Linear Quadratic Regulator

Contents

1 Introduction to the linear quadratic regulator problem

1.1 The general LQR problem

The linear quadratic regulator problem (LQR) is a powerful design method and the precursor of several control design procedures for linear multiple-input multiple-output (MIMO) systems, such as linear quadratic Gaussian (LQG), H_2, H_∞. The optimal controller ensures a stable closed-loop system, achieves guaranteed levels of stability robustness and is simple to compute (Levine, 1995). The optimal control problems is stated as follows.

Given the system dynamics:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \tag{1.1}$$

with $\mathbf{x}(t) = [x_1(t)\ x_2(t) \ldots\ x_n(t)]^T$ and $\mathbf{u}(t) = [u_1(t)\ u_2(t) \ldots\ u_m(t)]^T$, determine the optimal control law $\mathbf{u}^*(t)$ that minimizes the performance index:

$$J = \mathbf{x}^T(t_f)\mathbf{S}\mathbf{x}(t_f) + \int_{t_0}^{t_f} \left(\mathbf{x}^T(t)\mathbf{Q}\mathbf{x}(t) + \mathbf{u}^T(t)\mathbf{R}\mathbf{u}(t)\right) dt \tag{1.2}$$

with $\mathbf{S}$, $\mathbf{Q}$ positive semi-definite $n \times n$ and $\mathbf{R}$ positive definite $m \times m$ symmetric matrices.

The following assumptions hold:

1. The entire state vector is available for feedback / The system is observable.

2. The system is controllable (or $[\mathbf{A},\ \mathbf{B}]$ is stabilizable).

Obs1. *A system is controllable if the controllability matrix, defined as* $Co(A, B) = [B\ AB\ A^2B\ \ldots\ A^{n-1}B]$ *has full row-rank.*

Obs2. *A system is stabilizable if the uncontrollable states have stable dynamics.*

The state feedback configuration for the LQR problem is shown in Figure 1.1.

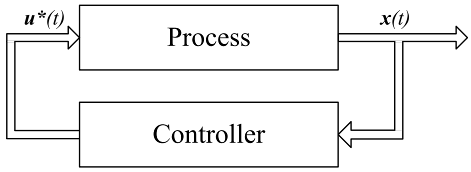

Figure 1.1: LQR state feedback configuration

Note the absence of a reference signal. The general objective is to make the measured states as close to zero as possible. It is called a *state regulator system.* In other words, the objective is to obtain a control law $\mathbf{u}^*(t)$ which takes the linear plant described by the state equations (1.1) from a nonzero state to zero state. This situation may arise when the plant is unstable, or it is subjected to disturbances.

The performance measure J is quadratic in states and inputs, with matrices $\mathbf{Q}$ and $\mathbf{R}$ positive semidefinite and positive definite, respectively. If, for example, the state vector and the matrix $\mathbf{Q}$ are:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \mathbf{Q} = \begin{bmatrix} q_{11} & 0 \\ 0 & q_{22} \end{bmatrix}, \quad q_{11}, q_{22} \geq 0$$

the quadratic form $\mathbf{x}^T\mathbf{Q}\mathbf{x}$ is:

$$\mathbf{x}^T\mathbf{Q}\mathbf{x} = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} q_{11} & 0 \\ 0 & q_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = q_{11}x_1^2 + q_{22}x_2^2$$

If $\mathbf{Q}$ is symmetric and positive semidefinite in the form:

$$\mathbf{Q} = \begin{bmatrix} q_{11} & q_{12} \\ q_{12} & q_{22} \end{bmatrix}, \quad q_{11}, q_{22}, q_{12} \geq 0$$

then the quadratic form is:

$$\mathbf{x}^T\mathbf{Q}\mathbf{x} = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} q_{11} & q_{12} \\ q_{12} & q_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = q_{11}x_1^2 + 2q_{12}x_1x_2 + q_{22}x_2^2$$

1.2 Choosing LQR weights

The choice of LQR weights $\mathbf{Q}$ and $\mathbf{R}$ is usually a trial-and-error process until a desired response is obtained. However, there are a few methods that are usually a staring point for the iterative design procedure aimed at obtaining desirable properties of the closed-loop system.

1. Choose $\mathbf{Q} = \mathbf{I}$, $\mathbf{R} = \rho\mathbf{I}$, (Murray, 2006). The terms under integral in relation (1.2) correspond to the energy of the controlled states and the control signal, respectively. Decreasing the energy of the controlled states will require a large control signal and a small control signal will lead to large controlled states. The role of the constant ρ is to establish a trade-of between these conflicting goals, (Hespanha, 2006):

   - If ρ is large, J may be decreased using a small control signal, at the expense of large controlled states.
   - If ρ is small, J decreases using a large control signal and small controlled states are obtained.

2. Choose $\mathbf{Q}$ and $\mathbf{R}$ as diagonal matrices with the elements according to Bryson's rule, (Hespanha, 2006):

$$q_{ii} = \frac{1}{\text{maximum acceptable value of } x_i^2}, \quad i = \overline{1, n}$$

$$r_{jj} = \frac{1}{\text{maximum acceptable value of } u_i^2}, \quad j = \overline{1, m}$$

   The Bryson's rule, mainly scales the variables that appear in J so that the maximum acceptable value for each term is one.

2

Discrete Linear Quadratic Regulator Problem

We will now apply the principle of optimality to find the optimal control of a linear state feedback from the discrete linear quadratic regulator problem.

The problem is stated as follows:

Given a discrete linear plant model:

$$\mathbf{x}_{k+1} = \mathbf{A}\mathbf{x}_k + \mathbf{B}\mathbf{u}_k, \quad k = 0, 1 \ldots N-1$$

with the specified initial condition $\mathbf{x}_0$. We wish to calculate the optimal control sequence $\mathbf{u}_0^*$, $\mathbf{u}_1^*$, $\ldots \mathbf{u}_{N-1}^*$ that minimizes the quadratic performance measure:

$$J = \frac{1}{2}\mathbf{x}_N^T \mathbf{H}_N \mathbf{x}_N + \frac{1}{2}\sum_{k=0}^{N-1}\left[\mathbf{x}_k^T \mathbf{Q}\mathbf{x}_k + \mathbf{u}_k^T \mathbf{R}\mathbf{u}_k\right]$$

where

- $\mathbf{H}_N$,$\mathbf{Q}$ are the real symmetric positive semi-definite $n \times n$ matrices
- $\mathbf{R}$ is a real symmetric positive definite $m \times m$ matrix

We assume that the components of the control vector are unconstrained.

To solve this problem we use dynamic programming.

Let

$$J_{NN}^*\left(\mathbf{x}_N\right) = \frac{1}{2}\mathbf{x}_N^T \mathbf{H}_N \mathbf{x}_N$$

The above is the penalty for being in a state $\mathbf{x}_N$ at a time N.

We now decrement k to N-1 to get

$$J_{N-1,N}\left(\mathbf{x}_{N-1}, \mathbf{u}_{N-1}\right) = \frac{1}{2}\mathbf{x}_N^T \mathbf{H}_N \mathbf{x}_N + \frac{1}{2}\mathbf{x}_{N-1}^T \mathbf{Q}\mathbf{x}_{N-1} + \frac{1}{2}\mathbf{u}_{N-1}^T \mathbf{R}\mathbf{u}_{N-1}$$

We use the state equation to eliminate $\mathbf{x}_N$ from $J_{N-1,N}$ to obtain

$$\begin{aligned} J_{N-1,N}\left(\mathbf{x}_{N-1},\mathbf{u}_{N-1}\right) &= \frac{1}{2}\left(\mathbf{A}\mathbf{x}_{N-1}+\mathbf{B}\mathbf{u}_{N-1}\right)^T\mathbf{H}_N\left(\mathbf{A}\mathbf{x}_{N-1}+\mathbf{B}\mathbf{u}_{N-1}\right)+ \\ &+ \frac{1}{2}\mathbf{x}_{N-1}^T\mathbf{Q}\mathbf{x}_{N-1}+\frac{1}{2}\mathbf{u}_{N-1}^T\mathbf{R}\mathbf{u}_{N-1} \end{aligned}$$

Because there are no constraints on control we apply the first-order necessary condition from the static optimization to find $\mathbf{u}_{N-1}^*$ as a function of $\mathbf{x}_{N-1}$.

$$\frac{\partial J_{N-1,N}}{\partial \mathbf{u}_{N-1}} = \left(\mathbf{A}\mathbf{x}_{N-1}+\mathbf{B}\mathbf{u}_{N-1}\right)^T\mathbf{H}_N\mathbf{B}+\mathbf{u}_{N-1}^T\mathbf{R}=0$$

Solving for control satisfying the first-order necessary condition we obtain:

$$\mathbf{u}_{N-1}^* = -\left(\mathbf{R}+\mathbf{B}^T\mathbf{H}_N\mathbf{B}\right)^{-1}\mathbf{B}^T\mathbf{H}_N\mathbf{A}\mathbf{x}_{N-1}$$

