20. Mathieu Functions

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Table 20.1. Characteristic Values, Joining Factors, Some Critical Values $(0 \leq q \leq \infty)$ . . . 748

Even Solutions

$$a_r,\ ce_r(0, q),\ ce_r\left(\frac{\pi}{2}, q\right),\ ce_r'\left(\frac{\pi}{2}, q\right),\ (4q)^{\frac{r}{2}} g_{e,r}(q),\ (4q)^r f_{e,r}(q)$$

Odd Solutions

$$b_r,\ se_r'(0, q),\ se_r\left(\frac{\pi}{2}, q\right),\ se_r'\left(\frac{\pi}{2}, q\right),\ (4q)^{\frac{r}{2}} g_{o,r}(q),\ (4q)^r f_{o,r}(q)$$

$$q=0(5)25, \quad 8D \text{ or } S$$

$$a_r+2q-(4r+2)\sqrt{q},\ b_r+2q-(4r-2)\sqrt{q}$$

$$q^{-\frac{1}{2}}=.16(-.04)0, \quad 8D$$

$$r=0, 1, 2, 5, 10, 15$$

Table 20.2. Coefficients A_m and B_m . . . 750

$$q=5, 25; r=0, 1, 2, 5, 10, 15, \quad 9D$$

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20. Mathieu Functions

Mathematical Properties

20.1. Mathieu's Equation

Canonical Form of the Differential Equation

20.1.1 $$\frac{d^2y}{dv^2}+(a-2q\cos 2v)y=0$$

Mathieu's Modified Differential Equation

20.1.2 $$\frac{d^2f}{du^2}-(a-2q\cosh 2u)f=0 \qquad (v=iu, y=f)$$

Relation Between Mathieu's Equation and the Wave Equation for the Elliptic Cylinder

The wave equation in Cartesian coordinates is

20.1.3 $$\frac{\partial^2W}{\partial x^2}+\frac{\partial^2W}{\partial y^2}+\frac{\partial^2W}{\partial z^2}+k^2W=0$$

A solution W is obtainable by separation of variables in elliptical coordinates. Thus, let

$$x=\rho\cosh u\cos v;\ y=\rho\sinh u\sin v;\ z=z;$$

ρ a positive constant; **20.1.3** becomes

20.1.4

$$*\ \frac{\partial^2W}{\partial z^2}+\frac{2}{\rho^2(\cosh 2u-\cos 2v)}\left(\frac{\partial^2W}{\partial u^2}+\frac{\partial^2W}{\partial v^2}\right)+k^2W=0$$

Assuming a solution of the form

$$W=\varphi(z)f(u)g(v)$$

and substituting the above into **20.1.4** one obtains, after dividing through by W,

$$\frac{1}{\varphi}\frac{d^2\varphi}{dz^2}+G=0$$

where

$$*\ G=\frac{2}{\rho^2(\cosh 2u-\cos 2v)}\left\{\frac{d^2f}{du^2}\frac{1}{f}+\frac{d^2g}{dv^2}\frac{1}{g}\right\}+k^2$$

Since z, u, v are independent variables, it follows that

20.1.5 $$\frac{d^2\varphi}{dz^2}+c\varphi=0$$

where c is a constant.

Again, from the fact that $G=c$ and that u, v are independent variables, one sets

20.1.6

$$*\ a=\frac{d^2f}{du^2}\frac{1}{f}+\frac{(k^2-c)}{2}\rho^2\cosh 2u$$

$$a=-\frac{d^2g}{dv^2}\frac{1}{g}+\frac{(k^2-c)}{2}\rho^2\cos 2v \qquad *$$

where a is a constant. The above are equivalent to **20.1.1** and **20.1.2.** The constants c and a are often referred to as *separation constants*, due to the role they play in **20.1.5** and **20.1.6.**

For some physically important solutions, the function g must be periodic, of period π or 2π. It can be shown that there exists a countably infinite set of *characteristic values* $a_r(q)$ which yield even periodic solutions of **20.1.1;** there is another countably infinite sequence of *characteristic values* $b_r(q)$ which yield odd periodic solutions of **20.1.1.**

It is known that there exist periodic solutions of period $k\pi$, where k is any positive integer. In what follows, however, the term *characteristic value* will be reserved for a value associated with solutions of period π or 2π only. These characteristic values are of basic importance to the general theory of the differential equation for arbitrary parameters a and q.

An Algebraic Form of Mathieu's Equation

20.1.7

$$(1-t^2)\frac{d^2y}{dt^2}-t\frac{dy}{dt}+(a+2q-4qt^2)y=0 \qquad (\cos v=t)$$

Relation to Spheroidal Wave Equation

20.1.8 $$(1-t^2)\frac{d^2y}{dt^2}-2(b+1)t\frac{dy}{dt}+(c-4qt^2)y=0 \qquad *$$

Thus, Mathieu's equation is a special case of **20.1.8,** with $b=-\frac{1}{2}$, $c=a+2q$.

20.2. Determination of Characteristic Values

A solution of **20.1.1** with v replaced by z, having period π or 2π is of the form

20.2.1 $$y=\sum_{m=0}^{\infty}(A_m\cos mz+B_m\sin mz)$$

where B_0 can be taken as zero. If the above is substituted into **20.1.1** one obtains

20.2.2

$$\sum_{m=-2}^{\infty}[(a-m^2)A_m-q(A_{m-2}+A_{m+2})]\cos mz$$
$$+\sum_{m=-1}^{\infty}[(a-m^2)B_m-q(B_{m-2}+B_{m+2})]\sin mz=0$$
$$A_{-m}, B_{-m}=0 \qquad m>0$$

*See page II.

Equation **20.2.2** can be reduced to one of four simpler types, given in **20.2.3** and **20.2.4** below

20.2.3 $$y_0=\sum_{m=0}^{\infty} A_{2m+p}\cos(2m+p)z, \qquad p=0 \text{ or } 1$$

20.2.4 $$y_1=\sum_{m=0}^{\infty} B_{2m+p}\sin(2m+p)z, \qquad p=0 \text{ or } 1$$

If $p=0$, the solution is of period π; if $p=1$, the solution is of period 2π.

Recurrence Relations Among the Coefficients

Even solutions of period π:

20.2.5 $$aA_0-qA_2=0$$

20.2.6 $$(a-4)A_2-q(2A_0+A_4)=0$$

20.2.7 $$(a-m^2)A_m-q(A_{m-2}+A_{m+2})=0 \qquad (m\geq 3)$$

Even solutions of period 2π:

20.2.8 $$(a-1)A_1-q(A_1+A_3)=0,$$

along with **20.2.7** for $m\geq 3$.

Odd solutions of period π:

20.2.9 $$(a-4)B_2-qB_4=0$$

* **20.2.10** $$(a-m^2)B_m-q(B_{m-2}+B_{m+2})=0 \qquad (m\geq 3)$$

Odd solutions of period 2π:

20.2.11 $$(a-1)B_1+q(B_1-B_3)=0,$$

along with **20.2.10** for $m\geq 3$.

Let

20.2.12 $$Ge_m=A_m/A_{m-2},\ Go_m=B_m/B_{m-2};$$

$G_m=Ge_m$ or Go_m when the same operations apply to both, and no ambiguity is likely to arise. Further let

20.2.13 $$V_m=(a-m^2)/q.$$

Equations **20.2.5–20.2.7** are equivalent to

20.2.14 $$Ge_2=V_0;\ Ge_4=V_2-\frac{2}{Ge_2}$$

20.2.15 $$G_m=1/(V_m-G_{m+2}) \qquad (m\geq 3),$$

for even solutions of period π.

Similarly

20.2.16 $V_1-1=Ge_3$; for even solutions of period 2π, along with **20.2.15**

20.2.17 $V_1+1=Go_3$, for odd solutions of period 2π, along with **20.2.15**

20.2.18 $V_2=Go_4$, for odd solutions of period π, along with **20.2.15**

These three-term recurrence relations among the coefficients indicate that every G_m can be developed into two types of continued fractions. Thus **20.2.15** is equivalent to

20.2.19

$$G_m=\frac{1}{V_m-G_{m+2}}=\frac{1}{V_m-}\,\frac{1}{V_{m+2}-}\,\frac{1}{V_{m+4}-}\cdots \quad (m\geq 3)$$

20.2.20

$$G_{m+2}=V_m-1/G_m$$
$$=V_m-\frac{1}{V_{m-2}-}\,\frac{1}{V_{m-4}-}\cdots\frac{\varphi_0}{V_{0+d}+\varphi_1} \qquad (m\geq 3)$$

where

$\varphi_1=d=0$; $\varphi_0=2$, if $G_{m+2}=A_{2s}/A_{2s-2}$

$\varphi_1=d=\varphi_0=0$, if $G_{m+2}=B_{2s}/B_{2s-2}$

$\varphi_1=-1$; $\varphi_0=d=1$, if $G_{m+2}=A_{2s+1}/A_{2s-1}$

$\varphi_1=d=\phi_0=1$, if $G_{m+2}=B_{2s+1}/B_{2s-1}$

The four choices of the parameters φ_1, φ_0, d correspond to the four types of solutions **20.2.3–20.2.4**. Hereafter, it will be convenient to separate the characteristic values a into two major subsets:

$a=a_r$, associated with even periodic solutions

$a=b_r$, associated with odd periodic solutions

If **20.2.19** is suitably combined with **20.2.13–20.2.18** there result four types of continued fractions, the roots of which yield the required characteristic values

20.2.21 $$V_0-\frac{2}{V_2-}\,\frac{1}{V_4-}\,\frac{1}{V_6-}\cdots=0 \qquad \text{Roots: } a_{2r}$$

20.2.22

$$V_1-1-\frac{1}{V_3-}\,\frac{1}{V_5-}\,\frac{1}{V_7-}\cdots=0 \qquad \text{Roots: } a_{2r+1}$$

20.2.23 $$V_2-\frac{1}{V_4-}\,\frac{1}{V_6-}\,\frac{1}{V_8-}\cdots=0 \qquad \text{Roots: } b_{2r}$$

20.2.24

$$V_1+1-\frac{1}{V_3-}\,\frac{1}{V_5-}\,\frac{1}{V_7-}\cdots=0 \qquad \text{Roots: } b_{2r+1}$$

If a is a root of **20.2.21–20.2.24**, then the corresponding solution exists and is an entire function of z, for general complex values of q.

If q is real, then the Sturmian theory of second order linear differential equations yields the

*See page II.

following:

(a) For a fixed real q, characteristic values a_r and b_r are real and distinct, if $q \neq 0$; $a_0 < b_1 < a_1 < b_2 < a_2 < \ldots$, $q > 0$ and $a_r(q)$, $b_r(q)$ approach r^2 as q approaches zero.

(b) A solution of **20.1.1** associated with a_r or b_r has r zeros in the interval $0 \leq z < \pi$, (q real).

(c) The form of **20.2.21** and **20.2.23** shows that if a_{2r} is a root of **20.2.21** and q is different from zero, then a_{2r} cannot be a root of **20.2.23**; similarly, no root of **20.2.22** can be a root of **20.2.24** if $q \neq 0$. It may be shown from other considerations that for a given point (a, q) there can be at most one periodic solution of period π or 2π if $q \neq 0$. This no longer holds for solutions of period $s\pi$, $s \geq 3$; for these all solutions are periodic, if one is.

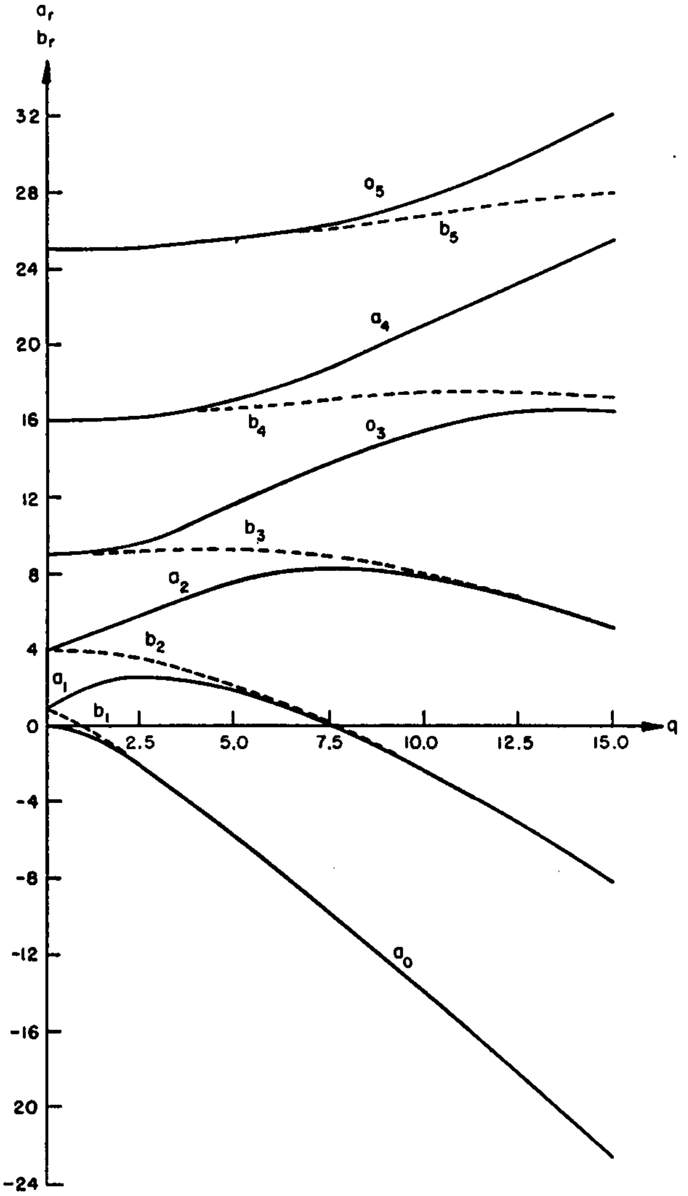

FIGURE 20.1. *Characteristic Values* a_r, b_r $r=0,1(1)5$

Power Series for Characteristic Values

20.2.25

$$a_0(q) = -\frac{q^2}{2} + \frac{7q^4}{128} - \frac{29q^6}{2304} + \frac{68687q^8}{18874368} + \ldots$$

$$\left.\begin{array}{l} a_1(-q) \\ b_1(q) \end{array}\right\} = 1 - q - \frac{q^2}{8} + \frac{q^3}{64} - \frac{q^4}{1536} - \frac{11q^5}{36864} + \frac{49q^6}{589824} - \frac{55q^7}{9437184} - \frac{83q^8}{35389440} + \ldots$$

$$b_2(q) = 4 - \frac{q^2}{12} + \frac{5q^4}{13824} - \frac{289q^6}{79626240} + \frac{21391q^8}{458647142400} + \ldots$$

$$a_2(q) = 4 + \frac{5q^2}{12} - \frac{763q^4}{13824} + \frac{1002401q^6}{79626240} - \frac{1669068401q^8}{458647142400} + \ldots$$

$$\left.\begin{array}{l} a_3(-q) \\ b_3(q) \end{array}\right\} = 9 + \frac{q^2}{16} - \frac{q^3}{64} + \frac{13q^4}{20480} + \frac{5q^5}{16384} - \frac{1961q^6}{23592960} + \frac{609q^7}{104857600} + \ldots$$

$$b_4(q) = 16 + \frac{q^2}{30} - \frac{317q^4}{864000} + \frac{10049q^6}{2721600000} + \ldots$$

$$a_4(q) = 16 + \frac{q^2}{30} + \frac{433q^4}{864000} - \frac{5701q^6}{2721600000} + \ldots$$

$$\left.\begin{array}{l} a_5(-q) \\ b_5(q) \end{array}\right\} = 25 + \frac{q^2}{48} + \frac{11q^4}{774144} - \frac{q^5}{147456} + \frac{37q^6}{891813888} + \ldots$$

$$b_6(q) = 36 + \frac{q^2}{70} + \frac{187q^4}{43904000} - \frac{5861633q^6}{92935987200000} + \ldots$$

$$a_6(q) = 36 + \frac{q^2}{70} + \frac{187q^4}{43904000} + \frac{6743617q^6}{92935987200000} + \ldots$$

For $r \geq 7$, and $|q|$ not too large, a_r is approximately equal to b_r, and the following approximation may be used

20.2.26

$$\left.\begin{array}{l} a_r \\ b_r \end{array}\right\} = r^2 + \frac{q^2}{2(r^2-1)} + \frac{(5r^2+7)q^4}{32(r^2-1)^3(r^2-4)} + \frac{(9r^4+58r^2+29)q^6}{64(r^2-1)^5(r^2-4)(r^2-9)} + \ldots$$

The above expansion is not limited to integral values of r, and it is a very good approximation for r of the form $n+\frac{1}{2}$ where n is an integer. In case of integral values of $r=n$, the series holds only up to terms not involving r^2-n^2 in the denominator. Subsequent terms must be derived specially (as shown by Mathieu). Mulholland and Goldstein [20.38] have computed characteristic values for purely imaginary q and found that a_0 and a_2 have a common real value for $|q|$ in the neighborhood of 1.468; Bouwkamp [20.5] has computed this number as $q_0=\pm i\ 1.46876852$ to 8 decimals. For values of $-iq>-iq_0$, a_0 and a_2 are conjugate complex numbers. From equation **20.2.25** it follows that the radius of convergence for the series defining a_0 is no greater than $|q_0|$. It is shown in [20.36], section **2.25** that the radius of convergence for $a_{2n}(q)$, $n\geq 2$ is greater than 3. Furthermore

$$a_r-b_r=O(q^r/r^{r-1}),\ r\rightarrow\infty.$$

Power Series in q for the Periodic Functions (for sufficiently small $|q|$)

20.2.27

$$ce_0(z,q)=2^{-\frac{1}{2}}\left[1-\frac{q}{2}\cos 2z+q^2\left(\frac{\cos 4z}{32}-\frac{1}{16}\right)-q^3\left(\frac{\cos 6z}{1152}-\frac{11\cos 2z}{128}\right)+\ldots\right]$$

$$ce_1(z,q)=\cos z-\frac{q}{8}\cos 3z+q^2\left[\frac{\cos 5z}{192}-\frac{\cos 3z}{64}-\frac{\cos z}{128}\right]-q^3\left[\frac{\cos 7z}{9216}-\frac{\cos 5z}{1152}-\frac{\cos 3z}{3072}+\frac{\cos z}{512}\right]+\ldots$$

$$se_1(z,q)=\sin z-\frac{q}{8}\sin 3z+q^2\left[\frac{\sin 5z}{192}+\frac{\sin 3z}{64}-\frac{\sin z}{128}\right]-q^3\left[\frac{\sin 7z}{9216}+\frac{\sin 5z}{1152}-\frac{\sin 3z}{3072}-\frac{\sin z}{512}\right]+\ldots$$

$$ce_2(z,q)=\cos 2z-q\left(\frac{\cos 4z}{12}-\frac{1}{4}\right)+q^2\left(\frac{\cos 6z}{384}-\frac{19\cos 2z}{288}\right)+\ldots$$

$$se_2(z,q)=\sin 2z-q\frac{\sin 4z}{12}+q^2\left(\frac{\sin 6z}{384}-\frac{\sin 2z}{288}\right)+\ldots$$

20.2.28

$$\begin{matrix}ce_r(z,q)\\ se_r(z,q)\end{matrix}=\cos(rz-p(\pi/2))-q\left\{\frac{\cos\left[(r+2)z-p\frac{\pi}{2}\right]}{4(r+1)}-\frac{\cos[(r-2)z-p(\pi/2)]}{4(r-1)}\right\}$$
$$+q^2\left\{\frac{\cos[(r+4)z-p(\pi/2)]}{32(r+1)(r+2)}+\frac{\cos[(r-4)z-p(\pi/2)]}{32(r-1)(r-2)}-\frac{\cos[rz-p(\pi/2)]}{32}\left[\frac{2(r^2+1)}{(r^2-1)^2}\right]\right\}+\ldots$$

with $p=0$ for $ce_r(z,q)$, $p=1$ for $se_r(z,q)$, $r\geq 3$.

