EXAMEN MA

PROBLEME ELEMENTARE

① VARIABILE ALEATOARE 1D DISCRETE SIMPLE

ⓐ FIE $X = \begin{pmatrix} -1 & 1 & 2 & 3 \\ 2p^2 & \frac{1}{4} & \frac{3}{8} & p \end{pmatrix}$ V.A. 1D SIMPLĂ

ⓘ $p = ?$

$\underbrace{p \in [0,1]}_{\text{DE CE?}}$

$$\boxed{2p^2 + \frac{1}{4} + \frac{3}{8} + p = 1}$$ → PROPRIETATE IMPORTANTĂ

$$2p^2 + p + \frac{5}{8} - 1 = 0 \quad \Big| \cdot 8$$

$$16p^2 + 8p - 3 = 0$$

$$p_{1,2} = \frac{-4 \pm 8}{16} \quad \begin{cases} p_1 = \frac{1}{4} \in [0,1] \\ p_2 = -\frac{3}{4} \notin [0,1] \end{cases} \Rightarrow \boxed{p = \frac{1}{4}}$$

$$X = \begin{pmatrix} -1 & 1 & 2 & 3 \\ \frac{1}{8} & \frac{1}{4} & \frac{3}{8} & \frac{1}{4} \end{pmatrix} \Rightarrow X^2 = \begin{pmatrix} 1 & 1 & 4 & 9 \\ \underbrace{\frac{1}{8} \quad \frac{1}{4}}_{+} & & \frac{3}{8} & \frac{1}{4} \end{pmatrix} = \begin{pmatrix} 1 & 4 & 9 \\ \frac{3}{8} & \frac{3}{8} & \frac{1}{4} \end{pmatrix}$$

ⓘⓘ $M(X) = E(X)$

$$M(X) = E(X) = \boxed{\sum_{i=1}^{4} p_i x_i} = -1 \cdot \frac{1}{8} + 1 \cdot \frac{1}{4} + 2 \cdot \frac{3}{8} + 3 \cdot \frac{1}{4} = \frac{13}{8} = \boxed{1.625}$$

ⓘⓘⓘ $D^2(X) = Var(X)$

$$D^2(X) = \boxed{M(X^2) - M^2(X)}$$

$$M(X^2) = \sum_{i=1}^{3} p_i x_i = 1 \cdot \frac{3}{8} + 4 \cdot \frac{3}{8} + 9 \cdot \frac{1}{4} = \frac{33}{8}$$

$$D^2 X = \frac{33}{8} - 1.625^2$$

$$\boxed{D^2 X \cong 1.49}$$

(iv) $\sigma(X) = ?$

$$\sigma(X) = \boxed{\sqrt{D^2(X)}} = \sqrt{\frac{95}{64}} = \frac{\sqrt{95}}{8} \approx \boxed{1.24}$$

$$\hookrightarrow = \sqrt{1.49}$$

(v) $P(\underbrace{\ln 2}_{0.7} < X \leq 2) \overset{}{\underset{\ln 2 = 0.7}{=\!=}} P(X=1) + P(X=2) = \frac{\cancel{1}^{2}}{4} + \frac{3}{8} = \frac{5}{8}$

! INTERVALUL E $0.7 < X \leq 2$
DECI DIN X POT LUA
DOAR 1 ȘI 2

$$X = \begin{pmatrix} -1 & 1 & 2 & 3 \\ \frac{1}{8} & \frac{1}{4} & \frac{3}{8} & \frac{1}{4} \end{pmatrix}$$

(vi) $F(e)$

$$F(e) = \boxed{P(X<e)} = P(X=-1) + P(X=1) + P(X=2) =$$

$\downarrow$

$X < 2.71$

$$= \frac{1}{8} + \frac{\cancel{1}^{2}}{4} + \frac{3}{8} = \frac{6}{8} = \boxed{\frac{3}{4}}$$

(vii) $M_0(X) = x_s \iff p_s = \boxed{\max\{p_i \; ; i \in I\}}$

$$\max\{p_i \; ; i \in I\} = \max\left\{\frac{1}{8} \;\; \frac{1}{4} \;\; \frac{3}{8} \;\; \frac{1}{4}\right\} = \frac{3}{8} \Rightarrow \boxed{M_0(X) = 2}$$

! DETERMIN MAXIMUL CA VAL. DE AICI ȘI $M_0(X)$ E POZIȚIA

(viii) ENTROPIA $H(X)$

$$H(X) = \boxed{\sum_{i=1}^{n} p_i \log_2 \frac{1}{x_i}} = \sum_{i=1}^{n} p_i \log_2 (x_i)^{-1} \qquad \boxed{H(X) \leq \log_2 n}$$

$$X = \begin{pmatrix} -1 & 1 & 2 & 3 \\ \frac{1}{8} & \frac{1}{4} & \frac{3}{8} & \frac{1}{4} \end{pmatrix}$$

$$H(X) = \frac{1}{8} \log_2\left(\frac{1}{8}\right)^{-1} + \frac{1}{4} \log_2\left(\frac{1}{4}\right)^{-1} + \frac{3}{8} \log_2\left(\frac{3}{4}\right)^{-1} + \frac{1}{4} \log_2\left(\frac{1}{4}\right)^{-1}$$

$$H(X) = \frac{5}{2} - \frac{3}{8} \log_2 3 \leq 2 = \log_2 4$$

$$\boxed{H(X) = \frac{1}{8}(20 - 3\log_2 3)}$$

② LANȚ MARKOV

ⓐ SE CONSIDERĂ LANȚUL MARKOV $(X_k)_{k\geq 0}$ ASOCIAT TRIPLETULUI $(S, p^{(0)}, \Pi)$ UNDE $S=\{1,2,3\}$

$$p^{(0)} = \left\{ a^2, a, \tfrac{5}{9} \right\}$$

$$\Pi = \begin{pmatrix} a & b & 1/2 \\ \frac{1}{10} & \frac{1}{2} & \frac{2}{5} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{pmatrix}$$

ⓘ DET $a, b \in \mathbb{R}$

$$\begin{cases} a^2 + a + \frac{5}{9} = 1 \;\Big|\cdot 9 \\ a + b + \frac{1}{2} = 1 \\ a, b \in [0,1] \end{cases} \qquad \begin{cases} 9a^2 + 9a + 5 = 9 \\ a + b + \frac{1}{2} = 1 \\ a, b \in [0,1] \end{cases} \Rightarrow a = \frac{-9 \pm \sqrt{81+144}}{18}$$

$$\Downarrow$$

$$a = \frac{-9-15}{18} < 0$$

$$\boxed{a = \frac{1}{3}}$$

$$\frac{1}{3} + b + \frac{1}{2} = 1 \qquad b = \overset{6}{1} - \overset{2}{\frac{1}{3}} - \overset{3}{\frac{1}{2}}$$

$$b = \frac{6-2-3}{6} = \boxed{\frac{1}{6}}$$

$S = \{1,2,3\}$

$$p^{(0)} = \left\{ \tfrac{1}{9}, \tfrac{1}{3}, \tfrac{5}{9} \right\}$$

$$\Pi = \begin{pmatrix} \frac{1}{3} & \frac{1}{6} & \frac{1}{2} \\ \frac{1}{10} & \frac{1}{2} & \frac{2}{5} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{pmatrix}$$

(ii) PROBABIL. CA LANȚUL SĂ EVOLUEZE PE TRAIECTORIA (2 1 3 3)

$$P(X_0=2, X_1=1, X_2=3, X_3=3)=$$

$$=P(X_0=2 \cap X_1=1 \cap X_2=3 \cap X_3=3)=$$

$$=P(X_0=2)\cdot P((X_1=1)/(X_0=2))\cdot P((X_2=3)/X_1=1,\cancel{X_0=2})\cdot P((X_3=3)/X_2=3,\cancel{X_1=1},\cancel{X_0=2})$$

SE REȚINE DOAR ULTIMA VALOARE

$$=P(X_0=2)\cdot P((X_1=1)/(X_0=2))\cdot P((X_2=3)/X_1=1)\cdot P((X_3=3)/X_2=3)=$$

$$=p_2^{(0)}\cdot \underbrace{p_{21}\cdot p_{13}\cdot p_{33}}_{\text{LE IAU DIN }\Pi}=$$

$$p^{(0)}=\left\{\underset{(1)}{\frac{1}{9}},\underset{(2)}{\frac{1}{3}},\underset{(3)}{\frac{5}{9}}\right\}$$

$$=\frac{1}{3}\cdot\frac{1}{10}\cdot\frac{1}{2}\cdot\frac{1}{2}=\frac{1}{120}$$

(iii) $$P(\underbrace{X_{200}=1, X_{199}=2, X_{198}=3}_{A}/\underbrace{X_{197}=2}_{B})=P(A/B)=$$

$$P(A/B)=\frac{P(A\cap B)}{P(B)}$$

$$P(B/A)=\frac{P(A\cap B)}{P(A)}$$

$$P(A/B)=\frac{P(A)\cdot P(B/A)}{P(B)}$$

$$=\frac{P(A\cap B)}{P(B)}=$$

$$=\frac{P(X_{200}=1, X_{199}=2, X_{198}=3, X_{197}=2)}{P(X_{197}=2)}$$

$$=\frac{P(X_{197}=2, X_{198}=3, X_{199}=2, X_{200}=1)}{P(X_{197}=2)}=$$

$$=\frac{\cancel{P(X_{197}=2)}\cdot P(X_{198}=3/X_{197}=2)\cdot P(X_{199}=2/X_{198}=3,\cancel{X_{197}=2})\cdot P(X_{200}=1/X_{199}=2,\cancel{\ldots})}{\cancel{P(X_{197}=2)}}$$

$$=p_{23}^{0}\cdot p_{32}^{0}\cdot p_{21}^{0}=\frac{2}{5}\cdot\frac{1}{4}\cdot\frac{1}{10}=\frac{1}{100}$$

(iv) $P(\underbrace{X_{1000}=1,\ X_{998}=2}_{A}\ /\ \underbrace{X_{997}=3}_{B}) = \dfrac{P(A\cap B)}{P(B)} =$

$$= \frac{P(X_{1000}=1,\ X_{998}=2,\ X_{997}=3)}{P(X_{997}=3)} = \frac{P(X_{997}=3,\ X_{998}=2,\ X_{1000}=1)}{P(X_{997}=3)}$$

$$= \frac{\cancel{P(X_{997}=3)}\cdot P(X_{998}=2\ /\ X_{997}=3)\cdot P(X_{1000}=1\ /\ X_{998}=2,\ \cancel{X_{997}=3})}{\cancel{P(X_{997}=3)}}$$

$$= P_{32}\cdot \boxed{P_{21}^{(1000-998)}} = P_{32}\cdot \underbrace{P_{21}^{2}}_{\Pi^2} = \frac{1}{4}\cdot(L_2+C_1) =$$

$$\Pi = \begin{pmatrix} \frac{1}{3} & \frac{1}{6} & \frac{1}{2} \\ \frac{1}{10} & \frac{1}{2} & \frac{2}{5} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{pmatrix}$$

$$= \frac{1}{4}\cdot \boxed{\begin{pmatrix} \frac{1}{10} & \frac{1}{2} & \frac{2}{5} \end{pmatrix}\cdot \begin{pmatrix} \frac{1}{3} \\ \frac{1}{10} \\ \frac{1}{4} \end{pmatrix}} = \frac{1}{4}\cdot\left(\frac{\overset{2)}{1}}{30} + \frac{\overset{3)}{1}}{20} + \frac{\overset{3)}{2}}{20}\right) =$$