Let

$$\mathbf{K}_{N-1} = \left(\mathbf{R}+\mathbf{B}^T\mathbf{H}_N\mathbf{B}\right)^{-1}\mathbf{B}^T\mathbf{H}_N\mathbf{A}$$

then

$$\mathbf{u}_{N-1}^* = -\mathbf{K}_{N-1}\mathbf{x}_{n-1}$$

Computing the optimal cost transferring the system from N-1 to N, yields:

$$\begin{aligned} J_{N-1,N}^*\left(\mathbf{x}_{N-1}\right) &= \frac{1}{2}\mathbf{x}_{N-1}^T\left(\mathbf{A}-\mathbf{B}\mathbf{K}_{N-1}\right)^T\mathbf{H}_N\left(\mathbf{A}-\mathbf{B}\mathbf{K}_{N-1}\right)\mathbf{x}_{N-1}+ \\ &+ \frac{1}{2}\mathbf{x}_{N-1}^T\left(\mathbf{K}_{N-1}^T\mathbf{R}\mathbf{K}_{N-1}+\mathbf{Q}\right)\mathbf{x}_{N-1} \end{aligned}$$

Let

$$\mathbf{H}_{N-1} = \left(\mathbf{A}-\mathbf{B}\mathbf{K}_{N-1}\right)^T\mathbf{H}_N\left(\mathbf{A}-\mathbf{B}\mathbf{K}_{N-1}\right)+\mathbf{K}_{N-1}^T\mathbf{R}\mathbf{K}_{N-1}+\mathbf{Q}$$

Then

$$J_{N-1,N}^*\left(\mathbf{x}_N\right) = \frac{1}{2}\mathbf{x}_{N-1}^T\mathbf{H}_{N-1}\mathbf{x}_{N-1}$$

Decrementing k to N-2, yields

$$J_{N-2,N}\left(\mathbf{x}_{N-2},\mathbf{u}_{N-2}\right) = \frac{1}{2}\mathbf{x}_{N-1}^T\mathbf{H}_{N-1}\mathbf{x}_{N-1}+\frac{1}{2}\mathbf{x}_{N-2}^T\mathbf{Q}\mathbf{x}_{N-2}+\frac{1}{2}\mathbf{u}_{N-2}^T\mathbf{R}_{N-1}\mathbf{u}_{N-2}$$

Note that $J_{N-2,N}$ has the same form as $J_{N-1,N}$.

Thus we obtain analogous optimal feedback gain where N is replaced with N-1. Continuing in this fashion we get the following results for each k=N-1, N-2, ... 0

$$\begin{aligned} \mathbf{K}_k &= \left(\mathbf{R}+\mathbf{B}^T\mathbf{H}_{k+1}\mathbf{B}\right)^{-1}\mathbf{B}^T\mathbf{H}_{k+1}\mathbf{A} \\ \mathbf{u}_k^* &= -\mathbf{K}_k\cdot\mathbf{x}_k \\ \mathbf{H}_k &= \left(\mathbf{A}-\mathbf{B}\mathbf{K}_k\right)^T\mathbf{H}_{k+1}\left(\mathbf{A}-\mathbf{B}\mathbf{K}_k\right)+\mathbf{K}_k^T\mathbf{R}\mathbf{K}_k+\mathbf{Q} \end{aligned} \tag{2.1}$$

and

$$J^*_{k,N}(\mathbf{x}_k) = \frac{1}{2}\mathbf{x}_k^T \mathbf{H}_k \mathbf{x}_k$$

The above control scheme can be implemented by computing the sequence of gain matrices $\{\mathbf{K}_k\}$ offline and stored.

Then we can implement the controller $\mathbf{u}^*_k = -\mathbf{K}_k \cdot \mathbf{x}_k$.

First, and most important observe that *the optimal control at each stage is a linear combination of the states*; therefore the optimal policy is linear state-variable feedback. Notice that *the feedback is time-varying*, even if A, B, Q, R are *all* constant matrices - this means that the controller for the optimal policy can be implemented by the m time varying summers each with n inputs.

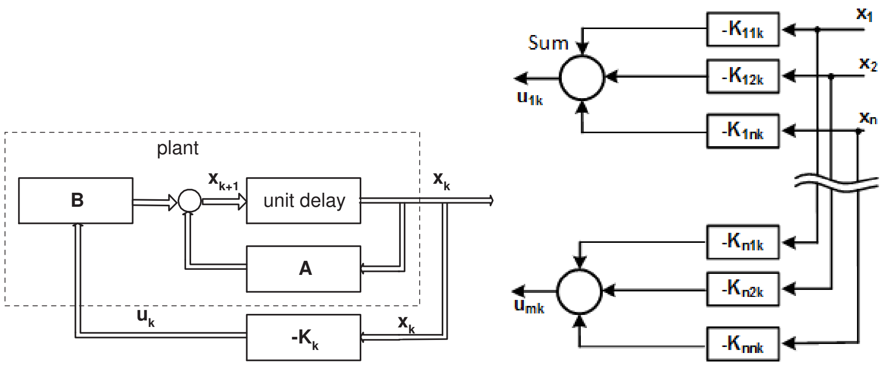

Figure 2.1: a) Left. Plant and linear time-varying feedback controller. b) Right. Controller configuration

Another important characteristic of the linear regulator problem is that if the system is completely controllable and time invariant, **H**=0, **R** and **Q** are constant matrices, then the optimal control law is time invariant for an infinite stage process; that is:

$\mathbf{K}_k \to \mathbf{K}$(a constant matrix) as $N \to \infty$

From a physical point of view this means that if a process is to be controlled for a large number of stages, the optimal control can be implemented by feedback of the states through a configuration of amplifiers-summers as shown in Figure 2.1.b, but with <u>fixed</u> gain factors.

One way to determine the constant matrix $\mathbf{K}$ is to solve the recurrence relations (1) for as many stages as required for $\mathbf{K}_k$ to converge to a constant matrix.

Specifically, let us consider a controllable system model:

$$\mathbf{x}_{k+1} = \mathbf{A}\mathbf{x}_k + \mathbf{B}\mathbf{u}_k; \quad k \geq 0; \quad \mathbf{x}_0$$

and the performance index:

$$J = \frac{1}{2}\sum_{k=0}^{\infty}\left\{\mathbf{x}_k^T\mathbf{Q}\mathbf{x} + \mathbf{u}_k^T\mathbf{R}\mathbf{u}_k\right\}$$

obtained from the previously considered performance index by letting $N \to \infty$ and setting $\mathbf{H}_N = 0$.

Then the optimal controller takes the form:

$$\mathbf{u}_k^* = -\mathbf{K}_\infty \mathbf{x}_k \text{ , k} \geq 0$$

where

$$\mathbf{K}_\infty = \left(\mathbf{R} + \mathbf{B}^T\mathbf{H}_\infty\mathbf{B}\right)^{-1}\mathbf{B}^T\mathbf{H}_\infty\mathbf{A} \tag{2.2}$$

and

$\mathbf{H}_\infty = \lim\limits_{k\to\infty} \mathbf{H}_k$ from recursion (2.1)

$$\mathbf{H}_\infty = \left(\mathbf{A} - \mathbf{B}\mathbf{K}_\infty\right)^T\mathbf{H}_\infty\left(\mathbf{A} - \mathbf{B}\mathbf{K}_\infty\right) + \mathbf{K}_\infty^T\mathbf{R}\mathbf{K}_\infty + \mathbf{Q} \tag{2.3}$$

where $\mathbf{K}_\infty$ is given in (2.2).

Substituting (2.2) in (2.3) yields

$$\begin{aligned}\mathbf{H}_\infty &= \left(\mathbf{A} - \mathbf{B}\left(\mathbf{R} + \mathbf{B}^T\mathbf{H}_\infty\mathbf{B}\right)^{-1}\mathbf{B}^T\mathbf{H}_\infty\mathbf{A}\right)^T\mathbf{H}_\infty\left(\mathbf{A} - \mathbf{B}\left(\mathbf{R} + \mathbf{B}^T\mathbf{H}_\infty\mathbf{B}\right)^{-1}\mathbf{B}^T\mathbf{H}_\infty\mathbf{A}\right) \\ &+ \left(\left(\mathbf{R} + \mathbf{B}^T\mathbf{H}_\infty\mathbf{B}\right)^{-1}\mathbf{B}^T\mathbf{H}_\infty\mathbf{A}\right)^T\mathbf{R}\left(\left(\mathbf{R} + \mathbf{B}^T\mathbf{H}_\infty\mathbf{B}\right)^{-1}\mathbf{B}^T\mathbf{H}_\infty\mathbf{A}\right) + \mathbf{Q}\end{aligned}$$

Thus we can compute $\mathbf{H}_\infty$ by solving the above algebraic equation. We note that the above formidable looking equation is just the discrete algebraic Riccati equation.