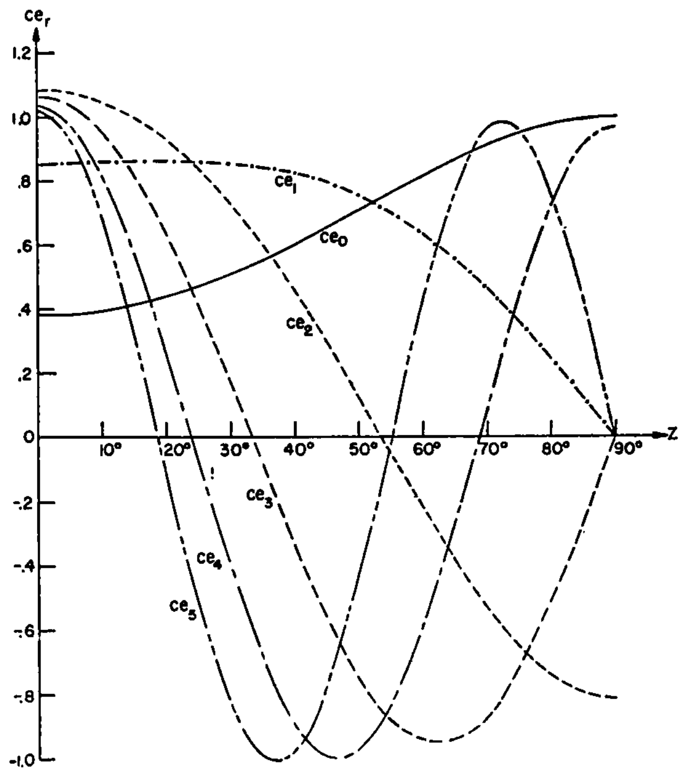

FIGURE 20.2. *Even Periodic Mathieu Functions, Orders 0–5* $q=1$.

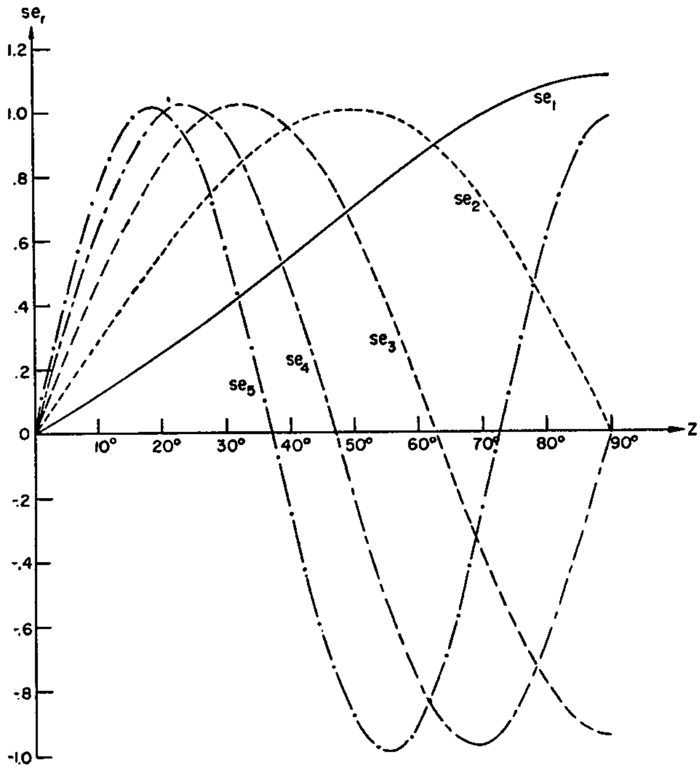

FIGURE 20.3. *Odd Periodic Mathieu Functions, Orders 1–5* $q=1$.

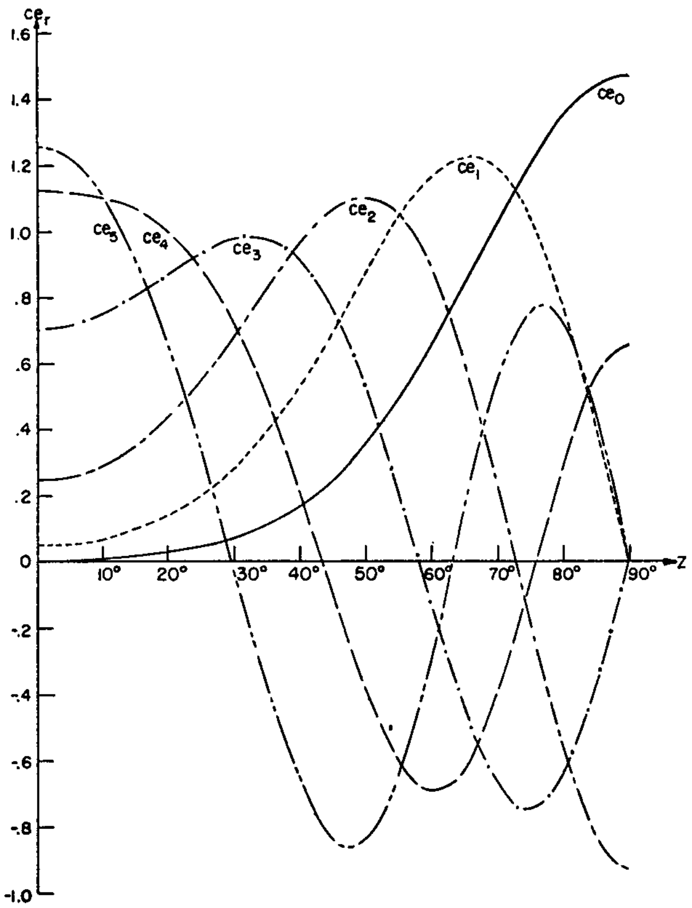

FIGURE 20.4. *Even Periodic Mathieu Functions, Orders 0–5* $q=10$.

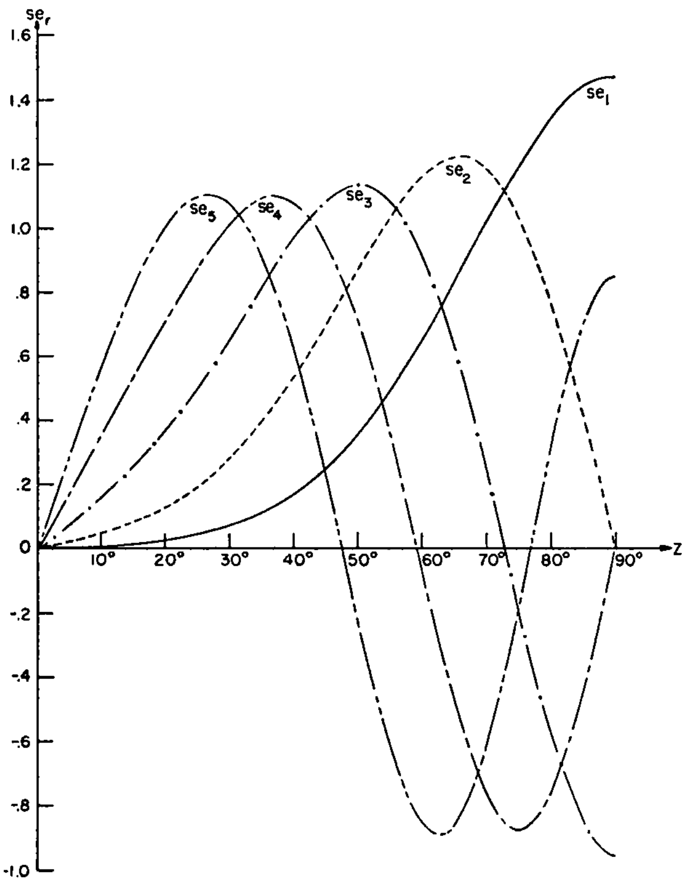

FIGURE 20.5. *Odd Periodic Mathieu Functions, Orders 1–5* $q=10$.

For coefficients associated with above functions

20.2.29

$$A_0^0(0)=2^{-\frac{1}{2}};\ A_r^r(0)=B_r^r(0)=1,\ r>0$$

$$A_{2s}^0=[(-1)^s q^s/s!\,s!\,2^{2s-1}]\,A_0^0+\ .\ .\ .,\ s>0$$

$$\left.\begin{matrix}A_{r+2s}^r\\ B_{r+2s}^r\end{matrix}\right\}=[(-1)^s r!\ q^s/4^s(r+s)!\ s!]\ C_r^r+\ .\ .\ .$$

$$rs>0,\ C_r^r=A_r^r \text{ or } B_r^r$$

$$A_{r-2s}^r \text{ or } B_{r-2s}^r=\frac{(r-s-1)!}{s!(r-1)!}\frac{q^s}{4^s}C_r^r+\ .\ .\ .$$

Asymptotic Expansion for Characteristic Values, $q\gg1$

Let $w=2r+1$, $q=w^4\varphi$, φ real. Then

20.2.30

$$a_r\sim b_{r+1}\sim -2q+2w\sqrt{q}-\frac{w^2+1}{8}-\frac{\left(w+\frac{3}{w}\right)}{2^7\sqrt{\varphi}}$$

$$-\frac{d_1}{2^{12}\varphi}-\frac{d_2}{2^{17}\varphi^{3/2}}-\frac{d_3}{2^{20}\varphi^2}-\frac{d_4}{2^{25}\varphi^{5/2}}-\ .\ .\ .$$

where

$$d_1=5+\frac{34}{w^2}+\frac{9}{w^4}$$

$$d_2=\frac{33}{w}+\frac{410}{w^3}+\frac{405}{w^5}$$

$$d_3=\frac{63}{w^2}+\frac{1260}{w^4}+\frac{2943}{w^6}+\frac{486}{w^8}$$

$$d_4=\frac{527}{w^3}+\frac{15617}{w^5}+\frac{69001}{w^7}+\frac{41607}{w^9}$$

20.2.31 $$b_{r+1}-a_r \sim 2^{4r+5}\sqrt{2/\pi}\,q^{\frac{1}{2}r+\frac{3}{4}}e^{-4\sqrt{q}}/r!, \qquad q\to\infty$$

(given in [20.36] without proof.)

20.3. Floquet's Theorem and Its Consequences

Since the coefficients of Mathieu's equation

20.3.1 $$y''+(a-2q\cos 2z)y=0$$

are periodic functions of z, it follows from the known theory relating to such equations that there exists a solution of the form

20.3.2 $$F_\nu(z)=e^{i\nu z}P(z),$$

where ν depends on a and q, and $P(z)$ is a periodic function, of the same period as that of the coefficients in **20.3.1,** namely π. (Floquet's theorem; see [20.16] or [20.22] for its more general form.) The constant ν is called the *characteristic exponent.* Similarly

20.3.3 $$F_\nu(-z)=e^{-i\nu z}P(-z)$$

satisfies **20.3.1** whenever **20.3.2** does. Both $F_\nu(z)$ and $F_\nu(-z)$ have the property

20.3.4

$$y(z+k\pi)=C^k y(z),\ y=F_\nu(z) \text{ or } F_\nu(-z),$$
$$C=e^{i\nu\pi} \text{ for } F_\nu(z),\ C=e^{-i\nu\pi} \text{ for } F_\nu(-z)$$

Solutions having the property **20.3.4** will hereafter be termed *Floquet* solutions. Whenever $F_\nu(z)$ and $F_\nu(-z)$ are linearly independent, the general solution of **20.3.1** can be put into the form

20.3.5 $$y=AF_\nu(z)+BF_\nu(-z)$$

If $AB\neq 0$, the above solution will *not be a Floquet solution.* It will be seen later, from the method for determining ν when a and q are given, that there is some ambiguity in the definition of ν; namely, ν can be replaced by $\nu+2k$, where k is an arbitrary integer. This is as it should be, since the addition of the factor $\exp\,(2ikz)$ in **20.3.2** still leaves a periodic function of period π for the coefficient of $\exp i\nu z$.

It turns out that when a belongs to the set of characteristic values a_r and b_r of **20.2,** then ν is zero or an integer. It is convenient to associate $\nu=r$ with $a_r(q)$, and $\nu=-r$ with $b_r(q)$; see [20.36]. In the special case when ν is an integer, $F_\nu(z)$ is proportional to $F_\nu(-z)$; the second, independent solution of **20.3.1** then has the form

20.3.6 $$y_2=z\,ce_r(z,q)+\sum_{k=0}^{\infty} d_{2k+p}\sin(2k+p)z,$$

associated with $ce_r(z,q)$

20.3.7 $$y_2=z\,se_r(z,q)+\sum_{k=0}^{\infty} f_{2k+p}\cos(2k+p)z,$$

associated with $se_r(z,q)$

The coefficients d_{2k+p} and f_{2k+p} depend on the corresponding coefficients A_m and B_m, respectively, of **20.2,** as well as on a and q. See [20.30], section (7.50)–(7.51) and [20.58], section V, for details.

If ν is not an integer, then the Floquet solutions $F_\nu(z)$ and $F_\nu(-z)$ are linearly independent. It is clear that **20.3.2** can be written in the form

20.3.8 $$F_\nu(z)=\sum_{k=-\infty}^{\infty} c_{2k}e^{i(\nu+2k)z}.$$

From **20.3.8** it follows that if ν is a proper fraction m_1/m_2, then every solution of **20.3.1** is periodic, and of period at most $2\pi m_2$. This agrees with results already noted in **20.2;** i.e., both independent solutions are periodic, if one is, provided the period is different from π and 2π.

Method of Generating the Characteristic Exponent

Define two linearly independent solutions of **20.3.1,** for fixed a, q by

20.3.9 $$y_1(0)=1;\ y_1'(0)=0.$$
$$y_2(0)=0;\ y_2'(0)=1.$$

Then it can be shown that

20.3.10 $$\cos\pi\nu-y_1(\pi)=0$$

20.3.11 $$\cos\pi\nu-1-2y_1'\left(\frac{\pi}{2}\right)y_2\left(\frac{\pi}{2}\right)=0$$

Thus ν may be obtained from a knowledge of $y_1(\pi)$ or from a knowledge of both $y_1'\left(\frac{\pi}{2}\right)$ and $y_2\left(\frac{\pi}{2}\right)$. For numerical purposes **20.3.11** may be more desirable because of the shorter range of integration, and hence the lesser accumulation of round-off errors. Either ν, $-\nu$, or $\pm\nu+2k$ (k an arbitrary integer) can be taken as the solution of **20.3.11.** Once ν has been fixed, the coefficients of **20.3.8** can be determined, except for an arbitrary multiplier which is independent of z.

The characteristic exponent can also be computed from a continued fraction, in a manner analogous to developments in **20.2,** if a sufficiently close first approximation to ν is available. For systematic tabulation, this method is considerably faster than the method of numerical integration. Thus, when **20.3.8** is substituted into **20.3.1**, there result the following recurrence relations:

20.3.12 $$V_{2n}c_{2n}=c_{2n-2}+c_{2n+2}$$

where

20.3.13 $V_{2n}=[a-(2n+\nu)^2]/q, \quad -\infty<n<\infty.$

When ν is complex, the coefficients V_{2n} may also be complex. As in **20.2**, it is possible to generate the ratios

$$G_m=c_m/c_{m-2} \text{ and } H_{-m}=c_{-m-2}/c_{-m}$$

from the continued fractions

20.3.14

$$G_m=\frac{1}{V_m-}\,\frac{1}{V_{m+2}-\ \ldots}, \qquad m\geq 0$$

$$H_{-m}=\frac{1}{V_{-m-2}-}\,\frac{1}{V_{-m-4}-\ \ldots}, \qquad m\geq 0.$$

From the form of **20.3.13** and the known properties of continued fractions it is assured that for sufficiently large values of $|m|$ both $|G_m|$ and $|H_{-m}|$ converge. Once values of G_m and H_{-m} are available for some sufficiently large value of m, then the finite number of ratios $G_{m-2}, G_{m-4}, \ldots, G_0$ can be computed in turn, if they exist. Similarly for $H_{-m+2}, \ldots, H_0$. It is easy to show that ν is the correct characteristic exponent, appropriate for the point (a, q), if and only if $H_0G_0=1$. An iteration technique can be used to improve the value of ν, by the method suggested in [20.3]. One coefficient c_j can be assigned arbitrarily; the rest are then completely determined. After all the c_j become available, a multiplier (depending on q but not on z) can be found to satisfy a prescribed normalization.

It is well known that continued fractions can be converted to determinantal form. Equation **20.3.14** can in fact be written as a determinant with an infinite number of rows—a special case of Hill's determinant. See [20.19], [20.36], [20.15], or [20.30] for details. Although the determinant has actually been used in computations where high-speed computers were available, the direct use of the continued fraction seems much less laborious.