$$= \frac{1}{4}\cdot\frac{11}{60} = \boxed{\frac{11}{240}}$$

(V) $P(X_{79}=2\ /\ X_{77}\neq 2,\ X_{77}\neq 1) =$ → DECI PRACTIC IAU O ALTĂ VALOARE

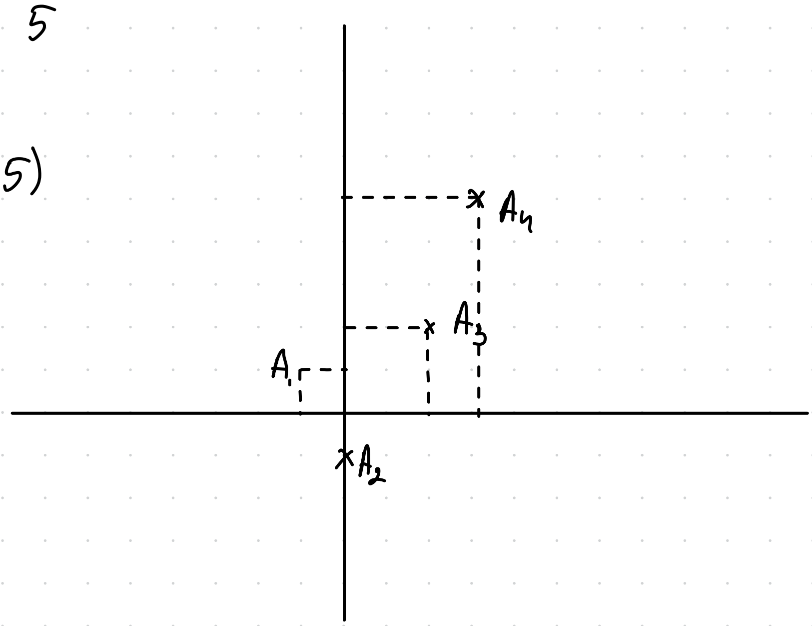
$$= P(X_{79}=2\ /\ X_{77}=3) = P_{32}^{(79-77)} = P_{32}^{2} = L_3 + C_2 =$$

$$= \begin{pmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{pmatrix}\cdot \begin{pmatrix} \frac{1}{6} \\ \frac{1}{2} \\ \frac{1}{4} \end{pmatrix} = \frac{1}{24} + \frac{1}{8} + \frac{1}{8} = \frac{1}{24} + \frac{\overset{6)}{1}}{4} = \frac{7}{24}$$

③ DREAPTA DE REGRESIE

ⓐ SĂ SE DETERMINE DREAPTA DE REGRESIE CORESPUNZĂTOARE SETULUI DE DATE EXPERIMENTALE

X	-1	0	2	3
y	1	-1	2	5



$A(-1,1)$ $B(0,-1)$ $C(2,2)$ $D(3,5)$

$d: ax+b \Rightarrow$ $B_1(-1,-a+b)$

$B_2(0,b)$

$B_3(2,2a+b)$

$B_4(3,3a+b)$

$$g(a,b) = A_1B_1^2 + A_2B_2^2 + A_3B_3^2 + A_4B_4^2$$

$$g(a,b) = (-a+b-1)^2 + (b+1)^2 + (2a+b-2)^2 + (3a+b-5)^2 \longrightarrow \text{MIN}$$

! SCAD Y DIN B ŞI A

DIN B_1 IAU $-a+b$ ŞI SCAD Y DIN A CARE E 1

DERIVEZ g ÎN FUNCŢIE DE a ŞI b

$$\begin{cases} \frac{1}{2}\cdot\frac{dg}{da} = (-a+b-1)(-1) + (2a+b-2)\cdot 2 + (3a+b-5)\cdot 3 = 0 \\ \frac{1}{2}\cdot\frac{dg}{db} = (-a+b-1) + (b+1) + (2a+b-2) + (3a+b-5) = 0 \end{cases}$$

$$\begin{cases} 14a + 4b = 18 \\ 4a + 4b = 7 \mid \cdot(-1) \end{cases} \Leftrightarrow \begin{cases} 10a = 11 \\ a+b = \frac{7}{4} \end{cases} \Rightarrow \begin{cases} a = \frac{4}{10} \\ b = \frac{13}{20} \end{cases}$$

$$\left.\begin{array}{l} a^* = \frac{11}{10} \\ b^* = \frac{13}{20} \end{array}\right\} \Rightarrow \boxed{d^*: \frac{11}{10}x + \frac{13}{20}}$$

4 LEGEA LUI GAUSS (REPARTIȚIA NORMALĂ)

a) FIE $X = N(3, 6)$. SĂ SE CALCULEZE

i) $P(0 \le x \le 7)$ $m = 3 \quad \sigma^2 = 6 \Rightarrow \sigma = \sqrt{6} = 2.45$

$$\boxed{P(a \le X \le b) = \phi\left(\frac{b-m}{\sigma}\right) - \phi\left(\frac{a-m}{\sigma}\right)}$$

$$P(0 \le x \le 7) = \phi\left(\frac{7-3}{\sqrt{6}}\right) - \phi\left(\frac{0-3}{\sqrt{6}}\right) = \phi\left(\frac{2\sqrt{6}}{3}\right) - \phi\left(-\frac{\sqrt{6}}{2}\right) =$$

$$= \phi(1.63) - (1 - \phi(1.22)) =$$

$$= \phi(1.63) + \phi(1.22) - 1 =$$

$$= 0.9484 + 0.8888 - 1 = 0.8372 = 83.72\%$$

ii) $P(|x| < 2)$

$$P(-2 < x < 2) = \phi\left(\frac{2-3}{\sqrt{6}}\right) - \phi\left(\frac{-2-3}{\sqrt{6}}\right) = \phi\left(-\frac{\sqrt{6}}{6}\right) - \phi\left(-\frac{5\sqrt{6}}{6}\right)$$

$$= 1 - \phi\left(\frac{\sqrt{6}}{6}\right) - \left(1 - \phi\left(\frac{5\sqrt{6}}{6}\right)\right) =$$

$$= -\phi\left(\frac{\sqrt{6}}{6}\right) + \phi\left(\frac{5\sqrt{6}}{6}\right) = 0.9773 - 0.6691 = 31.32\%$$

TIPURI DE PROBLEME COMPLEXE

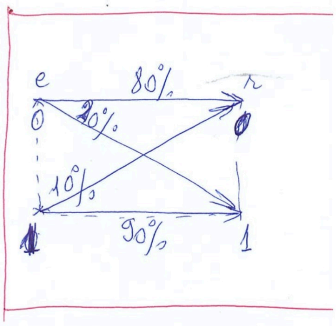
1 MODELAREA ZGOMOTULUI ÎN CANALE BINARE ȘI TERNARE

SEMINAR 2 EX. 5

CANAL BINAR

La intrarea unui canal binar de transmisie se emit semnalele 0 și 1, în raportul 4/5. În medie 20% din semnalele „0" se transmit eronat, iar 90% dintre semnalele „1" se transmit corect.
Să se determine:
(i) Probabilitatea recepționării semnalului „0"
(ii) Probabilitatea recepționării semnalului „1"
(iii) Probabilitatea ca, în cazul recepției unui „1", să se fi emis „0"
(iv) Probabilitatea recepției corecte
(v) Probabilitatea recepției eronate.

R. Schema modelării zgomotului este: →



Fie X_0, X_1 evenimentele care reprezintă emiterea lui „0", respectiv „1".
Fie Y_0, Y_1 evenimentele care reprezintă recepția lui „0" respectiv „1".
Din datele problemei avem $\frac{P(X_0)}{P(X_1)} = \frac{4}{5}$ și $P(X_0)+P(X_1)=1$, deoarece $\{X_0, X_1\}$ și $\{Y_0, Y_1\}$ reprezintă s.c.e.

De aici $\Rightarrow \frac{P(X_0)}{P(X_0)+P(X_1)} = \frac{4}{4+5} \Rightarrow \frac{P(X_0)}{1} = \frac{4}{9} \Rightarrow$

$\Rightarrow P(X_0) = \frac{4}{9},\ P(X_1) = \frac{5}{9}$ (1)

De asemenea, din schema de mai sus rezultă:

$$(2)\begin{cases} P(Y_0/X_0) = 80\% = \frac{80}{100} = \frac{4}{5} \\ P(Y_0|X_1) = 10\% = \frac{1}{10} \\ P(Y_1|X_0) = 20\% = \frac{20}{100} = \frac{1}{5} \\ P(Y_1|X_1) = 90\% = \frac{90}{100} = \frac{9}{10} \end{cases}$$

(i) ~~P(~~ Se cere $P(Y_0)$. Deoarece $\{X_0, X_1\}$ este un s.c.e, aplicând formula probabilității totale (f.p.t.) $\overset{(1)}{\underset{(2)}{\Longrightarrow}}$

$\overset{(1)}{\underset{(2)}{\Rightarrow}} P(Y_0) = P(X_0)\cdot P(Y_0|X_0) + P(X_1)\cdot P(Y_0/X_1) = \frac{4}{9}\cdot\frac{4}{5} + \frac{5}{9}\cdot\frac{1}{10} =$

$= \frac{16}{45} + \frac{5}{90} = \frac{37}{90} \simeq 0,4111\ldots \simeq 41,11\%$

(ii) Analog cu (i); $P(Y_1) = P(X_0)\cdot P(Y_1|X_0) + P(X_1)\cdot P(Y_1|X_1) =$

$= \frac{4}{9}\cdot\frac{1}{5} + \frac{5}{9}\cdot\frac{9}{10} = \frac{4}{45} + \frac{1}{2} = \frac{53}{90} \simeq 0.5888\ldots \simeq 58,88\%$

OBS. Deoarece $\{Y_0, Y_1\}$ este un s.c.e $\Rightarrow P(Y_1) = 1 - P(Y_0) = 1 - \frac{37}{90} = \frac{53}{90}$

(iii) Se cere $P(X_0/Y_1)$. Utilizăm formula lui Bayes, sau direct egalitatea $P(X_0 \cap Y_1) = \underline{P(X_0)\cdot P(Y_1/X_0) = P(Y_1)\cdot P(X_0/Y_1)} \Rightarrow$

$\Rightarrow P(X_0/Y_1) = \frac{P(X_0)\cdot P(Y_1|X_0)}{P(Y_1)} \overset{(1)}{\underset{(2)}{=}} \frac{\frac{4}{9}\cdot\frac{1}{5}}{\frac{53}{90}} = \frac{4}{45}\cdot\frac{90}{53} = \frac{8}{53} \simeq$

$\simeq 0.1594 = 15.94\%$

(iv) Emitem „0" și recepționăm „0" (condiționat de emiterea lui „0") sau emitem „1" și recepționăm „1" (condiționat de emiterea lui „1"), așa dar:

$P_{corect} = P((X_0 \cap Y_0) \cup (X_1 \cap Y_1))$. Deoarece $(X_0 \cap Y_0) \cap (X_1 \cap Y_1) = (X_0 \cap X_1) \cap (Y_0 \cap Y_1) = \phi \cap \phi = \phi$, deducem:

$$P_{corect} = P(X_0 \cap Y_0) + P(X_1 \cap Y_1) = P(X_0)\cdot P(Y_0|X_0) + P(X_1)P(Y_1|X_1) = \frac{4}{9}\cdot\frac{4}{5} + \frac{5}{9}\cdot\frac{9}{10} = \frac{16}{45} + \frac{1}{2} = \frac{77}{90} \simeq 0.8555 = 85,55\%$$

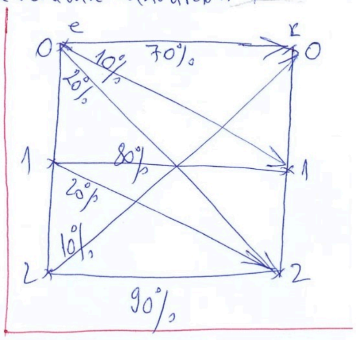
(v) $P_{eronat} = 1 - P_{corect} = 1 - \frac{77}{90} = \frac{13}{90} \simeq 0.1445 = 14,45\%$

