

Can One Shape the Sound of a Drum?

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Just 50 years ago, the Polish mathematician Kac [1] posed the intriguing question, ‘Can one hear the shape of a drum?’ It stimulated a variety of demonstrations that drums with differing irregular polygonal boundaries could have identical frequency spectra [2]. So, too, could circular drums, if only they could be made with inhomogeneous membranes having differing spatially-varying density or thickness profiles [3]. The answer to Kac was therefore an emphatic ‘No!’

This article derives from two related thoughts: first, that plastic membranes, now widely used as an alternative to traditional materials such as calfskin, could be made with prescribed thickness profiles by 3D printing and, secondly, that these profiles could shape the frequency spectrum.

A circular drum with a homogeneous membrane has a frequency spectrum determined by the zeros of Bessel functions. To an occidental ear, the ratios between frequencies so determined sound discordant. Could those ratios be made harmonious by shaping the profile of an inhomogeneous membrane? We shall show that the answer is a qualified ‘Yes’, with implications for orchestral drums.

A musical interlude

We have introduced some rather vague terms: ‘discordant’, ‘harmonious’, and ‘occidental ear’. Behind them lies an extensive musical background to which the encyclopaedic works by Scholes [4] and Latham [5] offer access. More specifically, Savage [6] has discussed the ratios between frequencies that have come to define those terms. For our purposes, it will suffice to say a little about tone colour (timbre), and thereby to introduce some particular ratios relevant to this article.

The sounds of musical instruments generally consist of a fundamental frequency accompanied by a spectrum of overtones, constituting its tone colour, and identifying the instrument to an experienced musical ear. A note played on a piano, for example, can be distinguished from a note with the same fundamental frequency played by plucking a violin string or by striking a drum. Subtler are the distinctions between the tone colours for a given fundamental played by bowing a cello, a viola or a violin. Instrument-makers influence tone colour through application of their arts to design and construction, and a few achieve the legendary status of a Stradivari. Players do so through their personal techniques. In particular, percussionists vary tone colour by their choices of beaters (for example, wooden sticks with or without felt, cloth, leather or cork covers, or brushes) and of where on the membrane a drum is struck.

Tone colour inspires adjectives worthy of a seed catalogue or a wine connoisseur: brilliant, coarse, mellow, muddy, lush, shrill, velvety, etc., together with an extensive palette of colours as nuanced as soft-red, red-purple, green-purple and emerald, perhaps influenced by synesthesia [4, pp. 202–210], [5, pp. 272–274]. Instruments differ in their capacity to vary the fundamental frequency and tone colour of the notes that they produce. The violin,

trombone and human voice, for example, can vary frequency continuously over limited ranges, whereas keyed instruments, such as the piano or clarinet, favour discrete sets of frequencies. Percussion instruments, such as drums, cymbals and bells, have frequency spectra determined by their shapes and sizes, and describable adequately in terms of ratios between a few overtones and the fundamental, so freeing a sober scientist from the intoxicating vocabulary of tone colour. What should those ratios be for a drum to sound harmonious?

For the sounds of musical instruments to be described, written down, composed with and reproduced, an agreed categorisation of the frequencies involved (as notes) is required. Within any such categorisation, it is to be expected that certain sequences (scales) and combinations of notes (chords), and of instruments (groups and orchestras) will be more appealing to the human ear (harmonious) than others. In different cultures, a variety of categorisations of

notes, scales and chords have evolved, determined by the capacities of their instruments, composers and performers. What has come to be regarded as harmonious to occidental ears over the last few centuries derives from a categorisation founded on a ‘well-tempered’ 12-note scale, with a frequency ratio between consecutive notes of $2^{1/12}$, described as one semitone [6].

Within this scale, three particular frequency ratios that are considered to be harmonious are the octave (= 2), the perfect fifth (= 3/2) and the perfect third (= 5/4). A prominent chord in musical composition is the major triad, consisting of a fundamental frequency and its perfect fifth and third. Happily, two notes in a 12-note scale approximate closely to the perfect fifth, $3/2 \approx 2^{7/12} = 1.498$ (the discrepancy being called the Pythagorean comma [5, p. 1019]), and to the perfect third, $5/4 \approx 2^{4/12} = 1.260$. As Savage points out [6], but for that ‘amazing mathematical fluke’, the beautiful music that we have learnt to enjoy would not exist, nor could instruments such as the piano be constructed.