Special Cases (a, q Real)

Corresponding to $q=0$, $y_1=\cos\sqrt{a}z$, $y_2=\sin\sqrt{a}z$; the Floquet solutions are $\exp(iaz)$ and $\exp(-iaz)$. As a, q vary continuously in the $q-a$ plane, ν describes curves; ν is real when (q, a), $q\geq 0$ lies in the region between $a_r(q)$ and $b_{r+1}(q)$ and

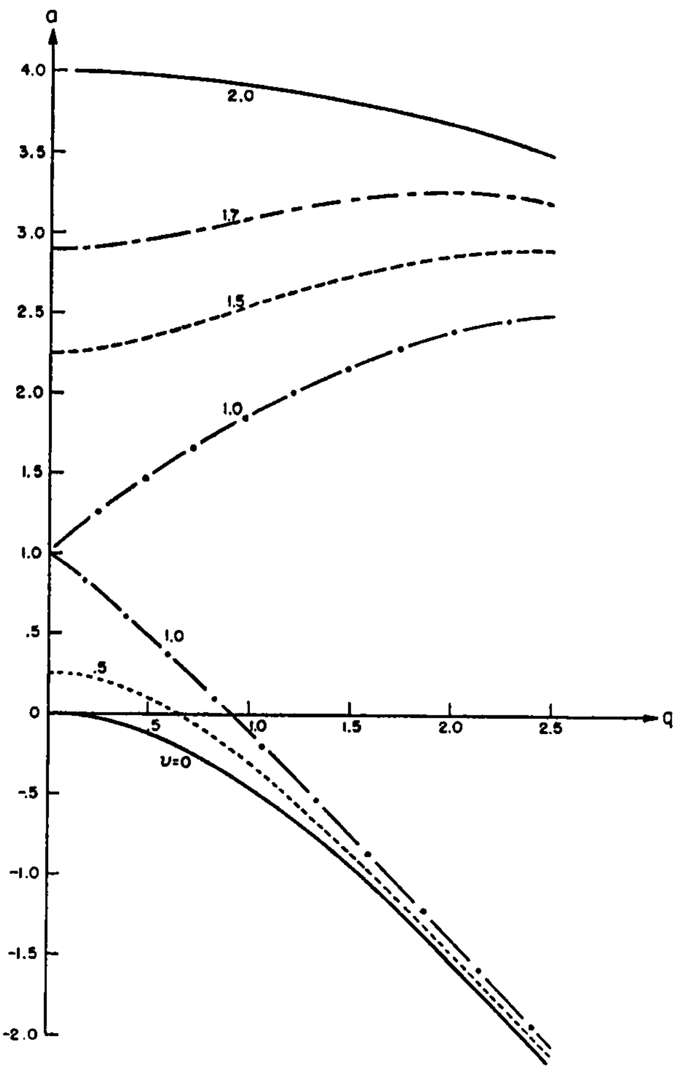

FIGURE 20.6. Characteristic Exponent-First Two Stable Regions $y=e^{i\nu x}P(x)$ where $P(x)$ is a periodic function of period π.

Definition of ν;
In first stable region, $0\leq\nu\leq 1$,
In second stable region, $1\leq\nu\leq 2$.

(Constructed from tabular values supplied by T. Tamir, Brooklyn Polytechnic Institute)

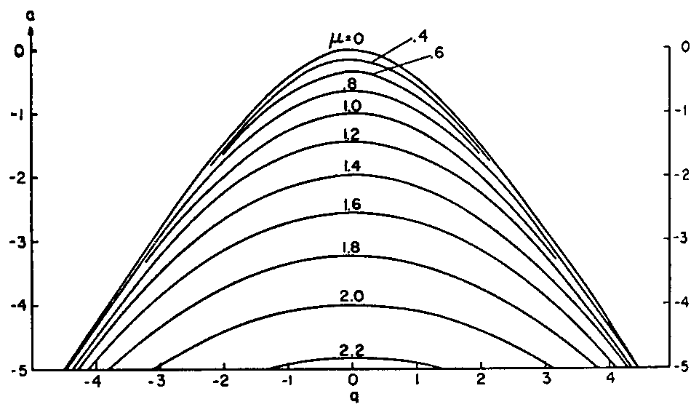

FIGURE 20.7. *Characteristic Exponent in First Unstable Region. Differential equation: $y''+(a-2q\cos 2x)y=0$. The Floquet solution $y=e^{i\nu x}P(x)$, where $P(x)$ is a periodic function of period π. In the first unstable region, $\nu=i\mu$; μ is given for $a\geq -5$.* (Constructed at NBS.)

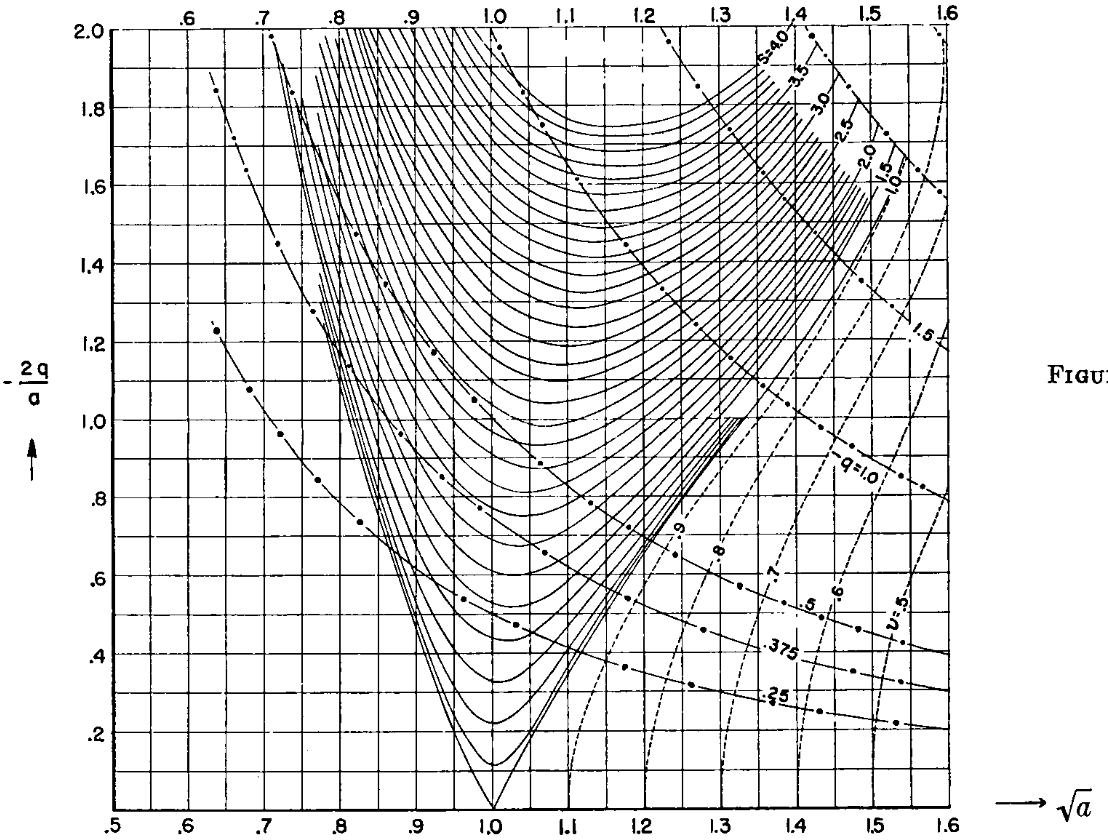

FIGURE 20.8

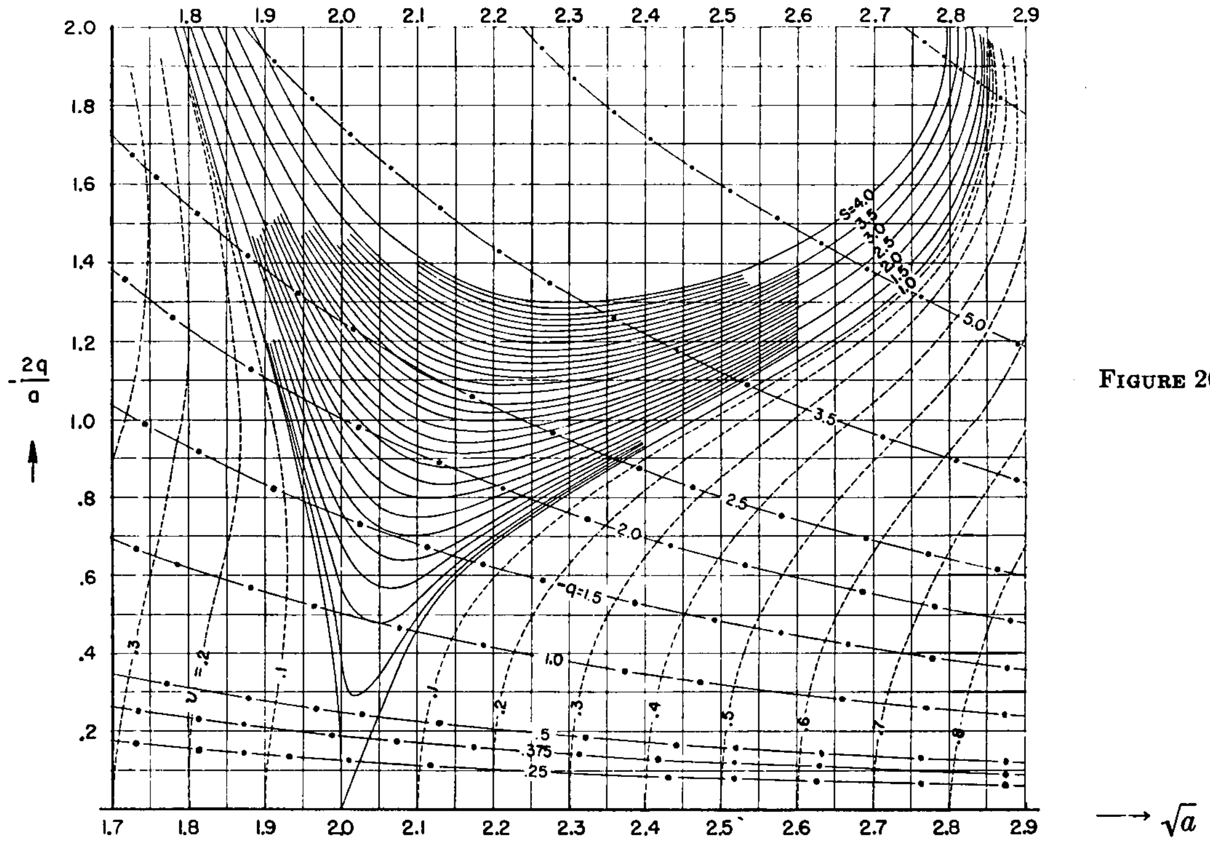

FIGURE 20.9

Charts of the Characteristic Exponent.

(From S. J. Zaroodny, An elementary review of the Mathieu-Hill equation of real variable based on numerical solutions, Ballistic Research Laboratory Memo. Rept. 878, Aberdeen Proving Ground, Md., 1955, with permission.)

——— $s=e^{i\nu\pi}$=*constant; in unstable regions*

– – – – ν=*constant; in stable regions*

–.–.– *Lines of constant values of* $-q$.

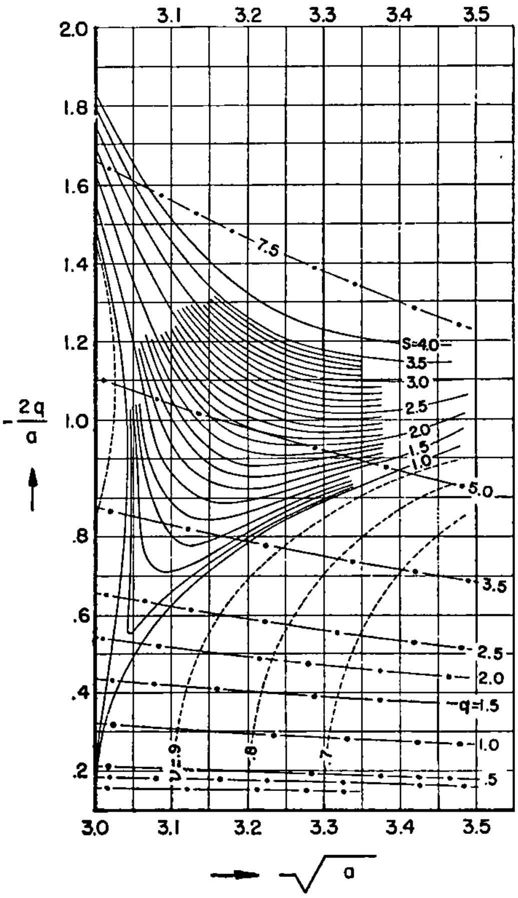

FIGURE 20.10. *Chart of the Characteristic Exponent.*

(From S. J. Zaroodny, An elementary review of the Mathieu-Hill equation of real variable based on numerical solutions, Ballistic Research Laboratory Memo. Rept. 878, Aberdeen Proving Ground, Md., 1955, with permission)

——— $s=e^{i\nu\pi}$=*constant; in unstable regions*
- - - - ν=*constant; in stable regions*
–.–.– *Lines of constant values of* $-q$.

all solutions of **20.1.1** for real z are therefore bounded (stable); ν is complex in regions between b_r and a_r; in these regions every solution becomes infinite at least once; hence these regions are termed "unstable regions". The characteristic curves a_r, b_r separate the regions of stability. For negative q, the stable regions are between b_{2r+1} and b_{2r+2}, a_{2r} and a_{2r+1}; the unstable regions are between a_{2r+1} and b_{2r+1}, a_{2r} and b_{2r}.

In some problems solutions are required for real values of z only. In such cases a knowledge of the characteristic exponent ν and the periodic function $P(z)$ is sufficient for the evaluation of the required functions. For complex values of z, however, the series defining $P(z)$ converges slowly. Other solutions will be determined in the next section; they all have the remarkable property that they depend on the same coefficients c_m developed in connection with Floquet's theorem (except for an arbitrary normalization factor).

Expansions for Small q ([20.36] chapter 2)

If ν, q are fixed:

20.3.15

$$a=\nu^2+\frac{q^2}{2(\nu^2-1)}+\frac{(5\nu^2+7)q^4}{32(\nu^2-1)^3(\nu^2-4)}+\frac{(9\nu^4+58\nu^2+29)q^6}{64(\nu^2-1)^5(\nu^2-4)(\nu^2-9)}+\ldots\quad(\nu\neq1,2,3).$$

For the coefficients c_{2j} of **20.3.8**

20.3.16

$$c_2/c_0=\frac{-q}{4(\nu+1)}-\frac{(\nu^2+4\nu+7)q^3}{128(\nu+1)^3(\nu+2)(\nu-1)}+\ldots\quad(\nu\neq1,2)$$

$$c_4/c_0=q^2/32(\nu+1)(\nu+2)+\ldots$$

$$c_{2s}/c_0=(-1)^s q^s\Gamma(\nu+1)/2^{2s}s!\Gamma(\nu+s+1)+\ldots$$

20.3.17

$$F_\nu(z)=c_0\left[e^{i\nu z}-q\left\{\frac{e^{i(\nu+2)z}}{4(\nu+1)}-\frac{e^{i(\nu-2)z}}{4(\nu-1)}\right\}\right]+\ldots$$

(ν not an integer)

For small values of a

20.3.18

$$\cos\nu\pi=\left(1-\frac{a\pi^2}{2}+\frac{a^2\pi^4}{24}+\ldots\right)-\frac{q^2\pi^2}{4}\left[1+a\left(1-\frac{\pi^2}{6}\right)+\ldots\right]+q^4\left(\frac{\pi^4}{96}-\frac{25\pi^2}{256}+\ldots\right)+\ldots$$

20.4. Other Solutions of Mathieu's Equation

Following Erdélyi [20.14], [20.15], define

20.4.1 $\varphi_k(z)=[e^{i\pi}\cos(z-b)/\cos(z+b)]^{\frac{1}{2}k}J_k(f)$

where

20.4.2 $f=2[q\cos(z-b)\cos(z+b)]^{\frac{1}{2}}$,

and $J_k(f)$ is the Bessel function of order k; b is a fixed, arbitrary complex number. By using the recurrence relations for Bessel functions the following may be verified:

20.4.3

$$\frac{d^2\varphi_k}{dz^2}-2q(\cos 2z)\varphi_k+q(\varphi_{k-2}+\varphi_{k+2})+k^2\varphi_k=0.$$

It follows that a formal solution of **20.1.1** is given by

20.4.4 $$y=\sum_{n=-\infty}^{\infty}c_{2n}\varphi_{2n+\nu}$$

where the coefficients c_{2n} are those associated with Floquet's solution. In the above, ν may be complex. Except for the special case when ν is an integer, the following holds:

$$\frac{\varphi_{2n+\nu-2}}{\varphi_{2n+\nu}} \sim \frac{\varphi_{-2n+\nu}}{\varphi_{-2n+\nu+2}} \sim \frac{-4n^2}{q[\cos(z-b)]^2} \qquad (n\to\infty)$$

If ν and n are integers, $J_{-2n+\nu}(f)=(-1)^\nu J_{2n-\nu}(f)$.

$$[\varphi_{2n+\nu}/\varphi_{2n+\nu-2}] \sim -[\cos(z-b)]^2 q/4n^2$$

$$[\varphi_{-2n+\nu}/\varphi_{-2n+\nu+2}] \sim -4n^2/q[\cos(z-b)]^2$$

On the other hand

$$\frac{c_{2n}}{c_{2n-2}} \sim \frac{c_{-2n}}{c_{-2n+2}} \sim \frac{-q}{4n^2} \qquad (n\to\infty)$$

It follows that **20.4.4** converges absolutely and uniformly in every closed region where

$|\cos(z-b)|>d_1>1$.

There are two such disjoint regions:

(I) $\mathscr{I}(z-b)>d_2>0$; $(|\cos(z-b)|>d_1>1)$

(II) $\mathscr{I}(z-b)<-d_2<0$; $(|\cos(z-b)|>d_1>1)$

If ν is an integer **20.4.4** converges for all values of z. Various representations are found by specializing b.

20.4.5

$$\text{If } b=0,\ y=e^{i\pi\nu/2}\sum_{n=-\infty}^{\infty} c_{2n}(-1)^n J_{2n+\nu}(2\sqrt{q}\cos z)$$

$$(|\cos z|>1,\ |\arg 2\sqrt{q}\cos z|\leq\pi)$$

20.4.6

$$\text{If } b=\frac{\pi}{2},\ y=\sum_{n=-\infty}^{\infty} c_{2n}J_{2n+\nu}(2i\sqrt{q}\sin z)$$

$$(|\sin z|>1,\ |\arg 2\sqrt{q}\sin z|\leq\pi)$$

If $b\to\infty i$, y reduces to a multiple of the solution **20.3.8.** The fact that **20.3.8, 20.4.5,** and **20.4.6** are special cases of **20.4.4** explains why it is that these apparently dissimilar expansions involve the same set of coefficients c_{2n}.

Since **20.4.4** results from the recurrence properties of Bessel functions, $J_k(f)$ can be replaced by $H_k^{(j)}(f)$, $j=1, 2$, where $H_k^{(j)}$ is the Hankel function, at least formally. Thus let

$$\psi_k^j=[e^{i\pi}\cos(z-b)/\cos(z+b)]^{\frac{1}{2}k}H_k^{(j)}(f)$$

where f satisfies **20.4.2.** An examination of the ratios $\psi_{2n+\nu}/\psi_{2n+\nu-2}$ shows that

$$y=\sum_{n=-\infty}^{\infty} c_{2n}\psi_{2n+\nu}^{(j)}$$

will be a solution provided

$$|\cos(z-b)|>1;\ |\cos(z+b)|>1.$$

The above two conditions are necessary even when ν is an integer. Once b is fixed, the regions in which the solutions converge can be readily established.