OBS. $P_{eronat} = P((X_0 \cap Y_1) \cup (X_1 \cap Y_0)) = P(X_0 \cap Y_1) + P(X_1 \cap Y_0) =$
$= P(X_0)\cdot P(Y_1|X_0) + P(X_1)\cdot P(Y_0|X_1) = \frac{4}{9}\cdot\frac{1}{5} + \frac{5}{9}\cdot\frac{1}{10} = \frac{13}{90} \simeq 0.1445$

La intrarea unui canal ternar de transmisiuni se emit semnalele 0, 1 şi 2, direct proporţional cu numerele 2, 3, 5 respectiv. Schema modelării zgomotului în canal este dată în schema alăturată. Să se determine:

(i) Probabilitatea recepţionării semnalului „0"
(ii) — ” — „1"
(iii) — ” — „2"
(iv) Probabilitatea ca, în cazul recepţiei unui „1", să se fi emis „2"
(v) Probabilitatea ca, în cazul recepţionării „2", să se fi emis „0" sau „1"
(vi) Probabilitatea recepţiei corecte
(vii) Probabilitatea recepţiei eronate

CANAL TERNAR



R. Fie X_0, X_1, X_2 (Y_0, Y_1, Y_2) evenimentele care reprezintă emiterea (recepţia) semnalelor „0", „1", „2", respectiv. Evident, $\{X_0, X_1, X_2\}$, $\{Y_0, Y_1, Y_2\}$ sunt s.c.e. Din datele problemei, avem:

$$\frac{P(X_0)}{2} = \frac{P(X_1)}{3} = \frac{P(X_2)}{5} \text{ şi } P(X_0)+P(X_1)+P(X_2) = 1 \text{ (s.c.e.)} \Rightarrow$$

$$\Rightarrow \frac{P(X_0)}{2} = \frac{P(X_1)}{3} = \frac{P(X_2)}{5} = \frac{P(X_0)+P(X_1)+P(X_2)}{2+3+5} = \frac{1}{10} \Rightarrow$$

$$\Rightarrow P(X_0) = \frac{1}{5};\ P(X_1) = \frac{3}{10};\ P(X_2) = \frac{1}{2} \quad (1)$$

De asemenea din schema modelării zgomotului avem:

$$(2)\begin{cases} P(Y_0/X_0) = 70\% = \frac{7}{10} \\ P(Y_0|X_1) = 0 \\ P(Y_0/X_2) = 10\% = \frac{1}{10} \end{cases} \quad \begin{vmatrix} P(Y_1/X_0) = 10\% = \frac{1}{10} \\ P(Y_1|X_1) = 80\% = \frac{4}{5} \\ P(Y_1/X_2) = 0 \end{vmatrix} \quad \begin{matrix} P(Y_2/X_0) = 20\% = \frac{1}{5} \\ P(Y_2|X_1) = 20\% = \frac{1}{5} \\ P(Y_2/X_2) = 90\% = \frac{9}{10} \end{matrix}$$

(i) $P(Y_0) \overset{FPT}{=} P(X_0)\cdot P(Y_0|X_0) + P(X_1)\cdot P(Y_0|X_1) + P(X_2)\cdot P(Y_0|X_2) =$
$= \frac{1}{5}\cdot\frac{7}{10} + \frac{3}{10}\cdot 0 + \frac{1}{2}\cdot\frac{1}{10} = \frac{7}{50} + \frac{1}{20} = \frac{19}{100} = 19\%$

(ii) $P(Y_1) \overset{FPT}{=} P(X_0)\cdot P(Y_1/X_0) + P(X_1)P(Y_1|X_1) + P(X_2)P(Y_1|X_2) =$
$= \frac{1}{5}\cdot\frac{1}{10} + \frac{3}{10}\cdot\frac{4}{5} + \frac{1}{2}\cdot 0 = \frac{1}{50} + \frac{12}{50} = \frac{13}{50} = 26\%$

(iii) $P(Y_2) \overset{s.c.e}{=} 1 - P(Y_0) - P(Y_1) = 1 - \frac{19}{100} - \frac{13}{50} = \frac{55}{100} = \frac{11}{20} = 55\%$

OBS. Aplicând FPT $\Rightarrow P(Y_2) = \frac{1}{5}\cdot\frac{1}{5} + \frac{3}{10}\cdot\frac{1}{5} + \frac{1}{2}\cdot\frac{9}{10} = \frac{1}{25} + \frac{3}{50} + \frac{9}{20} = \frac{55}{100} = \frac{11}{20}$

(iv) $P(X_0/Y_1) \overset{Bayes}{=} \frac{P(X_0)P(Y_1|X_0)}{P(Y_1)} = \frac{\frac{1}{5}\cdot\frac{1}{10}}{\frac{13}{50}} = \frac{1}{13} \simeq 0.0769 = 7,69\%$

(v) $P(X_0 \cup X_1/Y_2) \overset{X_0 \cap X_1 = \phi}{=} P(X_0/Y_2) + P(X_1/Y_2) \overset{Bayes}{=} \frac{P(X_0)\cdot P(Y_2|X_0)}{P(Y_2)} +$
$+ \frac{P(X_1)P(Y_2/X_1)}{P(Y_2)} = \frac{\frac{1}{5}\cdot\frac{1}{5} + \frac{3}{10}\cdot\frac{1}{5}}{\frac{11}{20}} = \frac{1}{10}\cdot\frac{20}{11} = \frac{2}{11} \simeq 0.1818 = 18,18\%$

(vi) $P_{corect} = P((X_0 \cap Y_0) \cup (X_1 \cap Y_1) \cup (X_2 \cap Y_2)) =$
$= P(X_0 \cap Y_0) + P(X_1 \cap Y_1) + P(X_2 \cap Y_2)$, deoarece

$(X_0 \cap Y_0) \cap (X_1 \cap Y_1) \cap (X_2 \cap Y_2) = (X_0 \cap X_1 \cap X_2) \cap (Y_0 \cap Y_1 \cap Y_2) = \emptyset \cap \emptyset = \emptyset$.

Din egalitatea $P(A \cap B) = P(A) \cdot P(B/A)$, rezultă:

$$P_{corect} = P(X_0)P(Y_0/X_0) + P(X_1)P(Y_1/X_1) + P(X_2) \cdot P(Y_2/X_2) =$$

$$= \frac{1}{5} \cdot \frac{7}{10} + \frac{3}{10} \cdot \frac{4}{5} + \frac{1}{2} \cdot \frac{9}{10} = \frac{7}{50} + \frac{6}{25} + \frac{9}{20} = \frac{83}{100} =$$

$$= 83\%$$

(vii) $P_{eronat} = 1 - P_{corect} = \frac{17}{100} = 17\%$.

<u>Obs.</u> $P_{eronat} = P((X_0 \cap Y_1) \cup (X_0 \cap Y_2) \cap (X_1 \cap Y_0) \cap (X_1 \cap Y_2) \cap (X_2 \cap Y_0) \cap (X_2 \cap Y_1))$

$= P(X_0 \cap Y_1) + P(X_0 \cap Y_2) + P(X_1 \cap Y_0) + P(X_1 \cap Y_2) + P(X_2 \cap Y_0) + P(X_2 \cap Y_1)$

ş.a.m.d.

② CARACTERISTICI NUMERICE (STATISTICE) ALE V.A.

CURS 2 PAG 8.

Exemple

Fie $X: \begin{pmatrix} -1 & 1 & 2 & 3 \\ 2p^2 & \frac{1}{4} & \frac{3}{8} & p \end{pmatrix}$ v.a. 1D simplă

(i) $p=?$; (ii) $M(X)=E(X)$; (iii) $D^2(X)=Var(X)$; (iv) $\sigma(X)$;

(v) $P(\ln 2 < X \le 2)$; (vi) $F(e)$; (vii) $Mo(X)$; (viii) Entropia $H(X)$

R. (i) $p \in [0,1]$; $2p^2 + \frac{1}{4} + \frac{3}{8} + p = 1 \Rightarrow 16p^2 + 8p - 3 = 0 \Rightarrow$

$\Rightarrow p_{1,2} = \frac{-4 \pm 8}{16}$ $\begin{cases} p_1 = -3/4 \notin [0,1] \\ p_2 = \frac{1}{4} \in [0,1] \end{cases} \Rightarrow p = \frac{1}{4} \Rightarrow$

$$\Rightarrow X: \begin{pmatrix} -1 & 1 & 2 & 3 \\ \frac{1}{8} & \frac{1}{4} & \frac{3}{8} & \frac{1}{4} \end{pmatrix} \Rightarrow X^2: \begin{pmatrix} (-1)^2 & 1^2 & 2^2 & 3^2 \\ \frac{1}{8} & \frac{1}{4} & \frac{3}{8} & \frac{1}{4} \end{pmatrix} = \begin{pmatrix} 1 & 4 & 9 \\ \frac{3}{8} & \frac{3}{8} & \frac{1}{4} \end{pmatrix}$$

(ii) $M(X) = E(X) = \sum_{i=1}^{4} p_i x_i = (-1)\cdot\frac{1}{8} + 1\cdot\frac{1}{4} + 2\cdot\frac{3}{8} + 3\cdot\frac{1}{4} = \frac{13}{8} \cong 1,625$

$(M_{aritm} = 1,25)$

(iii) $D^2(X) = M(X^2) - M^2(X)$

$M(X^2) = 1\cdot\frac{3}{8} + 4\cdot\frac{3}{8} + 9\cdot\frac{1}{4} = \frac{33}{8}$ $\Big\} \Rightarrow D^2(X) = \frac{33}{8} - \frac{169}{64} = \frac{264-169}{64} = \underbrace{\frac{95}{64}}_{\simeq 1,49}$

(iv) $\sigma(X) = \sqrt{D^2(X)} = \frac{\sqrt{95}}{8} \simeq 1.24$

(v) $P(\underbrace{\ln 2}_{0.7} < X \le 2) \xrightarrow[\ln 2 \simeq 0.7]{} P(X=1) + P(X=2) = \frac{1}{4} + \frac{3}{8} = \frac{5}{8}$

(vi) $F(e) = P(X<e) = P(X=-1) + P(X=1) + P(X=2) = 1 - P(X=3) = \frac{3}{4}$

(vii) $M_o(X) = x_s \Leftrightarrow p_s = \max\{p_i : i \in I\}$

$\max\{p_i : i \in I\} = \max\{\frac{1}{8}, \frac{1}{4}, \frac{3}{8}, \frac{1}{4}\} = \frac{3}{8} \Rightarrow M_o(X) = 2$

(viii) $H(X) = \sum_{i=1}^{n} p_i \log_2 \frac{1}{x_i} = \sum_{i=1}^{n} p_i \log_2 (x_i)^{-1}$; $H(X) \le \log_2 n$

$H(X) = \frac{1}{8}\log_2\left(\frac{1}{8}\right)^{-1} + \frac{1}{4}\log_2\left(\frac{1}{4}\right)^{-1} + \frac{3}{8}\log_2\left(\frac{3}{8}\right)^{-1} + \frac{1}{4}\log_2\left(\frac{1}{4}\right)^{-1} =$

$= \frac{1}{8}\log_2 8 + \frac{1}{4}\log_2 4 + \frac{3}{8}\log_2\frac{8}{3} + \frac{1}{4}\log_2 4 = \frac{1}{8}\cdot 3 + \frac{1}{4}\cdot 2 + \frac{3}{8}(3 - \log_2 3) + \frac{1}{4}\cdot 2 =$

$= \frac{3}{8} + \frac{1}{2} + \frac{1}{2} + \frac{9}{8} - \frac{3}{8}\log_2 3 = \frac{5}{2} - \frac{3}{8}\log_2 3 \le 2 = \log_2 4.$ $\boxed{H(X) = \frac{1}{8}(20 - 3\log_2 3)}$

$$P(a \le X < b) = F(b) - F(a)$$
$$P(a \le X \le b)$$
$$P(a \le X \le b) = F(b+0) - F(a)$$

$$H(X) \le \log_2 n$$

Ex. 1 Se consideră v.a. simplă $X:\begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ p^3 & p^2 & \frac{p}{2} & \frac{1}{4} & \frac{1}{8}\end{pmatrix}$. Să se determine

