Seminar 3 - MA

Variabile aleatoare 1D

Breviar teoretic $\begin{cases} (E, K, P) \text{ câmp de probabilitate} \\ X: E \to \mathbb{R}, \ \{\omega \in E / X(\omega) < x\} \in K, \ \forall x \in \mathbb{R} \end{cases}$

1. Funcția de repartiție $F: \mathbb{R} \to [0,1]$; $F(x) = P(X < x)$, $\forall x \in \mathbb{R}$

$$\begin{cases} F(-\infty) = 0; \ F(\infty) = 1; \ F \text{ monoton crescătoare}; \ F(x_0 - 0) = F(x_0) \le F(x_0 + 0) \\ P(a \le X < b) = F(b) - F(a); \ a, b \in \mathbb{R}, a < b \\ P(X = a) = F(a+0) - F(a) = \lim\limits_{\substack{x \to a \\ x > a}} F(x) - F(a) \end{cases}$$

2. $\boxed{\text{V.a. 1D discretă}}$: $X(E) =$ mulțime discretă $= \{x_i : i \in I\}$; $p_i = P(X = x_i)$

(i) V.a. 1D simplă $X: \begin{pmatrix} x_1 & x_2 & \dots & x_n \\ p_1 & p_2 & \dots & p_n \end{pmatrix}$; $p_i = P(X = x_i)$; $\boxed{\sum_{i=1}^{n} p_i = 1}$

(ii) V.a. 1D „numărabilă" $X: \begin{pmatrix} x_1 & x_2 & x_3 & \dots & x_n & \dots \\ p_1 & p_2 & p_3 & \dots & p_n & \dots \end{pmatrix}$; $\boxed{\sum_{n=1}^{\infty} p_n = 1}$

(iii) Funcția p.m.f. (masă de probabilitate)

$f: \{x_i : i \in I\} \to \mathbb{R}$, $\quad f(x_i) = p_i = P(X = x_i)$

$$F(x) = \sum_{x_i < x} p_i = \sum_{x_i < x} f(x_i)$$

3. Funcția p.d.f. (densitate de probabilitate)

Def. $f: \mathbb{R} \to \mathbb{R}$
- (i) f integrabilă pe $\mathbb{R}$
- (ii) $f(x) \ge 0$, $\forall x \in \mathbb{R}$
- (iii) $\boxed{\int_{-\infty}^{\infty} f(x)\,dx = 1}$

4. $\boxed{\text{V.a. 1D continuă}}$ Def. $F = F_X$ funcția de rep.; $\exists$ p.d.f. $f: \mathbb{R} \to \mathbb{R}$ a.î.

$$\boxed{F(x) = \int_{-\infty}^{x} f(t)\,dt, \ \forall t \in \mathbb{R}}$$

Proprietăți (i) $F'(x) = f(x)$, $\forall x \in \mathbb{R}$ unde f cont; (ii) $P(a \le X < b) = \int_a^b f(x)\,dx$;

(iii) $\forall a \in \mathbb{R}$: $P(X = a) = 0$.

5. Caracteristici numerice (statistice) ale r.a. 1D : $\begin{cases} X: \begin{pmatrix} x_i \\ p_i \end{pmatrix}_{i\in I} \\ X: \text{pdf } f(x) \end{cases}$

5.1. Valoarea medie $M(X) = E(X) = \sum_{i\in I} p_i x_i$ (r.a. 1D discretă)

$$M(X) = E(X) = \int_{-\infty}^{\infty} x f(x)\,dx \text{ (r.a. 1D continue)}$$

5.2. Moment de ordin $r \in \mathbb{N}^*$ $\begin{cases} M_r(X) = M(X^r) = \sum_{i\in I} x_i^r p_i & \to \text{discret} \\ M_r(X) = M(X^r) = \int_{-\infty}^{\infty} x^r f(x)\,dx & \to \text{cont.} \end{cases}$

5.3. Moment centrat de ordin $r \in \mathbb{N}^*$, $m = M(X)$, $\mu_r(X) = M_r(X-m) = M((X-m)^r)$,

5.4. Dispersia (Varianța) $D^2(X) = Var(X) = \mu_2(X) = M_2(X-m) = M((X-m)^2)$

- discret $D^2(X) = Var(X) = \sum_{i\in I} (x_i - m)^2 p_i$
- continuu $D^2(X) = Var(X) = \int_{-\infty}^{\infty} (x-m)^2 f(x)\,dx$

Formula dispersiei $D^2(X) = M(X^2) - (M(X))^2 = M_2(X) - M_1^2(X)$

5.5. Abaterea medie pătratică $\sigma(X) = \sqrt{D^2(X)} = \sqrt{Var(X)}$

5.6. Moda. Este abscisa punctului de maxim pentru p.m.f., respectiv pdf

(i) Discret $M_o(X) = x_s$, unde $p_s = \max\{p_i : i \in I\}$

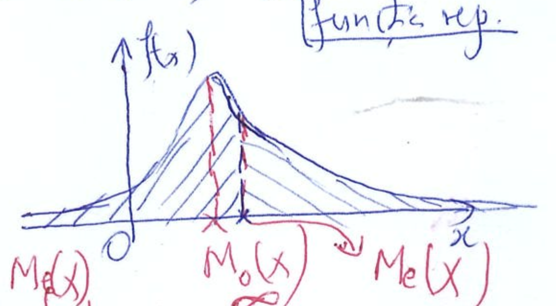
(ii) Continuu Se rezolvă ecuația $f'(x) = 0$ și se determină x_{max}.

5.7. Mediana $\boxed{Me(X)}$: $F(Me(X)) \le \frac{1}{2} \le F(Me(X)+0)$; $\begin{cases} F(x) = P(X<x) \\ \text{funcția rep.} \end{cases}$

Obs. F continuă $\Rightarrow F(Me(X)) = \frac{1}{2}$

5.8. Asimetria $A_s(X) = \dfrac{\mu_3(X)}{\sigma^3(X)}$

5.9. Excesul $E_x(X) = \dfrac{\mu_4(X)}{\sigma^4(X)} - 3$



$$\int_{-\infty}^{Me(X)} f(x)\,dx = \int_{Me(X)}^{\infty} f(x)\,dx = \frac{1}{2}$$

5.10 Entropia $H(X)$

Discret $\begin{cases} H(X) = \sum_{i\in I} p_i \log_2 \frac{1}{p_i} \\ H(X) = \int_{-\infty}^{\infty} f(x) \log_2 \frac{1}{f(x)}\,dx \end{cases}$ Cont.