Following [20.36] let

20.4.7

$$J_p(x)=Z_p^{(1)}(x);\quad Y_p(x)=Z_p^{(2)}(x);\quad H_p^{(1)}(x)=Z_p^{(3)}(x);\quad H_p^{(2)}(x)=Z_p^{(4)}(x)$$

If z is replaced by $-iz$ in **20.4.5** and **20.4.6** solutions of **20.1.2** are obtained. Thus

20.4.8

$$y_1^{(j)}(z)=\sum_{n=-\infty}^{\infty} c_{2n}(-1)^n Z_{2n+\nu}^{(j)}(2\sqrt{q}\cosh z)$$

$$(|\cosh z|>1)$$

20.4.9

$$y_2^{(j)}(z)=\sum_{n=-\infty}^{\infty} c_{2n}Z_{2n+\nu}^{(j)}(2\sqrt{q}\sinh z)$$

$$(|\sinh z|>1,\ j=1, 2, 3, 4)$$

The relation between $y_1^{(j)}(z)$ and $y_2^{(j)}(z)$ can be determined from the asymptotic properties of the Bessel functions for large values of argument. It can be shown that

20.4.10

$$y_1^{(j)}(z)/y_2^{(j)}(z)=[F_\nu(0)/F_\nu\left(\frac{\pi}{2}\right)]e^{i\nu\pi/2} \qquad (\mathscr{R}z>0):$$

When ν is not an integer, the above solutions do not vanish identically. See **20.6** for integral values of ν.

Solutions Involving Products of Bessel Functions

20.4.11

$$y_3^{(j)}(z)=\frac{1}{c_{2s}}\sum_{n=-\infty}^{\infty} c_{2n}(-1)^n Z_{n+\nu+s}^{(j)}(\sqrt{q}e^{iz})J_{n-s}(\sqrt{q}e^{-iz})$$

$$(j=1, 2, 3, 4)$$

satisfies **20.1.1,** where $Z_n^{(j)}(u)$ is defined in **20.4.7,** the coefficients c_{2n} belong to the Floquet solution, and s is an arbitrary integer, $c_{2s}\neq 0$. The solution converges over the entire complex z-plane if $q\neq 0$. Written with z replaced by $-iz$, one obtains solutions of **20.1.2.**

20.4.12

$$M_\nu^j(z,q)=\frac{1}{c_{2s}^\nu}\sum_{n=-\infty}^{\infty} c_{2n}^\nu(-1)^n Z_{n+\nu+s}^{(j)}(\sqrt{q}e^z)J_{n-s}(\sqrt{q}e^{-z})$$

It can be verified from **20.4.8** and **20.4.12** that

20.4.13 $$\frac{y_1^{(j)}(z)}{M_\nu^j(z,q)}=F_\nu(0), \qquad (\mathscr{R}z>0)$$

provided $c_{2s}\neq 0$. If $c_{2s}=0$, the coefficient of $1/c_{2s}$ in **20.4.11** vanishes identically. For details see [20.43], [20.15], [20.36].

If s is chosen so that $|c_{2s}|$ is the largest coefficient of the set $|c_{2j}|$, then rapid convergence of **20.4.12** is obtained, when $\mathscr{R}z>0$. Even then one must be on guard against the possible loss of significant figures in the process of summing the series, especially so when q is large, and $|z|$ small. (If $j\neq 1$, then the phase of the logarithmic terms occurring in **20.4.12** must be defined, to make the functions single-valued.)

20.5. Properties of Orthogonality and Normalization

If $a(\nu+2p, q)$, $a(\nu+2s, q)$ are simple roots of **20.3.10** then

20.5.1 $$\int_0^\pi F_{\nu+2p}(z)F_{\nu+2s}(-z)dz=0, \text{ if } p\neq s.$$

Define

20.5.2 $$ce_\nu(z,q)=\frac{1}{2}[F_\nu(z)+F_\nu(-z)];$$

$$se_\nu(z,q)=-i\,\frac{1}{2}[F_\nu(z)-F_\nu(-z)]$$

$ce_\nu(z,q)$, $se_\nu(z,q)$ are thus even and odd functions of z, respectively, for all ν (when not identically zero).

If ν is an integer, then $ce_\nu(z,q)$, $se_\nu(z,q)$ are either Floquet solutions or identically zero. The solutions $ce_r(z,q)$ are associated with a_r; $se_r(z,q)$ are associated with b_r; r an integer.

Normalization for Integral Values of ν and Real q

20.5.3 $$\int_0^{2\pi}[ce_r(z,q)]^2dz=\int_0^{2\pi}[se_r(z,q)]^2dz=\pi$$

For integral values of ν the summation in **20.3.8** reduces to the simpler forms **20.2.3–20.2.4**; on account of **20.5.3**, the coefficients A_m and B_m (for all orders r) have the property

20.5.4

$$2A_0^2+A_2^2+\ldots=A_1^2+A_3^2+\ldots$$
$$=B_1^2+B_3^2+\ldots=B_2^2+B_4^2+\ldots=1.$$

20.5.5

$$A_0^{2s}=\frac{1}{2\pi}\int_0^{2\pi}ce_{2s}(z,q)dz;\; A_n^r=\frac{1}{\pi}\int_0^{2\pi}ce_r(z,q)\cos nz dz$$

$$B_n^r=\frac{1}{\pi}\int_0^{2\pi}se_r(z,q)\sin nz dz \qquad n\neq 0$$

For integral values of ν, the functions $ce_r(z,q)$ and $se_r(z,q)$ form a complete orthogonal set for the interval $0\leq z\leq 2\pi$. Each of the four systems $ce_{2r}(z)$, $ce_{2r+1}(z)$, $se_{2r}(z)$, $se_{2r+1}(z)$ is complete in the smaller interval $0\leq z\leq \frac{1}{2}\pi$, and each of the systems $ce_r(z)$, $se_r(z)$ is complete in $0\leq z\leq\pi$.

If q is not real, there exist multiple roots of **20.3.10**; for such special values of $a(q)$, the integrals in **20.5.3** vanish, and the normalization is therefore impossible. In applications, the particular normalization adopted is of little importance, except possibly for obtaining quantitative relations between solutions of various types. For this reason the normalization of $F_\nu(z)$, for arbitrary complex values of a, q, will not be specified here. It is worth noting, however, that solutions

$$\alpha ce_r(z,q), \qquad \beta se_r(z,q)$$

defined so that

$$\alpha ce_r(0,q)=1; \qquad \left[\frac{d}{dz}\beta se_r(z,q)\right]_{z=0}=1$$

are always possible. This normalization has in fact been used in [20.59], and also in [20.58], where the most extensive tabular material is available. The tabulated entries in [20.58] supply the conversion factors $A=1/\alpha$, $B=1/\beta$, along with the coefficients. Thus conversion from one normalization to another is rather easy.

In a similar vein, no general normalization will be imposed on the functions defined in **20.4.8**.

20.6. Solutions of Mathieu's Modified Equation 20.1.2 for Integral ν (Radial Solutions)

Solutions of the first kind

20.6.1

$$Ce_{2r+p}(z,q)=ce_{2r+p}(iz,q)$$

$$=\sum_{k=0}^{\infty}A_{2k+p}^{2r+p}(q)\cosh(2k+p)z$$

associated with a_r

20.6.2 $$Se_{2r+p}(z,q)=-ise_{2r+p}(iz,q)=\sum_{k=0}^{\infty} B_{2k+p}^{2r+p}(q)\sinh(2k+p)z, \text{ associated with } b_r$$

writing $A_{2k+p}^{2r+p}(q)=A_{2k+p}$ for brevity; similarly for B_{2k+p}; $p=0, 1$,

20.6.3 $$Ce_{2r}(z,q)=\frac{ce_{2r}\left(\frac{\pi}{2},q\right)}{A_0^{2r}}\sum_{k=0}^{\infty}(-1)^k A_{2k}J_{2k}(2\sqrt{q}\cosh z)=\frac{ce_{2r}(0,q)}{A_0^{2r}}\sum_{k=0}^{\infty}A_{2k}J_{2k}(2\sqrt{q}\sinh z)$$

20.6.4 $$Ce_{2r+1}(z,q)=\frac{ce'_{2r+1}\left(\frac{\pi}{2},q\right)}{\sqrt{q}A_1^{2r+1}}\sum_{k=0}^{\infty}(-1)^{k+1}A_{2k+1}J_{2k+1}(2\sqrt{q}\cosh z)$$

$$=\frac{ce_{2r+1}(0,q)}{\sqrt{q}A_1^{2r+1}}\coth z\sum_{k=0}^{\infty}(2k+1)A_{2k+1}J_{2k+1}(2\sqrt{q}\sinh z)$$

20.6.5 $$Se_{2r}(z,q)=\frac{se'_{2r}\left(\frac{\pi}{2},q\right)\tanh z}{qB_2^{2r}}\sum_{k=1}^{\infty}(-1)^k 2kB_{2k}J_{2k}(2\sqrt{q}\cosh z)$$

$$=\frac{se'_{2r}(0,q)}{qB_2^{2r}}\coth z\sum_{k=1}^{\infty}2kB_{2k}J_{2k}(2\sqrt{q}\sinh z)$$

20.6.6 $$Se_{2r+1}(z,q)=\frac{se_{2r+1}\left(\frac{\pi}{2},q\right)}{\sqrt{q}B_1^{2r+1}}\tanh z\sum_{k=0}^{\infty}(-1)^k(2k+1)B_{2k+1}J_{2k+1}(2\sqrt{q}\cosh z)$$

$$=\frac{se'_{2r+1}(0,q)}{\sqrt{q}B_1^{2r+1}}\sum_{k=0}^{\infty}B_{2k+1}J_{2k+1}(2\sqrt{q}\sinh z)$$

See [20.30] for still other forms.

Solutions of the second kind, as well as solutions of the third and fourth kind (analogous to Hankel functions) are obtainable from **20.4.12.**

20.6.7 $$Mc_{2r}^{(j)}(z,q)=\sum_{k=0}^{\infty}(-1)^{r+k}A_{2k}^{2r}(q)[J_{k-s}(u_1)Z_{k+s}^{(j)}(u_2)+J_{k+s}(u_1)Z_{k-s}^{(j)}(u_2)]/\epsilon_s A_{2s}^{2r}$$

where $\epsilon_0=2$, $\epsilon_s=1$, for $s=1, 2, \ldots$; s arbitrary, associated with a_{2r}

20.6.8 $$Mc_{2r+1}^{(j)}(z,q)=\sum_{k=0}^{\infty}(-1)^{r+k}A_{2k+1}^{2r+1}(q)[J_{k-s}(u_1)\,Z_{k+s+1}^{(j)}(u_2)+J_{k+s+1}(u_1)Z_{k-s}^{(j)}(u_2)]/A_{2s+1}^{2r+1}$$

associated with a_{2r+1}

20.6.9 $$Ms_{2r}^{(j)}(z,q)=\sum_{k=1}^{\infty}(-1)^{k+r}B_{2k}^{2r}(q)[J_{k-s}(u_1)Z_{k+s}^{(j)}(u_2)-J_{k+s}(u_1)Z_{k-s}^{(j)}(u_2)]/B_{2s}^{2r}, \text{ associated with } b_{2r}$$

20.6.10 $$Ms_{2r+1}^{(j)}(z,q)=\sum_{k=0}^{\infty}(-1)^{k+r}B_{2k+1}^{2r+1}(q)[J_{k-s}(u_1)Z_{k+s+1}^{(j)}(u_2)-J_{k+s+1}(u_1)Z_{k-s}^{(j)}(u_2)]/B_{2s+1}^{2r+1}$$

associated with b_{2r+1}

where

$$u_1=\sqrt{q}e^{-z},\ u_2=\sqrt{q}e^{z},\ B_{2s+p}^{2r+p},\ A_{2s+p}^{2r+p}\neq 0,\ p=0, 1.$$

See **20.4.7** for definition of $Z_m^{(j)}(x)$.

Solutions **20.6.7–20.6.10** converge for all values of z, when $q\neq 0$. If $j=2, 3, 4$ the logarithmic terms entering into the Bessel functions $Y_m(u_2)$ must be defined, to make the functions single-valued. This can be accomplished as follows:

Define (as in [20.58])

20.6.11 $$\ln(\sqrt{q}e^{z})=\ln(\sqrt{q})+z$$

See [20.15] and [20.36], section **2.75** for derivation.

Other Expressions for the Radial Functions (Valid Over More Limited Regions)

20.6.12
$$Mc^{(j)}_{2r}(z,q)=[ce_{2r}(0,q)]^{-1}\sum_{k=0}^{\infty}(-1)^{k+r}A^{2r}_{2k}(q)Z^{(j)}_{2k}(2\sqrt{q}\cosh z)$$

$$Mc^{(j)}_{2r+1}(z,q)=[ce_{2r+1}(0,q)]^{-1}\sum_{k=0}^{\infty}(-1)^{k+r}A^{2r+1}_{2k+1}(q)Z^{(j)}_{2k+1}(2\sqrt{q}\cosh z)$$

20.6.13
$$Ms^{(j)}_{2r}(z,q)=[se'_{2r}(0,q)]^{-1}\tanh z\sum_{k=1}^{\infty}(-1)^{k+r}2kB^{2r}_{2k}(q)Z^{(j)}_{2k}(2\sqrt{q}\cosh z)$$

$$Ms^{(j)}_{2r+1}(z,q)=[se'_{2r+1}(0,q)]^{-1}\tanh z\sum_{k=0}^{\infty}(-1)^{k+r}(2k+1)B^{2r+1}_{2k+1}(q)Z^{(j)}_{2k+1}(2\sqrt{q}\cosh z)$$

Valid for $\mathscr{R}z>0$, $|\cosh z|>1$; if $j=1$, valid for all z. They agree with **20.6.7–20.6.10** if the Bessel functions $Y_m(2q^{\frac{1}{2}}\cosh z)$ are made single-valued in a suitable way. For example, let

$$Y_m(u)=\frac{2}{\pi}(\ln u)J_m(u)+\phi(u)$$

where $\phi(u)$ is single-valued for all finite values of u. With $u=2q^{\frac{1}{2}}\cosh z$, define

20.6.14
$$\ln(2q^{\frac{1}{2}}\cosh z)=\ln 2q^{\frac{1}{2}}+z+\ln\tfrac{1}{2}(1+e^{-2z})\qquad -\frac{\pi}{2}\leq\arg\tfrac{1}{2}(1+e^{-2z})\leq\frac{\pi}{2}.$$

(If q is not positive, the phase of $\ln 2q^{\frac{1}{2}}$ must also be specified, although this specification will not affect continuity with respect to z. If $Y_m(u)$ is defined from some other expression, the definition must be compatible with **20.6.14**.)

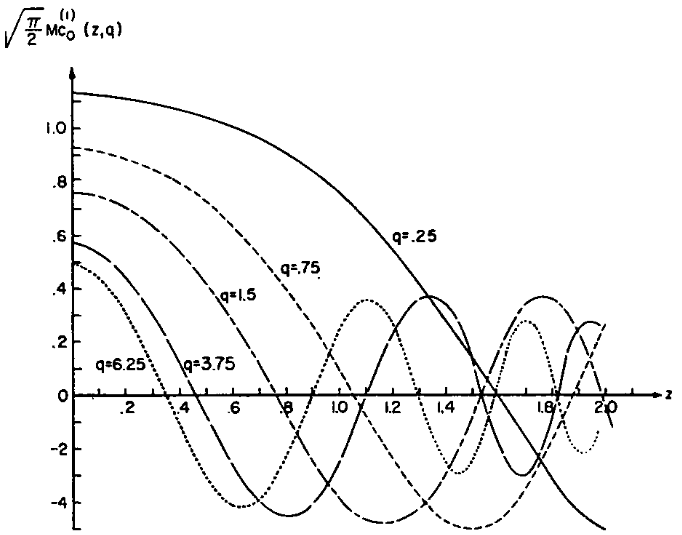

FIGURE 20.11. *Radial Mathieu Function of the First Kind.*

(From J. C. Wiltse and M. J. King, Values of the Mathieu functions, The Johns Hopkins Univ. Radiation Laboratory Tech. Rept. AF-53, 1958, with permission)

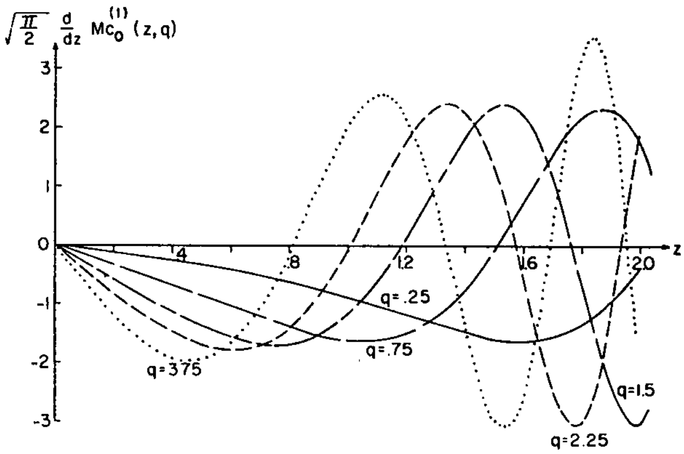

FIGURE 20.12. *Derivative of the Radial Mathieu Function of the First Kind.*

(From J. C. Wiltse and M. J. King, Derivatives, zeros, and other data pertaining to Mathieu functions, The Johns Hopkins Univ. Radiation Laboratory Tech. Rept. AF-57, 1958, with permission)

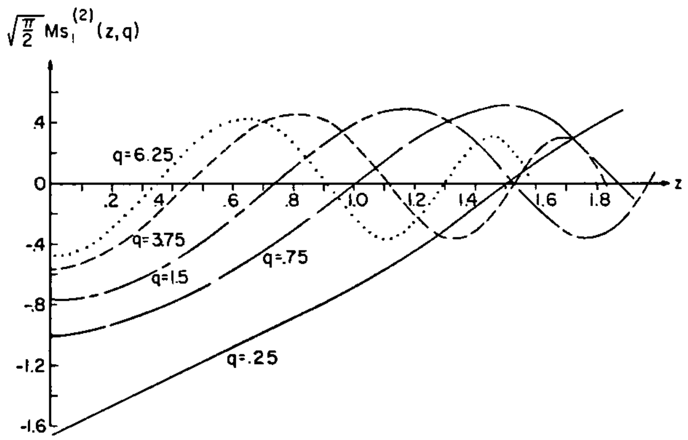

FIGURE 20.13. *Radial Mathieu Function of the Second Kind.*

(From J. C. Wiltse and M. J. King, Values of the Mathieu functions, The Johns Hopkins Univ. Radiation Laboratory Tech. Rept. AF-53, 1958, with permission)

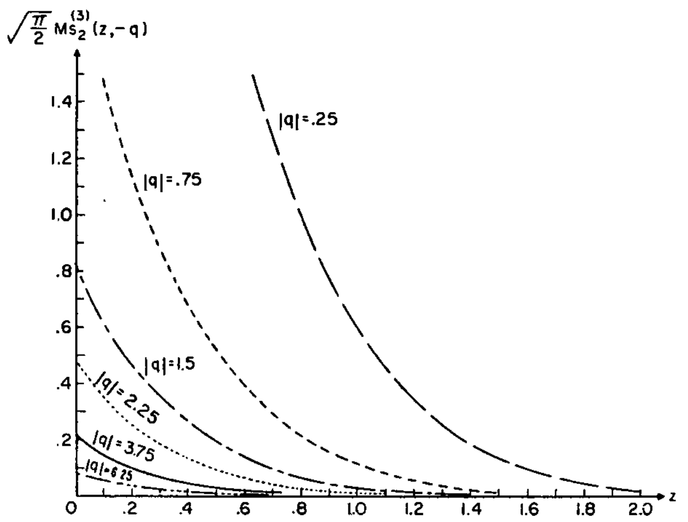

FIGURE 20.14. *Radial Mathieu Function of the Third Kind.*

(From J. C. Wiltse and M. J. King, Values of the Mathieu functions, The Johns Hopkins Univ. Radiation Laboratory Tech. Rept. AF-53, 1958, with permission)

If $j=1$, $Mc_{2r+p}^{(1)}$ and $Ms_{2r+p}^{(1)}, p=0, 1$ are solutions of the first kind, proportional to Ce_{2r+p} and Se_{2r+p}, respectively.

