem 5.2 (Cauchy Goursat)
If a function f is analytic at all points interior to and on a simple closed contour C, then we have

$$\int_C f(z)\,\mathrm{d}z = 0.$$

The proof of this is rather long, and involves more analysis techniques, which are kind of beyond the scope of this handout, so we're going to omit it.

Remark. The notation u_x, u_y, v_x, v_y is shorthand for the partial derivatives of u and v. They denote

$$\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \text{ and } \frac{\partial v}{\partial y},$$

respectively.

6 The Cauchy Integral Formula

This is another fundamental result.

Theorem 6.1 (Cauchy Integral Formula)
Suppose f is analytic both inside and on a simple closed contour C (taken in the positive sense). If z_0 is any point interior to C, then

$$f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)\,\mathrm{d}z)}{z - z_0}.$$

Proof. Let C_ρ denote a positively oriented circle $|z - z_0| = \rho$, such that C_ρ is interior to C. Since $\frac{f(z)}{z-z_0}$ is analytic between and on the two contours C and C_ρ, the contour integrals of this quotient on both C and C_ρ must be equal (this is called the **Principle of Deformation of Paths**). We may then write

$$\int_C \frac{f(z)\,\mathrm{d}z}{z - z_0} - f(z_0)\int_{C_\rho} \frac{\mathrm{d}z}{z - z_0} = \int_{C_\rho} \frac{f(z) - f(z_0)}{z - z_0}\,\mathrm{d}z.$$

We can evaluate

$$\int_{C_\rho} \frac{\mathrm{d}z}{z - z_0},$$

as it is just $2\pi i$. Since f must also be continuous, we have

$$|f(z) - f(z_0)| < \varepsilon \text{ whenever } |z - z_0| < \delta.$$

If we let $\rho < \delta$, then we can write

$$\left| \int_{C_\rho} \frac{f(z) - f(z_0)}{z - z_0} \,\mathrm{d}z \right| < \frac{\varepsilon}{\rho} 2\pi\rho = 2\pi\varepsilon.$$

Since the modulus of the integral is never negative, and ε is an arbitrarily small number, we can write

$$\int_C \frac{f(z)\,\mathrm{d}z}{z - z_0} - 2\pi i f(z_0) = 0 \iff \int_C \frac{f(z)\,\mathrm{d}z}{(z - z_0)} = 2\pi i f(z_0).$$

□

We can extend the Cauchy integral formula, to provide a representation of higher order derivatives:

Theorem 6.2 (Extended Cauchy Integral Formula)

Let f be analytic inside and on a simple closed contour C, taken in the positive sense. If z_0 is any point interior to C, then we have

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_C \frac{f(z)\,\mathrm{d}z}{(z - z_0)^{n+1}},$$

for some $n \in \mathbb{N}$.

Let's look at an example of how this can be used to evaluate an integral.

Example 6.3: Evaluate the following:

$$\int_C \frac{e^{2z}\,\mathrm{d}z}{z^4},$$

where C is the positively oriented unit circle.

Solution. Let $f(z) = e^{2z}$. Then we have

$$\int_C \frac{f(z)\,\mathrm{d}z}{z^4} = \frac{2\pi i}{3!} f'''(0) = \boxed{\frac{8\pi i}{3}}.$$

One consequence of this extension is that if f is an analytic function at a point z, its derivatives of all orders are also analytic at that point. This explains why all differentiable functions in $\mathbb{C}$ are infinitely differentiable – they must be analytic. This extends to the partial derivatives of u and v if $f(z) = u(x, y) + iv(x, y)$. We also have the following:

Lemma 6.4

If f is a continuous function on a domain D, and

$$\int_C f(z)\,\mathrm{d}z = 0$$

for every closed contour $C \in D$, then f must be analytic throughout D.

This is a nice way to check if a function is analytic, and follows rather nicely from the fact that f must have an antiderivative in D, and the previous result.

✾ 7 Liouville's Theorem and the Fundamental Theorem of Algebra

You may remember being told that you couldn't prove the Fundamental Theorem of Algebra when you first learned it. This is because we require machinery from complex analysis to prove that theorem. In this section, we'll finally prove the Fundamental Theorem of Algebra.