Example 2.1

Using the recursion (2.1) solve the following problem:

For a plant: $x_{k+1} = x_k + u_k$, k = 0,1, find the optimal control sequence u_k^*, $k = 0, 1$ that minimizes the performance measure:

$$J = x_2^2 + \sum_{k=0}^{1}\left(x_k^2 + 2u_k^2\right)$$

$\mathbf{A}$=1, $\mathbf{B}$=1, $\mathbf{H}_2 = \mathbf{H}_N = 1$, $\mathbf{Q}$=1, $\mathbf{R}$=1

$$\mathbf{H}_2 = 1$$

$\underline{\text{k=1}}$: $K_1 = (2 + 1\cdot 1\cdot 1)^{-1}\cdot 1\cdot 1\cdot 1 = \frac{1}{3}$

$$H_1 = \left(1 - 1\cdot\frac{1}{3}\right)\cdot 1\cdot\left(1-\frac{1}{3}\right) + \frac{1}{3}\cdot 2\cdot\frac{1}{3} + 1 = \frac{2}{3}\cdot\frac{2}{3} + \frac{2}{9} + 1 = \frac{15}{9} = \frac{5}{3}$$

$$u_1^* = -\frac{1}{3}x_1$$

$\underline{\text{k=0}}$: $K_0 = \left(2 + 1\cdot\frac{5}{3}\cdot 1\right)^{-1}\cdot 1\cdot\frac{5}{3}\cdot 1 = \left(2+\frac{5}{3}\right)^{-1}\cdot\frac{5}{3} = \left(\frac{11}{3}\right)^{-1}\cdot\frac{5}{3} = \frac{5}{11}$

$$u_0^* = -\frac{5}{11}x_0$$

2.1 The algorithm for computing LQR gains

Algorithm 1 Computing LQR gains

1: Set the system matrices $\mathbf{A}$, $\mathbf{B}$ and the LQR weights $\mathbf{H_N}$, $\mathbf{Q}$, $\mathbf{R}$
2: Set the number of time samples N.
3: **for** $k = N-1$ to 1 **do**
4: Compute the gain matrix K for time step k:

$$\mathbf{K}_k = \left(\mathbf{R} + \mathbf{B}^T\mathbf{H}_{k+1}\mathbf{B}\right)^{-1}\mathbf{B}^T\mathbf{H}_{k+1}\mathbf{A}$$

5: Compute the new matrix H_k:

$$\mathbf{H}_k = (\mathbf{A} - \mathbf{B}\mathbf{K}_k)^T\,\mathbf{H}_{k+1}\,(\mathbf{A} - \mathbf{B}\mathbf{K}_k) + \mathbf{K}_k^T\mathbf{R}\mathbf{K}_k + \mathbf{Q}$$

6: **end for**

Example 2.2 *Consider a sampled-data system described by the state equations:*

$$\mathbf{x}_{k+1} = \begin{bmatrix}1.01 & 0.01\\ 0.01 & 1.02\end{bmatrix}\mathbf{x}_k + \begin{bmatrix}0.01\\ 0\end{bmatrix}u_k,\quad k = \overline{1, N-1}$$

with the sampling time $dt = 0.01$. The problem is to find the optimal control sequence u_k, $k = \overline{1, N-1}$, that minimizes a quadratic cost function:

$$J = \mathbf{x}_N^T\mathbf{H}_N\mathbf{x}_N + \sum_{k=0}^{N-1}\left[\mathbf{x}_k^T\mathbf{Q}\mathbf{x}_k + \mathbf{u}_k^T\mathbf{R}\mathbf{u}_k\right]$$

where:

$$\mathbf{H}_N = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{Q} = \begin{bmatrix} 0.01 & 0 \\ 0 & 0.01 \end{bmatrix}, \quad R = 0.01$$

Since the state vector has two elements, the size of the gain matrix $\mathbf{K}$ *is* 1×2*:* $\mathbf{K} = \begin{bmatrix} k_1 & k_2 \end{bmatrix}$ *The algorithm 1 will compute the gains* k_1*,* k_2 *for each time sample between* $N-1$ *and* 1*. we shall obtain* $N-1$ *values for* k_1 *and for* k_2*, respectively. A plot of these values is given in Figure 2.2.*

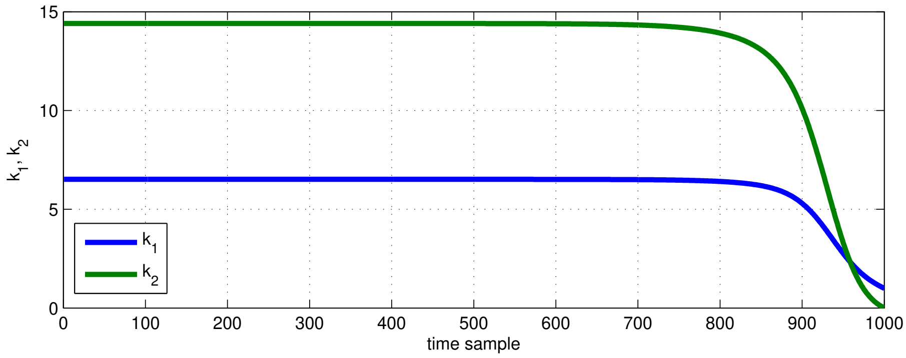

Figure 2.2: Variable gains

The gain matrix $\mathbf{K}_k$ *was computed starting from the final sample* $N-1$ *to* 1*. As illustrated in Figure 2.2, the elements of the gain matrix start from some initial values, but they converge to constant values as* $k \to 1$*.*

A simulation of the closed-loop system with the control law $u_k = -\mathbf{K}_k\mathbf{x}_k$*,* $k = \overline{1, N-1}$ *is shown in Figure 2.3. The states and control signal will converge at* 0*, as expected.*

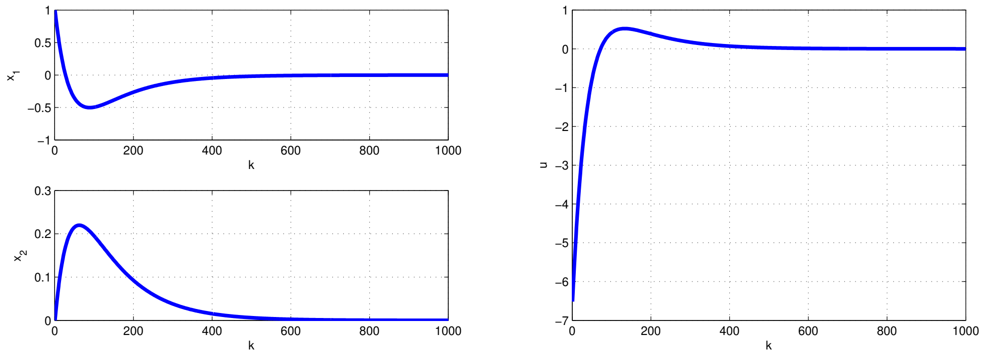

Figure 2.3: Optimal state trajectories (left) and optimal control (right)

3

Continuous LQR. Solution via the Hamilton-Jacobi-Bellman Equation

The problem is stated as follows:

Find an *admissible control* $\mathbf{u}^*(t)$ which causes the linear system:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \tag{3.1}$$

with n states and m control inputs to follow an *admissible trajectory* $\mathbf{x}^*(t)$ that *minimizes* the quadratic performance measure:

$$J = \mathbf{x}^T(t_f)\mathbf{S}\mathbf{x}(t_f) + \int_{t_0}^{t_f} \left(\mathbf{x}^T(t)\mathbf{Q}\mathbf{x}(t) + \mathbf{u}^T(t)\mathbf{R}\mathbf{u}(t)\right) dt \tag{3.2}$$

with $\mathbf{S}$, $\mathbf{Q}$ positive semi-definite $n\times n$ and $\mathbf{R}$ positive definite $m\times m$ symmetric matrices.

The problem is to solve the HJB equation:

$$0 = \frac{\partial J^*(\mathbf{x}(t), t)}{\partial t} + \min_{\mathbf{u}} H \tag{3.3}$$

subject to the boundary condition:

$$J^*(\mathbf{x}(t_f), t_f) = \mathbf{x}^T(t_f)\mathbf{S}\mathbf{x}(t_f) \tag{3.4}$$

Using the Hamilton-Jacobi-Bellman equation, we need to minimize the Hamiltonian H:

$$H = \mathbf{x}^T\mathbf{Q}\mathbf{x} + \mathbf{u}^T\mathbf{R}\mathbf{u} + J_x^{*T} \cdot (\mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}) \tag{3.5}$$

with respect to u. Minimize by setting the first derivative to zero:

$$\frac{\partial H}{\partial \mathbf{u}} = 2\mathbf{R}\mathbf{u} + \left(J_x^{*T} B\right)^T = 2\mathbf{R}\mathbf{u} + \mathbf{B}^T J_x^* = 0 \tag{3.6}$$

$$\mathbf{u}^* = -\frac{1}{2}\mathbf{R}^{-1}\mathbf{B}^T J_x^* \tag{3.7}$$

The optimal cost function is quadratic:

$$J^*(\mathbf{x}, t) = \mathbf{x}^T \mathbf{P}(t)\mathbf{x} \tag{3.8}$$

where $\mathbf{P}$ is symmetric. The first derivatives are:

$$J_t^* = \frac{\partial J^*(\mathbf{x}, t)}{\partial t} = \mathbf{x}^T \dot{\mathbf{P}}(t)\mathbf{x}, \qquad J_x^* = \frac{\partial J^*(\mathbf{x}, t)}{\partial \mathbf{x}} = 2\mathbf{P}(t)\mathbf{x} \tag{3.9}$$

By replacing (3.7) into (3.5) we obtain:

$$\begin{aligned} \min_{\mathbf{u}} H &= \mathbf{x}^T\mathbf{Q}\mathbf{x} + \left(-\frac{1}{2}\mathbf{R}^{-1}\mathbf{B}^T J_x^*\right)^T \mathbf{R}\left(-\frac{1}{2}\mathbf{R}^{-1}\mathbf{B}^T J_x^*\right) \\ &+ J_x^{*T} \cdot \left[\mathbf{A}\mathbf{x} + \mathbf{B}\left(-\frac{1}{2}\mathbf{R}^{-1}\mathbf{B}^T J_x^*\right)\right] \end{aligned} \tag{3.10}$$

We replace J_x^* into (3.10) and obtain:

$$\begin{aligned} \min_{\mathbf{u}} H &= \mathbf{x}^T\mathbf{Q}\mathbf{x} + \left(\mathbf{R}^{-1}\mathbf{B}^T\mathbf{P}\mathbf{x}\right)^T \mathbf{R}\left(\mathbf{R}^{-1}\mathbf{B}^T\mathbf{P}\mathbf{x}\right) \\ &+ 2(\mathbf{P}\mathbf{x})^T \cdot \left(\mathbf{A}\mathbf{x} - \mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{P}\mathbf{x}\right) \end{aligned} \tag{3.11}$$

Using the matrix property $(\mathbf{AB})^T = \mathbf{A}^T\mathbf{B}^T$, and because $\mathbf{R}\mathbf{R}^{-1} = \mathbf{I}$ the relation (3.11) becomes:

$$\begin{aligned} \min_{\mathbf{u}} H &= \mathbf{x}^T\mathbf{Q}\mathbf{x} + \mathbf{x}^T\mathbf{P}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{P}\mathbf{x} + 2\mathbf{x}^T\mathbf{P}\mathbf{A}\mathbf{x} \\ &- 2\mathbf{x}^T\mathbf{P}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{P}\mathbf{x} \\ &= \mathbf{x}^T\mathbf{Q}\mathbf{x} + 2\mathbf{x}^T\mathbf{P}\mathbf{A}\mathbf{x} - \mathbf{x}^T\mathbf{P}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{P}\mathbf{x} \end{aligned} \tag{3.12}$$

We also used the fact that the matrices $\mathbf{P}$ and $\mathbf{R}$ are symmetric ($\mathbf{P} = \mathbf{P}^T$, $\mathbf{R} = \mathbf{R}^T$).