OBS. $X: \begin{pmatrix} x_1 & x_2 & x_3 & \dots & x_n \\ p_1 & p_2 & p_3 & \dots & p_n \end{pmatrix} \Rightarrow H(X) = \sum_{i=1}^{n} p_i \log_2 \frac{1}{p_i} \le \log_2 n$

Seminar 3 - M4

Ex. 1 Se consideră v.a. simplă $X:\begin{pmatrix}0&1&2&3&4\\ p^3&p^2&\frac{p}{2}&\frac{1}{4}&\frac{1}{8}\end{pmatrix}$. Să se determine

1° $p=?$; 2° $M(X)=E(X)$; 3° $D^2(X)=\mathrm{Var}\,X$; 4° $\sigma(X)$; 5° $M_o(X)$;

6° $F(x)$; 7° $Me(X)$; 8° $P\left(\frac{1}{e}<X\le 3\right)$; 9° $H(X)$

Rezolvare 1° $p^3+p^2+\frac{p}{2}+\frac{1}{4}+\frac{1}{8}=1$; $p\in(0,1)\Rightarrow 8p^3+8p^2+4p-5=0$

	p^3	p^2	p^1	p^0
	8	8	4	-5
1/2	8	12	10	0

; $\left(p-\frac{1}{2}\right)(8p^2+12p+10)=0 \Leftrightarrow$ $p-\frac{1}{2}=0 \Rightarrow p_1=\frac{1}{2}$; $2(4p^2+6p+5)=0$; $\Delta<0$, $p_{2,3}\notin\mathbb{R}$

Deci $p=\frac{1}{2}$ $\Rightarrow X:\begin{pmatrix}0&1&2&3&4\\ \frac{1}{8}&\frac{1}{4}&\frac{1}{4}&\frac{1}{4}&\frac{1}{8}\end{pmatrix}$

2° $M(X)=\frac{1}{8}\cdot 0+1\cdot\frac{1}{4}+2\cdot\frac{1}{4}+3\cdot\frac{1}{4}+4\cdot\frac{1}{8}=2$

$X^2:\begin{pmatrix}0^2&1^2&2^2&3^2&4^2\\ \frac{1}{8}&\frac{1}{4}&\frac{1}{4}&\frac{1}{4}&\frac{1}{8}\end{pmatrix}$

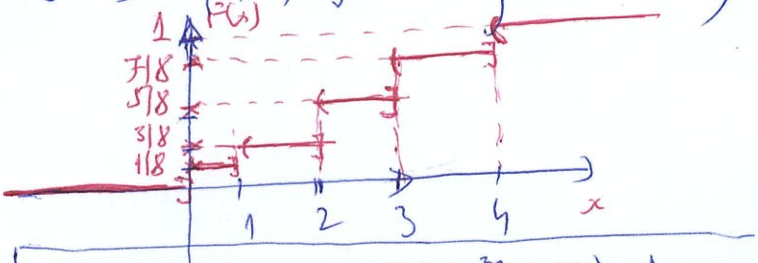
3° $D^2(X)=\mathrm{Var}(X)=M(X^2)-M^2(X)=$

$=0\cdot\frac{1}{8}+1\cdot\frac{1}{4}+4\cdot\frac{1}{4}+9\cdot\frac{1}{4}+16\cdot\frac{1}{8}-2^2=\frac{3}{2}=1.5$;

4° $\sigma(X)=\sqrt{D^2(X)}=\sqrt{\frac{3}{2}}=\frac{\sqrt{6}}{2}\simeq 1.22$

5° $\max\left\{\frac{1}{8},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{8}\right\}=\frac{1}{4}\Rightarrow M_o(X)\in\{1,2,3\}$ (v.a. plurimodală)

6° $F(x)=\begin{cases}0; & x\le 0\\ 1/8; & 0<x\le 1\\ 1/8+1/4=3/8; & 1<x\le 2\\ 1/8+1/4+1/4=5/8; & 2<x\le 3\\ 7/8; & 3<x\le 4\\ 1; & x>4\end{cases}$



7° $\begin{cases}F(Me(X))\le\frac{1}{2}\le F(Me(X)+0)\\ Me(X)=2.\end{cases}$

$F(a)\le\frac{1}{2}\le F(a+0)$

$\begin{cases}\frac{1}{2}\in\left(\frac{3}{8},\frac{5}{8}\right)\\ \frac{1}{2}\in(F(2),F(3))\end{cases}$

Deci $a=2$

$a=2$: $F(2)\le\frac{1}{2}\le F(2+0) \Leftrightarrow$ $\Leftrightarrow \frac{3}{8}\le\frac{1}{2}\le\frac{5}{8}$ ✓

$a=3$: $F(3)\le\frac{1}{2}\le F(3+0)$; $5/8\le\frac{1}{2}$ Fals

$a\in(2,3)$: $\frac{5}{8}\le\frac{1}{2}$ Fals

$a\in(1,2)$: $\frac{3}{8}\le\frac{1}{2}\le\frac{3}{8}$ Fals

8° $P\left(\frac{1}{e}<X\le 3\right)=P(X=1)+P(X=2)+P(X=3)=$ ($\frac{1}{e}\sim 0{,}4$)

$=\frac{1}{4}+\frac{1}{4}+\frac{1}{4}=3/4$

OBS. $P\left(\frac{1}{e}<X\le 3\right)=F(3+0)-F\left(\frac{1}{e}+0\right)=\frac{7}{8}-\frac{1}{8}=\frac{3}{4}$.

9° $H(X)=\frac{1}{8}\log_2\frac{1}{1/8}+\frac{1}{4}\log_2\frac{1}{1/4}+\frac{1}{4}\log_2\frac{1}{1/4}+\frac{1}{8}\log_2\frac{1}{1/8}=$

$=2\cdot\frac{1}{8}\log_2 8+3\cdot\frac{1}{4}\log_2 4=\frac{1}{4}\cdot 3+\frac{3}{4}\cdot 2=\frac{9}{4}=2.25$

OBS. $H(X)\le\log_2 n=\log_2 5$; Într-adevăr; $\frac{9}{4}\le\log_2 5\Leftrightarrow 9\le\log_2 5^4\Leftrightarrow$

$\Leftrightarrow 2^9\le 5^4\Leftrightarrow 512\le 625$ ✓ (A)