1° $p=?$; 2° $M(X)=E(X)$; 3° $D^2(X)=Var\,X$; 4° $\sigma(X)$; 5° $M_o(X)$;

6° $F(x)$; 7° $Me(X)$; 8° $P(\frac{1}{e}<X\leq 3)$; 9° $H(X)$

Rezolvare

1° $p^3+p^2+\frac{p}{2}+\frac{1}{4}+\frac{1}{8}=1$; $p\in(0,1) \Rightarrow 8p^3+8p^2+4p-5=0$

	p^3	p^2	p^1	p^0
	8	8	4	-5
1/2	8	12	10	0

; $(p-\frac{1}{2})(8p^2+12p+10)=0 \Leftrightarrow$ $p-\frac{1}{2}=0 \Rightarrow p_1=\frac{1}{2}$; $2(4p^2+6p+5)=0$; $\Delta<0$ $p_{2,3}\notin\mathbb{R}$

Deci $p=\frac{1}{2} \Rightarrow X:\begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ \frac{1}{8} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{8}\end{pmatrix}$

2° $M(X)=\frac{1}{8}\cdot 0+1\cdot\frac{1}{4}+2\cdot\frac{1}{4}+3\cdot\frac{1}{4}+4\cdot\frac{1}{8}=2$

$X^2:\begin{pmatrix} 0^2 & 1^2 & 2^2 & 3^2 & 4^2 \\ \frac{1}{8} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{8}\end{pmatrix}$

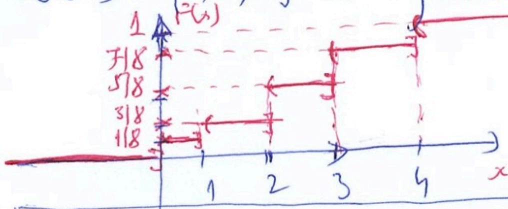
3° $D^2(X)=Var(X)=M(X^2)-M^2(X)=$

$=0\cdot\frac{1}{8}+1\cdot\frac{1}{4}+4\cdot\frac{1}{4}+9\cdot\frac{1}{4}+16\cdot\frac{1}{8}-2^2=\frac{3}{2}=1.5$;

4° $\sigma(X)=\sqrt{D^2(X)}=\sqrt{\frac{3}{2}}=\frac{\sqrt{6}}{2}\simeq 1.22$

5° $\max\{\frac{1}{8},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{8}\}=\frac{1}{4} \Rightarrow M_o(X)\in\{1,2,3\}$ (v.a. plurimodală)

6° $F(x)=\begin{cases} 0; & x\leq 0 \\ 1/8; & 0<x\leq 1 \\ 1/8+1/4=3/8; & 1<x\leq 2 \\ 1/8+1/4+1/4=5/8; & 2<x\leq 3 \\ 7/8; & 3<x\leq 4 \\ 1; & x>4 \end{cases}$



7° $\begin{cases} F(Me(X))\leq\frac{1}{2}\leq F(Me(X)+0) \\ Me(X)=2. \end{cases}$

$F(a)\leq\frac{1}{2}\leq F(a+0)$

$\begin{cases} \frac{1}{2}\in(\frac{3}{8},\frac{5}{8}) \\ \frac{1}{2}\in(F(2),F(3)) \end{cases}$

Deci $a=2$

$a=2$: $F(2)\leq\frac{1}{2}\leq F(2+0) \Leftrightarrow$ $\Leftrightarrow \frac{3}{8}\leq\frac{1}{2}\leq\frac{5}{8}$ ✓

$a=3$: $F(3)\leq\frac{1}{2}\leq F(3+0)$; $5/8\leq\frac{1}{2}$ Fals

$a\in(2,3)$: $\frac{5}{8}\leq\frac{1}{2}$ Fals

$a\in(1,2)$: $\frac{3}{8}\leq\frac{1}{2}\leq\frac{3}{8}$ Fals

8° $P(\underset{\sim 0,4}{\frac{1}{e}}<X\leq 3)=P(X=1)+P(X=2)+P(X=3)=$

$=\frac{1}{4}+\frac{1}{4}+\frac{1}{4}=3/4$

OBS. $P(\frac{1}{e}<X\leq 3)=F(3+0)-F(\frac{1}{e}+0)=\frac{7}{8}-\frac{1}{8}=\frac{3}{4}$.

9° $H(X)=\frac{1}{8}\log_2\frac{1}{1/8}+\frac{1}{4}\log_2\frac{1}{1/4}+\frac{1}{4}\log_2\frac{1}{1/4}+\frac{1}{8}\log_2\frac{1}{1/8}$

$=2\cdot\frac{1}{8}\log_2 8+3\cdot\frac{1}{4}\log_2 4=\frac{1}{4}\cdot 3+\frac{3}{4}\cdot 2=\frac{9}{4}=2.25$

OBS. $H(X)\leq\log_2 n=\log_2 5$; Într-adevăr, $\frac{9}{4}\leq\log_2 5 \Leftrightarrow 9\leq\log_2 5^4 \Leftrightarrow$ $\Leftrightarrow 2^9\leq 5^4 \Leftrightarrow 512\leq 625$ ✓ (A)

Ex. 2 Fie $X: \begin{pmatrix} -1 & 0 & 1 \\ a+\frac{1}{6} & 2b+\frac{1}{3} & \frac{1}{3} \end{pmatrix}$, $Y: \begin{pmatrix} b & 0 & 1 \\ \frac{1}{4} & 2(a+b) & 15a^2 \end{pmatrix}$

v.a. simple, independente, ~~...~~ . Să se determine:

1° a, b; 2° $M(X)$; 3° $D^2(X+Y)$; 4° $P(-1 \le X+Y \le 1)$;
5° X^n; 6° Y^m; 7° $M_o(X)$; 8° $X^{10}+Y^{25}$; 9° $X^{11}Y^{16}$; 10° $H(X)$

Rezolvare 1° $\begin{cases} a+\frac{1}{6}+2b+\frac{1}{3}+\frac{1}{3}=1 \;;\; 0 \le a+\frac{1}{6} \le 1 \;;\; 0 \le 2b+\frac{1}{3} \le 1; \\ \frac{1}{4}+2(a+b)+15a^2=1 \;;\; 0 \le 2(a+b) \le 1 \;;\; 0 \le 15a^2 \le 1. \end{cases}$

$\begin{cases} a+2b=\frac{1}{6} \;|\cdot(-1) \\ 2a+2b+15a^2=\frac{3}{4} \end{cases} \Leftrightarrow \begin{cases} a+2b=1/6 \\ 15a^2+a=7/12 \end{cases} \Leftrightarrow \begin{cases} a+2b=1/6 \\ \Delta=1+4\cdot 15\cdot\frac{7}{12}=1+35=36 \end{cases} \quad a_{1,2}=\frac{-1\pm 6}{30}$

$\Leftrightarrow \begin{cases} a=1/6 \\ b=0 \end{cases}$, $X: \begin{pmatrix} -1 & 0 & 1 \\ 1/3 & 1/3 & 1/3 \end{pmatrix}$; $Y: \begin{pmatrix} 0 & 0 & 1 \\ \frac{1}{4} & \frac{1}{3} & \frac{5}{12} \end{pmatrix} \equiv \begin{pmatrix} 0 & 1 \\ \frac{7}{12} & \frac{5}{12} \end{pmatrix}$

2° $M(X)=-\frac{1}{3}+0\cdot\frac{1}{3}+1\cdot\frac{1}{3}=0$; 3° $X+Y: \begin{pmatrix} -1+0 & -1+1 & 0+0 & 0+1 & 1+0 & 1+1 \\ \frac{1}{3}\cdot\frac{7}{12} & \frac{1}{3}\cdot\frac{5}{12} & \frac{1}{3}\cdot\frac{7}{12} & \frac{1}{3}\cdot\frac{5}{12} & \frac{1}{3}\cdot\frac{7}{12} & \frac{1}{3}\cdot\frac{5}{12} \end{pmatrix}$

$X+Y: \begin{pmatrix} -1 & 0 & 0 & 1 & 1 & 2 \\ 7/36 & 5/36 & 7/36 & \frac{5}{36} & \frac{7}{36} & \frac{5}{36} \end{pmatrix} \equiv \begin{pmatrix} -1 & 0 & 1 & 2 \\ \frac{7}{36} & \frac{1}{3} & \frac{1}{3} & \frac{5}{36} \end{pmatrix}$; $M(X+Y)=-\frac{7}{36}+\frac{1}{3}+\frac{5\cdot 2}{36}=\frac{5}{12}$

$(X+Y)^2: \begin{pmatrix} (-1)^2 & 0^2 & 1^2 & 2^2 \\ 7/36 & 1/3 & 1/3 & 5/36 \end{pmatrix} \equiv \begin{pmatrix} 0 & 1 & 4 \\ \frac{1}{3} & \frac{19}{36} & \frac{5}{36} \end{pmatrix}$ | $\frac{131}{144} \simeq 0.91$

$D^2(X+Y)=M\left((X+Y)^2\right)-M^2(X+Y)=0\cdot\frac{1}{3}+1\cdot\frac{19}{36}+4\cdot\frac{5}{36}-\left(\frac{5}{12}\right)^2=\frac{39}{36}-\frac{25}{144}$ ~~$\simeq 1.08$~~

Metoda 2 X, Y indep $\Rightarrow D^2(X+Y)=D^2(X)+D^2(Y)=M(X^2)-M^2(X)+M(Y^2)-M^2(Y)=$
$=(-1)^2\cdot\frac{1}{3}+1^2\cdot\frac{1}{3}-0^2+1^2\cdot\frac{5}{12}-\left(\frac{5}{12}\right)^2=\frac{2}{3}+\frac{5}{12}-\frac{25}{144}=\frac{131}{144}\simeq 0.91.$

4° $P(-1 \le X+Y \le 1)=P(X+Y=-1)+P(X+Y=0)+P(X+Y=1)=\frac{7}{36}+\frac{1}{3}+\frac{1}{3}=\frac{31}{36}$

5° $X^2: \begin{pmatrix} (-1)^2 & 0^2 & 1^2 \\ 1/3 & 1/3 & 1/3 \end{pmatrix}=\begin{pmatrix} 0 & 1 \\ 1/3 & 2/3 \end{pmatrix} \Rightarrow X^{2n}=\begin{pmatrix} 0 & 1 \\ 1/3 & 2/3 \end{pmatrix}$; $X^{2n+1}=X$

6° $Y^m=Y$; 7° $M_o(X)\in\{-1,0,1\}$; 8° $X^{10}+Y^{25}=\begin{pmatrix} 0 & 1 \\ 1/3 & 2/3 \end{pmatrix}+\begin{pmatrix} 0 & 1 \\ 7/12 & \frac{5}{12} \end{pmatrix}$
$=\begin{pmatrix} 0 & 1 & 2 \\ \frac{1}{3}\cdot\frac{7}{12} & \frac{1}{3}\cdot\frac{5}{12}+\frac{2}{3}\cdot\frac{7}{12} & \frac{2}{3}\cdot\frac{5}{12} \end{pmatrix}=\begin{pmatrix} 0 & 1 & 2 \\ \frac{7}{36} & \frac{19}{36} & \frac{5}{18} \end{pmatrix}$

9° $X^{11}Y^{16}: \begin{pmatrix} -1 & 0 & 1 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}\cdot\begin{pmatrix} 0 & 1 \\ \frac{7}{12} & \frac{5}{12} \end{pmatrix}=\begin{pmatrix} -1\cdot 0 & -1\cdot 1 & 0\cdot 0 & 0\cdot 1 & 1\cdot 0 & 1\cdot 1 \\ \frac{7}{36} & \frac{5}{36} & \frac{7}{36} & \frac{5}{36} & \frac{7}{36} & \frac{5}{36} \end{pmatrix}$

$X^{11}Y^{16}: \begin{pmatrix} -1 & 0 & 1 \\ \frac{5}{36} & \frac{13}{18} & \frac{5}{36} \end{pmatrix}$;

10° $H(X)=\frac{1}{3}\log_2 3+\frac{1}{3}\log_2 3+\frac{1}{2}\log_2 3=\log_2 3$

Ex.3 $X: \begin{pmatrix} 1 & 3 & 5 & \dots & 2n+1 & \dots \\ \frac{p}{2} & p^2 & \frac{3}{2}p^3 & \dots & \frac{n}{2}p^n & \dots \end{pmatrix} = \begin{pmatrix} 2n-1 \\ \frac{n}{2}p^n \end{pmatrix}_{n\geq 1}$

Să se determine: 1° $p=?$ 2° $M(X)$; 3° $D^2(X)$;

4° m minim, $m\in\mathbb{N}$ a.î. $P(X\geq m)\leq 0.2$.