In what follows, we shall respond to our question, ‘Can one shape the sound of a drum?’, by attempting to design an inhomogeneous membrane whose overtones relate to the fundamental through these three harmonious ratios.

Testing the test

To assess the feasibility of what we are attempting, we require the simplest plausible model for the membrane, and an adequate test for the effect of a specified inhomogeneity profile on a given overtone. To this end, we start from the Helmholtz equation for propagation across a stretched flexible membrane [7, p. 739]:

$$\nabla^2\eta + (\omega^2\sigma/T)\eta = 0,$$

where η is the displacement of the membrane normal to its plane, ω is the frequency of vibration, σ is the areal mass density at a given point within the membrane and T is the areal surface tension, assumed uniform.



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Multiplying this equation by η , integrating over the area, G , of the membrane and imposing $\eta = 0$ at the boundary gives

$$D[\eta] - (\omega^2/T)H[\sigma, \eta] = 0,$$

$$D[\eta] = \int_G |\nabla \eta|^2 dG,$$

$$H[\sigma, \eta] = \int_G \sigma \eta^2 dG.$$

This expression can be shown to be variational [7, pp. 790–793], so substitution for η of a trial function satisfying the boundary conditions will give an approximate value for ω^2 exceeding its exact value. As a trial function for an inhomogeneous membrane, we shall use the exact solution to the Helmholtz equation for a homogeneous membrane with areal mass density σ_0 , for which $\eta = \eta_0$ and $\omega = \omega_0$, and we have

$$\frac{\omega_0^2}{T} = \frac{D[\eta_0]}{H[\sigma_0, \eta_0]}, \quad \frac{\omega^2}{T} = \frac{D[\eta]}{H[\sigma, \eta]} \leq \frac{D[\eta_0]}{H[\sigma_0, \eta_0]},$$

which reduce to

$$\frac{\omega_0^2}{\omega^2} \geq \frac{H[\sigma, \eta_0]}{H[\sigma_0, \eta_0]}.$$

This expression relates ω and σ , but will only be an adequate test of the effect on ω of varying σ if there is approximate equality between the two sides. Without solving the Helmholtz equation exactly, some reassurance on this point can be inferred by employing a remarkable result that has lain idle in the literature on the Kac question for several decades [3]. First, we note that if the conformal transformation $w = f(z)$ is applied to the Helmholtz equation for a homogeneous membrane, expressed in rectangular coordinates (u, v) in the w -plane,

$$\frac{\partial^2 \eta}{\partial u^2} + \frac{\partial^2 \eta}{\partial v^2} + (\omega^2 \sigma_0/T)\eta = 0,$$

we obtain

$$\frac{\partial^2 \eta}{\partial x^2} + \frac{\partial^2 \eta}{\partial y^2} + (\omega^2 \sigma_0/T)|f'(z)|^2 \eta = 0,$$

expressed in rectangular coordinates (x, y) in the z -plane [8, p. 242]. This equation can be interpreted as describing a membrane with an inhomogeneous areal mass density profile, but having a frequency spectrum and mass identical to that of the homogeneous membrane.

Gottlieb's key step in [3] was to choose

$$w = f(z) = \frac{z - a}{1 - az},$$

where the parameter a lies in the range $0 < a < 1$. This transformation maps the unit disc onto itself, and maps concentric circles of radius R in the w -plane into nested circles of radius r_a , with centres displaced by x_a from the origin, in the z -plane:

$$u^2 + v^2 = R^2,$$

$$(x - x_a)^2 + y^2 = r_a^2,$$

$$x_a = a \frac{1 - R^2}{1 - a^2 R^2},$$

$$r_a = R \frac{1 - a^2}{1 - a^2 R^2}.$$

The areal density inhomogeneity profile in the z -plane is

$$\sigma(x, y) = \sigma_0 |f'(z)|^2 = \sigma_0 \frac{(1 - a^2)^2}{((1 - ax)^2 + a^2 y^2)^2},$$

$$\sigma(r, \theta) = \sigma_0 \frac{(1 - a^2)^2}{(1 - 2ar \cos \theta + a^2 r^2)^2}.$$