$\boxed{\text{Ex. 2}}$ Fie $X:\begin{pmatrix}-1 & 0 & 1\\ a+\frac{1}{6} & 2b+\frac{1}{3} & \frac{1}{3}\end{pmatrix}$, $Y:\begin{pmatrix}b & 0 & 1\\ \frac{1}{4} & 2(a+b) & 15a^2\end{pmatrix}$

v.a. simple, independente, ~~a + b < 0, P(X+Y = 0) = 1/4~~. Să se determine:

1° a, b ; 2° $M(X)$; 3° $D^2(X+Y)$; 4° $P(-1 \le X+Y \le 1)$;

5° X^n ; 6° Y^m ; 7° $M_0(X)$; 8° $X^{10}+Y^{25}$; 9° $X^{11}Y^{16}$; 10° $H(X)$

Rezolvare 1° $\begin{cases} a+\frac{1}{6}+2b+\frac{1}{3}+\frac{1}{3}=1 & ; 0\le a+\frac{1}{6}\le 1 ;\ 0\le 2b+\frac{1}{3}\le 1;\\ \frac{1}{4}+2(a+b)+15a^2=1 & ; 0\le 2(a+b)\le 1;\ 0\le 15a^2\le 1.\end{cases}$

$\begin{cases} a+2b=\frac{1}{6} \ \Big| -1\\ 2a+2b+15a^2=\frac{3}{4}\end{cases} \Leftrightarrow \begin{cases} a+2b=1/6\\ 15a^2+a=7/12\end{cases} \Leftrightarrow \begin{cases} a+2b=1/6\\ \Delta=1+4\cdot 15\cdot\frac{7}{12}=1+35=36\end{cases}$ $a_{1,2}=\frac{-1\pm 6}{30}$

$\Leftrightarrow \begin{cases} a=1/6\\ b=0\end{cases}$, $X:\begin{pmatrix}-1 & 0 & 1\\ 1/3 & 1/3 & 1/3\end{pmatrix}$; $Y:\begin{pmatrix}0 & 0 & 1\\ \frac{1}{4} & \frac{1}{3} & \frac{5}{12}\end{pmatrix}\equiv\begin{pmatrix}0 & 1\\ \frac{7}{12} & \frac{5}{12}\end{pmatrix}$

2° $M(X)=-\frac{1}{3}+0\cdot\frac{1}{3}+1\cdot\frac{1}{3}=0$; 3° $X+Y:\begin{pmatrix}-1+0 & -1+1 & 0+0 & 0+1 & 1+0 & 1+1\\ \frac{1}{3}\cdot\frac{7}{12} & \frac{1}{3}\cdot\frac{5}{12} & \frac{1}{3}\cdot\frac{7}{12} & \frac{1}{3}\cdot\frac{5}{12} & \frac{1}{3}\cdot\frac{7}{12} & \frac{1}{3}\cdot\frac{5}{12}\end{pmatrix}$

$X+Y:\begin{pmatrix}-1 & 0 & 0 & 1 & 1 & 2\\ 7/36 & 5/36 & 7/36 & \frac{5}{36} & \frac{7}{36} & \frac{5}{36}\end{pmatrix}\equiv\begin{pmatrix}-1 & 0 & 1 & 2\\ \frac{7}{36} & \frac{1}{3} & \frac{1}{3} & \frac{5}{36}\end{pmatrix}$; $M(X+Y)=-\frac{7}{36}+\frac{1}{3}+\frac{5\cdot 2}{36}=\frac{5}{12}$

$(X+Y)^2:\begin{pmatrix}(-1)^2 & 0^2 & 1^2 & 2^2\\ 7/36 & 1/3 & 1/3 & 5/36\end{pmatrix}\equiv\begin{pmatrix}0 & 1 & 4\\ \frac{1}{3} & \frac{19}{36} & \frac{5}{36}\end{pmatrix}$ $\frac{131}{144}\simeq 0.91$

$D^2(X+Y)=M\left((X+Y)^2\right)-M^2(X+Y)=0\cdot\frac{1}{3}+1\cdot\frac{19}{36}+4\cdot\frac{5}{36}-\left(\frac{5}{12}\right)^2=\frac{39}{36}-\frac{25}{144}$ ~~≃ 1.08~~

Metoda 2 X, Y indep $\Rightarrow D^2(X+Y)=D^2(X)+D^2(Y)=M(X^2)-M^2(X)+M(Y^2)-M^2(Y)=$

$=(-1)^2\cdot\frac{1}{3}+1^2\cdot\frac{1}{3}-0^2+1^2\cdot\frac{5}{12}-\left(\frac{5}{12}\right)^2=\frac{2}{3}+\frac{5}{12}-\frac{25}{144}=\frac{131}{144}\simeq 0.91.$

4° $P(-1\le X+Y\le 1)=P(X+Y=-1)+P(X+Y=0)+P(X+Y=1)=\frac{7}{36}+\frac{1}{3}+\frac{1}{3}=\frac{31}{36}$

5° $X^2:\begin{pmatrix}(-1)^2 & 0^2 & 1^2\\ 1/3 & 1/3 & 1/3\end{pmatrix}=\begin{pmatrix}0 & 1\\ 1/3 & 2/3\end{pmatrix}\Rightarrow X^{2n}=\begin{pmatrix}0 & 1\\ 1/3 & 2/3\end{pmatrix}$; $X^{2n+1}=X$

6° $Y^m=Y$; 7° $M_0(X)\in\{-1,0,1\}$; 8° $X^{10}+Y^{25}=\begin{pmatrix}0 & 1\\ 1/3 & 2/3\end{pmatrix}+\begin{pmatrix}0 & 1\\ 7/12 & \frac{5}{12}\end{pmatrix}$

$=\begin{pmatrix}0 & 1 & 2\\ \frac{1}{3}\cdot\frac{7}{12} & \frac{2}{3}\cdot\frac{5}{12}+\frac{2}{3}\cdot\frac{7}{12} & \frac{2}{3}\cdot\frac{5}{12}\end{pmatrix}=\begin{pmatrix}0 & 1 & 2\\ \frac{7}{36} & \frac{19}{36} & \frac{5}{18}\end{pmatrix}$

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(9°) $X^{11}Y^{16}: \begin{pmatrix} -1 & 0 & 1 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ \frac{7}{12} & \frac{5}{12} \end{pmatrix} = \begin{pmatrix} -1\cdot 0 & -1\cdot 1 & 0\cdot 0 & 0\cdot 1 & 1\cdot 0 & 1\cdot 1 \\ \frac{7}{36} & \frac{5}{36} & \frac{7}{36} & \frac{5}{36} & \frac{7}{36} & \frac{5}{36} \end{pmatrix}$

$X^{11}Y^{16}: \begin{pmatrix} -1 & 0 & 1 \\ \frac{5}{36} & \frac{13}{18} & \frac{5}{36} \end{pmatrix}$;

(10°) $H(X) = \frac{1}{3}\log_2 3 + \frac{1}{3}\log_2 3 + \frac{1}{2}\log_2 3 = \log_2 3$

Ex. 3 $X: \begin{pmatrix} 1 & 3 & 5 & \dots & 2n+1 & \dots \\ \frac{p}{2} & p^2 & \frac{3}{2}p^3 & \dots & \frac{n}{2}p^n & \dots \end{pmatrix} = \begin{pmatrix} 2n-1 \\ \frac{n}{2}p^n \end{pmatrix}_{n\geq 1}$

Să se determine: (1°) $p = ?$ (2°) $M(X)$; (3°) $D^2(X)$;

(4°) m minim, $m \in \mathbb{N}$ a.î. $P(X \geq m) \leq 0.2$.