Rezolvare 1° $\sum_{n=1}^{\infty}\frac{n}{2}p^n=1 \iff \frac{1}{2}\sum_{n=1}^{\infty}np^n=1 \iff \sum_{n=0}^{\infty}np^n=2 \iff$

$\iff \sum_{n=0}^{\infty}\frac{n}{\left(\frac{1}{p}\right)^n}=2 \iff \mathcal{Z}\{n\}\left(\frac{1}{p}\right)=2 \iff$

$\iff \frac{1/p}{\left(\frac{1}{p}-1\right)^2}=2 \iff p=2(1-p)^2 \iff 2p^2-5p+2=0$

$\iff p_1=1/2;\ p_2=2 \xLeftrightarrow{p\in(0,1)} \boxed{p=\frac{1}{2}}$

$$\boxed{\begin{aligned} \mathcal{Z}\{x(n)\}(z)&=\sum_{n=0}^{\infty}\frac{x(n)}{z^n} \\ \mathcal{Z}\{n\}(z)&=\frac{z}{(z-1)^2} \\ \mathcal{Z}\{n^2\}(z)&=\frac{z(z+1)}{(z-1)^3} \\ \mathcal{Z}\{n^3\}(z)&=\frac{z(z^2+4z+1)}{(z-1)^4} \end{aligned}}$$

$X: \begin{pmatrix} 2n-1 \\ \frac{n}{2^{n+1}} \end{pmatrix}_{n\geq 1} = \begin{pmatrix} 1 & 3 & 5 & \dots & 2n-1 & \dots \\ \frac{1}{4} & \frac{1}{4} & \frac{3}{16} & \dots & \frac{n}{2^{n+1}} & \dots \end{pmatrix}$

2° $M(X)=\sum_{n=1}^{\infty}(2n-1)\frac{n}{2^{n+1}}=\sum_{n=1}^{\infty}\frac{2n^2-n}{2^{n+1}}=\sum_{n=0}^{\infty}\frac{2n^2-n}{2^n\cdot 2}=$

$=\frac{1}{2}\left(2\sum_{n=0}^{\infty}\frac{n^2}{2^n}-\sum_{n=0}^{\infty}\frac{n}{2^n}\right)=\frac{1}{2}\left(2\mathcal{Z}\{n^2\}(2)-\mathcal{Z}\{n\}(2)\right)=$

$=\frac{1}{2}\left(2\frac{z(z+1)}{(z-1)^3}-\frac{z}{(z-1)^2}\right)\Big|_{z=2}=\frac{1}{2}\left(2\frac{2\cdot 3}{(2-1)^3}-\frac{2}{(2-1)^2}\right)=\frac{1}{2}10=5$

3° $D^2(X)=M(X^2)-M^2(X)$; $X^2: \begin{pmatrix} (2n-1)^2 \\ n/2^{n+1} \end{pmatrix}_{n\geq 1}$

$M(X^2)=\sum_{n=1}^{\infty}(2n-1)^2\cdot\frac{n}{2^{n+1}}=\frac{1}{2}\sum_{n=0}^{\infty}\frac{n(2n-1)^2}{2^n}=\frac{1}{2}\mathcal{Z}\{n(2n-1)^2\}(2)=$

$=\frac{1}{2}\mathcal{Z}\{4n^3-4n^2+n\}(2)=\frac{1}{2}\left[4\mathcal{Z}\{n^3\}(2)-4\mathcal{Z}\{n^2\}(2)+\mathcal{Z}\{n\}(2)\right]=$

$=\frac{1}{2}\left[4\frac{z(z^2+4z+1)}{(z-1)^4}-4\cdot\frac{z(z+1)}{(z-1)^3}+\frac{z}{(z-1)^2}\right]_{z=2}=$

$=\frac{1}{2}\left(4\cdot 2\cdot 13-4\cdot 2\cdot 3+2\right)=52-8+1=45$

$D^2(X)=45-5^2=20.$

4° $1-P(X<m)\leq 0.2 \Rightarrow P(X<m)\geq 0.8$

Şirul $a_m=P(X<m)$ este strict crescător; Avem $a_2=P(X<2)=$
$=P(X=1)=\frac{1}{4}<0.8$; $a_3=P(X<3)=P(X=1)+P(X=2)=\frac{1}{4}+\frac{1}{4}=\frac{1}{2}<0.8$
$a_4=P(X<4)=\frac{1}{4}+\frac{1}{4}+\frac{3}{16}=\frac{11}{16}<0.8$; $a_5=P(X<5)=\frac{11}{16}+\frac{1}{8}=\frac{13}{16}>0.8$,
deci $m=5$.

EX.

$$X:\begin{pmatrix} 1 & 5 & 9 & \dots & 4n-3 & \dots \\ p & 4p^2 & 16p^3 & \dots & 4^{n-1}\cdot p^n & \dots \end{pmatrix}$$

$$\Downarrow$$

$$X:\begin{pmatrix} 4n-3 \\ 4^{n-1}\cdot p^n \end{pmatrix}$$

(i) $p=?$

$$\sum_{n=1}^{\infty} 4^{n-1}\cdot p^n = 1 \quad \Rightarrow \quad 4^{-1}\cdot \sum_{n=1}^{\infty} (4p)^n = 1$$

$$\Rightarrow \frac{1}{\cancel{4}}\cdot\frac{\cancel{4}p}{1-4p} = 1$$

$$\Rightarrow p = 1-4p$$

$$5p = 1 \qquad \boxed{p=\frac{1}{5}}$$

$$\boxed{\sum_{n=m}^{\infty} q^n = \frac{q^m}{1-q} \qquad |q|<1}$$

$$X:\begin{pmatrix} 1 & 5 & 9 & \dots & 4n-3 \\ \frac{1}{5} & \frac{4}{25} & \frac{16}{125} & \dots & \frac{4^{n-1}}{5^n} \end{pmatrix}$$

(ii) $M(X) = \sum_{n=1}^{\infty} (4n-3)\cdot\frac{1}{4}\cdot\left(\frac{4}{5}\right)^n$

VEZI SEMINAR 3 ADA

TREBUIE FOLOSITE TRANSFORMATELE Z

EX. 5 FUNCȚIA $f: \mathbb{R} \to \mathbb{R}$, $f(x) = a x^2 e^{-2|x|}$ ESTE PDF. PT $\forall$ V.A. ID CONTINUĂ, NOTATĂ X.

$$\mathcal{L}\{f(x)\}(s) = \int_0^\infty f(x) \cdot e^{-sx} dx$$

$$\mathcal{L}\{x^n\}(s) = \frac{n!}{s^{n+1}}$$

(i) $a = ?$ $a \in \mathbb{R}$

$$\int_{-\infty}^{\infty} f(x)\, dx = 1 \overset{f \text{ PARĂ}}{\underset{f(-x)=f(x)}{\Longleftrightarrow}} 2a \cdot \int_0^{\infty} f(x)\, dx = 1$$

$$2a \int_0^\infty x^2 e^{-2x} dx = 1 \Rightarrow$$

$$2a \cdot \mathcal{L}\{x^2\}(2) = 1 \Rightarrow 2a \cdot \left.\frac{2!}{s^3}\right|_{s=2} = 1 \Rightarrow 2a \cdot \frac{2}{8} = 1$$

$$\Rightarrow \frac{4a}{8} = 1 \Rightarrow \boxed{a = 2}$$

(ii) $M_n(X) = \int_{-\infty}^{\infty} x^n f(x) dx = a \int_{-\infty}^{\infty} x^{n+2} e^{-2|x|} dx$ OBS. $M_0(X) = 1$

n impar $g(x) = x^{n+2} e^{-2|x|}$ impară deoarece $g(-x) = a(-x)^{n+2} e^{-2|-x|} =$
$= -a x^{n+2} e^{-2|x|} = -g(-x) \Longrightarrow M_n(X) = a \int_{-\infty}^{\infty} g(x) dx = 0$

n par $g(x) = x^{n+2} e^{-2|x|}$ este funcție pară deoarece $g(-x) = g(x) \Rightarrow$
$\Rightarrow M_n(X) = 2 \int_{-\infty}^{\infty} x^{n+2} e^{-2|x|} dx = 2 \cdot 2 \int_0^\infty x^{n+2} e^{-2x} dx = 4 \cdot \mathcal{L}\{x^{n+2}\}(2) =$
$= 2^2 \cdot \left.\frac{(n+2)!}{s^{n+3}}\right|_{s=3} = 2^2 \cdot \frac{(n+2)!}{2^{n+3}} = \frac{(n+2)!}{2^{n+1}}$; Verif. $M_0(X) = \frac{2!}{2^1} = 1$

(iii) $D^2(X) = Var(X) = M(X^2) - M^2(X) = M_2(X) - M_1^2(X) \overset{(ii)}{\underset{n=1;\, n=2}{=}}$
$\overset{(ii)}{=} \frac{4!}{2^3} - \left(\frac{3!}{2^2}\right)^2 = 3 - \left(\frac{3}{2}\right)^2 = 3 - \frac{9}{4} = \frac{3}{4}.$

(iv) $M_o(X)$. $x \geq 0 \Rightarrow f(x) = 2x^2 e^{-2x} \Rightarrow f'(x) = 2(2x e^{-2x} - x^2 \cdot 2 e^{-2x})$
$\Rightarrow f'(x) = 4x e^{-2x}(1-x)$
$x < 0 \Rightarrow f(x) = 2x^2 e^{2x} \Rightarrow f'(x) = 2(2x e^{2x} + x^2 \cdot 2 e^{2x}) = 4x e^{2x}(1+x)$
OBS. f pară $\Rightarrow f'$ impară, deci pt. $x < 0 \Rightarrow f'(x) = -f'(-x) = -4 \cdot (-x) e^{2x}(1+x) =$
$= 4x e^{2x}(1+x)$.