Candombe drummers
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Taking $a = 0.3$, for example, at the centre and cardinal points on the boundary of the unit disc, we have, respectively,

$$\begin{aligned}\sigma(0, \theta)/\sigma_0 &= (91/100)^2, \\ \sigma(1, \pi)/\sigma_0 &= (7/13)^2, \\ \sigma(1, \pi/2)/\sigma_0 &= (91/109)^2 = \sigma(1, 3\pi/2)/\sigma_0, \\ \sigma(1, 0)/\sigma_0 &= (13/7)^2 = (13/7)^4 \sigma(1, \pi)/\sigma_0 \\ &\approx 12\sigma(1, \pi)/\sigma_0.\end{aligned}$$

Exact solutions, η_0 , for the homogeneous membrane in the w -plane may now be used in the z -plane as trial functions, $J_n(j_{n,k}r) \cos n\theta$, to obtain ω/ω_0 approximately, where $j_{n,k}$ is the k th root of the J_n Bessel function, and $\omega_0 = \omega_{n,k}$. The fundamental and third overtones are axisymmetric ($n = 0$) with $j_{0,1} = 2.4048$ and $j_{0,2} = 5.5201$. Substituting in the test expression above, and assuming exact equality, we calculate:

$a = 0$	0.1	0.2	0.3	0.4	0.5
$\omega/\omega_{0,1} = 1$	1.0057	1.0234	1.0551	1.1050	1.1811
$\omega/\omega_{0,2} = 1$	1.0038	1.0154	1.0358	1.0666	1.1115

Even for inhomogeneities as strong as $a = 0.3$, with its 12-fold variation in areal mass density around the membrane, these calculations approximate closely the exact values of unity that would have been obtained if the exact solutions for η in the z -plane had been used in the test expression. We shall therefore assume that

it is adequate for our purposes. The closeness of these approximations must also be interpreted, however, as a warning that substantial inhomogeneities may be required to tune the frequencies of overtones by only small amounts.

Tuning the overtones

With that pessimistic remark in mind, we start by considering only the fundamental and first three overtones. The first and second vary azimuthally as $J_1(j_{1,1}r) \cos \theta$ and $J_2(j_{2,1}r) \cos 2\theta$, where $j_{1,1} = 3.8317$, $j_{2,1} = 5.1356$, and the third, $J_0(j_{0,2}r)$, is axisymmetric. Since the frequencies $\omega_{n,k}$ are proportional to the $j_{n,k}$, the frequencies of these overtones normalise to the fundamental, as: $\omega_{1,1}/\omega_{0,1} = 1.5933 (= 1.0622 \times (3/2))$, $\omega_{2,1}/\omega_{0,1} = 2.1355 (= 1.0678 \times 2)$, $\omega_{0,2}/\omega_{0,1} = 2.2954 (= 1.0202 \times (3/2)^2)$. The first ratio is close to $3/2$ (perfect fifth), the second to 2 (octave), and the third to $(3/2)^2$, which would also qualify as harmonious [6].

How might we tune these overtones? Reducing the mass of a homogeneous membrane will cause all of its frequencies to rise. We shall consider the feasibility of loading a homogeneous membrane of reduced mass with additional masses in such a way that its fundamental frequency is lowered to that of the higher mass, while its overtones are lowered differentially to frequencies related harmoniously to that frequency.

We shall assign unit mass to an unloaded homogeneous unit disc membrane, so that $\sigma_0 = 1/\pi$, and load a homogeneous membrane, of mass m , with masses, m_i , each of which is a uniform

flexible ring of mean radius r_i , centred on the origin, and of rectangular section with radial width δr_i and thickness $1/\delta r_i$.

To sharpen the criteria for feasibility, we shall consider first the limit of delta-function rings ($\delta r_i \rightarrow 0$), for each of which the areal density contribution is

$$\sigma_i = \frac{m_i \delta(r - r_i)}{2\pi r_i}.$$

For a particular trial function, $J_n(j_{n,k}r) \cos n\theta$, and a single delta-function ring, $m_i = m_{n,k}$, substitution in the test expression yields

$$\begin{aligned} \frac{\omega_{n,k}^2}{\omega_{T(n,k)}^2} &\geq m + \frac{\int_0^1 \frac{m_{n,k}\delta(r-r_i)}{2\pi r_i} J_n(j_{n,k}r)^2 r dr}{\int_0^1 \frac{1}{\pi} J_n(j_{n,k}r)^2 r dr} \\ &= m + m_{n,k} \frac{J_n(j_{n,k}r_i)^2}{J_{n+1}(j_{n,k})^2}, \end{aligned}$$

where $\omega_{T(n,k)}$ is the frequency to which provision of the delta-function ring tunes $\omega_{n,k}$. The axisymmetric delta-function ring affects only the radial variation of η .