Thus

20.6.15

$$Ce_{2r}(z,q)=\frac{ce_{2r}\left(\frac{\pi}{2},q\right)ce_{2r}(0,q)}{(-1)^r A_0^{2r}}Mc_{2r}^{(1)}(z,q)$$

$$Ce_{2r+1}(z,q)=\frac{ce'_{2r+1}\left(\frac{\pi}{2},q\right)ce_{2r+1}(0,q)}{(-1)^{r+1}\sqrt{q}A_1^{2r+1}}Mc_{2r+1}^{(1)}(z,q)$$

$$Se_{2r}(z,q)=\frac{se'_{2r}(0,q)se'_{2r}\left(\frac{\pi}{2},q\right)}{(-1)^r qB_2^{2r}}Ms_{2r}^{(1)}(z,q)$$

$$Se_{2r+1}(z,q)=\frac{se'_{2r+1}(0,q)se_{2r+1}\left(\frac{\pi}{2},q\right)}{(-1)^r\sqrt{q}B_1^{2r+1}}Ms_{2r+1}^{(1)}(z,q)$$

The Mathieu-Hankel functions are

20.6.16

$$M_r^{(3)}(z,q)=M_r^{(1)}(z,q)+iM_r^{(2)}(z,q)$$
$$M_r^{(4)}(z,q)=M_r^{(1)}(z,q)-iM_r^{(2)}(z,q)$$
$$M_r^{(j)}=Mc_r^{(j)} \text{ or } Ms_r^{(j)}.$$

From **20.6.7–20.6.11** and the known properties of Bessel functions one obtains

20.6.17

$$M_{2r+p}^{(2)}(z+in\pi,q)=(-1)^{np}[M_{2r+p}^{(2)}(z,q)+2niM_{2r+p}^{(1)}(z,q)]$$

$$M_{2r+p}^{(3)}(z+in\pi,q)=(-1)^{np}[M_{2r+p}^{(3)}(z,q)-2nM_{2r+p}^{(1)}(z,q)]$$

$$M_{2r+p}^{(4)}(z+in\pi,q)=(-1)^{np}[M_{2r+p}^{(4)}(z,q)+2nM_{2r+p}^{(1)}(z,q)]$$

where $M=Mc$ or Ms throughout any of the above equations.

Other Properties of Characteristic Functions, q Real (Associated With a_r and b_r)

Consider

20.6.18

$$X_1=Mc_r^{(2)}(z,q)+Mc_r^{(2)}(-z,q);$$
$$X_2=Ms_r^{(2)}(z,q)-Ms_r^{(2)}(-z,q)$$

Since X_1 is an even solution it must be proportional to $Mc_r^{(1)}(z,q)$; for **20.1.2** admits of only one even solution (aside from an arbitrary constant factor). Similarly, X_2 is proportional to $Ms_r^{(1)}(z,q)$. The proportionality factors can be found by considering values of the functions at $z=0$. Define, therefore,

20.6.19

$$Mc_r^{(2)}(-z,q)=-Mc_r^{(2)}(z,q)-2f_{e,r}Mc_r^{(1)}(z,q)$$

20.6.20

$$Ms_r^{(2)}(-z,q)=Ms_r^{(2)}(z,q)-2f_{o,r}Ms_r^{(1)}(z,a)$$

where

20.6.21

$$f_{e,r}=-Mc_r^{(2)}(0,q)/Mc_r^{(1)}(0,q)$$
$$f_{o,r}=\left[\frac{d}{dz}Ms_r^{(2)}(z,q)/\frac{d}{dz}Ms_r^{(1)}(z,q)\right]_{z=0}$$

See [20.58].

In particular the above equations can be used to extend solutions of **20.6.12–20.6.13** when $\mathscr{R}z<0$. For although the latter converge for $\mathscr{R}z<0$, provided only $|\cosh z|>1$, they do not represent the same functions as **20.6.9–20.6.10**.

20.7. Representations by Integrals and Some Integral Equations

Let

20.7.1 $$G(u)=\oint_C K(u,t)V(t)dt$$

be defined for u in a domain U and let the contour C belong to the region T of the complex t-plane, with $t=\gamma_0$ as the starting point of the contour and $t=\gamma_1$ as its end-point. The kernel $K(u,t)$ and the function $V(t)$ satisfy **20.7.3** and the hypotheses in **20.7.2.**

20.7.2 $K(u,t)$ and its first two partial derivatives with respect to u and t are continuous for t on C and u in U; V and $\frac{dV}{dt}$ are continuous in t.

20.7.3

$$\left[\frac{\partial K}{\partial t}V-\frac{dV}{dt}K\right]_{\gamma_0}^{\gamma_1}=0;\quad\frac{d^2V}{dt^2}+(a-2q\cos 2t)V=0.$$

If K satisfies

20.7.4 $$\frac{\partial^2K}{\partial u^2}+\frac{\partial^2K}{\partial t^2}+2q(\cosh 2u-\cos 2t)K=0$$

then $G(u)$ is a solution of Mathieu's modified equation **20.1.2.**

If $K(u,t)$ satisfies

20.7.5 $$\frac{\partial^2K}{\partial u^2}+\frac{\partial^2K}{\partial t^2}+2q(\cos 2u-\cos 2t)K=0$$

then $G(u)$ is a solution of Mathieu's equation **20.1.1,** with u replacing v.

Kernels $K_1(z, t)$ and $K_2(z, t)$

20.7.6 $K_1(z, t)=Z_\nu^{(j)}(u)[M(z, t)]^{-\nu/2}, \qquad (\mathscr{R}z>0)$

where

20.7.7 $u=\sqrt{2q(\cosh 2z+\cos 2t)}$

20.7.8 $M(z, t)=\cosh (z+it)/\cosh (z-it)$

To make $M^{-\frac{1}{2}\nu}$ single-valued, define

20.7.9

$$\cosh (z+i\pi)=e^{i\pi}\cosh z$$
$$\cosh (z-i\pi)=e^{-i\pi}\cosh z$$
$$M(z, 0)=1$$
$$[M(z, \pi)]^{-\frac{1}{2}\nu}=e^{-i\nu\pi}M(z, 0)$$

Let

20.7.10 $G(z, q)=\frac{1}{\pi}\int_0^\pi K_1(u, t)F_\nu(t)dt, \qquad (\mathscr{R}z>0)$

where $F_\nu(t)$ is defined in **20.3.8.** It may be verified that K_1F_ν satisfies **20.7.3,** K satisfies **20.7.2** and **20.7.4.** Hence G is a solution of **20.1.2** (with z replacing u). It can be shown that K_1 may be replaced by the more general function

20.7.11

$K_2(z, t)=Z_{\nu+2s}^{(j)}(u)[M(z, t)]^{-\frac{1}{2}\nu+s}$, s any integer.

See **20.4.7** for definition of $Z_{\nu+2s}^{(j)}(u)$.

From the known expansions for $Z_{\nu+2s}^{(j)}(u)$ when $\mathscr{R}z$ is large and positive it may be verified that

20.7.12

$$M_\nu^{(j)}(z, q)=\frac{(-1)^s}{\pi c_{2s}}\int_0^\pi Z_{\nu+2s}^{(j)}(u)\left[\frac{\cosh z+it}{\cosh z-it}\right]^{-\frac{1}{2}\nu-s}F_\nu(t)dt$$
$$(\mathscr{R}z>0, \mathscr{R}(\nu+\tfrac{1}{2})>0)$$

where $M_\nu^{(j)}(z, q)$ is given by **20.4.12,** $s=0, 1, \ldots$, $c_{2s}\neq 0$, and $F_\nu(t)$ is the Floquet solution, **20.3.8.**

Kernel $K_3(z, t, a)$

20.7.13 $K_3(z, t, a)=e^{2i\sqrt{q}\,w}$

where

20.7.14 $w=\cosh z\cos a\cos t+\sinh z\sin a\sin t$

20.7.15 $G(z, q, a)=\frac{1}{\pi}\oint_C e^{2i\sqrt{q}\,w}F_\nu(t)dt$

where $F_\nu(t)$ is the Floquet solution **20.3.8.** The path C is chosen so that $G(z, t, a)$ exists, and **20.7.2, 20.7.3** are satisfied. Then it may be verified that $K_3(z, t, a)$, considered as a function of z and t, satisfies **20.7.4;** also, considered as a function of a and t, K_3 satisfies **20.7.5.** Consequently $G(z, q, a)=Y(z, q)y(a, q)$, where Y and y satisfy **20.1.2** and **20.1.1,** respectively.

Choice of Path C. Three paths will be defined:

20.7.16

Path C_3: from $-d_1+i\infty$ to $d_2-i\infty$, d_1, d_2 real

$$-d_1<\arg [\sqrt{q}\{\cosh (z+ia)\pm 1\}]<\pi-d_1$$
$$-d_2<\arg [\sqrt{q}\{\cosh (z-ia)\pm 1\}]<\pi-d_2$$

20.7.17

Path C_4: from $d_2-i\infty$ to $2\pi+i\infty-d_1$

(same d_1, d_2 as in **20.7.16**)

20.7.18

$$F_\nu(a)M_\nu^{j}(z, q)=\frac{e^{-i\nu\frac{\pi}{2}}}{\pi}\oint_{C_j} e^{2i\sqrt{q}\,w}F_\nu(t)dt \qquad j=3, 4$$

where $M_\nu^{j}(z, q)$ is also given by **20.4.12.**

20.7.19 Path C_1: from $-d_1+i\infty$ to $2\pi-d_1+i\infty$

$$F_\nu(a)M_\nu^{(1)}(z, q)=\frac{e^{-i\nu\frac{\pi}{2}}}{2\pi}\oint_{C_1} e^{2i\sqrt{q}\,w}F_\nu(t)dt$$

See [20.36], section **2.68.**

If ν is an integer the paths can be simplified; for in that case $F_\nu(t)$ is periodic and the integrals exist when the path is taken from 0 to 2π. Still further simplifications are possible, if z is also real.

The following are among the more important integral representations for the periodic functions $ce_r(z, q)$, $se_r(z, q)$ and for the associated radial solutions.

Let $r=2s+p$, $p=0$ or 1

20.7.20

$$ce_r(z, q)=\rho_r\int_0^{\pi/2}\cos\left(2\sqrt{q}\cos z\cos t-p\frac{\pi}{2}\right)ce_r(t, q)dt$$

20.7.21 $$ce_r(z,q)=\sigma_r\int_0^{\pi/2}\cosh\,(2\sqrt{q}\sin z\sin t)[(1-p)+p\cos z\cos t]ce_r(t,q)dt$$

20.7.22 $$se_r(z,q)=\rho_r\int_0^{\pi/2}\sin\left(2\sqrt{q}\cos z\cos t+p\frac{\pi}{2}\right)\sin z\sin t\; se_r(t,q)dt$$

20.7.23 $$se_r(z,q)=\sigma_r\int_0^{\pi/2}\sinh\,(2\sqrt{q}\sin z\sin t)[(1-p)\cos z\cos t+p]se_r(t,q)dt$$

where

20.7.24 $$\rho_r=\frac{2}{\pi}ce_{2s}\left(\frac{\pi}{2},q\right)/A_0^{2s}(q);\;p=0\,\rho_r=\frac{-2}{\pi}ce'_{2s+1}\left(\frac{\pi}{2},q\right)/\sqrt{q}A_1^{2s+1}(q)\text{ if }p=1,\text{ for functions }ce_r(z,q)$$

$$\rho_r=\frac{-4}{\pi}se'_{2s}\left(\frac{\pi}{2},q\right)/\sqrt{q}B_2^{2s}(q);\;\rho_r=\frac{4}{\pi}se_{2s+1}\left(\frac{\pi}{2},q\right)/B_1^{2s+1}(q),\text{ for functions }se_r(z,q)$$

$$\sigma_r=\frac{2}{\pi}ce_{2s}(0,q)/A_0^{2s}(q)\text{ if }p=0;\qquad\sigma_r=\frac{4}{\pi}ce_{2s+1}(0,q)/A_1^{2s+1}(q),\text{ if }p=1;\text{ associated with functions }ce_r(z,q)$$

$$\sigma_r=\frac{4}{\pi}se'_{2s}(0,q)/\sqrt{q}B_2^{2s}(q),\text{ if }p=0;\qquad\sigma_r=\frac{2}{\pi}se'_{2s+1}(0,q)/\sqrt{q}B_1^{2s+1}(q),\text{ if }p=1;\text{ associated with }se_r(z,q)$$

Integrals Involving Bessel Function Kernels

Let

20.7.25 $$u=\sqrt{2q(\cosh 2z+\cos 2t)},\;(\mathscr{R}\cosh 2z>1;\text{ if }j=1,\text{ valid also when }z=0)$$

20.7.26

$$Mc_{2r}^{(j)}(z,q)=\frac{(-1)^r2}{\pi A_0^{2r}}\int_0^{\frac{\pi}{2}}Z_0^{(j)}(u)ce_{2r}(t,q)dt;\;Mc_{2r+1}^{(j)}(z,q)=\frac{(-1)^r8\sqrt{q}\cosh z}{\pi A_1^{2r+1}}\int_0^{\frac{\pi}{2}}\frac{Z_1^{(j)}(u)\cos t}{u}ce_{2r+1}(t,q)dt$$

20.7.27 $$Ms_{2r}^{(j)}(z,q)=\frac{(-1)^{r+1}8q\sinh 2z}{\pi B_2^{2r}}\int_0^{\frac{\pi}{2}}\frac{Z_2^{(j)}(u)\sin 2t\; se_{2r}(t,q)dt}{u^2}$$

$$Ms_{2r+1}^{(j)}(z,q)=\frac{(-1)^r8\sqrt{q}\sinh z}{\pi B_1^{2r+1}}\int_0^{\frac{\pi}{2}}\frac{Z_1^{(j)}(u)\sin t\; se_{2r+1}(t,q)dt}{u}$$

In the above the j-convention of **20.4.7** applies and the functions Mc, Ms are defined in **20.5.1–20.5.4.** (These solutions are normalized so that they approach the corresponding Bessel-Hankel functions as $\mathscr{R}z\to\infty$.)

Other Integrals for $Mc_r^{(1)}(z,q)$ and $Ms_r^{(1)}(z,q)$

20.7.28 $$Mc_r^{(1)}(z,q)=\frac{(-1)^s2}{\pi ce_r(0,q)}\int_0^{\frac{\pi}{2}}\cos\left(2\sqrt{q}\cosh z\cos t-p\frac{\pi}{2}\right)ce_r(t,q)dt$$

20.7.29 $$Mc_r^{(1)}(z,q)=\tau_r\int_0^{\frac{\pi}{2}}[(1-p)+p\cosh z\cos t]\cos\,(2\sqrt{q}\sinh z\sin t)ce_r(t,q)dt$$

$$r=2s+p,\,p=0,1;\;\tau_r=\frac{2}{\pi}(-1)^s/ce_{2s}\left(\frac{\pi}{2},q\right),\text{ if }p=0;\;\tau_r=\frac{2}{\pi}(-1)^{s+1}2\sqrt{q}/ce'_{2s+1}\left(\frac{\pi}{2},q\right)$$

20.7.30 $$Ms_{2r+1}^{(1)}(z,q)=\frac{2}{\pi}\frac{(-1)^r}{se_{2r+1}\left(\frac{\pi}{2},q\right)}\int_0^{\frac{\pi}{2}}\sin\,(2\sqrt{q}\sinh z\sin t)se_{2r+1}(t,q)dt$$

20.7.31 $$Ms_{2r+1}^{(1)}(z,q)=\frac{4}{\pi}\frac{\sqrt{q}(-1)^r}{se'_{2r+1}(0,q)}\int_0^{\frac{\pi}{2}}\sinh z\sin t\cos\,(2\sqrt{q}\cosh z\cos t)se_{2r+1}(t,q)dt$$

20.7.32 $$Ms_{2r}^{(1)}(z,q)=\frac{4}{\pi}\sqrt{q}\frac{(-1)^{r+1}}{se'_{2r}(0,q)}\int_0^{\frac{\pi}{2}}\sin\,(2\sqrt{q}\cosh z\cos t)\,[\sinh z\sin t\; se_{2r}(t,q)]dt$$

20.7.33 $$Ms_{2r}^{(1)}(z,q)=\frac{4}{\pi}\frac{(-1)^r\sqrt{q}}{se'_{2r}\left(\frac{\pi}{2},q\right)}\int_0^{\frac{\pi}{2}}\sin\,(2\sqrt{q}\sinh z\sin t)[\cosh z\cos t\; se_{2r}(t,q)]dt$$

Further with $w=\cosh z \cos \alpha \cos t+\sinh z \sin \alpha \sin t$

20.7.34 $$ce_r(\alpha, q)Mc_r^{(1)}(z, q)=\frac{(-1)^s(i)^{-p}}{2\pi}\int_0^{2\pi} e^{2i\sqrt{q}\,w}ce_r(t, q)dt$$

20.7.35 $$se_r(\alpha, q)Ms_r^{(1)}(z, q)=\frac{(-1)^s(-i)^p}{2\pi}\int_0^{2\pi} e^{2i\sqrt{q}\,w}se_r(t, q)dt.$$

The above can be differentiated with respect to α, and we obtain

20.7.36 $$ce_r'(\alpha, q)Mc_r^{(1)}(z, q)=\frac{(-1)^s(i)^{-p+1}\sqrt{q}}{\pi}\int_0^{2\pi} e^{2i\sqrt{q}\,w}\frac{\partial w}{\partial\alpha}ce_r(t, q)dt$$

20.7.37 $$se_r'(\alpha, q)Ms_r^{(1)}(z, q)=\frac{(-1)^{s+p}(i)^{-p+1}\sqrt{q}}{\pi}\int_0^{2\pi} e^{2i\sqrt{q}\,w}\frac{\partial w}{\partial\alpha}se_r(t, q)dt$$

Integrals With Infinite Limits

$$r=2s+p$$

In **20.7.38–20.7.41** below, z and q are positive.