Deci $f'(x) = \begin{cases} 4x e^{2x}(1+x); & x < 0 \\ 4x e^{2x}(1-x); & x \geq 0 \end{cases}$. Observăm că $f'_s(0) = f'_d(0) = 0$
deci $f'(0) = 0$

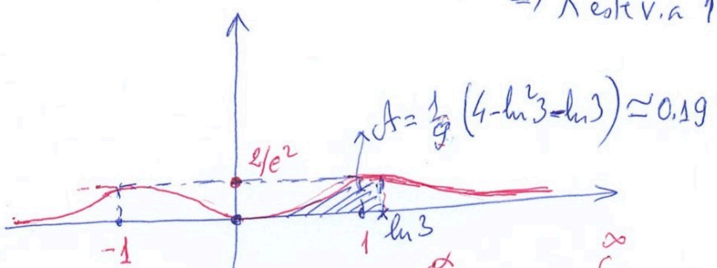
$f'(x) = 0 \Rightarrow \left\langle \begin{array}{l} x < 0 \Rightarrow 1 + x = 0 \Rightarrow x = -1 \\ x \geq 0 \Rightarrow x = 0 \text{ sau } x = 1 \end{array} \right\} \Rightarrow x \in \{-1, 0, 1\}$

x	$-\infty$		-1		0		1		$+\infty$
$f'(x)$		$+$	0	$-$	0	$+$	0	$-$	
$f(x)$	0	$\nearrow$	$\frac{2}{e^2}$	$\searrow$	0	$\nearrow$	$\frac{2}{e^2}$	$\searrow$	0

Deci $f(x)$ are 2 puncte de maxim $x_1=-1$ și $x_2=1$ $\Rightarrow$

$\Rightarrow Mo(X) \in \{-1, 1\}$

$\Rightarrow X$ este v.a 1D continuă plurimodală



$$\boxed{\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx}$$

(v) Notăm $\alpha = Me(X) \Rightarrow \int_{-\infty}^{\alpha} f(x)dx = \int_{\alpha}^{\infty} f(x)dx = \frac{1}{2} \Leftrightarrow F(\alpha)=\frac{1}{2}$, unde $F(x) = \int_{-\infty}^{x} f(t)dt$. Deoarece $F'(x) = f(x) > 0 \Rightarrow F$ este strict crescătoare pe $\mathbb{R}$, $0 \le F(x) \le 1$, deci ecuația $F(\alpha) = \frac{1}{2}$ are soluție unică. Deoarece f este pară $\Rightarrow G_f$ este simetric față de $Oy \Rightarrow \alpha = 0 \Rightarrow Me(X) = 0$

(vi) $P(0 \le X \le \ln 3) = \int_0^{\ln 3} f(x)dx = 2\int_0^{\ln 3} x^2 e^{-2x}dx = 2\int_0^{\ln 3} x^2\left(\frac{e^{-2x}}{-2}\right)'dx =$

$= \cancel{2}\left[x^2 \cdot \frac{e^{-2x}}{-\cancel{2}}\Big|_0^{\ln 3} + \frac{2}{\cancel{2}}\int_0^{\ln 3} x e^{-2x}dx\right]$. Deoarece $e^{-2\ln 3} = (e^{\ln 3})^{-2} = 3^{-2} = \frac{1}{9}$,

obținem: $P(0 \le X \le \ln 3) = -\frac{1}{9}\ln^2 3 + \cancel{2}\int_0^{\ln 3} x\left(\frac{e^{-2x}}{-\cancel{2}}\right)'dx =$

$= -\frac{1}{9}\ln^2 3 - xe^{-2x}\Big|_0^{\ln 3} + \int_0^{\ln 3} \underbrace{x' e^{-2x}}_{e^{-2x}} dx = -\frac{1}{9}\ln^2 3 - \frac{1}{9}\ln 3 - \frac{e^{-2x}}{2}\Big|_0^{\ln 3} =$

$= -\frac{1}{9}\ln^2 3 - \frac{1}{9}\ln 3 - \frac{1}{18} + \frac{1}{2} = \frac{1}{18}(8 - 2\ln 3 - 2\ln^2 3) = \frac{1}{9}(4 - \ln 3 - \ln^2 3)$

$\boxed{Ex. 6}$ Funcția $f: \mathbb{R} \to \mathbb{R}$, $f(x) = \begin{cases} (a+b)\cos ax; & 0 < x < \frac{\pi}{8} \\ b & ; \text{în rest} \end{cases}$; $a, b \in$

este p.d.f. pentru o v.a. 1Dimensională, notată X. Se cere:
(i) Să se determine a, b
(ii) $M(X) = E(X)$; (iii) $D^2(X) = Var(X)$; (iv) $F(x)$; (v) $Me(X)$; (vi) $Mo(X)$
(vii) $P(-1 \le X \le \frac{\pi}{6})$; (viii) Funcțiile de repartiție și p.d.f. pentru v.a. 1D, $Y = 3X + \pi$; $Z = X^4$; $W = e^X$

Ex. 6

R (i) Din $\int_{-\infty}^{\infty} f(x)\,dx = 1 \Rightarrow \int_{-\infty}^{0} \underbrace{f(x)}_{b}dx + \int_{0}^{\pi/8} f(x)\,dx + \int_{\pi/8}^{\infty} \underbrace{f(x)}_{b}\,dx = 1 \Leftrightarrow$

$\Leftrightarrow bx\Big|_{-\infty}^{0} + bx\Big|_{0}^{\infty} + (a+b)\int_0^{\pi/8}\cos(ax)\,dx = b\cdot\infty + b\cdot\infty + \frac{a+b}{a}\sin ax\Big|_0^{\pi/8} - b\frac{\pi}{8}\cdot$

$= 1 \Rightarrow b = 0$ și $\sin a\frac{\pi}{8} = 1 \Rightarrow b = 0$ și $a\frac{\pi}{8} = \frac{\pi}{2} + 2n\pi \Big| \frac{8}{\pi} \Rightarrow$

$\Rightarrow b = 0$ și $a = 16n+4$, $n \in \mathbb{Z}$, deci $f(x) = (16n+4)\cos(16n+4)x$, $0 < x < \frac{\pi}{8}$

Deoarece $f(x) \ge 0$, $\forall x \in \mathbb{R} \Rightarrow (16n+4)x \in [0, \frac{\pi}{2}]$, $\forall x \in (0, \frac{\pi}{8}) \Rightarrow$

$\Rightarrow 0 \le (16n+4)\frac{\pi}{8} \le \frac{\pi}{2} \Rightarrow 0 \le 16n+4 \le 4 \Rightarrow n = 0$, deci

$a = 16n+4 = 4 \Rightarrow \underline{a = 4,\ b = 0}$, $\Rightarrow$ $f(x) = \begin{cases} 4\cos(4x); & 0 < x < \frac{\pi}{8} \\ 0 & ; \text{în rest.} \end{cases}$

(ii) $M(X) = E(X) = \int_{-\infty}^{\infty} x f(x)\,dx = 4\int_0^{\pi/8} x\cos 4x\,dx = 4\int_0^{\pi/8} x\cdot\left(\frac{\sin 4x}{4}\right)'dx =$

$= x\sin 4x\Big|_0^{\pi/8} - \int_0^{\pi/8} 1\cdot\sin 4x\,dx = \frac{\pi}{8}\underbrace{\sin\frac{\pi}{2}}_{=1} + \frac{\cos 4x}{4}\Big|_0^{\pi/8} = \frac{\pi}{8} + \frac{1}{4}(\underbrace{\cos\frac{\pi}{2}}_{0} - \underbrace{\cos 0}_{1})$

Deci $M(X) = E(X) = \frac{\pi}{8} - \frac{1}{4} = \frac{1}{8}(\pi - 2)$

(iii) $D^2(X) = M(X^2) - M^2(X)$

$M(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx = 4\int_0^{\pi/8} x^2\cos 4x\,dx = 4\int_0^{\pi/8} x^2\cdot\left(\frac{\sin 4x}{4}\right)'dx =$

$= x^2\sin 4x\Big|_0^{\pi/8} - 2\int_0^{\pi/8} x\sin 4x\,dx = \left(\frac{\pi}{8}\right)^2\sin\frac{\pi}{2} - 2\int_0^{\pi/8} x\cdot\left(\frac{\cos 4x}{-4}\right)'dx =$

$= \frac{\pi^2}{64}\cdot 1 + \frac{1}{2}\left(x\cos 4x\Big|_0^{\pi/8} - \int_0^{\pi/8} 1\cdot\cos 4x\,dx\right) = \frac{\pi^2}{64} + \frac{1}{2}\left(\frac{\pi}{8}\underbrace{\cos\frac{\pi}{2}}_{0} - 0\right) -$

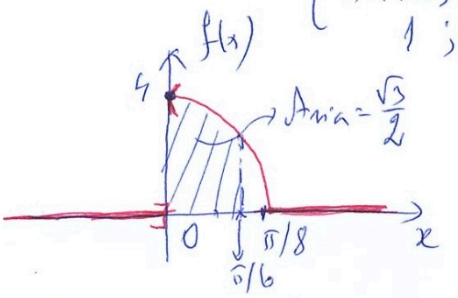
$- \frac{1}{2}\cdot\frac{\sin 4x}{4}\Big|_0^{\pi/8} = \frac{\pi^2}{64} - \frac{1}{8}(\underbrace{\sin\frac{\pi}{2}}_{1} - 0) = \frac{\pi^2 - 8}{64}$.

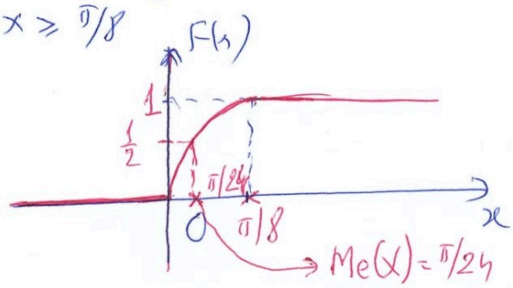
Deci $D^2(X) = \frac{\pi^2 - 8}{64} - \frac{(\pi-2)^2}{64} = \frac{4\pi - 12}{64} = \frac{\pi - 3}{16}$

(iv) $F(x) = \int_{-\infty}^{x} f(t)\,dt$

- Dacă $x \le 0 \Rightarrow F(x) = 0$
- Dacă $0 < x < \frac{\pi}{8} \Rightarrow F(x) = \int_{-\infty}^{0} 0\,dt + \int_0^x 4\cos 4t\,dt = \sin 4x$
- Dacă $x \ge \frac{\pi}{8} \Rightarrow F(x) = \int_{-\infty}^{0} 0\,dt + \int_0^{\pi/8} 4\cos 4t\,dt + \int_{\pi/8}^{x} 0\,dt = 4\cdot\frac{\sin 4x}{4}\Big|_0^{\pi/8} = 1$

Deci $F(x)=\begin{cases}0; & x\le 0\\ \sin 4x, & \text{dacă } 0<x<\frac{\pi}{8}\\ 1; & \text{dacă } x\ge \pi/8\end{cases}$





(v) $\underline{Me(X)=\alpha} \iff F(\alpha)=\frac{1}{2} \iff \sin 4\alpha=\frac{1}{2} \Rightarrow 4\alpha=\frac{\pi}{6} \Rightarrow \alpha=\pi/24 \Rightarrow$

$\Rightarrow Me(X)=\pi/24$

(vi) $Mo(X)$ = abscisa punctului de maxim pt. $f(x)$, dacă $\exists$.
Din grafic se observă că $\sup\{f(x) : x\in\mathbb{R}\}=4$, dar $\nexists x_0\in\mathbb{R}$ a.î. $f(x_0)=4$,
deci $Mo(X)$ NU EXISTĂ.