The mass $m_{n,k}$ will exert its strongest tuning effect if it is located at radius $r_i = r_{n,k}$, where $r_{n,k} > 0$ is the smallest radius for which $J_n(j_{n,k}r_i)^2$ maximises. The optimised test expression can then be used, assuming approximate equality:

$$\frac{\omega_{n,k}^2}{\omega_{T(n,k)}^2} \approx m + m_{n,k} \frac{J_n(j_{n,k}r_{n,k})^2}{J_{n+1}(j_{n,k})^2}.$$

For the fundamental to be unchanged in frequency, we require

$$\frac{\omega_{0,1}^2}{\omega_{T(0,1)}^2} = 1 \approx m + m_{n,k} \frac{J_0(j_{0,1}r_{n,k})^2}{J_1(j_{0,1})^2}.$$

Prescription of $\omega_{T(n,k)}$ enables the masses m and $m_{n,k}$ to be estimated.

The first three overtones

It is straightforward to extend the analysis for a single delta-function ring by adding two more. We locate them at the maximising radii: mass $m_{1,1}$ at $r_{1,1} = 0.4805$, mass $m_{2,1}$ at $r_{2,1} = 0.5947$, and mass $m_{0,2}$ at $r_{0,2} = 0.6941$. After evaluating the corresponding Bessel function coefficients, we obtain

$$\begin{aligned} \frac{\omega_{0,1}^2}{\omega_{T(0,1)}^2} &= 1 \\ &\approx m + 1.7820m_{1,1} + 1.1241m_{2,1} + 0.6412m_{0,2}, \\ \frac{\omega_{1,1}^2}{\omega_{T(1,1)}^2} &= \frac{1.5933^2}{(3/2)^2} = 1.1283 \\ &\approx m + 2.0871m_{1,1} + 1.8216m_{2,1} + 1.2690m_{0,2}, \\ \frac{\omega_{2,1}^2}{\omega_{T(2,1)}^2} &= \frac{2.1355^2}{2^2} = 1.1401 \\ &\approx m + 1.6889m_{1,1} + 2.0514m_{2,1} + 1.7549m_{0,2}, \\ \frac{\omega_{0,2}^2}{\omega_{T(0,2)}^2} &= \frac{2.2954^2}{(3/2)^4} = 1.0408 \\ &\approx m + 0.1267m_{1,1} + 1.0012m_{2,1} + 1.4011m_{0,2}. \end{aligned}$$

Regrettably, the solution obtained by assuming exact equality in each equation is a nonphysical mixture of positive and negative masses. Can the first two overtones, at least, be tuned?

The first two overtones

With $m_{0,2} = 0$, the first three equations above can be solved satisfactorily to give $m = 0.7144 \approx 0.71$, $m_{1,1} = 0.0611 \approx 0.06$ and $m_{2,1} = 0.1572 \approx 0.16$, all of which are positive. Because of the optimised locations of the two delta-function rings, the total mass is less than unity.

It is important to distinguish here between precision and accuracy: once a trial function has been specified for use in the test expression, calculation of the coefficients in the equations above can be arbitrarily precise, to four decimal places, for example. The accuracy of the masses calculated depends, however, on how closely the trial function simulates the exact solution. We recognise this distinction by approximating the masses to two decimal places, which seems a plausible level of accuracy to infer from our tests on isospectral membranes.

The third overtone

We now check whether or not the solution for the first two overtones tunes the third overtone fortuitously to a harmonious frequency. Substituting these masses in the fourth equation above yields

$$\frac{\omega_{0,2}^2}{\omega_{T(0,2)}^2} = \frac{2.2954^2}{X^2} \approx m + 0.1267m_{1,1} + 1.0012m_{2,1} = 0.8795,$$

from which it follows that $X \approx 2.4475$, a ratio close to $(5/4)^4 = 2.4414$. Savage [6] classifies this within the perfect third criterion, so we need search no further for harmonious frequencies with $m_{0,2} > 0$.

Summing up, our original aim for the third overtone (to have $X = (3/2)^2$ with $m_{0,2} > 0$) has fallen outside the tuning window with positive masses. That fatal defenestration brings to mind President Truman's injunction to the heat-averse, 'get out of the kitchen'. Having demonstrated that the two overtones most likely to be excited could be tuned by providing two delta-function rings, $m_{1,1}$ and $m_{2,1}$, we do, however, emerge from the 'kitchen' (as the percussion section is familiarly known) with some satisfaction.