20.7.38 $$Mc_r^{(1)}(z, q)=\gamma_r\int_0^\infty \sin\left(2\sqrt{q}\cosh z\cosh t+p\frac{\pi}{2}\right)Mc_r^{(1)}(t, q)dt$$

$$\gamma_r=2ce_{2s}\left(\frac{\pi}{2}, q\right)/\pi A_0^{2s},\text{ if } p=0 \qquad \gamma_r=2ce_{2s+1}'\left(\frac{\pi}{2}, q\right)/\sqrt{q}\,\pi A_1^{2s+1},\text{ if } p=1$$

20.7.39 $$Ms_r^{(1)}(z, q)=\gamma_r\int_0^\infty \sinh z\sinh t\left[\cos\left(2\sqrt{q}\cosh z\cosh t-p\frac{\pi}{2}\right)\right]Ms_r^{(1)}(t, q)dt$$

$$\gamma_r=-4se_{2s}'\left(\frac{\pi}{2}, q\right)/\sqrt{q}\pi B_2^{2s},\text{ if } p=0 \qquad \gamma_r=-4se_{2s+1}\left(\frac{\pi}{2}, q\right)/\pi B_1^{2s+1},\text{ if } p=1$$

20.7.40 $$Mc_r^{(2)}(z, q)=\gamma_r\int_0^\infty \cos\left(2\sqrt{q}\cosh z\cosh t-p\frac{\pi}{2}\right)Mc_r^{(1)}(t, q)dt$$

$$\gamma_r=-2ce_{2s}(\tfrac{1}{2}\pi, q)/\pi A_0^{2s},\text{ if } p=0 \qquad \gamma_r=2ce_{2s+1}'(\tfrac{1}{2}\pi, q)/\pi\sqrt{q}A_1^{2s+1},\text{ if } p=1$$

20.7.41 $$Ms_r^{(2)}(z, q)=\gamma_r\int_0^\infty \sin\left(2\sqrt{q}\cosh z\cosh t+p\frac{\pi}{2}\right)\sinh z\sinh t\, Ms_r^{(1)}(t, q)dt$$

$$\gamma_r=-4se_{2s}'(\tfrac{1}{2}\pi, q)/\sqrt{q}\,\pi B_2^{2s},\text{ if } p=0 \qquad \gamma_r=4se_{2s+1}(\tfrac{1}{2}\pi, q)/\pi B_1^{2s+1},\text{ if } p=1$$

Additional forms in [20.30], [20.36], [20.15].

20.8. Other Properties

Relations Between Solutions for Parameters q and $-q$

Replacing z by $\frac{1}{2}\pi-z$ in **20.1.1** one obtains

20.8.1 $$y''+(a+2q\cos 2z)y=0$$

Hence if $u(z)$ is a solution of **20.1.1** then $u(\frac{1}{2}\pi-z)$ satisfies **20.8.1.** It can be shown that

20.8.2

$$a(-\nu, q)=a(\nu, -q)=a(\nu, q),\ \nu\text{ not an integer}$$

$$c_{2m}^{\nu}(-q)=\rho(-1)^m c_{2m}^{\nu}(q),\ \nu\text{ not an integer}$$

(c_{2m} defined in **20.3.8**) and ρ depending on the normalization;

$$F_\nu(z, -q)=\rho e^{-i\nu\pi/2}F_\nu\left(z+\frac{\pi}{2}, q\right)=\rho e^{i\nu\pi/2}F_\nu\left(z-\frac{\pi}{2}, q\right)$$

20.8.3

$$a_{2r}(-q)=a_{2r}(q);\ b_{2r}(-q)=b_{2r}(q),\ \text{for integral } \nu$$
$$a_{2r+1}(-q)=b_{2r+1}(q),\ b_{2r+1}(-q)=a_{2r+1}(q)$$

20.8.4

$$ce_{2r}(z,\ -q)=(-1)^r ce_{2r}(\tfrac{1}{2}\pi-z,\ q)$$
$$ce_{2r+1}(z,\ -q)=(-1)^r se_{2r+1}(\tfrac{1}{2}\pi-z,\ q)$$
$$se_{2r+1}(z,\ -q)=(-1)^r ce_{2r+1}(\tfrac{1}{2}\pi-z,\ q)$$
$$se_{2r}(z,\ -q)=(-1)^{r-1} se_{2r}(\tfrac{1}{2}\pi-z,\ q)$$

For the coefficients associated with the above solutions for integral ν:

20.8.5

$$A_{2m}^{2r}(-q)=(-1)^{m-r}A_{2m}^{2r}(q);\qquad B_{2m}^{2r}(-q)=(-1)^{m-r}B_{2m}^{2r}(q)$$
$$A_{2m+1}^{2r+1}(-q)=(-1)^{m-r}B_{2m+1}^{2r+1}(q);\qquad B_{2m+1}^{2r+1}(-q)=(-1)^{m-r}A_{2m+1}^{2r+1}(q).$$

For the corresponding modified equation

20.8.6 $$y''-(a+2q\cosh 2z)y=0$$

20.8.7

$$M_\nu^{(j)}(z,\ -q)=M_\nu^{(j)}\left(z+i\frac{\pi}{2},\ q\right),$$

$M_\nu^{(j)}(z,\ q)$ defined in **20.4.12.**

For integral values of ν let

20.8.8

$$Ie_{2r}(z,\ q)=\sum_{k=0}^{\infty}(-1)^{k+s}A_{2k}[I_{k-s}(u_1)I_{k+s}(u_2)+I_{k+s}(u_1)I_{k-s}(u_2)]/A_{2s}\epsilon_s$$

$$Io_{2r}(z,\ q)=\sum_{k=1}^{\infty}(-1)^{k+s}B_{2k}[I_{k-s}(u_1)I_{k+s}(u_2)-I_{k+s}(u_1)I_{k-s}(u_2)]/B_{2s}$$

$$Ie_{2r+1}(z,\ q)=\sum_{k=0}^{\infty}(-1)^{k+s}B_{2k+1}[I_{k-s}(u_1)I_{k+s+1}(u_2)+I_{k+s+1}(u_1)I_{k-s}(u_2)]/B_{2s+1}$$

$$Io_{2r+1}(z,\ q)=\sum_{k=0}^{\infty}(-1)^{k+s}A_{2k+1}[I_{k-s}(u_1)I_{k+s+1}(u_2)-I_{k+s+1}(u_1)I_{k-s}(u_2)]/A_{2s+1}$$

20.8.9

$$Ke_{2r}(z,\ q)=\sum_{k=0}^{\infty}A_{2k}[I_{k-s}(u_1)K_{k+s}(u_2)+I_{k+s}(u_1)K_{k-s}(u_2)]/A_{2s}\epsilon_s$$

* $$Ko_{2r}(z,\ q)=\sum_{k=0}^{\infty}B_{2k}[I_{k-s}(u_1)K_{k+s}(u_2)-I_{k+s}(u_1)K_{k-s}(u_2)]/B_{2s}$$ *

$$Ke_{2r+1}(z,\ q)=\sum_{k=0}^{\infty}B_{2k+1}[I_{k-s}(u_1)K_{k+s+1}(u_2)-I_{k+s+1}(u_1)K_{k-s}(u_2)]/B_{2s+1}$$

$$Ko_{2r+1}(z,\ q)=\sum_{k=0}^{\infty}A_{2k+1}[I_{k-s}(u_1)K_{k+s+1}(u_2)+I_{k+s+1}(u_1)K_{k-s}(u_2)]/A_{2s+1}$$

where $I_m(x)$, $K_m(x)$ are the modified Bessel functions, u_1, u_2 are defined below **20.6.10.** Superscripts are omitted, $\epsilon_s=2$, if $s=0$, $\epsilon_s=1$ if $s\neq 0$.

Then for functions of first kind:

20.8.10

$$Mc_{2r}^{(1)}(z,\ -q)=(-1)^r Ie_{2r}(z,\ q)$$
$$Ms_{2r}^{(1)}(z,\ -q)=(-1)^r Io_{2r}(z,\ q)$$
$$Mc_{2r+1}^{(1)}(z,\ -q)=(-1)^r i Ie_{2r+1}(z,\ q)$$
$$Ms_{2r+1}^{(1)}(z,\ -q)=(-1)^r i Io_{2r+1}(z,\ q)$$

For the Mathieu-Hankel function of first kind:

20.8.11

$$Mc_{2r}^{(3)}(z,\ -q)=(-1)^{r+1}i\frac{2}{\pi}Ke_{2r}(z,\ q)$$
$$Ms_{2r}^{(3)}(z,\ -q)=(-1)^{r+1}i\frac{2}{\pi}Ko_{2r}(z,\ q)$$
$$Mc_{2r+1}^{(3)}(z,\ -q)=(-1)^{r+1}\frac{2}{\pi}Ke_{2r+1}(z,\ q)$$
$$Ms_{2r+1}^{(3)}(z,\ -q)=(-1)^{r+1}\frac{2}{\pi}Ko_{2r+1}(z,\ q)$$

For $M_r^{(j)}(z,\ -q)$, $j=2,\ 4$, one may use the definitions

$$M_r^{(2)}=-i(M_r^{(3)}-M_r^{(1)});\ M_r=Mc_r \text{ or } Ms_r$$

also

$$M_r^{(4)}(z,\ -q)=2M_r^{(1)}(z,\ -q)-M_r^{(3)}(z,\ -q)$$

$M=Mc$ or Ms; for real z, q, $M_r^{(j)}(z,\ -q)$ are in general complex if $j=2,\ 4$.

Zeros of the Functions for Real Values of q.

See [20.36], section **2.8** for further results.

Zeros of $ce_r(z,\ q)$ and $se_r(z,\ q)$, $Mc_r^{(1)}(z,\ q)$, $Ms_r^{(1)}(z,\ q)$.

In $0\leq z<\pi$, $ce_r(z,\ q)$ and $se_r(z,\ q)$ have r real zeros. *

There are complex zeros if $q>0$.

If $z_0=x_0+iy_0$ is any zero of $ce_r(z,\ q)$, $se_r(z,\ q)$ in

$$-\frac{\pi}{2}<x_0<\frac{\pi}{2},\ \text{then } k\pi\pm z_0,\ k\pi\pm\bar{z}_0$$

are also zeros, k an integer.

*See page II.

In the strip $-\frac{\pi}{2}<x_0<\frac{\pi}{2}$, the imaginary zeros of $ce_r(z, q)$, $se_r(z, q)$ are the real zeros of $Ce_r(z, q)$, $Se_r(z, q)$, hence also the real zeros of $Mc_r^{(1)}(z, q)$ and $Ms_r^{(1)}(z, q)$, respectively.

For small q, the large zeros of $Ce_r(z, q)$, $Se_r(z, q)$ approach the zeros of $J_r(2\sqrt{q} \cosh z)$.

Tabulation of Zeros

Ince [20.56] tabulates the first "non-trivial" zero $\left(\text{i.e. different from } 0, \frac{\pi}{2}, \pi\right)$ for $ce_r(z)$, $se_r(z)$, $r=2(1)5$ and for $se_6(z)$ to within $°10^{-4}$, for $q=0(1)10(2)40$. He also gives the "turning" points (zeros of the derivative) and also expansions for them for small q. Wiltse and King [20.61,2] tabulate the first two (non-trivial) zeros of $Mc_r^{(1)}(z, q)$ and $Ms_r^{(1)}(z, q)$ and of their derivatives $r=0$, 1, 2 for 6 or 7 values of q between .25 and 10. The graphs reproduced here indicate their location.

Between two real zeros of $Mc_r^{(1)}(z, q)$, $Ms_r^{(1)}(z, q)$ there is a zero of $Mc_r^{(2)}(z, q)$, $Ms_r^{(2)}(z, q)$, respectively. No tabulation of such zeros exists yet.

Available tables are described in the References. The most comprehensive tabulation of the characteristic values a_r, b_r (in a somewhat different notation) and of the coefficients proportional to A_m and B_m as defined in **20.5.4** and **20.5.5** can be found in [20.58]. In addition, the table contains certain important "joining factors", with the aid of which it is possible to obtain values of $Mc_r^{(j)}(z, q)$ and $Ms_r^{(j)}(z, q)$ as well as their derivatives, at $x=0$. Values of the functions $ce_r(x, q)$ and $se_r(x, q)$ for orders up to five or six can be found in [20.56]. Tabulations of less extensive character, but important in some aspects, are outlined in the other references cited. In this chapter only representative values of the various functions are given, along with several graphs.

Special Values for Arguments 0 and $\frac{\pi}{2}$

20.8.12

$$ce_{2r}\left(\frac{\pi}{2}, q\right)=(-1)^r g_{e,2r}(q)A_0^{2r}(q)\sqrt{\frac{\pi}{2}}$$

$$ce'_{2r+1}\left(\frac{\pi}{2}, q\right)=(-1)^{r+1} g_{e,2r+1}(q)A_1^{2r+1}(q)\sqrt{\frac{\pi}{2}q}$$

$$se'_{2r}\left(\frac{\pi}{2}, q\right)=(-1)^r g_{o,2r}(q)B_2^{2r}(q)\cdot q\sqrt{\frac{\pi}{2}}$$

$$se_{2r+1}\left(\frac{\pi}{2}, q\right)=(-1)^r g_{o,2r+1}(q)B_1^{2r+1}(q)\sqrt{\frac{\pi}{2}q}$$

$$Mc_r^{(1)}(0, q)=\sqrt{\frac{2}{\pi}}\frac{1}{g_{e,r}(q)}$$

$$Mc_r^{(2)}(0, q)=-\sqrt{\frac{2}{\pi}}f_{e,r}(q)/g_{e,r}(q)$$

$$\frac{d}{dz}[Mc_r^{(2)}(z, q)]_{z=0}=\sqrt{\frac{2}{\pi}}\,g_{e,r}(q)$$

$$\frac{d}{dz}[Ms_r^{(1)}(z, q)]_{z=0}=\sqrt{\frac{2}{\pi}}\frac{1}{g_{o,r}(q)}$$

$$\frac{d}{dz}\left[Ms_r^{(2)}(z, q)\right]_{z=0}=\sqrt{\frac{2}{\pi}}f_{o,r}(q)/g_{o,r}(q)$$

$$Ms_r^{(2)}(z, q)=-g_{o,r}(q)\sqrt{\frac{2}{\pi}}$$

The functions $f_{o,r}$, $g_{o,r}$, $f_{e,r}$, $g_{e,r}$ are tabulated in [20.58] for $q\leq 25$.

20.9. Asymptotic Representations

The representations given below are applicable to the *characteristic solutions*, for real values of q, unless otherwise noted. The Floquet exponent ν is defined below, as in [20.36] to be as follows:

In solutions associated with a_r: $\nu=r$
In solutions associated with b_r: $\nu=-r$.

For the functions defined in **20.6.7–20.6.10:**

20.9.1

$$\begin{aligned}&Mc_r^{(3)}(z, q)\\&(-1)^r Ms_r^{(3)}(z, q)\end{aligned}\sim\frac{e^{i\left(2\sqrt{q}\cosh z-\frac{\nu\pi}{2}-\frac{\pi}{4}\right)}}{\pi^{\frac{1}{2}}q^{1/4}(\cosh z-\sigma)^{\frac{1}{2}}}\sum_{m=0}^{\infty}\frac{D_m}{[-4i\sqrt{q}(\cosh z-\sigma)]^m}$$

where $D_{-1}=D_{-2}=0$; $D_0=1$, and the coefficients D_m are obtainable from the following recurrence formula:

20.9.2

$$(m+1)D_{m+1}+\left[\left(m+\frac{1}{2}\right)^2-\left(m+\frac{1}{4}\right)8i\sqrt{q}\,\sigma\right.$$
$$\left.+2q-a\right]D_m+\left(m-\frac{1}{2}\right)[16q(1-\sigma^2)-8i\sqrt{q}\,\sigma m]D_{m-1}$$
$$+4q(2m-3)(2m-1)(1-\sigma^2)D_{m-2}=0$$

20.9.3

$$\begin{aligned}&Mc_r^{(4)}(z, q)\\&(-1)^r Ms_r^{(4)}(z, q)\end{aligned}\sim\frac{e^{-i\left[2\sqrt{q}\cosh z-\frac{1}{2}\nu\pi-\frac{1}{4}\pi\right]}}{\pi^{\frac{1}{2}}q^{1/4}(\cosh z-\sigma)^{\frac{1}{2}}}\sum_{m=0}^{\infty}\frac{d_m}{[4i\sqrt{q}(\cosh z-\sigma)]^m}$$

$d_{-1}=d_{-2}=0$; $d_0=1$, and

20.9.4

$$(m+1)d_{m+1}+\left[\left(m+\frac{1}{2}\right)^2+\left(m+\frac{1}{4}\right)8i\sqrt{q}\,\sigma\right.$$
$$\left.+2q-a\right]d_m+\left(m-\frac{1}{2}\right)[16q(1-\sigma^2)+8i\sqrt{q}\,\sigma m]d_{m-1}$$
$$+4q(2m-3)(2m-1)(1-\sigma^2)d_{m-2}=0.$$

In the above

$$-2\pi<\arg\sqrt{q}\cosh z<\pi$$
$$|\cosh z-\sigma|>|\sigma\pm 1|,\ \mathscr{R}z>0,$$

but σ is otherwise arbitrary. If $\sigma^2=1$, **20.9.2** and **20.9.4** become three-term recurrence relations.