Rezolvare (1°) $\sum_{n=1}^{\infty} \frac{n}{2} p^n = 1 \iff \frac{1}{2}\sum_{n=1}^{\infty} n p^n = 1 \iff \sum_{n=0}^{\infty} n p^n = 2 \iff$

$\iff \sum_{n=0}^{\infty} \frac{n}{\left(\frac{1}{p}\right)^n} = 2 \iff \mathcal{Z}\{n\}\left(\frac{1}{p}\right) = 2 \iff$

$\iff \frac{1/p}{\left(\frac{1}{p}-1\right)^2} = 2 \iff p = 2(1-p)^2 \iff 2p^2 - 5p + 2 = 0$

$\iff p_1 = 1/2;\ p_2 = 2 \xrightarrow{p\in(0,1)} \boxed{p = \frac{1}{2}}$.

$X: \begin{pmatrix} 2n-1 \\ \frac{n}{2^{n+1}} \end{pmatrix}_{n\geq 1} = \begin{pmatrix} 1 & 3 & 5 & \dots & 2n-1 & \dots \\ \frac{1}{4} & \frac{1}{4} & \frac{3}{16} & \dots & \frac{n}{2^{n+1}} & \dots \end{pmatrix}$

$\mathcal{Z}\{x(n)\}(z) = \sum_{n=0}^{\infty} \frac{x(n)}{z^n}$

$\mathcal{Z}\{n\}(z) = \frac{z}{(z-1)^2}$

$\mathcal{Z}\{n^2\}(z) = \frac{z(z+1)}{(z-1)^3}$

$\mathcal{Z}\{n^3\}(z) = \frac{z(z^2+4z+1)}{(z-1)^4}$

(2°) $M(X) = \sum_{n=1}^{\infty}(2n-1)\frac{n}{2^{n+1}} = \sum_{n=1}^{\infty} \frac{2n^2-n}{2^{n+1}} = \sum_{n=0}^{\infty} \frac{2n^2-n}{2^n\cdot 2} =$

$= \frac{1}{2}\left(2\sum_{n=0}^{\infty}\frac{n^2}{2^n} - \sum_{n=0}^{\infty}\frac{n}{2^n}\right) = \frac{1}{2}\left(2\mathcal{Z}\{n^2\}(2) - \mathcal{Z}\{n\}(2)\right) =$

$= \frac{1}{2}\left(2\frac{z(z+1)}{(z-1)^3} - \frac{z}{(z-1)^2}\right)\Big|_{z=2} = \frac{1}{2}\left(2\frac{2\cdot 3}{(2-1)^3} - \frac{2}{(2-1)^2}\right) = \frac{1}{2}10 = 5$

(3°) $D^2(X) = M(X^2) - M^2(X)$; $X^2: \begin{pmatrix} (2n-1)^2 \\ n/2^{n+1} \end{pmatrix}_{n\geq 1}$

$$M(X^2)=\sum_{n=1}^{\infty}(2n-1)^2\cdot\frac{n}{2^{n+1}}=\frac{1}{2}\sum_{n=0}^{\infty}\frac{n(2n-1)^2}{2^n}=\frac{1}{2}Z\{n(2n-1)^2\}(2)=$$

$$=\frac{1}{2}Z\{4n^3-4n^2+n\}(2)=\frac{1}{2}\left[4Z\{n^3\}(2)-4Z\{n^2\}(2)+Z\{n\}(2)\right]=$$

$$=\frac{1}{2}\left[4\frac{z(z^2+4z+1)}{(z-1)^4}-4\cdot\frac{z(z+1)}{(z-1)^3}+\frac{z}{(z-1)^2}\right]_{z=2}=$$

$$=\cancel{\frac{1}{2}(4\cdot 6-4}=\frac{1}{2}\left(4\cdot 2\cdot 13-4\cdot 2\cdot 3+2\right)=52-8+1=45$$

$D^2(X)=45-5^2=20.$

4° $1-P(X<m)\le 0.2 \Rightarrow P(X<m)\ge 0.8$

Şirul $a_m=P(X<m)$ este strict crescător; Avem $a_2=P(X<2)=$ $=P(X=1)=\frac{1}{4}<0.8$; $a_3=P(X<3)=P(X=1)+P(X=2)=\frac{1}{4}+\frac{1}{4}=\frac{1}{2}<0.8$

$a_4=P(X<4)=\frac{1}{4}+\frac{1}{4}+\frac{3}{16}=\frac{11}{16}<0.8$; $a_5=P(X<5)=\frac{11}{16}+\frac{1}{8}=\frac{13}{16}>0.8$, deci $m=5$.

Ex. 4 Temă

4.1. $X:\begin{pmatrix}-1 & 0 & 2 & 3\\ \frac{2}{5} & p/2 & \frac{1}{2} & p^2\end{pmatrix}$ (i) Sem. $p=1/3$

(ii) $M(X)=E(X)$; (iii) $D^2(X)=\mathrm{Var}\,X$ şi $\sigma(X)$;

(iii) $F(\sqrt{3})$; (iv) $P(-\frac{1}{5}\le X<2)$

(v) $Mo(X)$; (vi) $H(X)$

~ Ex 1

4.2. $X:\begin{pmatrix}1 & 2 & 3 \dots & n \dots\\ p & 4p^2 & 16p^3\dots & 2^{n-1}p^n\dots\end{pmatrix}=\begin{pmatrix}n\\ 2^{n-1}p^n\end{pmatrix}_{n\ge 1}$

(i) Sem. $p=1/5$; (ii) $M(X)=E(X)$; (iii) $\sigma(X)$; (iv) $P(X\ge e)$

4.3. V.a. $X:\begin{pmatrix}1 & 2 & 3 & 4\\ p^2 & \frac{5}{4}p & \frac{1}{3} & a\end{pmatrix}$ are $M(X)=\frac{125}{48}$

(i) Sem. $p=1/4$ şi $a=1/6$; (ii) $D^2(X)=\mathrm{Var}(X)$; (iii) $Mo(X)$; (iv) $H(X)$

4.4. Fie $X:\begin{pmatrix}a & 2a & 3a\dots & na\dots\\ \frac{2}{q} & \frac{2}{q^2} & \frac{2}{q^3}\dots & \frac{2}{q^n}\dots\end{pmatrix}=\begin{pmatrix}na\\ 2q^{-n}\end{pmatrix}_{n\ge 1}$ a.î. $M(X)=3$.