Practicality

Flexible rectangular section rings, in the delta-function limit of infinite thickness and infinitesimal width, have served to demonstrate mathematically the feasibility of tuning the first two overtones, but are of course impractical. A flexible rectangular section ring of finite width δr_i , thickness, $1/\delta r_i$ and mean radius r_i , is physically realisable, however, and contributes an areal mass density profile of

$$\sigma_i = m_i / 2\pi r_i \delta r_i, \quad r_i - \delta r_i/2 \leq r \leq r_i + \delta r_i/2.$$

We must now recalculate the coefficients in the equations obtained above for two delta-function masses. For a given trial function, $J_n(j_{n,k}r) \cos n\theta$, the relevant integrals are of the form

$$\int_{r_{n,k}-\delta r_{n,k}/2}^{r_{n,k}+\delta r_{n,k}/2} \frac{m_{n,k}}{2\pi r_{n,k} \delta r_{n,k}} J_n(j_{n,k}r)^2 r dr.$$

Since the radial separation between the two rings, $r_{2,1} - r_{1,1} = 0.5947 - 0.4805 \approx 0.11$, we may take $\delta r_{2,1} = 0.1 = \delta r_{1,1}$ as an

example without overlap. The integrations yield

$$\begin{aligned}\frac{\omega_{0,1}^2}{\omega_{T(0,1)}^2} &= 1 \\ &\approx m + 1.7725m_{1,1} + 1.1202m_{2,1}, \\ \frac{\omega_{1,1}^2}{\omega_{T(1,1)}^2} &= \frac{1.5933^2}{(3/2)^2} = 1.1283 \\ &\approx m + 2.0692m_{1,1} + 1.8029m_{2,1}, \\ \frac{\omega_{2,1}^2}{\omega_{T(2,1)}^2} &= \frac{2.1355^2}{2^2} = 1.1401 \\ &\approx m + 1.6851m_{1,1} + 2.0259m_{2,1}.\end{aligned}$$

Assuming exact equality, we obtain

$$\begin{aligned}m &= 0.7090 \quad (0.7144), \\ m_{1,1} &= 0.0626 \quad (0.0611), \\ m_{2,1} &= 0.1607 \quad (0.1572),\end{aligned}$$

which agree with the delta-function solutions (shown in parentheses) to two decimal places, confirming that the first two overtones can be tuned.

The corresponding areal mass densities are

$$\sigma/\sigma_0 \approx 0.71, \quad \sigma_{1,1}/\sigma_0 \approx 0.65, \quad \sigma_{2,1}/\sigma_0 \approx 1.16,$$

implying a variation in thickness over the membrane of a modest 2.6 (= 1.87/0.71). This should be realisable with a 3D printer without unduly stiffening the membrane.

The rings should preferably be located on the undersides of membranes, so that the variations in thickness do not give them an appearance distractingly different from homogeneous membranes. It might, however, aid the percussionist in varying tone colour if circles with radii corresponding to the displacement maxima ($r_{1,1}$ and $r_{2,1}$) were marked on the membrane.

Conclusions

We began with the plausible hypothesis that at least the first few overtones of a circular drum could be tuned harmoniously by suitably prescribed inhomogeneities in its membrane. ‘The tragedy of science,’ wrote Thomas Huxley, is ‘a beautiful hypothesis slain by an ugly fact.’ We end with the practical conclusion that only the first two overtones can be tuned (to a perfect fifth and an octave, respectively).

Our recompense for the Kac-induced headaches that we have suffered (his name aptly translates as ‘hangover’!) is, however, substantial: these are the overtones which influence most strongly the tone colour of orchestral drums. In modern orchestras, a percussionist controls several timpani [5, p. 1275]. Although they are used mainly for repeated notes and drum rolls, composers have demonstrated that use of these timpani, combined with operation of their tuning pedals, enables melodies to be contributed by skilled percussionists [4, pp. 775–782]. Tuned membranes would offer materially enhanced melodic potential, and new opportunities to composers and percussionists.

As soon as plastic membranes with prescribed inhomogeneities are made available by 3D printing, composers and percussionists will be able to decide whether or not to exploit these new opportunities. On that crucial judgement, will depend the fairy-tale outcome of an obscure opsimath becoming the Stradivari of drums, or the grim prospect of his being hunted down by demented percussionists, baffled by the augmented complexity of their timpani.

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