With (3.9) and (3.12), the HJB equation (3.3) becomes:

$$0 = \mathbf{x}^T\dot{\mathbf{P}}(t)\mathbf{x} + \mathbf{x}^T\mathbf{Q}\mathbf{x} + 2\mathbf{x}^T\mathbf{P}\mathbf{A}\mathbf{x} - \mathbf{x}^T\mathbf{P}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{P}\mathbf{x} \tag{3.13}$$

or

$$0 = \mathbf{x}^T\left(\dot{\mathbf{P}} + \mathbf{Q} + 2\mathbf{P}\mathbf{A} - \mathbf{P}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{P}\right)\mathbf{x} \tag{3.14}$$

The term $2\mathbf{PA}$ in the relation (3.14) can be written as a sum of a symmetric part and an unsymmetric part, (Kirk, 2004; Lewis et al., 2012):

$$2\mathbf{PA} = \underbrace{\mathbf{PA} + (\mathbf{PA})^T}_{\textit{symmetric}} + \underbrace{\mathbf{PA} - (\mathbf{PA})^T}_{\textit{skew symmetric}} \tag{3.15}$$

Obs1. *A skew-symmetric matrix* $\mathbf{M}$ *is a square matrix whose negative equals its transpose:* $-\mathbf{M} = \mathbf{M}^T$*. This can be easily checked for the previous relation.*

Obs2. *For any skew-symmetric matrix* $\mathbf{M}$*,* $\mathbf{x}^T\mathbf{M}\mathbf{x} = 0$*. This can be easily verified in a two-by-two case.*

Obs3. *If* $f(\mathbf{x}) = \mathbf{x}^T\mathbf{N}\mathbf{x}$ *where* $\mathbf{N}$ *is symmetric, then* $f(\mathbf{x}) = 0$ *for all* $\mathbf{x}$ *if and only if* $\mathbf{N} = 0$*.*

Considering the observations above and the relation (3.15), we notice that only the symmetric part of $2\mathbf{PA}$ contributes to (3.14), and the HBJ equation becomes:

$$\dot{\mathbf{P}}(t) + \mathbf{Q} + \mathbf{A}^T P(t) + \mathbf{P}(t)\mathbf{A} - \mathbf{P}(t)\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{P}(t) = 0 \tag{3.16}$$

with the boundary condition:

$$\mathbf{x}^T(t_f)\mathbf{S}\mathbf{x}(t_f) = \mathbf{x}^T(t_f)\mathbf{P}(t_f)\mathbf{x}(t_f) \text{ or } \mathbf{P}(t_f) = \mathbf{S} \tag{3.17}$$

which gives the differential matrix Ricatti equation and final boundary condition:

$$\begin{aligned} -\dot{\mathbf{P}}(t) &= \mathbf{A}^T\mathbf{P}(t) + \mathbf{P}(t)\mathbf{A} + \mathbf{Q} - \mathbf{P}(t)\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{P}(t) & (3.18) \\ \mathbf{P}(t_f) &= \mathbf{S} & (3.19) \end{aligned}$$

Solving this nonlinear matrix differential equation is non-trivial. The matrix $\mathbf{P}(t)$ is the solution of Riccati matrix differential equation.

From (3.7) and (3.9) the optimal state-feedback control law is given by:

$$\mathbf{u}^*(t) = -\frac{1}{2}\mathbf{R}^{-1}\mathbf{B}^T(2\mathbf{P}(t)\mathbf{x}) = -\mathbf{R}^{-1}\mathbf{B}^T\mathbf{P}(t)\mathbf{x} = -\mathbf{K}(t)\mathbf{x} \tag{3.20}$$

where

$$\mathbf{K}(t) = \mathbf{R}^{-1}\mathbf{B}^T\mathbf{P}(t) \tag{3.21}$$

The performance measure starting from a point $\mathbf{x}$ and time t to the final time is also given by $\mathbf{P}(t)$ in:

$$J^*(\mathbf{x}, t) = \mathbf{x}^T\mathbf{P}(t)\mathbf{x}$$

So if we can solve the Riccati equation for the time-varying matrix $\mathbf{P}(t)$, then we have solved the optimal control problem.

In summary, we are given: system information (matrices $\mathbf{A}$, $\mathbf{B}$), relative state and control move cost information (matrices $\mathbf{Q}$, $\mathbf{R}$), termination cost, if any, (matrix $\mathbf{S}$) and we want to establish: $\mathbf{P}(t)$, $\mathbf{K}(t)$, J^*.

In practice the optimal time-varying controller is "mostly" constant, which means we can use the steady-state solution, $\mathbf{P}_\infty$ and $\mathbf{K}_\infty$. This has the following advantages: much easier to compute, (no matrix differential equations), less parameters to store, less on-line computation (one matrix multiplication rather than solving a matrix differential equation), good approximation for most cases, one can use the Matlab functions.

As the optimization horizon approaches infinity, the optimal matrices become constant, i.e. $\dot{\mathbf{P}} = 0$, and we now need to solve the time invariant algebraic matrix Ricatti equation:

$$\mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A} + \mathbf{Q} - \mathbf{P}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{P} = 0 \tag{3.22}$$

for $\mathbf{P} = \mathbf{P}_\infty$, and the state feedback is now:

$$\mathbf{u}^*(t) = -\mathbf{R}^{-1}\mathbf{B}^T\mathbf{P}_\infty\mathbf{x} = -\mathbf{K}_\infty\mathbf{x} \tag{3.23}$$

Note:

- System must be controllable so that the closed loop control law is stable.
- Steady-state solutions, $\mathbf{P}_\infty$, is independent of termination cost, $\mathbf{S}$ as expected.

Solution techniques:

- Iterate difference approximation of differential equation until steady-state (Euler method starting from the final time t_f and calculating backwards the solution of matrix differential Riccati equation (3.18))
- Matlab routines *lqr* or *care*.

3.1 The algorithm for computing LQR gains

Example 3.1 *Consider a continuous system described by the state equations:*

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t), \quad t \in [0, 10]$$

Algorithm 2 Computing LQR gains

1: Set the system matrices $\mathbf{A}$, $\mathbf{B}$ and the LQR weights $\mathbf{S}$, $\mathbf{Q}$, $\mathbf{R}$

2: Set the number of time interval $t \in [0, t_f]$.

3: Set the final condition

$$\mathbf{P}(t_f) = \mathbf{S}$$

4: Integrate backwards from $t = t_f$ to $t = 0$ (using a numerical method as Euler, Runge-Kutta, etc.) the Riccati equation:

$$-\dot{\mathbf{P}}(t) = \mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A} + \mathbf{Q} - \mathbf{P}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{P}$$

and compute the gain matrix:

$$\mathbf{K}(t_k) = \mathbf{R}^{-1}\mathbf{B}^T\mathbf{P}(t_k)$$

The problem is to find the optimal control law $u^*(t)$, that minimizes a quadratic cost function:

$$J = \mathbf{x}(10)^T\mathbf{S}\mathbf{x}(10) + \int_0^{10} \left[\mathbf{x}(t)^T\mathbf{Q}\mathbf{x}(t) + \mathbf{u}(t)^T\mathbf{R}\mathbf{u}(t)\right]$$

where:

$$\mathbf{S} = \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}, \quad \mathbf{Q} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad R = 1$$

Since the state vector has two elements, the size of the gain matrix $\mathbf{K}$ *is* 1×2*:* $\mathbf{K} = \begin{bmatrix} k_1 & k_2 \end{bmatrix}$ *The algorithm 2 will compute the gains* k_1*,* k_2 *for each time moment between* $t_f = 10$ *and* $t_0 = 0$*. A plot of these values is given in Figure 3.1.*