(vii) $P\left(-1\le X\le \frac{\pi}{6}\right)=\int_{-1}^{\pi/6} f(x)dx = \int_0^{\pi/6} 4\cos 4x\,dx = \sin 4x\Big|_0^{\pi/6} = \sin\frac{2\pi}{3}=\frac{\sqrt{3}}{2}$

(viii) • $\underline{\text{Pentru } Y=3X+\pi}$. $G(x)=F_Y(x)=P(Y<x)=P(3X+\pi<x)=P\left(X<\frac{x-\pi}{3}\right)=$

$=F_X\left(\frac{x-\pi}{3}\right)=F\left(\frac{x-\pi}{3}\right) \Rightarrow G(x)=\begin{cases}0; & \frac{x-\pi}{3}\le 0\\ \sin 4\frac{x-\pi}{3}; & 0<\frac{x-\pi}{3}<\frac{\pi}{8}\\ 1 & ; \frac{x-\pi}{3}\ge \pi/8\end{cases} \Rightarrow$

$\Rightarrow G(x)=\begin{cases}0; & x\le\pi\\ \sin\frac{4}{3}(x-\pi); & \pi<x<\frac{11\pi}{8}\\ 1 & ; x\ge\frac{11\pi}{8}\end{cases} \Rightarrow g(x)=G'(x)=\begin{cases}0; & x<\pi\\ \frac{4}{3}\cos\frac{4}{3}(x-\pi); & \pi<x<\frac{11\pi}{8}\\ 0; & x>11\pi/8\end{cases}$

• $\underline{\text{Pentru } Z=X^4}$ $\Rightarrow H(x)=F_Z(x)=P(Z<x)=P(X^4<x)=\begin{cases}0; & x\le 0\\ P(X<\sqrt[4]{x}); & x>0\end{cases}$

$H(x)=\begin{cases}0; & x\le 0\\ F(\sqrt[4]{x}); & x>0\end{cases}=\begin{cases}0; & x\le 0\\ \sin(4\sqrt[4]{x}); & 0<x<\frac{\pi^4}{32^4}\\ 1; & x\ge\frac{\pi^4}{1024}\end{cases}$

$h(x)=H'(x)=\begin{cases}0; & x<0 \text{ sau } x>\pi^4/1024\\ \frac{1}{\sqrt[4]{x^3}}\cos(4\sqrt[4]{x}); & 0<x<\pi^4/1024\end{cases}$

• $\underline{\text{Pentru } W=e^X}$. $L(x)=F_W(x)=P(W<x)=P(e^X<x)=\begin{cases}0; & x\le 0\\ P(X<\ln x); & x>0\end{cases}=$

$=\begin{cases}0; & x\le 0\\ F(\ln x); & x>0\end{cases}=\begin{cases}0; & x\le 0 \text{ sau } (\ln x\le 0, x>0)\\ \sin(4\ln x); & (0<\ln x<\pi/8; x>0\\ 1 & ; \ln x\ge\pi/8; x>0\end{cases}=\begin{cases}0; & x\le 1\\ \sin(4\ln x); & 1<x<e^{\pi/8}\\ 1; & x\ge e^{\pi/8}\end{cases}$

$l(x)=L'(x)=\begin{cases}0; & x<1 \text{ sau } x\ge e^{\pi/8}\\ (4/x)\cos(4\ln x); & 1<x<e^{\pi/8}\end{cases}$

Ex $f:\mathbb{R}\to\mathbb{R}$

$$f(x)=\begin{cases}0; & \le 0\\ a x e^{-4x}; & x>0\end{cases}$$

PDF. PT. O V.A. ID CONTINUĂ NOTATĂ X

REZOLVARE

(i) $a=?$

$$\int_{-\infty}^{\infty} f(x)\,dx=1$$

$$\underbrace{\int_{-\infty}^{0} 0\,dx}_{=0}+a\int_{0}^{\infty} \overset{f(x)}{x}\, e^{-\overset{s}{4}x}\,dx=1$$

$$a\cdot \mathcal{L}\{x\}(4)=1$$

$$a\cdot \frac{1}{s^2}\Big|_{s=4}=1 \implies \boxed{a=16}$$

SEMINAR 3 ADA

$$\int_{-\infty}^{\infty} f(x)\,dx=1$$

$$M_n(X)=\int_{-\infty}^{\infty} x^n f(x)\,dx$$

$$M_n(X)=M(X^n)$$

$$D^2(X)=M(X^2)-M^2(X)=M_2(X)-M_1^2(X)$$

$$F(x)=\int_{-\infty}^{x} f(t)\,dt$$

$$P(a\le X\le b)=\int_a^b f(x)\,dx=F(b)-F(a)$$

$$\mathcal{L}\{f(x)\}(s)=\int_0^{\infty} f(x)\cdot e^{-sx}\,dx$$

$$\mathcal{L}\{x^n\}(s)=\frac{n!}{s^{n+1}}$$

Ex $f:\mathbb{R}\to\mathbb{R}$ PDF 1D CONTINUĂ

$$f(t) = \begin{cases} a(2t+3)\,e^{-4t}, & t\geq 0 \\ 0, & t<0 \end{cases}$$

i) $a=?$

ii) $M_n(X)=?$

iii) $Var(X)$

iv) $R(t)$

v) $r(t)$

vi) $P(|X|\leq \ln 2)$

SEMINAR 4 ADA

③ LEGI CLASICE DE PROBABILITATE. TEOREMA LIMITĂ CENTRALĂ.

APLICAȚII ÎN TEORIA AȘTEPTĂRII, TEORIA FIABILITĂȚII, PROBLEME DE SONDAJ

$\boxed{\text{Ex. 1}}$ Legea de probabilitate (repartiția) Gamma

Fie $\alpha>0$, $s>0$ date. O v.a. 1D continuă, notată X are repartiția (distribuția) Gamma de parametri α și s dacă p.d.f. (X) este:

$$f(x)=\begin{cases}0; & x\le 0\\ \dfrac{s^{\alpha}}{\Gamma(\alpha)}x^{\alpha-1}e^{-sx}; & x>0\end{cases}\qquad \text{Notație } X=\Gamma(\alpha;s)$$

Să se determine: 1° Funcția caracteristică $\varphi(t)$; 2° $M_n(X)$; 3° $D^2(X)$

4° $Mo(X)$ 5° Pentru legea $X=\Gamma(2;3)$ să se determine:

(i) funcția de fiabilitate și rata de hazard. ~~...~~

(ii) $t\in\mathbb{N}$ minim, pentru care $R(t)\ge 20e^{-3t}$.

(iii) $t>0$ a.i. $r(t)=1$

R.

Rezolvare 1° $\varphi(t)=\int_{-\infty}^{\infty}f(x)e^{jtx}dx=\frac{s^\alpha}{\Gamma(\alpha)}\int_0^\infty x^{\alpha-1}e^{-sx}e^{jtx}dx=$

$=\frac{s^\alpha}{\Gamma(\alpha)}\int_0^\infty x^{\alpha-1}e^{-(s-jt)x}dx=\frac{s^\alpha}{\Gamma(\alpha)}\mathcal{L}\{x^{\alpha-1}\}(s-jt)=$

$=\frac{\Gamma(\alpha-1+1)}{(s-jt)^{\alpha-1+1}}\frac{s^\alpha}{\Gamma(\alpha)}=\frac{s^\alpha}{(s-jt)^\alpha}=\left(\frac{s}{s-jt}\right)^\alpha$

$\mathcal{L}\{f(x)\}(s)=\int_0^\infty f(x)e^{-sx}dx$

$\mathcal{L}\{x^\alpha\}(s)=\frac{\Gamma(\alpha+1)}{s^{\alpha+1}}$; $(\alpha>-1)$

$\mathcal{L}\{x^n\}(s)=\frac{n!}{s^{n+1}}$; $n\in\mathbb{N}$

2° Metoda 1 $\boxed{M_n(X)}=j^{-n}\varphi^{(n)}(0)=$

$=j^{-n}\cdot s^\alpha\left[(s-jt)^{-\alpha}\right]^{(n)}(0)=$

$=j^{-n}s^\alpha\underbrace{(-\alpha)(-\alpha-1)\dots(-\alpha-n+1)}(-j)^n(s-jt)^{-\alpha-n}\Big|_{t=0}=$

$=j^{-n}s^\alpha(-1)^n\cdot\alpha(\alpha+1)\dots(\alpha+n-1)\cdot(-1)^n j^n s^{-\alpha-n}=$

$=\frac{1}{s^n}\alpha(\alpha+1)(\alpha+2)\dots(\alpha+n-1)=\boxed{\frac{\Gamma(n+\alpha)}{\Gamma(\alpha)}s^{n}}$

$\left[(at+b)^\alpha\right]^{(n)}=$

$=\alpha(\alpha-1)\dots(\alpha-n+1)\cdot a^n(at+b)^{\alpha-n}$

$\Gamma(\alpha)=\int_0^\infty x^{\alpha-1}e^{-x}dx$; $\alpha>0$

$\Gamma(\alpha+1)=\alpha\Gamma(\alpha)$; $\alpha>0$

$\cancel{\Gamma(\alpha+1)}=\begin{cases}\Gamma(\alpha)=(\alpha-1)\Gamma(\alpha-1)\\ \alpha>+1\end{cases}$

$\cancel{\Gamma(n)}\begin{cases}\Gamma(n+\alpha)=(n+\alpha-1)(n+\alpha-2)\dots\\ \dots\alpha\Gamma(\alpha)\\ \Gamma(n+\alpha)=\alpha(\alpha+1)(\alpha+2)\dots(\alpha+n-1)\cdot\Gamma(\alpha)\end{cases}$

Metoda 2 $M_n(X)=\int_{-\infty}^{\infty}f(x)\cdot x^n dx=$

$=\frac{s^\alpha}{\Gamma(\alpha)}\int_0^\infty x^{n+\alpha-1}e^{-sx}dx=\frac{s^\alpha}{\Gamma(\alpha)}\mathcal{L}\{x^{n+\alpha-1}\}(s)=$

$=\frac{s^\alpha}{\Gamma(\alpha)}\cdot\frac{\Gamma(n+\alpha-1+1)}{s^{n+\alpha-1+1}}=\frac{s^\alpha}{s^{n+\alpha}}\cdot\frac{\Gamma(n+\alpha)}{\Gamma(\alpha)}=\boxed{\frac{1}{s^n}\frac{\Gamma(n+\alpha)}{\Gamma(\alpha)}}$

$\begin{cases}\Gamma(n)=(n-1)!; & n\in\mathbb{N}^*\\ \Gamma(1/2)=\sqrt{\pi}\end{cases}$

3° $\boxed{D^2(X)}=\boxed{Var(X)}=M(X^2)-M^2(X)=$

$=M_2(X)-M_1^2(X)=\frac{1}{s^2}\cdot\frac{\Gamma(\alpha+2)}{\Gamma(\alpha)}-\left(\frac{1}{s}\cdot\frac{\Gamma(\alpha+1)}{\Gamma(\alpha)}\right)^2=$

$=\frac{1}{s^2}\cdot\frac{(\alpha+1)\alpha\cancel{\Gamma(\alpha)}}{\cancel{\Gamma(\alpha)}}-\frac{1}{s^2}\left(\frac{\alpha\cancel{\Gamma(\alpha)}}{\cancel{\Gamma(\alpha)}}\right)^2=$

$=\frac{1}{s^2}(\alpha^2+\alpha-\alpha^2)=\boxed{\frac{\alpha}{s^2}}$

$\left\{\begin{array}{l}\Gamma(n+\frac12)=\frac{(2n)!}{2^{2n}n!}\sqrt{\pi}\\ \Gamma(n+\frac12)=\frac{(n+1)(n+2)\dots 2n}{2^{2n}}\sqrt{\pi}\end{array}\right.$

4° $x>0\Rightarrow f'(x)=\frac{s^\alpha}{\Gamma(\alpha)}(x^{\alpha-1}e^{-sx})'=\frac{s^\alpha}{\Gamma(\alpha)}e^{-sx}\left[(\alpha-1)x^{\alpha-2}-s\cdot x^{\alpha-1}\right]=$

$=\frac{s^\alpha}{\Gamma(\alpha)}e^{-sx}\cdot x^{\alpha-2}\left[(\alpha-1)-sx\right]$; $f'(x)=0\Leftrightarrow \underline{x=\frac{\alpha-1}{s}}>0\Leftrightarrow\alpha>1$

4° (I) $\underline{0<\alpha<1} \Rightarrow f'(x)<0, \forall x>0$

x	0		∞
$f(x)$	∞	$\searrow$	0

$\Rightarrow f(x)$ nu are punct de maxim $\Rightarrow \nexists M_o(X)$.