(i) Sem. $a=2$; $q=3$; (ii) $D^2(X)=\mathrm{Var}(X)$; (iii) $P(3<X\le 6)$; (iv) $P(X\ge 5)$

Ex. 5 Funcția $f:\mathbb{R}\to\mathbb{R}$, $f(x)=ax^2e^{-2|x|}$ este p.d.f. pentru o v.a. 1D continuă, notată X.

(i) Să se determine $a\in\mathbb{R}$; (ii) ~~M(X)~~ $M_n(X)$, $n\in\mathbb{N}$; (iii) $D^2(X)=Var(X)$

(iv) $Mo(X)$; (vi) $P(0\le X\le \ln 3)$; (v) $Me(X)$

R. (i) $\int_{-\infty}^{\infty} f(x)\,dx=1 \underset{f(-x)=f(x)}{\overset{f\text{ pară}}{\Longleftrightarrow}} 2a\int_0^{\infty} f(x)\,dx=1 \Leftrightarrow$

$$\begin{cases}\mathcal{L}\{f(x)\}(s)=\int_0^{\infty} f(x)e^{-sx}dx\\ \mathcal{L}\{x^n\}(s)=\frac{n!}{s^{n+1}}\end{cases}$$

$\Leftrightarrow 2a\int_0^{\infty} x^2e^{-2x}dx=1 \Leftrightarrow 2a\mathcal{L}\{x^2\}(2)=1 \Leftrightarrow$

$\Leftrightarrow 2a\left.\frac{2!}{s^3}\right|_{s=2}=1 \Leftrightarrow \frac{a}{2}=1 \Leftrightarrow \boxed{a=2}$.

(ii) $M_n(X)=\int_{-\infty}^{\infty} x^n f(x)\,dx = a\int_{-\infty}^{\infty} x^{n+2}e^{-2|x|}dx$ OBS. $M_0(X)=1$

n impar $g(x)=x^{n+2}e^{-2|x|}$ impară deoarece $g(-x)=a(-x)^{n+2}e^{-2|-x|}=$

$=-ax^{n+2}e^{-2|x|}=-g(-x) \Longrightarrow M_n(X)=a\int_{-\infty}^{\infty} g(x)\,dx=0$

n par $g(x)=x^{n+2}e^{-2|x|}$ este funcție pară deoarece $g(-x)=g(x) \Rightarrow$

$\Rightarrow M_n(X)=2\int_{-\infty}^{\infty} x^{n+2}e^{-2|x|}dx = 2\cdot 2\int_0^{\infty} x^{n+2}e^{-2x}dx = 4\cdot\mathcal{L}\{x^{n+2}\}(2)=$

$=2^2\cdot\left.\frac{(n+2)!}{s^{n+3}}\right|_{s=3} = 2^2\cdot\frac{(n+2)!}{2^{n+3}} = \frac{(n+2)!}{2^{n+1}}$; Verif. $M_0(X)=\frac{2!}{2^1}=1$

(iii) $D^2(X)=Var(X)=M(X^2)-M^2(X)=M_2(X)-M_1^2(X) \xrightarrow[n=1;\,n=2]{(ii)}$

$\overset{(ii)}{=} \frac{4!}{2^3}-\left(\frac{3!}{2^2}\right)^2=3-\left(\frac{3}{2}\right)^2=3-\frac{9}{4}=\frac{3}{4}$.

(iv) $Mo(X)$, $x\ge 0 \Rightarrow f(x)=2x^2e^{-2x} \Rightarrow f'(x)=2(2xe^{-2x}-x^2\cdot 2e^{-2x})$

$\Rightarrow f'(x)=4xe^{-2x}(1-x)$

$x<0 \Rightarrow f(x)=2x^2e^{2x} \Rightarrow f'(x)=2(2xe^{2x}+x^2\cdot 2e^{2x})=4xe^{2x}(1+x)$

OBS. f pară $\Rightarrow f'$ impară, deci pt. $x<0 \Rightarrow f'(x)=-f'(-x)=-4\cdot(-x)e^{2x}(1+x)=$
$=4xe^{2x}(1+x)$

Deci $f'(x)=\begin{cases}4xe^{2x}(1+x); & x<0\\ 4xe^{2x}(1-x); & x\ge 0\end{cases}$. Observăm că $f'_s(0)=f'_d(0)=0$
deci $f'(0)=0$

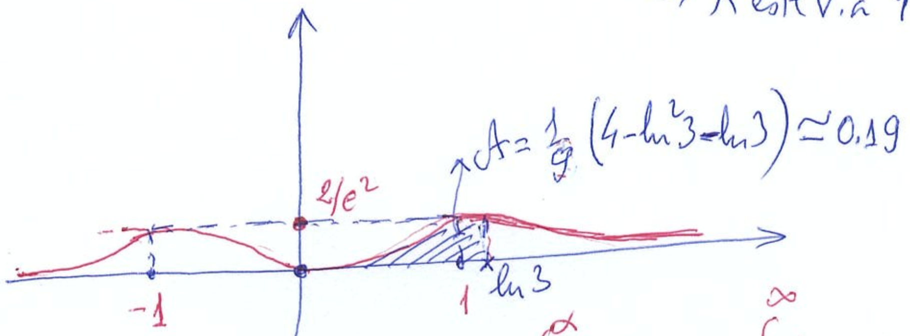
$f'(x)=0 \Rightarrow \begin{cases} x<0 \Rightarrow 1+x=0 \Rightarrow x=-1\\ x\ge 0 \Rightarrow x=0 \text{ sau } x=1\end{cases} \Rightarrow x\in\{-1,0,1\}$

x	$-\infty$		-1		0		1		$+\infty$
$f'(x)$		$+$	0	$-$	0	$+$	0	$-$	
$f(x)$	0	$\nearrow$	$\frac{2}{e^2}$	$\searrow$	0	$\nearrow$	$\frac{2}{e^2}$	$\searrow$	0