First, we must state the following important result:

Theorem 7.1 (Liouville's Theorem)
If a function f is entire and bounded in the comlex plane, then it must be constant throughout the plane.

Proof. We begin with the following lemma:

Lemma (Cauchy's Inequality)
Suppose that a function f is analytic inside and on a positively oriented circle C_R, centered at z_0 with radius R. Let M_R denote the maximum value of $|f(z)|$ on C_R. Then we have

$$\left|f^{(n)}(z_0)\right| \leq \frac{n!M_R}{R^n},$$

for $n \in \mathbb{N}$. The proof of this immediately follows from the extended Cauchy integral formula, and a bounding of the modulus of the contour integral.

Note that f is an entire function, and thus we can choose any z_0 and R for the above lemma. Taking $n = 1$, we have

$$|f'(z_0)| \leq \frac{M}{R},$$

where R can be arbitrarily large. Since M is independent of R, this can only hold when $|f'(z_0)| = 0$. As z_0 was arbitrary, this means $|f'(z)| = 0$ for all $z \in \mathbb{C}$, and this means f must be a constant function, and we are done. □

With that out of the way, (and by the way, this is a rather powerful result), we can prove the fundamental theorem of algebra.

Theorem 7.2 (Fundamental Theorem of Algebra)
Any polynomial

$$P(z) = \sum_{k=0}^{n} a_k z^k,$$

where $a_n \neq 0$ of degree $n \geq 1$ has at least one root. That is, there exists at least one point $z_0 \in \mathbb{C}$ such that $P(z_0) = 0$.

Proof. Assume, for the sake of contradiction, that $P(z)$ is not zero for any $z \in \mathbb{C}$. Then $\frac{1}{P(z)}$ is clearly an entire function. There exists a positive number R such that

$$\left|\frac{1}{P(z)}\right| < \frac{2}{|a_n|R^n}, \text{ whenever } |z| > R,$$

meaning that $\frac{1}{P(z)}$ must be bounded in the region exterior to the disk $|z| \leq R$. As $\frac{1}{P(z)}$ is continuous on said closed disk, it must be bounded there too, and is thus bounded in the entire plane.

As $\frac{1}{P(z)}$ is bounded in the entire plane, and entire, it must be a constant, by Liouville's theorem. This means that $P(z)$ must also be constant, which is a contradiction, as $\deg P(z) \geq 1$. Thus $P(z) = 0$ for some $z \in \mathbb{C}$, and we are done. □

From this, we can show that $P(z)$ must not have more than n distinct roots. The theorem above guarantees that we have at least one root, say z_1. Since we are guaranteed one root, we may write $P(z)$ as

$$P(z) = (z - z_1)Q(z),$$

for some polynomial $Q(z)$. But the same argument applies to $Q(z)$, and every polynomial we generate after that, until we reach a constant. Thus we can write $P(z)$ as a product of n linear factors, up to a constant:

$$P(z) = c\prod_{k=1}^{n}(z - z_k).$$

This guarantees that $P(z)$ cannot have more than n distinct roots. (Of course, some of the z_k may not be distinct, in which we have algebraic multiplicity of roots)

And with that, we're done! This handout does not cover everything related to contour integrals, namely some more inequalities and the concept of representations of contours. You can find out more about these topics from any good complex analysis text.

✾ 8 Sources

I used a few sources for this handout. They're listed here:

1. Brown, James Ward, Churchill, Ruel. *Complex Variables and Applications, 9ed.*
2. *Fundamental Theorem of Algebra.* Wikipedia. https://en.wikipedia.org/wiki/Fundamental_theorem_of_algebra.
3. *Green's Theorem.* Wikipedia. https://en.wikipedia.org/wiki/Green%27s_theorem.
4. *Jordan Curve Theorem.* Wikipedia. https://en.wikipedia.org/wiki/Jordan_curve_theorem.
5. *Cauchy's integral theorem.* Wikipedia. https://en.wikipedia.org/wiki/Cauchy%27s_integral_theorem.

I also dipped into my own notes from my complex analysis class, as well as the class lecture notes.