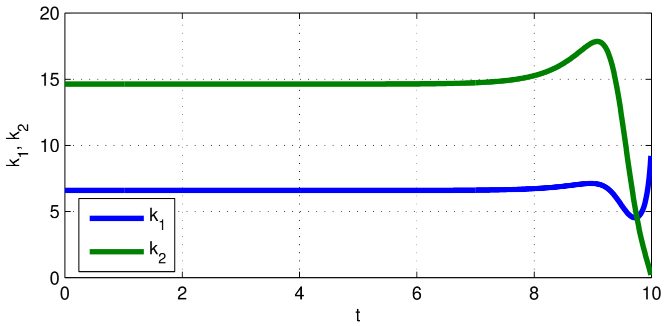

Figure 3.1: Variable gains

The gain matrix $\mathbf{K}(t)$ *was computed starting from the final time* $t_f = 10$ *to* 0*. As illustrated in Figure 3.1, the elements of the gain matrix start from some initial values, but they converge to constant values as* $t \to 0$*.*

A simulation of the closed-loop system with the control law $u^*(t) = -\mathbf{K}(t)\mathbf{x}(t)$, $t \in [0, 10]$*is shown in Figure 3.2. The states and control signal will converge at* 0*, as expected.*

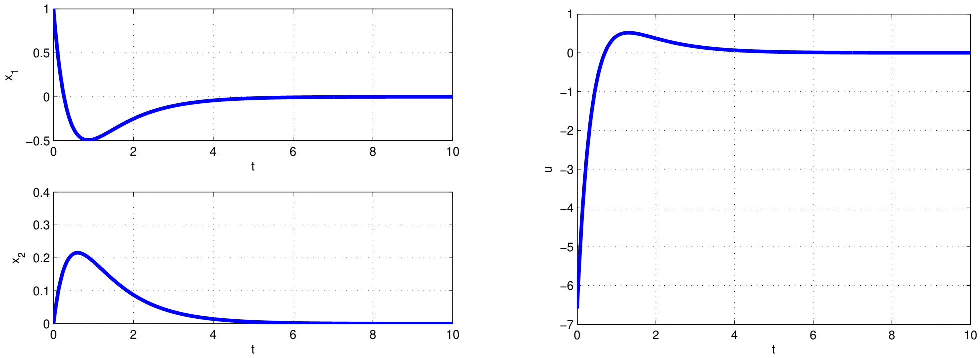

Figure 3.2: Optimal state trajectories (left) and optimal control (right)

4

LQR tracking problem

4.1 Tracking a constant reference input

LQR is a control law that returns the system states to 0 while balancing the amount of control used. However, we are also interested in tracking a reference command $r(t)$, so that the output $y(t) = r(t)$ as $t \to \infty$.

Consider a linear system described by:

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) && (4.1) \\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t) && (4.2) \end{aligned}$$

where the number of states is n, the number of control inputs is m and the number of outputs is p. Thus the system matrices will have the sizes: $\mathbf{A}_{n\times n}$, $\mathbf{B}_{n\times m}$, $\mathbf{C}_{n\times p}$.

A simple way to implement a reference tracker is to modify the LQR controller from

$$\mathbf{u}(t) = -\mathbf{K}\ \mathbf{x}(t)$$

to

$$\mathbf{u}(t) = \mathbf{N}\ \mathbf{r}(t) - \mathbf{K}\ \mathbf{x}(t) \tag{4.3}$$

In this case, the closed-loop system will be described by:

$$\begin{aligned} \dot{\mathbf{x}}(t) &= (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x}(t) + \mathbf{B}\mathbf{N}\mathbf{r}(t) && (4.4) \\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t) && (4.5) \end{aligned}$$

and the block diagram for implementation is presented in Figure 4.1.

The reference signal has been multiplied by the gain $\mathbf{N}$ to be chosen so that for a step reference input $\mathbf{r}$(t)=$\mathbf{R}$, the steady-state of the output is $\mathbf{R}$:

$$\lim_{t\to\infty} \mathbf{y}(t) = \mathbf{R}$$

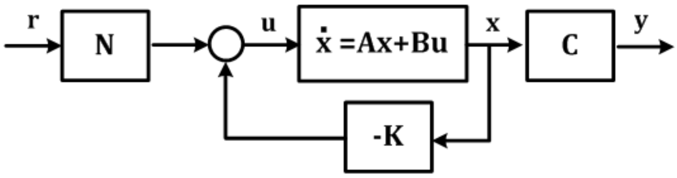

Figure 4.1: Implementation of LQR tracker

For the constant reference input steady-state corresponds to an equilibrium condition for the closed-loop state equation involving an equilibrium state denoted by $\mathbf{x}_{ss}$. Thus, the state equation satisfies:

$$\dot{\mathbf{x}}_{ss}(t) = 0 = (\mathbf{A} - \mathbf{BK})\mathbf{x}_{ss} + \mathbf{BNR} \tag{4.6}$$

The steady-state for the output is obtained for:

$$\mathbf{y}_{ss}(t) = \mathbf{C}\mathbf{x}_{ss} = -\mathbf{C}(\mathbf{A} - \mathbf{BK})^{-1}\mathbf{BNR} = \mathbf{R} \tag{4.7}$$

Thus, the gain $\mathbf{N}$ will result as:

$$\mathbf{N} = -(\mathbf{C}(\mathbf{A} - \mathbf{BK})^{-1}\mathbf{B})^{-1} = -(\mathbf{C}\mathbf{A}_{cl}^{-1}\mathbf{B})^{-1} \tag{4.8}$$

where $\mathbf{A}_{cl} = \mathbf{A} - \mathbf{BK}$.

Note that:

- if the number of outputs p is equal to the number of inputs m, the matrix $\mathbf{C}\mathbf{A}_{cl}^{-1}\mathbf{B}$ is square, otherwise its size will be $p \times m$ and cannot be inverted.
- if the number of outputs is greater than the number of inputs, $p > m$, the relation (4.7) will provide a solution for some particular cases. There are not enough degrees of freedom in the control to steer the output to any desired set point.
- if the number of outputs is less than the number of inputs, $p < m$, there will be multiple solutions to (4.7). We can choose one of them, or augment the output to $\mathbf{y}_a$ such that $p = m$.

4.2 LQR-servo problem

We will design an LQR control law where the aim is the system output will follow a time-dependent reference signal $\mathbf{r}(t)$.

The state equation of the system are:

$$
\begin{aligned}
\dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) && (4.9)\\
\mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t) && (4.10)
\end{aligned}
$$

A different approach to ensuring zero steady-state error is to use what is often called an *LQR-servo*. The approach is to add a new states $\mathbf{z}(t)$ to the system that integrates the tracking error:

$$
\dot{\mathbf{z}}(t) = \mathbf{e}(t) = \mathbf{r}(t) - \mathbf{y}(t) = \mathbf{r}(t) - \mathbf{C}\mathbf{x}(t) \tag{4.11}
$$

The extended dynamics are given by:

$$
\begin{bmatrix} \dot{\mathbf{x}}(t) \\ \dot{\mathbf{z}}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{A} & 0 \\ \mathbf{-C} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x}(\mathrm{t}) \\ \mathbf{z}(\mathrm{t}) \end{bmatrix} + \begin{bmatrix} \mathbf{B} \\ \mathbf{0} \end{bmatrix} \mathbf{u} + \begin{bmatrix} \mathbf{0} \\ \mathbf{I} \end{bmatrix} \mathbf{r} \tag{4.12}
$$

and define $\overline{\mathbf{x}}(t) = [\mathbf{x}(t) \;\; \mathbf{z}(t)]^T$. In steady state, $\dot{\mathbf{z}} = 0$, i.e. $\mathbf{r} = \mathbf{y}$ or $\mathbf{r} = \mathbf{C}\mathbf{x}$.

The optimal feedback computed for the cost:

$$
J = \int_0^\infty \left[\overline{\mathbf{x}}^T \mathbf{Q}_x \overline{\mathbf{x}} + \mathbf{u}^\mathbf{T} \mathbf{R}_\mathbf{u} \mathbf{u}\right] dt \tag{4.13}
$$

is full state feedback and given by:

$$
\mathbf{u}(t) = -\left[\mathbf{K}_x \;\; \mathbf{K}_z\right] \cdot \begin{bmatrix} \mathbf{x}(\mathrm{t}) \\ \mathbf{z}(\mathrm{t}) \end{bmatrix} \tag{4.14}
$$

using the LQR design procedure with the extended system matrices:

$$
\mathbf{A}_e = \begin{bmatrix} \mathbf{A} & 0 \\ \mathbf{-C} & 0 \end{bmatrix}, \quad \mathbf{B}_e = \begin{bmatrix} \mathbf{B} \\ \mathbf{0} \end{bmatrix} \tag{4.15}
$$

$(\mathbf{A}_e, \mathbf{B}_e)$ should be a controllable pair.

Once we have the gains $\mathbf{K_x}$ and $\mathbf{K_z}$, a typical implementation is presented in Figure 4.2.

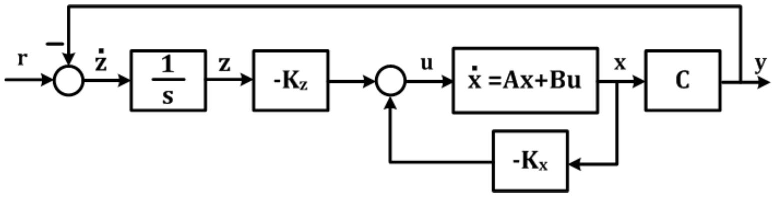

Figure 4.2: Implementation of LQR servo

5

State estimation for state feedback control systems

5.1 Observer design

5.1.1 Full-state observers

The LQR design procedure presented before assumed that all the states are available for feedback at all times. This means that the states can be measured with one or more sensors. However, in many cases, only a part of the state variables can be measured and are available for feedback. A similar situation occurs when the control system has to be implemented with a minimum number of sensors. If, for a given set of outputs, the system is completely observable, it is possible to estimate the states from available measurements (Dorf and Bishop, 2011; Franklin et al., 2010).