Deci $f(x)$ are 2 puncte de maxim $x_1=-1$ şi $x_2=1$ $\Rightarrow$

$\Rightarrow Mo(X)\in\{-1,1\}$

$\Rightarrow X$ este v.a 1D continuă plurimodală



$$\boxed{\int f(x)g'(x)dx = f(x)g(x)-\int f'(x)g(x)dx}$$

(v) Notăm $\alpha = Me(X) \Rightarrow \int_{-\infty}^{\alpha} f(x)dx=\int_{\alpha}^{\infty} f(x)dx=\frac{1}{2} \Longleftrightarrow F(\alpha)=\frac{1}{2}$, unde $F(x)=\int_{-\infty}^{x} f(t)dt$. Deoarece $F'(x)=f(x)>0 \Rightarrow F$ este strict crescătoare pe $\mathbb{R}$, $0\le F(x)\le 1$, deci ecuaţia $F(\alpha)=\frac{1}{2}$ are soluţie unică.

Deoarece f este pară $\Rightarrow G_f$ este simetric faţă de $Oy \Rightarrow \alpha=0 \Rightarrow Me(X)=0$

(vi) $P(0\le X\le \ln 3)=\int_0^{\ln 3} f(x)dx = 2\int_0^{\ln 3} x^2 e^{-2x}dx = 2\int_0^{\ln 3} x^2\left(\frac{e^{-2x}}{-2}\right)'dx =$

$= \cancel{2}\left[x^2\cdot\frac{e^{-2x}}{-\cancel{2}}\Big|_0^{\ln 3}+\frac{2}{\cancel{2}}\int_0^{\ln 3} x e^{-2x}dx\right]$. Deoarece $e^{-2\ln 3}=(e^{\ln 3})^{-2}=3^{-2}=\frac{1}{9}$,

obţinem: $P(0\le X\le \ln 3)=-\frac{1}{9}\ln^2 3+\cancel{2}\int_0^{\ln 3} x\cdot\left(\frac{e^{-2x}}{-\cancel{2}}\right)'dx =$

$=-\frac{1}{9}\ln^2 3 - xe^{-2x}\Big|_0^{\ln 3}+\int_0^{\ln 3}\underbrace{x'e^{-2x}}_{e^{-2x}}dx = -\frac{1}{9}\ln^2 3-\frac{1}{9}\ln 3-\frac{e^{-2x}}{2}\Big|_0^{\ln 3}=$

$=-\frac{1}{9}\ln^2 3-\frac{1}{9}\ln 3-\frac{1}{18}+\frac{1}{2}=\frac{1}{18}(8-2\ln 3-2\ln^2 3)=\frac{1}{9}(4-\ln 3-\ln^2 3)$

Ex. 6 Funcţia $f:\mathbb{R}\to\mathbb{R}$, $f(x)=\begin{cases}(a+b)\cos ax; & 0<x<\frac{\pi}{8}\\ b & ;\text{ în rest}\end{cases}$; $a,b\in\mathbb{R}$

este p.d.f. pentru o v.a. 1D continuă, notată X. Se cere:

(i) Să se determine a, b

(ii) $M(X)=E(X)$; (iii) $D^2(X)=Var(X)$; (iv) $F(x)$; (v) $Me(X)$; (vi) $Mo(X)$

(vii) $P(-1\le X\le\frac{\pi}{6})$; (viii) Funcţiile de repartiţie şi p.d.f. pentru v.a. 1D, $Y=3X+\pi$; $Z=X^4$; $W=e^X$

R (i) Din $\int_{-\infty}^{\infty} f(x)\,dx = 1 \Rightarrow \int_{-\infty}^{0} \underbrace{f(x)}_{b}\,dx + \int_{0}^{\pi/8} f(x)\,dx + \int_{\pi/8}^{\infty} \underbrace{f(x)}_{b}\,dx = 1 \Leftrightarrow$

$\Leftrightarrow bx\Big|_{-\infty}^{0} + bx\Big|_{0}^{\infty} + (a+b)\int_0^{\pi/8}\cos(ax)\,dx = b\cdot\infty + b\cdot\infty + \frac{a+b}{a}\sin ax\Big|_0^{\pi/8} - b\frac{\pi}{8} =$

$= 1 \Rightarrow b = 0$ și $\sin a\frac{\pi}{8} = 1 \Rightarrow b = 0$ și $a\frac{\pi}{8} = \frac{\pi}{2} + 2n\pi \,\Big|\, \frac{8}{\pi} \Rightarrow$

$\Rightarrow b = 0$ și $a = 16n+4,\ n\in\mathbb{Z}$, deci $f(x) = (16n+4)\cos(16n+4)x,\ 0<x<\frac{\pi}{8}$

Deoarece $f(x)\ge 0,\ \forall x\in\mathbb{R} \Rightarrow (16n+4)x \in \left[0,\frac{\pi}{2}\right],\ \forall x\in\left(0,\frac{\pi}{8}\right) \Rightarrow$

$\Rightarrow 0 \le (16n+4)\frac{\pi}{8} \le \frac{\pi}{2} \Rightarrow 0 \le 16n+4 \le 4 \Rightarrow n = 0$, deci

$a = 16n+4 = 4 \Rightarrow \underline{a = 4,\ b = 0} \Rightarrow \boxed{f(x) = \begin{cases} 4\cos(4x); & 0<x<\frac{\pi}{8} \\ 0; & \text{în rest.} \end{cases}}$

(ii) $M(X) = E(X) = \int_{-\infty}^{\infty} x f(x)\,dx = 4\int_0^{\pi/8} x\cos 4x\,dx = \cancel{4}\int_0^{\pi/8} x\cdot\left(\frac{\sin 4x}{\cancel{4}}\right)'dx =$

$= x\sin 4x\Big|_0^{\pi/8} - \int_0^{\pi/8} 1\cdot\sin 4x\,dx = \frac{\pi}{8}\underbrace{\sin\frac{\pi}{2}}_{=1} + \frac{\cos 4x}{4}\Big|_0^{\pi/8} = \frac{\pi}{8} + \frac{1}{4}\Big(\underbrace{\cos\frac{\pi}{2}}_{0} - \underbrace{\cos 0}_{1}\Big)$