Formulas **20.9.1** and **20.9.3** are valid for arbitrary a, q, provided ν is also known; they give multiples of **20.4.12,** normalized so as to approach the corresponding Hankel functions $H_\nu^{(1)}(\sqrt{q}e^z)$, $H_\nu^{(2)}(\sqrt{q}e^z)$, as $z\to\infty$. See [20.36], section **2.63.** The formula is especially useful if $|\cosh z|$is large and q is not too large; thus if $\sigma=-1$, the absolute ratio of two successive terms in the expansion is essentially

$$\left|\left(\frac{\sqrt{q}}{m}+\frac{m}{4\sqrt{q}}+2\right)/(\cosh z+1)\right|.$$

If a, q, z, ν are real, the real and imaginary components of $Mc_r^{(3)}(z,q)$ are $Mc_r^{(1)}(z,q)$ and $Mc_r^{(2)}(z,q)$, respectively; similarly for the components of $Ms_r^{(3)}(z,q)$. If the parameters are complex

20.9.5 $$Mc_r^{(1)}(z,q)=\tfrac{1}{2}\,[Mc_r^{(3)}(z,q)+Mc_r^{(4)}(z,q)]$$

20.9.6 $$Mc_r^{(2)}(z,q)=-\frac{i}{2}\,[Mc_r^{(3)}(z,q)-Mc_r^{(4)}(z,q)]$$

Replacing c by s in the above will yield corresponding relations among $Ms_r^{(j)}(z,q)$.

Formulas in which the parameter a does not enter explicitly:

Goldstein's Expansions

20.9.7

$$Mc_r^{(3)}(z,q)\sim iMs_{r+1}^{(3)}(z,q)$$
$$\approx[F_0(z)-iF_1(z)]e^{i\phi}/\pi^{\frac{1}{2}}q^{\frac{1}{4}}(\cosh z)^{\frac{1}{2}}$$

where

20.9.8

$$\phi=2\sqrt{q}\sinh z-\frac{1}{2}(2r+1)\arctan\sinh z,$$
$$\mathscr{R}z>0,\ q>>1,\ w=2r+1$$

20.9.9

$$F_0(z)\sim 1+\frac{w}{8\sqrt{q}\cosh^2 z}$$
$$+\frac{1}{2048q}\left[\frac{w^4+86w^2+105}{\cosh^4 z}-\frac{w^4+22w^2+57}{\cosh^2 z}\right]$$
$$+\frac{1}{16384q^{3/2}}\left[\frac{-(w^5+14w^3+33w)}{\cosh^2 z}\right.$$
$$\left.-\frac{(2w^5+124w^3+1122w)}{\cosh^4 z}+\frac{3w^5+290w^3+1627w}{\cosh^6 z}\right]+\dots$$

20.9.10

$$F_1(z)\sim\frac{\sinh z}{\cosh^2 z}\left[\frac{w^2+3}{32\sqrt{q}}+\frac{1}{512q}\left(w^3+3w+\frac{4w^3+44w}{\cosh^2 z}\right)\right.$$
$$+\frac{1}{16384q^{3/2}}\left\{5w^4+34w^2+9\right.$$
$$-\frac{(w^6-47w^4+667w^2+2835)}{12\cosh^2 z}$$
$$\left.\left.+\frac{(w^6+505w^4+12139w^2+10395)}{12\cosh^4 z}\right\}\right]+\dots$$

See [20.18] for details and an added term in $q^{-5/2}$; a correction to the latter is noted in [20.58].

The expansions **20.9.7** are especially useful when q is large and z is bounded away from zero. The order of magnitude of $Mc_r^{2}(0,q)$ cannot be obtained from the expansion. The expansion can also be used, with some success, for $z=ix$, when q is large, if $|\cos x|>>0$; they fail at $x=\frac{1}{2}\pi$. Thus, if q, x are real, one obtains

20.9.11

$$ce_r(x,q)\sim\frac{ce_r(0,q)2^{r-\frac{1}{2}}}{F_0(0)}\{W_1[P_0(x)-P_1(x)]$$
$$+W_2[P_0(x)+P_1(x)]\}$$

20.9.12

$$se_{r+1}(x,q)\sim se'_{r+1}(0,q)\tau_{r+1}\{W_1[P_0(x)-P_1(x)]$$
$$-W_2[P_0(x)+P_1(x)]\}$$

In the above, $P_0(x)$ and $P_1(x)$ are obtainable from $F_0(z)$, $F_1(x)$ in **20.9.9–20.9.10** by replacing $\cosh z$ with $\cos x$ and $\sinh z$ with $\sin x$. Thus $P_0(x)=F_0(ix)$; $P_1(x)=-iF_1(ix)$:

20.9.13

$$W_1=e^{2\sqrt{q}\sin x}\,[\cos(\tfrac{1}{2}x+\tfrac{1}{4}\pi)]^{2r+1}/(\cos x)^{r+1}$$

$$W_2=e^{-2\sqrt{q}\sin x}\,[\sin(\tfrac{1}{2}x+\tfrac{1}{4}\pi)]^{2r+1}/(\cos x)^{r+1}$$

20.9.14

$$\tau_{r+1}\sim 2^{r-\frac{1}{2}}\Big/\left[2\sqrt{q}-\tfrac{1}{4}w-\frac{(2w^2+3)}{64\sqrt{q}}-\frac{(7w^3+47w)}{1024q}-\ldots\right]$$

See **20.9.23–20.9.24** for expressions relating to $ce_r(0, q)$ and $se_r'(0, q)$. When $|\cos x|>\sqrt{4r+2}/q^{\frac{1}{4}}$, **20.9.11–20.9.12** are useful. The approximations become poorer as r increases.

Expansions in Terms of Parabolic Cylinder Functions

(Good for angles close to $\frac{1}{2}\pi$, for large values of q, especially when $|\cos x|<2^{\frac{1}{2}}/q^{\frac{1}{4}}$.) Due to Sips [20.44–20.46].

20.9.15 $\quad ce_r(x, q)\sim C_r[Z_0(\alpha)+Z_1(\alpha)]$

20.9.16

$$se_{r+1}(x, q)\sim S_r[Z_0(\alpha)-Z_1(\alpha)]\sin x, \qquad \alpha=2q^{\frac{1}{4}}\cos x.$$

Let $D_k=D_k(\alpha)=(-1)^k e^{\frac{1}{4}\alpha^2}\dfrac{d^k}{d\alpha^k}e^{-\frac{1}{4}\alpha^2}$.

20.9.17

$$Z_0(\alpha)\sim D_r+\frac{1}{4q^{\frac{1}{2}}}\left[-\frac{D_{r+4}}{16}+\frac{3}{2}\binom{r}{4}D_{r-4}\right]$$
$$+\frac{1}{16q}\left[\frac{D_{r+8}}{512}-\frac{(r+2)D_{r+4}}{16}+\frac{3}{2}(r-1)\binom{r}{4}D_{r-4}\right.$$
$$\left.+\frac{315}{4}\binom{r}{8}D_{r-8}\right]+\ldots$$

20.9.18

$$Z_1(\alpha)\sim\frac{1}{4q^{\frac{1}{2}}}\left[-\frac{1}{4}D_{r+2}-\frac{r(r-1)}{4}D_{r-2}\right]$$
$$+\frac{1}{16q}\left[\frac{D_{r+6}}{64}+\frac{(r^2-25r-36)}{64}D_{r+2}\right.$$
$$\left.+\frac{r(r-1)(-r^2-27r+10)}{64}D_{r-2}-\frac{45}{4}\binom{r}{6}D_{r-6}+\ldots\right]$$

20.9.19

$$C_r\sim\left(\frac{\pi}{2}\right)^{\frac{1}{4}}q^{\frac{1}{8}}/(r!)^{\frac{1}{2}}\left[1+\frac{2r+1}{8q^{\frac{1}{2}}}\right.$$
$$\left.+\frac{r^4+2r^3+263r^2+262r+108}{2048q}+\frac{f_1}{16384q^{\frac{3}{2}}}+\ldots\right]^{-\frac{1}{2}}$$
$$f_1=6r^5+15r^4+1280r^3+1905r^2+1778r+572$$

20.9.20

$$S_r\sim\left(\frac{\pi}{2}\right)^{\frac{1}{4}}q^{\frac{1}{8}}/(r!)^{\frac{1}{2}}\left[1-\frac{2r+1}{8q^{\frac{1}{2}}}\right.$$
$$\left.+\frac{r^4+2r^3-121r^2-122r-84}{2048q}+\frac{f_2}{16384q^{\frac{3}{2}}}+\ldots\right]^{-\frac{1}{2}}$$
$$f_2=2r^5+5r^4-416r^3-629r^2-1162r-476$$

It should be noted that **20.9.15** is also valid as an approximation for $se_{r+1}(x, q)$, but **20.9.16** may give slightly better results. See [20.4.]

Explicit Expansions for Orders 0, 1, to Terms in $q^{-3/2}$ (q Large)

20.9.21 For $r=0$:

$$Z_0\sim D_0-\frac{D_4}{64\sqrt{q}}+\frac{1}{16q}\left(-\frac{D_4}{8}+\frac{D_8}{512}\right)\ *$$
$$+\frac{1}{64q^{3/2}}\left(-\frac{99D_4}{256}+\frac{3D_8}{256}-\frac{D_{12}}{24576}\right)+\ldots$$
$$Z_1\sim\frac{-D_2}{16\sqrt{q}}+\frac{1}{16q}\left(-\frac{9D_2}{16}+\frac{D_6}{64}\right)$$
$$+\frac{1}{64q^{3/2}}\left(-\frac{61D_2}{32}+\frac{25D_6}{256}-\frac{5D_{10}}{10240}\right)+\ldots$$

20.9.22 For $r=1$:

$$Z_0\sim D_1-\frac{D_5}{64\sqrt{q}}+\frac{1}{16q}\left(-\frac{3D_5}{16}+\frac{D_9}{512}\right)$$
$$+\frac{1}{64q^{3/2}}\left(-\frac{207D_5}{256}+\frac{D_9}{64}-\frac{D_{13}}{24576}\right)+\ldots$$
$$Z_1\sim\frac{-D_3}{16\sqrt{q}}+\frac{1}{16q}\left(-\frac{15D_3}{16}+\frac{D_7}{64}\right)$$
$$+\frac{1}{64q^{3/2}}\left(-\frac{153D_3}{32}+\frac{35D_7}{256}-\frac{D_{11}}{2048}\right)+\ldots$$

Formulas Involving $ce_r(0, q)$ and $se_r(0, q)$

20.9.23

$$\frac{ce_0(0, q)}{ce_0(\frac{1}{2}\pi, q)}\sim 2\sqrt{2}\,e^{-2\sqrt{q}}\left(1+\frac{1}{16\sqrt{q}}+\frac{9}{256q}+\ldots\right)$$
$$\frac{ce_2(0, q)}{ce_2(\frac{1}{2}\pi, q)}\sim -32q\sqrt{2}\,e^{-2\sqrt{q}}\left(1-\frac{1}{16\sqrt{q}}+\frac{29}{128q}+\ldots\right)$$

*See page II.

$$\frac{ce_1(0,q)}{ce_1'(\frac{1}{2}\pi,q)}\sim -4\sqrt{2}e^{-2\sqrt{q}}\left(1+\frac{3}{16\sqrt{q}}+\frac{45}{256q}+\ldots\right)$$

$$\frac{ce_3(0,q)}{ce_3'(\frac{1}{2}\pi,q)}\sim \frac{64}{3}q\sqrt{2}e^{-2\sqrt{q}}\left(1-\frac{3}{16\sqrt{q}}+\frac{47}{128q}+\ldots\right)$$

20.9.24

$$\frac{se_1'(0,q)}{se_1(\frac{1}{2}\pi,q)}\sim 4q\sqrt{2}e^{-2\sqrt{q}}\left(1-\frac{3}{16\sqrt{q}}-\frac{11}{256q}+\ldots\right)$$

$$\frac{se_3'(0,q)}{se_3(\frac{1}{2}\pi,q)}\sim -64q\sqrt{2}e^{-2\sqrt{q}}\left(1-\frac{21}{16\sqrt{q}}-\frac{17}{128q}+\ldots\right)$$

$$\frac{se_2'(0,q)}{se_2'(\frac{1}{2}\pi,q)}\sim -8q\sqrt{2}e^{-2\sqrt{q}}\left(1-\frac{9}{16\sqrt{q}}-\frac{39}{256q}+\ldots\right)$$

$$\frac{se_4'(0,q)}{se_4'(\frac{1}{2}\pi,q)}\sim \frac{128}{3}q\sqrt{2}e^{-2\sqrt{q}}\left(1-\frac{31}{16\sqrt{q}}-\frac{15}{128q}+\ldots\right)$$

For higher orders, these ratios are increasingly more difficult to obtain. One method of estimating values at the origin is to evaluate both **20.9.11** and **20.9.15** for some x where both expansions are satisfactory, and so to use **20.9.11** as a means to solve for $ce_r(0,q)$; similarly for $se_r'(0,q)$.

Other asymptotic expansions, valid over various regions of the complex z-plane, for real values of a, q, have been given by Langer [20.25]. It is not always easy, however, to determine the linear combinations of Langer's solutions which coincide with those defined here.

20.10. Comparative Notations

	This Volume	[20.58] NBS	[20.59] Stratton-Morse, etc.	[20.36] Meixner and Schäfke	[20.30] McLachlan	[20.15] Bateman Manuscript	Comments
Parameters in **20.1.1**	a	$b=a+2q$	b	λ	a	h	
	q	$s=4q$	$c=2\sqrt{q}$	h^2	q	θ	
	a_r	$be_r=a_r+2q$	$b_r=a_r+2q$	a_r	a_r	a_r	
	b_r	$bo_r=b_r+2q$	$b'_r=b_r+2q$	b_r	b_r	b_r	
Periodic Solutions, of **20.1.1**:							
Even	$ce_r(z,q)$	$A^r Se_r(s,x)$ *	$A^r Se_r^{(1)}(c,\cos x)$ *	$ce_r(z,h^2)$ *	$ce_r(z,q)$	$ce_r(z,\theta)$	See Note 1.
Odd	$se_r(z,q)$	$B^r So_r(s,x)$ *	$A^r So_r^{(1)}(c,\cos x)$ *	$se_r(z,h^2)$ *	$se_r(z,q)$	$se_r(z,\theta)$	
Coefficients in Periodic Solutions:							
Even	$A_m^r(q)$	$A^r De_m^r(s)$ *	$A^r D_m^r$ *	A_m^r	A_m^r	A_m^r	
Odd	$B_m^r(q)$	$B^r Do_m^r(s)$ *	$B^r F_m^r$ *	B_m^r	B_m^r	B_m^r	
$\frac{1}{\pi}\int_0^{2\pi} y^2\,dx$, y is the Standard Solution of **20.1.1.**	1	$(A^r)^{-2}$ or $(B^r)^{-2}$	$(A^r)^{-2}$ or $(B^r)^{-2}$	1	1	1	See Note 1.
Floquet's Solutions **20.3.8**	$F_\nu(z)$			$me_\nu(z,h^2)$	$\phi(z)$		
Characteristic Exponent	ν	$\mu=i\nu$		ν	$\mu=i\nu$	$\mu=i\nu$	
Normalizations of Floquet's Solutions.	Unspecified			$\frac{1}{\pi}\int_0^{\pi}(me_\nu(z,h^2)me_{-\nu}(z,h^2)=1$			
Solutions of Modified Equation **20.1.2.**	$Ce_r(z,q)$	$Ag_{e,r}(s)Je_r(s,q)$	$Ag_{e,r}(s)Je_r(c,\cosh x)$	$Ce_r(z,q)$	$Ce_r(z,q)$	$Ce_r(z,\theta)$	
	$Se_r(z,q)$	$Bg_{o,r}(s)Jo_r(s,q)$	$Bg_{e,r}(s)Jo_r(c,\cosh x)$	$Se_r(z,q)$	$Se_r(z,q)$	$Se_r(z,\theta)$	
	$Mc_r^{(1)}(z,q)$	$\sqrt{\frac{2}{\pi}}\,Je_r(s,z)$	$\sqrt{\frac{2}{\pi}}\,Je_r(c,\cosh z)$	$Mc_r^{(1)}(z,h)$	$\sqrt{\frac{2}{\pi}}\,Ce_r(z,q)/Ag_{e,r}(q)$	$\sqrt{\frac{2}{\pi}}\,Ce_r(z,\theta)/Ag_{e,r}(q)$	
	$Ms_r^{(1)}(z,q)$	$\sqrt{\frac{2}{\pi}}\,Jo_r(s,z)$	$\sqrt{\frac{2}{\pi}}\,Jo_r(c,\cosh z)$	$Ms_r^{(1)}(z,h)$	$\sqrt{\frac{2}{\pi}}\,Se_r(z,q)/Bg_{o,r}(q)$	$\sqrt{\frac{2}{\pi}}\,Se_r(z,\theta)/Bg_{o,r}(q)$	
	$Mc_r^{(2)}(z,q)$	$\sqrt{\frac{2}{\pi}}\,Ne_r(s,z)$	$\sqrt{\frac{2}{\pi}}\,Ne_r(c,\cosh z)$	$Mc_r^{(2)}(z,h)$	$\sqrt{\frac{2}{\pi}}\,Fey_r(z,q)/Ag_{e,r}(q)$	$\sqrt{\frac{2}{\pi}}\,Fey_r(z,\theta)/Ag_{e,r}(q)$	
	$Ms_r^{(2)}(z,q)$	$\sqrt{\frac{2}{\pi}}\,No_r(s,z)$	$\sqrt{\frac{2}{\pi}}\,No_r(c,\cosh z)$	$Ms_r^{(2)}(z,h)$	$\sqrt{\frac{2}{\pi}}\,Gey_r(z,q)/Bg_{o,r}(q)$	$\sqrt{\frac{2}{\pi}}\,Gey_r(z,\theta)/Bg_{o,r}(q)$	
Joining Factors	$\sqrt{2/\pi}/Mc_r^{(1)}(0,q)$	$g_{e,r}(s)$	$\sqrt{2\pi}\,\lambda_r^{(e)}$	$\sqrt{2/\pi}/Mc_r^{(1)}(0,h)$	$(-1)^r p_r\sqrt{\frac{2}{\pi}}/A$	Same as [20.30]	See Note 2.
	$\sqrt{2/\pi}/\frac{d}{dz}[Ms_r^{(1)}(z,q)]_{z=0}$	$g_{o,r}(s)$	$\sqrt{2\pi}\,\lambda_r^{(0)}$	$\sqrt{2/\pi}/\frac{d}{dz}[Ms_r^{(1)}(z,h)]_{z=0}$	$(-1)^r s_r\sqrt{\frac{2}{\pi}}/B$		
	$-Mc_r^{(2)}(0,q)/Mc_r^{(1)}(0,q)$	$f_{e,r}(s)$	$-\frac{2}{\pi}\frac{K_1'}{K_1}$	$-Mc_r^{(2)}(0,h)/Mc_r^{(1)}(0,h)$	$\frac{-Fey_r(0,q)}{Ce_r(0,q)}$	Same as [20.30]	See Note 3.
	$\left[\frac{\frac{d}{dz}Ms_r^{(2)}(z,q)}{\frac{d}{dz}Ms_r^{(1)}(z,q)}\right]_{z=0}$	$f_{o,r}(s)$	$\frac{2}{\pi}\frac{K_3'}{K_3}$	Same as this volume	$\left[\frac{\frac{d}{dz}Gey_r(z,q)}{\frac{d}{dz}Se_r(z,q)}\right]_{z=0}$	Same as [20.30]	

NOTE: 1. The conversion factors A^r and B^r are tabulated in [20.58] along with the coefficients.
2. The multipliers p_r and s_r are defined in [20.30], Appendix **1**, section **3**, equations **3**, **4**, **5**, **6**.
3. See [20.59], sections **(5.3)** and **(5.5)**. In eq. **(316)** of **(5.5)**, the first term should have a minus sign.