In this section only full-state observers will be presented, i.e. the observer will provide an estimation of all states. Since some states can be measured directly, it is possible to design an observer that estimates only the states that are not measured directly. These are known as reduced-order observers and details can be obtained from (Franklin et al., 2010; Ogata, 2002).

Consider the linear system:

$$\begin{cases} \dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t) \\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t) \end{cases} \tag{5.1}$$

If the estimated state is denoted by $\hat{\mathbf{x}}(t)$, and the estimated output is $\hat{y}(t)$, the full-state observer of the system (5.1) is given by:

$$\begin{cases} \dot{\hat{\mathbf{x}}}(t) &= \mathbf{A}\hat{\mathbf{x}}(t) + \mathbf{B}u(t) + \mathbf{L}\left(\mathbf{y}(t) - \hat{\mathbf{y}}(t)\right) \\ \hat{\mathbf{y}}(t) &= \mathbf{C}\hat{\mathbf{x}}(t) \end{cases} \tag{5.2}$$

The term involving the matrix $\mathbf{L}$ - *the observer gain matrix* - is introduced to correct the estimated state $\hat{\mathbf{x}}(t)$ so that it will approach the real state $\mathbf{x}(t)$ as $t \to \infty$. In other words, the observer estimation error, given by:

$$\mathbf{e}(t) = \mathbf{x}(t) - \hat{\mathbf{x}}(t) \tag{5.3}$$

must approach 0 as $t \to \infty$.

One important result of systems theory is that if the system is completely observable, the matrix $\mathbf{L}$ can always be determined to make the tracking error asymptotically stable (Dorf and Bishop, 2011).

A block diagram of the closed-loop full-state observer is presented in Figure 5.1.

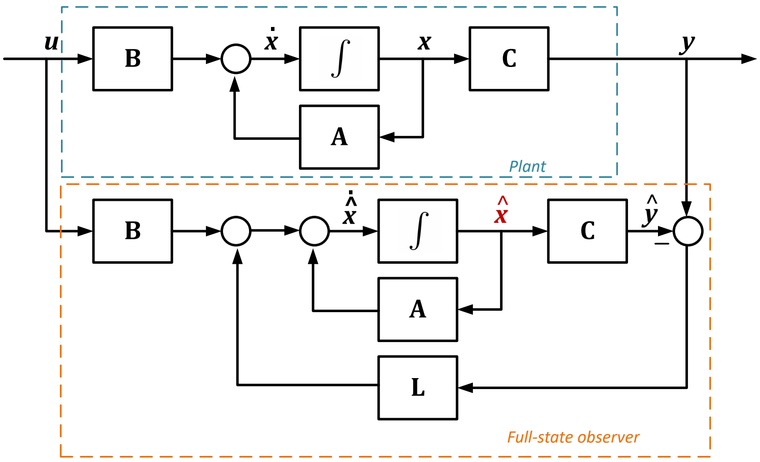

Figure 5.1: System and full-state observer, (Ogata, 2002)

Taking the derivative of the estimation error (5.3) and using the relations (5.1) and (5.2), we obtain:

$$\dot{\mathbf{e}} = \dot{\mathbf{x}} - \dot{\hat{\mathbf{x}}} = \mathbf{Ax} + \mathbf{Bu} - \mathbf{A}\hat{\mathbf{x}} - \mathbf{Bu} - \mathbf{L}\left(\mathbf{Cx} - \mathbf{C}\hat{\mathbf{x}}\right) = \mathbf{A}\left(\mathbf{x} - \hat{\mathbf{x}}\right) - \mathbf{LC}\left(\mathbf{x} - \hat{\mathbf{x}}\right)$$

or, the error dynamics is described by:

$$\dot{\mathbf{e}}(t) = (\mathbf{A} - \mathbf{LC})\mathbf{e}(t) \tag{5.4}$$

Analyzing the differential equation (5.4), we can state that the error $\mathbf{e}(t) \to 0$ as $t \to \infty$, for any initial conditions $\mathbf{e}(t_0)$ if all eigenvalues of the system matrix $\mathbf{A} - \mathbf{LC}$ are located in the left half-plane, or the system (5.4) is asymptotically stable.

The matrix $\mathbf{L}$ can be computed, for example, by pole placement. The desired location of the estimator error poles is chosen $\begin{bmatrix} ep_1 & ep_2 & \cdots & ep_n \end{bmatrix}$ as equal to the eigenvalues of the error system matrix and solving:

$$\det(\lambda\mathbf{I} - (\mathbf{A} - \mathbf{LC})) = (\lambda_1 - ep_1)(\lambda_2 - ep_2)\cdots(\lambda_n - ep_n)$$

The speed at which the estimation error decays is clearly determined by the estimator poles $(ep_1, ..., ep_n)$. Therefore, they have to be placed in the left half-plane and, in general, must be faster than the system poles. Although various results are available in the literature for choosing the best location considering also the system poles, an indication may be as follows: the estimator poles can be placed about 5 times on the left of system's dominant poles.

5.1.2 Duality controllability-observability

Consider the following systems:

$$(S1): \begin{cases} \dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t) \\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t) \end{cases} \qquad (S2): \begin{cases} \dot{\mathbf{x}}(t) &= \mathbf{A}^T\mathbf{x}(t) + \mathbf{C}^T u(t) \\ \mathbf{y}(t) &= \mathbf{B}^T\mathbf{x}(t) \end{cases}$$

The controllability (Co) and observability (Ob) matrices for these systems are:

$$Co_1 = \begin{bmatrix} \mathbf{B} & \mathbf{AB} & \cdots & \mathbf{A}^{n-1}\mathbf{B} \end{bmatrix}, \qquad Ob_1 = \begin{bmatrix} \mathbf{C}^T & \mathbf{A}^T\mathbf{C}^T & \cdots & \mathbf{A}^{T^{n-1}}\mathbf{C}^T \end{bmatrix}$$

$$Co_2 = \begin{bmatrix} \mathbf{C}^T & \mathbf{A}^T\mathbf{C}^T & \cdots & \mathbf{A}^{T^{n-1}}\mathbf{C}^T \end{bmatrix}, \qquad Ob_2 = \begin{bmatrix} \mathbf{B} & \mathbf{AB} & \cdots & \mathbf{A}^{n-1}\mathbf{B} \end{bmatrix}$$

The systems S_1 and S_2 are controllable/observable if the matrices Co/Ob have full row rank. It is clear that the system S_1 is controllable if and only if the system S_2 is observable and S_2 is controllable if and only if the system S_1 is observable.

The problem of observer design for a system is in fact a problem of controller design for its dual system.

Using this property, the observer gain matrix $\mathbf{L}^T$ can be obtained by pole placement with a similar procedure to a standard regulator problem with the system matrix $(\mathbf{A} - \mathbf{BK})$ and the replacements: $\mathbf{A} \to \mathbf{A}^T$, $\mathbf{B} \to \mathbf{C}^T$ and $\mathbf{K} \to \mathbf{L}^T$.

5.2 Separation principle

Consider the completely controllable and observable system

$$\begin{cases} \dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t) \\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t) \end{cases} \tag{5.5}$$

The state feedback control law based on estimated state $\hat{\mathbf{x}}(t)$ is:

$$\mathbf{u}(t) = -\mathbf{K}\hat{\mathbf{x}}(t) \tag{5.6}$$

where the feedback gain matrix $\mathbf{K}$ is computed by any state-feedback control method, including LQR.

We will verify that using this control law we retain the stability of the closed-loop system and the observer, (Ogata, 2002).

With the control law (5.6), the state equation is:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) - \mathbf{B}\mathbf{K}\hat{\mathbf{x}}(t)$$

and the observer (5.2) becomes:

$$\dot{\hat{\mathbf{x}}}(t) = \mathbf{A}\hat{\mathbf{x}}(t) - \mathbf{B}\mathbf{K}\hat{\mathbf{x}}(t) + \mathbf{L}\mathbf{C}\left(\mathbf{x}(t) - \hat{\mathbf{x}}(t)\right)$$

The estimation error is $\mathbf{e}(t) = \mathbf{x}(t) - \hat{\mathbf{x}}(t)$ and its dynamics is described by (5.3):

$$\dot{\mathbf{e}} = \dot{\mathbf{x}} - \dot{\hat{\mathbf{x}}} = \mathbf{A}\mathbf{x} - \mathbf{B}\mathbf{K}\hat{\mathbf{x}} - \mathbf{A}\hat{\mathbf{x}} - \mathbf{B}\mathbf{K}\hat{\mathbf{x}} - \mathbf{L}\mathbf{C}(\mathbf{x} - \hat{\mathbf{x}})$$

or

$$\dot{\mathbf{e}}(t) = (\mathbf{A} - \mathbf{L}\mathbf{C})\mathbf{e}(t) \tag{5.7}$$

Because $\hat{\mathbf{x}}(t) = \mathbf{x}(t) - \mathbf{e}(t)$, the state equation can be written:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) - \mathbf{B}\mathbf{K}(\mathbf{x}(t) - \mathbf{e}(t)) = (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x}(t) + \mathbf{B}\mathbf{K}\mathbf{e}(t)$$