Deci $M(X) = E(X) = \frac{\pi}{8} - \frac{1}{4} = \frac{1}{8}(\pi-2)$

(iii) $D^2(X) = M(X^2) - M^2(X)$

$M(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx = 4\int_0^{\pi/8} x^2\cos 4x\,dx = \cancel{4}\int_0^{\pi/8} x^2\cdot\left(\frac{\sin 4x}{\cancel{4}}\right)'dx =$

$= x^2\sin 4x\Big|_0^{\pi/8} - 2\int_0^{\pi/8} x\sin 4x\,dx = \left(\frac{\pi}{8}\right)^2\sin\frac{\pi}{2} - \cancel{2}\int_0^{\pi/8} x\cdot\left(\frac{\cos 4x}{-\cancel{4}2}\right)'dx =$

$= \frac{\pi^2}{64}\cdot 1 + \frac{1}{2}\left(x\cos 4x\Big|_0^{\pi/8} - \int_0^{\pi/8} 1\cdot\cos 4x\,dx\right) = \frac{\pi^2}{64} + \frac{1}{2}\Big(\frac{\pi}{8}\underbrace{\cos\frac{\pi}{2}}_{0} - 0\Big) -$

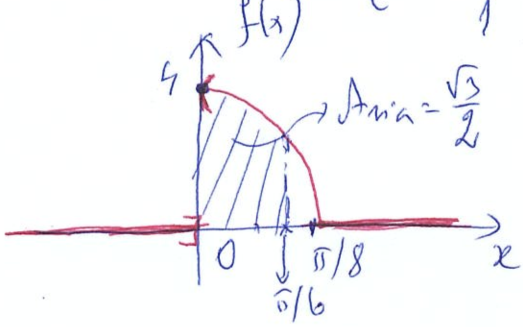
$- \frac{1}{2}\cdot\frac{\sin 4x}{4}\Big|_0^{\pi/8} = \frac{\pi^2}{64} - \frac{1}{8}\Big(\underbrace{\sin\frac{\pi}{2}}_{1} - 0\Big) = \frac{\pi^2-8}{64}.$

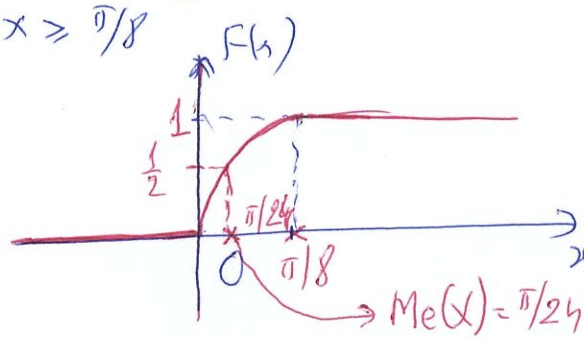
Deci $D^2(X) = \frac{\pi^2-8}{64} - \frac{(\pi-2)^2}{64} = \frac{4\pi-12}{64} = \frac{\pi-3}{16}$

(iv) $F(x) = \int_{-\infty}^{x} f(t)\,dt$

- Dacă $x \le 0 \Rightarrow F(x) = 0$
- Dacă $0 < x < \frac{\pi}{8} \Rightarrow F(x) = \int_{-\infty}^{0} 0\,dt + \int_0^x 4\cos 4t\,dt = \sin 4x$
- Dacă $x \ge \frac{\pi}{8} \Rightarrow F(x) = \int_{-\infty}^{0} 0\,dt + \int_0^{\pi/8} 4\cos 4t\,dt + \int_{\pi/8}^{x} 0\,dt = \cancel{4}\cdot\frac{\sin 4x}{\cancel{4}}\Big|_0^{\pi/8} = 1$

Deci $F(x) = \begin{cases} 0; & x \le 0 \\ \sin 4x, & \text{dacă } 0 < x < \frac{\pi}{8} \\ 1; & \text{dacă } x \ge \pi/8 \end{cases}$





(v) $\underline{Me(X) = \alpha} \iff F(\alpha) = \frac{1}{2} \iff \sin 4\alpha = \frac{1}{2} \Rightarrow 4\alpha = \frac{\pi}{6} \Rightarrow \alpha = \pi/24 \Rightarrow$

$\Rightarrow Me(X) = \pi/24$

(vi) $Mo(X)$ = abscisa punctului de maxim pt. $f(x)$, dacă $\exists$.

Din grafic ~~se~~ observă că $\sup\{f(x) : x \in \mathbb{R}\} = 4$, dar $\nexists\, x_0 \in \mathbb{R}$ a.i. $f(x_0) = 4$,

deci $Mo(X)$ NU EXISTĂ.

(vii) $P(-1 \le X \le \frac{\pi}{6}) = \int_{-1}^{\pi/6} f(x)dx = \int_0^{\pi/6} 4\cos 4x\, dx = \sin 4x \Big|_0^{\pi/6} = \sin \frac{2\pi}{3} = \frac{\sqrt{3}}{2}$

(viii) • Pentru $Y = 3X + \pi$. $G(x) = F_Y(x) = P(Y < x) = P(3X + \pi < x) = P(X < \frac{x - \pi}{3}) =$

$= F_X\left(\frac{x-\pi}{3}\right) = F\left(\frac{x-\pi}{3}\right) \Rightarrow G(x) = \begin{cases} 0; & \frac{x-\pi}{3} \le 0 \\ \sin 4\frac{x-\pi}{3}; & 0 < \frac{x-\pi}{3} < \frac{\pi}{8} \\ 1; & \frac{x-\pi}{3} > \pi/8 \end{cases} \Rightarrow$

$\Rightarrow G(x) = \begin{cases} 0; & x \le \pi \\ \sin \frac{4}{3}(x - \pi); & \pi < x < \frac{11\pi}{8} \\ 1; & x \ge \frac{11\pi}{8} \end{cases} \Rightarrow g(x) = G'(x) = \begin{cases} 0; & x < \pi \\ \frac{4}{3}\cos\frac{4}{3}(x-\pi); & \pi < x < \frac{11\pi}{8} \\ 0; & x > 11\pi/8 \end{cases}$

• Pentru $Z = X^4 \Rightarrow H(x) = F_Z(x) = P(Z < x) = P(X^4 < x) = \begin{cases} 0; & x \le 0 \\ P(X < \sqrt[4]{x}); & x > 0 \end{cases}$

$H(x) = \begin{cases} 0; & x \le 0 \\ F(\sqrt[4]{x}); & x > 0 \end{cases} = \begin{cases} 0; & x \le 0 \\ \sin(4\sqrt[4]{x}); & 0 < x < \frac{\pi^4}{324} \\ 1; & x \ge \frac{\pi^4}{1024} \end{cases}$