*See page II.

References

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See also [20.18]. It contains, among other tabulations, values of a_r, b_r and coefficients for $ce_r(x,q)$, $se_r(x,q)$, $q=40(20)100(50)200$; 5D, $r\leq 2$.

*See page II.

Table 20.1 CHARACTERISTIC VALUES, JOINING FACTORS, SOME CRITICAL VALUES

EVEN SOLUTIONS

r	q	a_r	$ce_r(0, q)$	$ce_r(\frac{1}{2}\pi, q)$	$(4q)^{\frac{1}{2}r}g_{e,r}(q)$	$(4q)^r f_{e,r}(q)$
0	0	0.00000 000	(-1)7.07106 781	(-1) 7.07106 78	(-1)7.97884 56	∞
	5	5.80004 602	(-2)4.48001 817	1.33484 87	1.97009 00	(- 3)1.86132 97
	10	- 13.93697 996	(-3)7.62651 757	1.46866 05	2.40237 95	(- 5)5.54257 96
	15	- 22.51303 776	(-3)1.93250 832	1.55010 82	2.68433 53	(- 6)3.59660 89
	20	- 31.31339 007	(-4)6.03743 829	1.60989 09	2.90011 25	(- 7)3.53093 01
	25	- 40.25677 955	(-4)2.15863 018	1.65751 03	3.07743 91	(- 8)4.53098 68
2	0	4.00000 000	1.00000 000	-1.00000 00	(1)1.27661 53	(1)8.14873 31
	5	7.44910 974	(-1)7.35294 308	(-1)-7.24488 15	(1)2.63509 89	(2)1.68665 79
	10	7.71736 985	(-1)2.45888 349	(-1)-9.26759 26	(1)7.22275 58	(1)6.89192 56
	15	5.07798 320	(-2)7.87928 278	-1.01996 62	(2)1.32067 71	(1)1.73770 48
	20	+ 1.15428 288	(-2)2.86489 431	-1.07529 32	(2)1.98201 14	4.29953 32
	25	- 3.52216 473	(-2)1.15128 663	-1.11627 90	(2)2.69191 26	1.11858 69
10	0	100.00000 000	1.00000 000	-1.00000 00	(12)1.51800 43	(23)2.30433 72
	5	100.12636 922	1.02599 503	(-1)-9.75347 49	(12)1.48332 54	(23)2.31909 77
	10	100.50677 002	1.05381 599	(-1)-9.51645 32	(12)1.45530 39	(23)2.36418 54
	15	101.14520 345	1.08410 631	(-1)-9.28548 06	(12)1.43299 34	(23)2.44213 04
	20	102.04891 602	1.11778 863	(-1)-9.05710 78	(12)1.41537 24	(23)2.55760 55
	25	103.23020 480	1.15623 992	(-1)-8.82691 92	(12)1.40118 52	(23)2.71854 15

r	q	a_r	$ce_r(0, q)$	$ce'_r(\frac{1}{2}\pi, q)$	$(4q)^{\frac{1}{2}r}g_{e,r}(q)$	$(4q)^r f_{e,r}(q)$
1	0	1.00000 000	1.00000 000	-1.00000 00	1.59576 91	2.54647 91
	5	+ 1.85818 754	(-1)2.56542 879	-3.46904 21	7.26039 84	1.02263 46
	10	- 2.39914 240	(-2)5.35987 478	-4.85043 83	(1)1.35943 49	(- 2)9.72660 12
	15	- 8.10110 513	(-2)1.50400 665	-5.76420 64	(1)1.91348 51	(- 2)1.19739 95
	20	- 14.49130 142	(-3)5.05181 376	-6.49056 58	(1)2.42144 01	(- 3)1.84066 20
	25	- 21.31489 969	(-3)1.91105 151	-7.10674 15	(1)2.89856 94	(- 4)3.33747 55
5	0	25.00000 000	1.00000 000	-5.00000 00	(4)4.90220 27	(8)4.80631 83
	5	25.54997 175	1.12480 725	-5.39248 61	(4)4.43075 22	(8)5.11270 71
	10	27.70376 873	1.25801 994	-5.32127 65	(4)4.19827 66	(8)6.83327 77
	15	31.95782 125	1.19343 223	-5.11914 99	(4)5.25017 04	(9)1.18373 72
	20	36.64498 973	(-1)9.36575 531	-5.77867 52	(4)8.96243 97	(9)1.85341 57
	25	40.05019 099	(-1)6.10694 310	-7.05988 45	(5)1.71582 55	(9)2.09679 12
15	0	225.00000 000	1.00000 000	(1) 1.50000 00	(20)5.60156 72	(40)2.09183 70
	5	225.05581 248	1.01129 373	(1) 1.51636 57	(20)5.54349 84	(40)2.09575 00
	10	225.22335 698	1.02287 828	(1) 1.53198 84	(20)5.49405 67	(40)2.10754 45
	15	225.50295 624	1.03479 365	(1) 1.54687 43	(20)5.45287 72	(40)2.12738 84
	20	225.89515 341	1.04708 434	(1) 1.56102 79	(20)5.41964 26	(40)2.15556 69
	25	226.40072 004	1.05980 044	(1) 1.57444 72	(20)5.39407 68	(40)2.19249 18

Compiled from National Bureau of Standards, Tables relating to Mathieu functions, Columbia Univ. Press, New York, N.Y., 1951 (with permission).

$$a_r + 2q - (4r+2)\sqrt{q}$$

$q^{-\frac{1}{2}} \backslash r$	0	1	2	5	10	15	$\langle q \rangle$
0.16	-0.25532 994	-1.30027 212	-3.45639 483	-17.84809 551	-76.04295 314	- 80.93485 048	39
0.12	-0.25393 098	-1.28658 972	-3.39777 782	-16.92019 225	-76.84607 855	-141.64507 841	69
0.08	-0.25257 851	-1.27371 191	-3.34441 938	-16.25305 645	-63.58155 264	-162.30500 052	156
0.04	-0.25126 918	-1.26154 161	-3.29538 745	-15.70968 373	-58.63500 546	-132.08298 271	625
0.00	-0.25000 000	-1.25000 000	-3.25000 000	-15.25000 000	-55.25000 000	-120.25000 000	∞

For $g_{e,r}$ and $f_{e,r}$ see **20.8.12.**

$\langle q \rangle$ = nearest integer to q.

Compiled from G. Blanch and I. Rhodes, Table of characteristic values of Mathieu's equation for large values of the parameter, Jour. Wash. Acad. Sci., **45**, 6, 1955 (with permission).

CHARACTERISTIC VALUES, JOINING FACTORS, SOME CRITICAL VALUES — Table 20.1

ODD SOLUTIONS

r	q	b_r	$se_r'(0, q)$	$se_r'(\frac{1}{2}\pi, q)$	$(4q)^{\frac{1}{2}r}g_{o,r}(q)$	$(4q)^r f_{o,r}(q)$
2	0	4.00000 000	2.00000 00	-2.00000 00	6.38307 65	(1)8.14873 31
	5	+ 2.09946 045	(-1)7.33166 22	-3.64051 79	(1)1.24474 88	(1)2.24948 08
	10	- 2.38215 824	(-1)2.48822 84	-4.86342 21	(1)1.86133 36	3.91049 85
	15	- 8.09934 680	(-2)9.18197 14	-5.76557 38	(1)2.42888 57	(- 1)7.18762 28
	20	- 14.49106 325	(-2)3.70277 78	-6.49075 22	(1)2.95502 89	(- 1)1.47260 95
	25	- 21.31486 062	(-2)1.60562 17	-7.10677 19	(1)3.44997 83	(- 2)3.33750 27
10	0	100.00000 000	(1)1.00000 00	(1)-1.00000 00	(11)1.51800 43	(23)2.30433 72
	5	100.12636 922	9.73417 32	(1)-1.02396 46	(11)1.56344 50	(23)2.31909 77
	10	100.50676 946	9.44040 54	(1)-1.04539 48	(11)1.62453 03	(23)2.36418 52
	15	101.14517 229	9.11575 13	(1)-1.06429 00	(11)1.70421 18	(23)2.44211 78
	20	102.04839 286	8.75554 51	(1)-1.08057 24	(11)1.80695 19	(23)2.55740 30
	25	103.22568 004	8.35267 84	(1)-1.09413 54	(11)1.93959 86	(23)2.71681 11

r	q	b_r	$se_r'(0, q)$	$se_r(\frac{1}{2}\pi, q)$	$(4q)^{\frac{1}{2}r}g_{o,r}(q)$	$(4q)^r f_{o,r}(q)$
1	0	+ 1.00000 000	1.00000 00	1.00000 00	1.59576 91	2.54647 91
	5	- 5.79008 060	(-1)1.74675 40	1.33743 39	2.27041 76	(- 2)3.74062 82
	10	- 13.93655 248	(-2)4.40225 66	1.46875 57	2.63262 99	(- 3)2.21737 88
	15	- 22.51300 350	(-2)1.39251 35	1.55011 51	2.88561 87	(- 4)2.15798 83
	20	- 31.31338 617	(-3)5.07788 49	1.60989 16	3.08411 21	(- 4)2.82474 71
	25	- 40.25677 898	(-3)2.04435 94	1.65751 04	3.24945 50	(- 6)4.53098 74
5	0	25.00000 000	5.00000 00	1.00000 00	(3)9.80440 55	(8)4.80631 83
	5	25.51081 605	4.33957 00	(-1) 9.06077 93	(4)1.14793 21	(8)5.05257 20
	10	26.76642 636	3.40722 68	(-1) 8.46038 43	(4)1.52179 77	(8)5.46799 57
	15	27.96788 060	2.41166 65	(-1) 8.37949 34	(4)2.20680 20	(8)5.27524 17
	20	28.46822 133	1.56889 69	(-1) 8.63543 12	(4)3.27551 12	(8)4.26215 66
	25	28.06276 590	(-1)9.64071 62	(-1) 8.99268 33	(4)4.76476 62	(8)2.94147 89
15	0	225.00000 000	(1)1.50000 00	-1.00000 00	(19)3.73437 81	(40)2.09183 70
	5	225.05581 248	(1)1.48287 89	(-1)-9.88960 70	(19)3.78055 49	(40)2.09575 00
	10	225.22335 698	(1)1.46498 60	(-1)-9.78142 35	(19)3.83604 43	(40)2.10754 45
	15	225.50295 624	(1)1.44630 01	(-1)-9.67513 70	(19)3.90140 52	(40)2.12738 84
	20	225.89515 341	(1)1.42679 46	(-1)-9.57045 25	(19)3.97732 29	(40)2.15556 69
	25	226.40072 004	(1)1.40643 73	(-1)-9.46708 70	(19)4.06462 83	(40)2.19249 18

$$b_r+2q-(4r-2)\sqrt{q}$$

$q^{-\frac{1}{2}}\backslash r$	1	2	5	10	15	$\langle q\rangle$
0.16	-0.25532 994	-1.30027 164	-11.53046 855	-51.32546 875	- 55.93485 112	39
0.12	-0.25393 098	-1.28658 971	-11.12574 983	-56.10964 961	-108.31442 060	69
0.08	-0.25257 851	-1.27371 191	-10.78895 146	-51.15347 975	-132.59692 424	156
0.04	-0.25126 918	-1.26154 161	-10.50135 748	-47.72149 533	-114.76358 461	625
0.00	-0.25000 000	-1.25000 000	-10.25000 000	-45.25000 000	-105.25000 000	∞

For $g_{o,r}$ and $f_{o,r}$ see **20.8.12.**

$\langle q\rangle$ = nearest integer to q.

Table 20.2

COEFFICIENTS A_m AND B_m

A_m

$q=5$

$m\backslash r$	0	2	10
0	+0.54061 2446	+0.43873 7166	+0.00000 1679
2	-0.62711 5414	+0.65364 0260	+0.00003 3619
4	+0.14792 7090	-0.42657 8935	+0.00064 2987
6	-0.01784 8061	+0.07588 5673	+0.01078 4807
8	+0.00128 2863	-0.00674 1769	+0.13767 5121
10	-0.00006 0723	+0.00036 4942	+0.98395 5640
12	+0.00000 2028	-0.00001 3376	-0.11280 6780
14	-0.00000 0050	+0.00000 0355	+0.00589 2962
16	+0.00000 0001	-0.00000 0007	-0.00018 9166
18			+0.00000 4226
20			-0.00000 0071
22			+0.00000 0001

$m\backslash r$	1	5	15
1	+0.76246 3686	+0.07768 5798	0.00000 0000
3	-0.63159 6319	+0.30375 1030	+0.00000 0002
5	+0.13968 4806	+0.92772 8396	+0.00000 0106
7	-0.01491 5596	-0.20170 6148	+0.00000 4227
9	+0.00094 4842	+0.01827 4579	+0.00014 8749
11	-0.00003 9702	-0.00095 9038	+0.00428 1393
13	+0.00000 1189	+0.00003 3457	+0.08895 2014
15	-0.00000 0027	-0.00000 0839	+0.99297 4092
17	+0.00000 0001	+0.00000 0016	-0.07786 7946
19			+0.00286 6409
21			-0.00006 6394
23			+0.00000 1092
25			-0.00000 0014

$q=25$

$m\backslash r$	0	2	10
0	+0.42974 1038	+0.33086 5777	+0.00502 6361
2	-0.69199 9610	-0.04661 4551	+0.02075 4891
4	+0.36554 4890	-0.64770 5862	+0.07232 7761
6	-0.13057 5523	+0.55239 9372	+0.23161 1726
8	+0.03274 5863	-0.22557 4897	+0.55052 4391
10	-0.00598 3606	+0.05685 2843	+0.63227 5658
12	+0.00082 3792	-0.00984 6277	-0.46882 9197
14	-0.00008 7961	+0.00124 8919	+0.13228 7155
16	+0.00000 7466	-0.00012 1205	-0.02206 0893
18	-0.00000 0514	+0.00000 9296	+0.00252 2374
20	+0.00000 0029	-0.00000 0578	-0.00021 3672
22	-0.00000 0001	+0.00000 0030	+0.00001 4078
24		-0.00000 0001	-0.00000 0746
26			+0.00000 0032
28			-0.00000 0001

$m\backslash r$	1	5	15
1	+0.39125 2265	+0.65659 0398	+0.00000 4658
3	-0.74048 2467	+0.36900 8820	+0.00003 7337
5	+0.50665 3803	-0.19827 8625	+0.00032 0026
7	-0.19814 2336	-0.48837 4067	+0.00254 0806
9	+0.05064 0536	+0.37311 2810	+0.01770 9603
11	-0.00910 8920	-0.12278 1866	+0.10045 8755
13	+0.00121 2864	+0.02445 3933	+0.40582 7402
15	-0.00012 4121	-0.00335 1335	+0.83133 2650
17	+0.00001 0053	+0.00033 9214	-0.35924 8831
19	-0.00000 0660	-0.00002 6552	+0.06821 6074
21	+0.00000 0036	+0.00000 1661	-0.00802 4550
23	-0.00000 0002	-0.00000 0085	+0.00066 6432
25		+0.00000 0004	-0.00004 1930
27			+0.00000 2090
29			-0.00000 0085
31			+0.00000 0003

B_m

$q=5$

$m\backslash r$	2	10
2	+0.93342 9442	+0.00003 3444
4	-0.35480 3915	+0.00064 2976
6	+0.05296 3730	+0.01078 4807
8	-0.00429 5885	+0.13767 5120
10	+0.00021 9797	+0.98395 5640
12	-0.00000 7752	-0.11280 6780
14	+0.00000 0200	+0.00589 2962
16	-0.00000 0004	-0.00018 9166
18		+0.00000 4227
20		-0.00000 0070
22		+0.00000 0001

$m\backslash r$	1	5	15
1	+0.94001 9024	+0.05038 2462	0.00000 0000
3	-0.33654 1963	+0.29736 5513	+0.00000 0002
5	+0.05547 7529	+0.93156 6997	+0.00000 0106
7	-0.00508 9553	-0.20219 3638	+0.00000 4227
9	+0.00029 3879	+0.01830 5721	+0.00014 8749
11	-0.00001 1602	-0.00096 0277	+0.00428 1392
13	+0.00000 0332	+0.00003 3493	+0.08895 2014
15	-0.00000 0007	-0.00000 0842	+0.99297 4092
17		+0.00000 0017	-0.07786 7946
19			+0.00286 6409
21			-0.00006 6394
23			+0.00000 1093
25			-0.00000 0013

$q=25$

$m\backslash r$	2	10
2	+0.65743 9912	+0.01800 3596
4	-0.66571 9990	+0.07145 6762
6	+0.33621 0033	+0.23131 0990
8	-0.10507 3258	+0.55054 4783
10	+0.02236 2380	+0.63250 8750
12	-0.00344 2304	-0.46893 3949
14	+0.00040 0182	+0.13230 9765
16	-0.00003 6315	-0.02206 3990
18	+0.00000 2640	+0.00252 2676
20	-0.00000 0157	-0.00021 3694
22	+0.00000 0008	+0.00001 4079
24		-0.00000 0746
26		+0.00000 0033

$m\backslash r$	1	5	15
1	+0.81398 3846	+0.30117 4196	+0.00000 3717
3	-0.52931 0219	+0.62719 8468	+0.00003 7227
5	+0.22890 0813	+0.17707 1306	+0.00032 0013
7	-0.06818 2972	-0.60550 5349	+0.00254 0804
9	+0.01453 0886	+0.33003 2984	+0.01770 9603
11	-0.00229 5765	-0.09333 5984	+0.10045 8755
13	+0.00027 7422	+0.01694 2545	+0.40582 7403
15	-0.00002 6336	-0.00217 7430	+0.83133 2650
17	+0.00000 2009	+0.00021 0135	-0.35924 8830
19	-0.00000 0126	-0.00001 5851	+0.06821 6074
21	+0.00000 0007	+0.00000 0962	-0.00802 4551
23		-0.00000 0048	+0.00066 6432
25		+0.00000 0002	-0.00004 1930
27			+0.00000 2090
29			-0.00000 0086
31			+0.00000 0003

For A_m and B_m see **20.2.3–20.2.11**

Compiled from National Bureau of Standards, Tables relating to Mathieu functions, Columbia Univ. Press, New York, N.Y., 1951 (with permission).