Thus, the state equation and the error dynamics are:

$$\left\{ \begin{array}{lcl} \dot{\mathbf{x}}(t) & = & (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x}(t) + \mathbf{B}\mathbf{K}\mathbf{e}(t) \\ \dot{\mathbf{e}}(t) & = & (\mathbf{A} - \mathbf{L}\mathbf{C})\mathbf{e}(t) \end{array} \right. \tag{5.8}$$

which can be written in a matrix form, giving the dynamics of the observed-state feedback control system:

$$\begin{bmatrix} \dot{\mathbf{x}}(t) \\ \dot{\mathbf{e}}(t) \end{bmatrix} = \underbrace{\begin{bmatrix} \mathbf{A} - \mathbf{B}\mathbf{K} & \mathbf{B}\mathbf{K} \\ \mathbf{0} & \mathbf{A} - \mathbf{L}\mathbf{C} \end{bmatrix}}_{\mathbf{A}_{cl}} \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{e}(t) \end{bmatrix} \tag{5.9}$$

The characteristic equation of (5.9) is:

$$\det(\lambda\mathbf{I} - \mathbf{A}_{cl}) = 0$$

or:

$$\begin{vmatrix} \lambda\mathbf{I} - (\mathbf{A} - \mathbf{B}\mathbf{K}) & -\mathbf{B}\mathbf{K} \\ \mathbf{0} & \lambda\mathbf{I} - (\mathbf{A} - \mathbf{L}\mathbf{C}) \end{vmatrix} = |\lambda\mathbf{I} - \mathbf{A} + \mathbf{B}\mathbf{K}| \cdot |\lambda\mathbf{I} - \mathbf{A} + \mathbf{L}\mathbf{C}| = 0$$

This means that the eigenvalues of the closed-loop observed feedback control system consist of the poles due to the controller given by: $|\lambda\mathbf{I} - \mathbf{A} + \mathbf{B}\mathbf{K}| = 0$ and the poles due to observer given by $|\lambda\mathbf{I} - \mathbf{A} + \mathbf{L}\mathbf{C}| = 0$. Thus, the controller and the observer can be designed independently because they are not dependent of each other.

5.3 Full state feedback design

The block diagram of a stabilizing closed-loop control system with full-state observer is presented in Figure 5.2, (Ogata, 2002). Notice that the control law $\mathbf{u} = -\mathbf{K}\hat{\mathbf{x}}$ is using the estimated state for feedback and not the real state of the process. A simplified block diagram is given in Figure 5.3. The

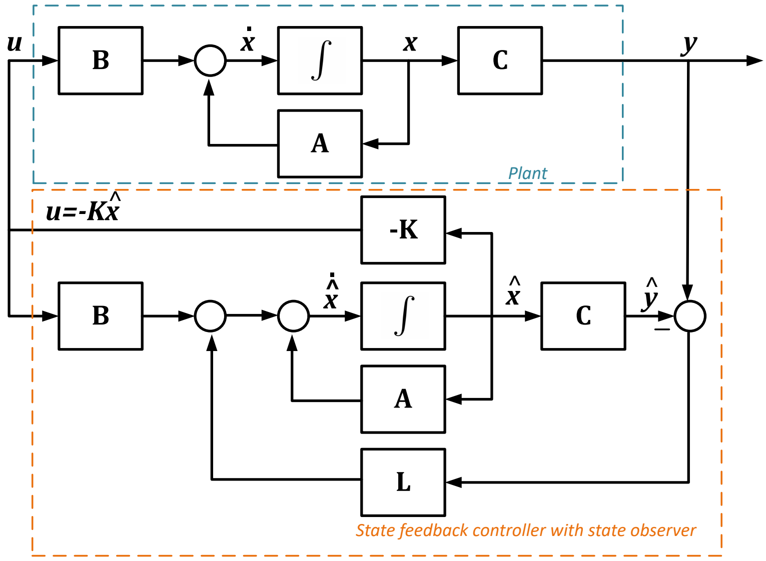

Figure 5.2: Control system with full-state observer

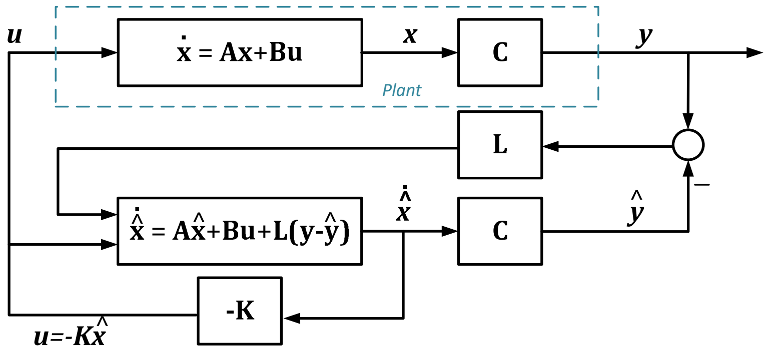

Figure 5.3: Control system with full-state observer. Simplified block diagram

state-space model for the whole control system, including the observer can be obtained as follows. The state equations of the plant and observer are:

$$
\begin{aligned}
\dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \\
\dot{\hat{\mathbf{x}}}(t) &= \mathbf{A}\hat{\mathbf{x}}(t) + \mathbf{B}\mathbf{u}(t) + \mathbf{L}\mathbf{C}\left(\mathbf{x}(t) - \hat{\mathbf{x}}(t)\right)
\end{aligned}
$$

and including the control law $\mathbf{u}(t) = -\mathbf{K}\hat{\mathbf{x}}(t)$, they become:

$$\begin{aligned}
\dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) - \mathbf{B}\mathbf{K}\hat{\mathbf{x}}(t) \\
\dot{\hat{\mathbf{x}}}(t) &= \mathbf{A}\hat{\mathbf{x}}(t) - \mathbf{B}\mathbf{K}\hat{\mathbf{x}}(t) + \mathbf{L}\mathbf{C}\left(\mathbf{x}(t) - \hat{\mathbf{x}}(t)\right) = \mathbf{L}\mathbf{C}\mathbf{x}(t) + (\mathbf{A} - \mathbf{B}\mathbf{K} - \mathbf{L}\mathbf{C})\hat{\mathbf{x}}(t)
\end{aligned}$$

The output equations for the plant and observer are:

$$\begin{aligned}
\mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t) \\
\hat{\mathbf{y}}(t) &= \mathbf{C}\hat{\mathbf{x}}(t)
\end{aligned}$$

The state and output equations can be written in a matrix form:

$$\begin{bmatrix} \dot{\mathbf{x}}(t) \\ \dot{\hat{\mathbf{x}}}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{A} & -\mathbf{B}\mathbf{K} \\ \mathbf{L}\mathbf{C} & \mathbf{A} - \mathbf{B}\mathbf{K} - \mathbf{L}\mathbf{C} \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) \\ \hat{\mathbf{x}}(t) \end{bmatrix} \tag{5.10}$$

$$\begin{bmatrix} \mathbf{y}(t) \\ \hat{\mathbf{y}}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{C} & \mathbf{0} \\ \mathbf{0} & \mathbf{C} \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) \\ \hat{\mathbf{x}}(t) \end{bmatrix} \tag{5.11}$$

Summarizing the previous sections, the design of the state feedback control system with state estimation can follow, for example, the next steps:

- Consider the open-loop system defined by the state-space matrices: $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{D} = \mathbf{0}$. (The general case when $\mathbf{D} \neq \mathbf{0}$ can be addressed in a similar manner as presented before. This simpler case is presented here for readability).

- Compute the controllability and observability matrices and verify that the system is controllable and observable.

- Design the state-feedback gain matrix $\mathbf{K}$. If the design method is LQR, choose the matrices $\mathbf{Q}$ and $\mathbf{R}$ and compute $\mathbf{K}$.

- Select the observer poles ep_i, $i = \overline{1, n}$. It is important to avoid that the observer makes the closed-loop system slower. The observer poles should be faster than the closed-loop system. A rule of thumb can be: select the observer poles around 5 to 10 times higher than the closed-loop system poles. If the matrix $\mathbf{K}$ is already computed, the closed-loop poles are the eigenvalues of $\mathbf{A} - \mathbf{B}\mathbf{K}$.

- Compute the observer gain matrix $\mathbf{L}$ by pole placement. Taking advantage of the duality controllability-observability, the matrix $\mathbf{L}$ can be determined similar to computing a state-feedback control gain matrix for a system with matrices ($\mathbf{A}^T$, $\mathbf{C}^T$).

  For example, the Matlab statement for obtaining the observer gain is: L = place(A', C', [ep1 ep2...])'. *Notice that the result of* place *must be transposed.*

- Initial conditions. In general, if the states cannot be measured, the initial conditions for the states are not well known. In case no other information is available, the initial values for the estimated states can be chosen as zero $\hat{\mathbf{x}}_0 = \mathbf{0}$. The initial conditions for the process states are usually, not equal to $\hat{\mathbf{x}}_0$. The observer should be able to make the estimated states approach the process states as time increases, starting from any initial conditions.

- Implement the state-feedback control system with observer according to the block diagram from Figure 5.2, or use the state-space model for the closed-loop system (5.10), (5.11) to simulate from the initial conditions $\mathbf{x}_0$, $\hat{\mathbf{x}}_0$.

Example 5.1 *Consider a linear system:*

$$\begin{aligned} \dot{\mathbf{x}} &= \begin{bmatrix} 2 & -1 \\ 3 & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u \\ y &= \begin{bmatrix} 0.1 & 0.2 \end{bmatrix} \mathbf{x} \end{aligned} \quad , \quad \mathbf{x}_0 = [-10 \ \ 10]^T$$

We will design an LQR controller with a full state observer to stabilize the system.