$h(x) = H'(x) = \begin{cases} 0; & x < 0 \text{ sau } x > \pi^4/1024 \\ \frac{1}{\sqrt[4]{x^3}}\cos(4\sqrt[4]{x}); & 0 < x < \pi^4/1024 \end{cases}$

• Pentru $W = e^X$. $L(x) = F_W(x) = P(W < x) = P(e^X < x) = \begin{cases} 0; & x \le 0 \\ P(X < \ln x); & x > 0 \end{cases} =$

$= \begin{cases} 0; & x \le 0 \\ F(\ln x); & x > 0 \end{cases} = \begin{cases} 0; & x \le 0 \text{ sau } (\ln x \le 0, x > 0) \\ \sin(4\ln x); & (0 < \ln x < \pi/8; x > 0) \\ 1; & \ln x \ge \pi/8; x > 0 \end{cases} = \begin{cases} 0; & x \le 1 \\ \sin(4\ln x); & 1 < x < e^{\pi/8} \\ 1; & x \ge e^{\pi/8} \end{cases}$

$l(x) = L'(x) = \begin{cases} 0; & x < 1 \text{ sau } x \ge e^{\pi/8} \\ (4/x)\cos(4\ln x); & 1 < x < e^{\pi/8} \end{cases}$

Ex. 7 Temă ($\sim$ Ex. 5, Ex. 6)

7.1. $f(x) = \begin{cases} 0; & x \le 0 \\ a x e^{-5x}; & x > 0 \end{cases}$ (i) $a = ?$; (ii) $M_n(X)$; (iii) $D^2(X) = Var(X)$; (iv) $M_o(X)$; (v) $P(|X| < \ln 2)$; (vi) $Ex(X)$

7.2. $f(x) = \begin{cases} 0; & x \le 0 \\ a(2x+5) e^{-2x}; & x > 0 \end{cases}$ (i) $a = ?$; (ii) $M_n(X)$; (iii) $\sigma(X)$; (iv) $M_o(X)$; (v) $P(0 \le X \le \ln 3)$; (vi) $As(X)$; (vii) $Ex(X)$

7.3. $f(x) = \begin{cases} a \sin 3x; & 0 \le x \le \pi/3 \\ 0; & \text{în rest} \end{cases}$; (i) $a = ?$; (ii) $M_n(X) = E(X)$; (iii) $\sqrt{Var(X)}$ $Var(X) = D^2(X)$; (iv) $M_o(X)$; (v) $Me(X)$; (vi) $P\left(-1 < X < \frac{\pi}{12}\right)$

7.4. $f(x) = a \sqrt{x}\, e^{-3x} u(x) = \begin{cases} 0; & x \le 0 \\ a\sqrt{x}\, e^{-3x}; & x > 0 \end{cases}$ (i) $a = ?$; (ii) $M_n(X)$; (iii) $\sigma(X)$; (iv) $M_o(X)$; (v) $Ex(X)$

7.5. $f(x) = \begin{cases} a \ln \frac{2}{x}; & 0 < x < 2 \\ 0; & \text{în rest} \end{cases}$; (i) $a = ?$; (ii) $M_n(X)$; (iii) $\sigma(X)$; (iv) $M_o(X)$; (v) $P(0 < X < \sqrt{e})$

REMEMBER.

TRANSFORMATA LAPLACE

$f \in \mathcal{O} \Rightarrow F(s) = \mathcal{L}\{f(t)\}(s) = \int_0^\infty f(t) e^{-st}\, dt$ $(\operatorname{Re} s > \sigma_0(f))$

$\mathcal{L}\{t^\alpha\}(s) = \frac{\Gamma(\alpha+1)}{s^{\alpha+1}}$; $\alpha > -1$; $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x}\, dx$, $\alpha > 0$

$\Gamma(n) = (n-1)!$; $n \in \mathbb{N}^*$

$\Gamma(\alpha) = (\alpha-1)\Gamma(\alpha-1)$, $\alpha > 1$

$\mathcal{L}\{t^n\}(s) = \frac{n!}{s^{n+1}}$; $\operatorname{Re} s > 0$

$\mathcal{L}\{e^{at}\}(s) = \frac{1}{s-a}$

$$\begin{cases} \mathcal{L}\{t^{n+1/2}\}(s) = \dfrac{\Gamma\left(n+\frac{3}{2}\right)}{s^{n+3/2}} \\ \Gamma\left(n+\frac{1}{2}\right) = \dfrac{(2n)!}{2^{2n} n!}\sqrt{\pi} \\ \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \end{cases}$$

$\mathcal{L}\{t^n e^{at}\}(s) = \frac{n!}{(s-a)^{n+1}}$

$\mathcal{L}\{\sin at\}(s) = \frac{a}{s^2+a^2}$

$\mathcal{L}\{\cos at\}(s) = \frac{s}{s^2+a^2}$

TRANSFORMATA „Z"

$Z\{x(n)\}(z) = X(z) = \sum_{n=0}^{\infty} \frac{x(n)}{n!}$ $(x \in S_d^+)$

$Z\{x(n)\}(z) = \sum_{n=-\infty}^{\infty} \frac{x(n)}{n!}$ $x \in S_d$

Dirac $\delta_k(n) = \begin{cases} 1; & n = k \\ 0; & n \ne k \end{cases}$ $\delta_0(n) = \delta(n) = \begin{cases} 1; & n = 0 \\ 0; & n \ne 0 \end{cases}$

Heaviside $u(n) = \begin{cases} 0; & n < 0 \\ 1; & n \ge 0 \end{cases}$

$Z\{a^n u(n)\}(z) = \frac{z}{z-a} \Leftrightarrow Z^{-1}\left\{\frac{z}{z-a}\right\}(n) = a^n u(n)$

$Z\{n a^n u(n)\}(z) = \frac{az}{(z-a)^2} \Leftrightarrow Z^{-1}\left\{\frac{z}{(z-a)^2}\right\}(n) = n a^{n-1} u(n)$

$Z\{\delta_k(n)\}(z) = z^{-k} \Leftrightarrow Z^{-1}\{z^k\}(n) = \delta_{-k}(n)$

$Z\{n\} = \frac{z}{(z-1)^2}$; $Z\{n^2\} = \frac{z(z+1)}{(z-1)^3}$; $Z\{n^3\} = \frac{z(z^2+4z+1)}{(z-1)^4}$