1. *Check controllability and observability of the system. For*

$$\mathbf{A} = \begin{bmatrix} 2 & -1 \\ 3 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 0.1 & 0.2 \end{bmatrix}$$

the controllability matrix is:

$$Co = \begin{bmatrix} \mathbf{B} & \mathbf{AB} \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \quad \text{and} \quad rank(Co) = 2$$

and the observability matrix:

$$Ob = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \end{bmatrix} = \begin{bmatrix} 0.1 & 0.2 \\ 0.8 & -0.1 \end{bmatrix} \quad \text{and} \quad rank(Ob) = 2$$

The system is thus, fully controllable and observable.

Notice also that the open-loop system is unstable because the eigenvalues of matrix $\mathbf{A}$ *are* $1 \pm \sqrt{2}j$. *The objective of this problem is to find a controller* $\mathbf{K}$ *, combined with a full state observer that will stabilize the system.*

2. *Only for comparison purposes, we will consider that all the states can be measured and an observer is not required. We select some matrices* $\mathbf{Q}$ *and* $\mathbf{R}$ *and compute the gain* $\mathbf{K}$*, as for example:*

$$\mathbf{Q} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad R = 1 \quad \overset{K = lqr(A,B,Q,R)}{\Rightarrow} \quad \mathbf{K} = \begin{bmatrix} 4.7359 & 0.4142 \end{bmatrix}$$

The control law is $u = -\mathbf{K}\mathbf{x}$ *and the closed-loop system is described by:*

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u = (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x} \quad (and) \quad y = \mathbf{C}\mathbf{x}$$

By simulation, starting from the initial conditions $\mathbf{x}_0$ *we obtain the results from Figures 5.4, 5.5 and 5.6.*

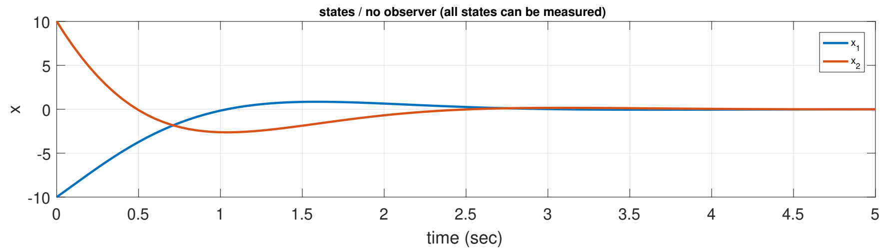

Figure 5.4: Optimal state trajectories (without observer)

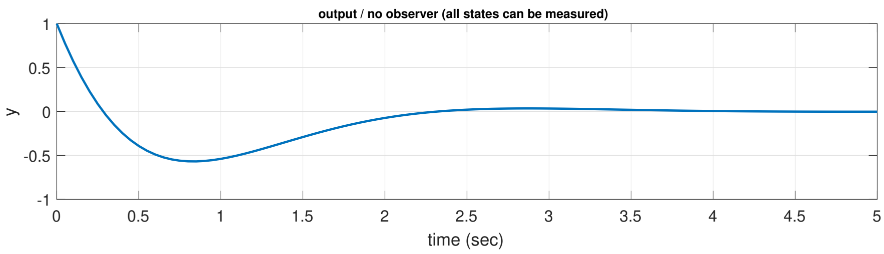

Figure 5.5: Closed-loop system output (without observer)

As seen in the figures, the closed-loop system states and output approach the zero value in about 3 seconds. The control signal has a significantly large value at the initial time. This may be corrected by altering the matrices $\mathbf{Q}$ *and* R*, but it is not the purpose of this example. We will compare these results with the ones we'll obtain when an observer is introduced.*

3. *Observer design. The controller and the observer can be designed independently, but we will start with the controller so that we obtain the closed-loop poles due to the controller first. They will be used to select the observer poles.*

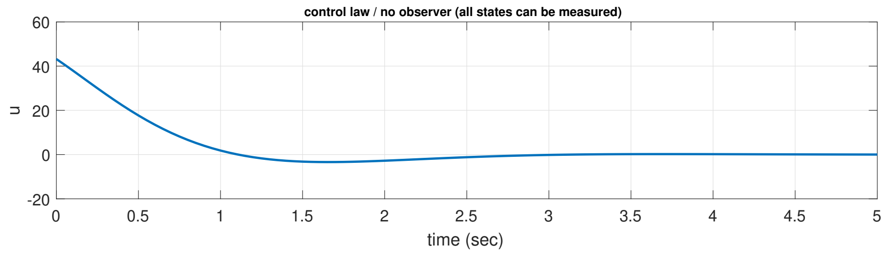

Figure 5.6: Control signal (without observer)

(a) *The controller gain* $\mathbf{K}$ *is calculated as in the previous step:* K = lqr(A,B,Q,R)*, with the same matrices* $\mathbf{Q}$ *and* R*. The closed-loop system matrix* $\mathbf{A} - \mathbf{BK}$ *(ignoring the observer) has the eigenvalues* $\lambda_{1,2} = -1.37 \pm 1.54i$*. The estimator poles will be chosen, for example, as about* $5 - 6$ *times the real part of* $\lambda_{1,2}$*:* $ep_1 = 5 \cdot real(\lambda_1)$*,* $ep_2 = 6 \cdot real(\lambda_2)$*.*

(b) *The estimator gain matrix* $\mathbf{L}^T$ *may be computed by pole placement using matrices* $\mathbf{A}^T$ *and* $\mathbf{C}^T$*:* L = place(A', C', [ep1 ep2])'*.*

4. *The initial conditions for the observer are set to 0:* $\hat{\mathbf{x}}_0 = [0 \;\; 0]^T$ *and the initial conditions for the process states are* $\mathbf{x}_0 = [-10 \;\; 10]^T$*.*

5. *The closed-loop control system with full-state estimation can be simulated either by implementing the block diagram from Figure 5.2, or from the state and output equations (5.10), (5.11).*

6. *The results of simulation are shown in Figures 5.7 (the process states and estimated states), 5.8 (the process output and the estimated output) and 5.9 (the control signal* $u = -\mathbf{K}\hat{\mathbf{x}}$*).*

Although the process states and estimated states were simulated from different initial conditions, the results in Figure 5.7 show that the estimated ones converge to $\mathbf{x}$ *and the system is stabilized in about* 3 *seconds. The observer is fast enough so it will not make the system slower than in the case with no observer. The shape of the signals was altered comparing to the case with no observer (they exhibit more oscillations), but the system becomes stable in about the same time.*

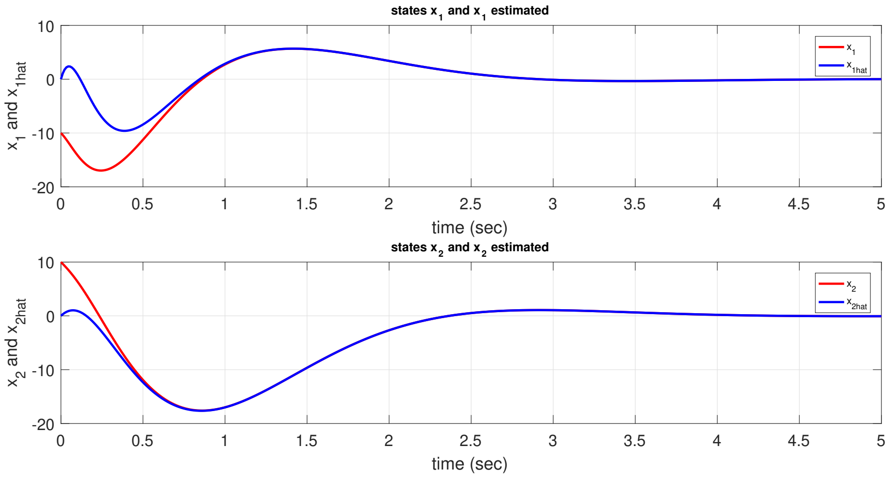

Figure 5.7: Optimal state trajectories (with observer)

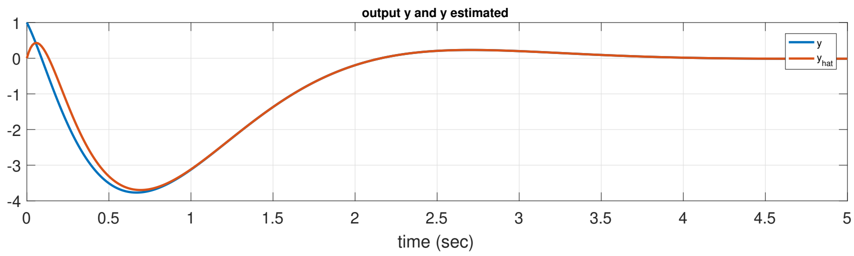

Figure 5.8: Closed-loop system output (with observer)

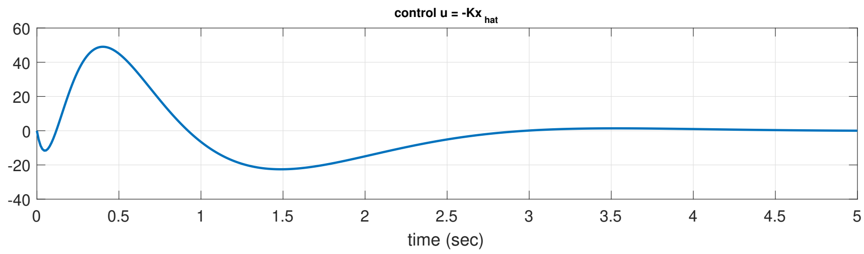

Figure 5.9: Control signal (with observer)

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