(II) $\underline{\alpha=1} \Rightarrow \begin{cases} f'(x) = s e^{-sx}(-s) = -e^{-sx}<0 \\ f(x) = s e^{-sx} \end{cases}$

x	$-\infty$	$0 \mid 0$	∞
$f(x)$	0	$\nexists \mid s>0$	

$\sup\{f(x): x\in\mathbb{R}\} = s \neq f(0)=0 \Rightarrow \nexists M_o(X)$

(III) $\underline{\alpha>1}$

x	0		$(\alpha-1)/s$		∞
$f'(x)$	$\cancel{0}$	$+$	0	$-$	
$f(x)$	0	$\nearrow$	Max	$\searrow$	

$$\boxed{M_o(X) = \frac{\alpha-1}{s}}$$

$$f_{max} = f(M_o(X)) = \cancel{\frac{s^\alpha}{\Gamma(\alpha)}} = \frac{s^\alpha}{\Gamma(\alpha)}\cdot\frac{(\alpha-1)^{\alpha-1}}{s^{\alpha-1}}\cdot e^{1-\alpha} = \frac{s}{\Gamma(\alpha)}\left(\frac{\alpha-1}{e}\right)^{\alpha-1}$$

5° $X=\Gamma(2;3) \Rightarrow f(x) = \begin{cases} 0; & x\le 0 \\ \frac{3^2}{\Gamma(2)}\cdot x^{2-1}e^{-3x}; & x>0 \end{cases}$

$$\boxed{f(x) = \begin{cases} 0; & x\le 0 \\ 9xe^{-3x}; & x>0 \end{cases}} \Rightarrow F(x) = \int_{-\infty}^{x} f(t)\,dt = \begin{cases} 0; & x\le 0 \\ \int_0^x 9te^{-3t}\,dt; & x>0 \end{cases}$$

Avem $\int_0^x te^{-3t}dt = \int_0^x t\cdot\left(\frac{e^{-3t}}{-3}\right)'dt = t\cdot\frac{e^{-3t}}{-3}\Big|_0^x + \frac{1}{3}\int_0^x t'\cdot e^{-3t}dt =$

$= -\frac{1}{3}xe^{-3x} - \frac{1}{9}e^{-3t}\Big|_0^x = -\frac{1}{3}xe^{-3x} - \frac{1}{9}e^{-3x} + \frac{1}{9} = \frac{1}{9}\left(1-3xe^{-3x}-e^{-3x}\right)$

Deci $F(x) = \begin{cases} 0; & x\le 0 \\ 1-e^{-3x}(3x+1); & x>0 \end{cases}$ sau $F(t) = \begin{cases} 0; & t\le 0 \\ 1-e^{-3t}(3t+1); & t>0 \end{cases}$

(i) Astfel, $R(t) = 1-F(t) = \begin{cases} 1; & t\le 0 \\ e^{-3t}(3t+1); & t>0 \end{cases}$; $f(t) = \begin{cases} 0; & t\le 0 \\ 9te^{-3t}; & t>0 \end{cases}$

$$r(t) = \frac{f(t)}{R(t)} = \begin{cases} \frac{0}{1} = 0; & t\le 0 \\ \frac{9t}{3t+1}; & t>0 \end{cases}$$

(ii) $R(t) \ge 20e^{-3t} \Rightarrow$

- $\underline{t\le 0}$: $1\ge 20e^{-3t} \Rightarrow e^{-3t}\le\frac{1}{20}$ fals, deoarece $e^{-3t}\ge e^0=1$
- $\underline{t>0}$: $e^{-3t}(3t+1)\ge 20e^{-3t} \Rightarrow (3t+1)\ge 20 \Rightarrow$
  $\Rightarrow t\ge 19/3 = 6,(3) \Rightarrow \underline{t_{min} = 7,\ t\in\mathbb{N}}$

(iii) $r(t) = 1 \Leftrightarrow 9t = 3t+1 \Rightarrow \underline{t=1/6}$.

OBS. $r'(t) = 9/(3t+1)^2 \Rightarrow 0\le r(t) < \lim_{t\to\infty} r(t) \Rightarrow \underline{0\le r(t)<3}$.

$\boxed{\text{Ex. 2}}$ Funcția de fiabilitate asociată v. a. T, care reprezintă timpul de bună funcționare a unui tub catodic, este

$$R(t) = P(T>t) = \begin{cases} 1; & t \le 0 \\ e^{-\lambda t}; & t>0 \end{cases} \; ; \lambda > 0 \text{ dat.}$$

1° Să se arate că T urmează o lege de probabilitate de tip exponențial-negativ, i.e. $T = \Gamma(1; \lambda)$.

2° Să se determine $M_n(T)$, $D^2(T)$, $\sigma(T)$, $Me(T)$, $A_0(T)$, $Ex(T)$.

3° Să se determine rata de hazard (defecțiune) $r(t)$ pentru T.

4° Dacă $\lambda = 0.001$, să se determine durata de bună funcționare a tubului catodic, pentru care fiabilitatea este cel puțin 95%.

5° Știind că durata medie de viață a unui tub catodic este 1600 h, să se determine:

(i) probabilitatea ca tubul să iasă din funcțiune în 1200 h

(ii) probabilitatea ca tubul să iasă din funcțiune într-un interval de timp cuprins între 1500 h și 2000 h

6° Două tuburi catodice (care funcționează independent) au duratele medii de viață 1000 h, respectiv 2000 h. Să se determine probabilitatea ca tuburile să iasă din funcțiune în intervalul (1500 h, 2500 h).

Rezolvare

1° Observăm că $R(t)$ este o funcție continuă (în $t=0$: $l_s = l_d = R(0) = 1$)

Avem $F(t) = 1 - R(t) = \begin{cases} 0; & t \le 0 \\ 1-e^{-\lambda t}; & t>0 \end{cases} \Rightarrow f(t) = \begin{cases} 0; & t \le 0 \\ \lambda e^{-\lambda t}; & t>0 \end{cases}$,

deoarece $f(t) = F'(t)$ și admitem $f(0) = 0$. Se observă că $f(t)$ este p.d.f. pentru legea $\Gamma(1;\lambda)$, deci $T = \Gamma(1;\lambda)$.

2° Din $\boxed{\text{Ex. 1}}$, luând $\alpha = 1$; $\beta = \lambda \Rightarrow M_n(X) = \frac{1}{\lambda^n} \cdot \frac{\Gamma(n+1)}{\Gamma(1)} = \frac{n!}{\lambda^n}$;

$M(X) = M_1(X) = \frac{1}{\lambda}$; $D^2(X) = Var(X) = \frac{1}{\lambda^2}$; $\sigma(X) = \sqrt{D^2(X)} = \frac{1}{\lambda}$.

Reținem $\boxed{M(T) = E(T) = \sigma(T) = \sqrt{D^2(T)} = \sqrt{Var(T)} = \frac{1}{\lambda}}$

pentru o lege exponențial-negativă $T = \Gamma(1;\lambda);\ \lambda > 0$

$\underline{Me(X) = d} \iff F(d) = \frac{1}{2} \iff 1 - e^{-\lambda d} = \frac{1}{2} \iff e^{-\lambda d} = \frac{1}{2} \iff$

$\iff -\lambda d = \ln\frac{1}{2} \iff -\lambda d = -\ln 2 \iff \boxed{d = Me(T) = \frac{\ln 2}{\lambda}}$

$\underline{As(T) = \frac{\mu_3(T)}{\sigma^3(T)}}\ ;\ \mu_3(T) = M_3(T - m),\ m = M(T) = \frac{1}{\lambda} \Rightarrow$

$\Rightarrow \mu_3(T) = M\left(\left(T - \frac{1}{\lambda}\right)^3\right) = M\left(T^3 - 3\frac{1}{\lambda}T^2 + 3\frac{1}{\lambda^2}T - \frac{1}{\lambda^3}\right) =$

$= M(T^3) - \frac{3}{\lambda}M(T^2) + \frac{3}{\lambda^2}M(T) - \frac{1}{\lambda^3}M(1) = M_3(T) - \frac{3}{\lambda}M_2(T) +$

$+ \frac{3}{\lambda^2}M(T) - \frac{1}{\lambda^3}M(T^0) = \frac{3!}{\lambda^3} - \frac{3}{\lambda}\cdot\frac{2!}{\lambda^2} + \frac{3}{\lambda^2}\frac{1!}{\lambda} - \frac{1}{\lambda^3}\cdot 1 =$

$= \frac{1}{\lambda^3}(6 - 6 + 3 - 1) = \frac{2}{\lambda^3} \Rightarrow \underline{As(T) = \frac{2/\lambda^3}{1/\lambda^3} = 2}$

$\underline{Ex(T) = \frac{\mu_4(T)}{\sigma^4(T)} - 3} \qquad \mu_4(T) = M_3(T - m) = M\left((T - m)^4\right) =$

$= M_4(T) - 4mM_3(T) + 6m^2M_2(T) - 4m^3M(T) + m^4 M(T^0) =$

$= \frac{4!}{\lambda^4} - \frac{4}{\lambda}\cdot\frac{3!}{\lambda^3} + \frac{6}{\lambda^2}\frac{2!}{\lambda^2} - \frac{4}{\lambda^3}\cdot\frac{1!}{\lambda} + \frac{1}{\lambda^4}\cdot 1 = \frac{1}{\lambda^4}(24 - 24 + 12 - 4 + 1) =$

$= 9/\lambda^4 \Rightarrow \underline{Ex(X) = \frac{\mu_4(X)}{\sigma^4(X)} - 3 = \frac{9/\lambda^4}{1/\lambda^4} - 3 = 6}$

3° $r(t) = \frac{f(t)}{R(t)} = \begin{cases} \frac{0}{1};\ t \le 0 \\ \frac{\lambda e^{-\lambda t}}{e^{-\lambda t}};\ t > 0 \end{cases} = \begin{cases} 0;\ t \le 0 \\ \lambda;\ t > 0 \end{cases} \Rightarrow \underline{r(t) = \text{const}}$

OBS. Reciproc, dacă o v.a. 1D, notată T, are $r_T(t) = \text{const} \Rightarrow$
$\Rightarrow T$ urmează o lege de probabilitate exponențial-negativă.

4° $R(t) \ge \frac{95}{100} \iff t \le 0$ sau $\left(t > 0 \text{ și } e^{-\lambda t} \ge \frac{19}{20}\right) \iff$

$\iff t \le 0$ și $\left(-\lambda t \ge \ln\frac{19}{20},\ t > 0\right) \iff \left(t \le 0 \text{ și } \begin{cases} t \le \frac{1}{\lambda}\ln\frac{20}{19} \\ t > 0 \end{cases}\right) \xLeftrightarrow{\lambda = 10^{-3}}$

$\Leftrightarrow t \leq 0$ şi $\left(t>0,\ t \leq 1000(\ln 20 - \ln 19) \simeq 1000 \cdot 0.0512 = 51,2\right)$

Deci $t \leq 51.2$ h. Luăm t_{max} (întreg) $= 51$ h.

5° $M(T) = \frac{1}{\lambda} = 1600 \Rightarrow \lambda = 1/1600$

(i) $P(T<1200) = F(1200) = 1 - e^{-\frac{1200}{1600}} = 1 - e^{-3/4} = 1 - \frac{1}{\sqrt[4]{e^3}} =$
$= 1 - \frac{\sqrt[4]{e}}{e} \simeq 0,53$

(ii) $P(1500 < T < 2000) = F(2000) - F(1500) = R(1500) - R(2000) =$
$= e^{-1500/1600} - e^{-2000/1500} = e^{-15/16} - e^{-4/3} = \frac{\sqrt[16]{e}}{e} - \frac{1}{\sqrt[3]{e^4}} = \frac{\sqrt[16]{e}}{e} - \frac{\sqrt[3]{e^2}}{e^2} =$
$= \frac{e\sqrt[16]{e} - \sqrt[3]{e^2}}{e^2}$

6° Fie T_1, T_2 v.a. asociate celor două tuburi

$M(T_1) = \cancel{\lambda}\ \frac{1}{\lambda_1} = 1000 \Rightarrow \lambda_1 = \frac{1}{1000}$; $M(T_2) = \frac{1}{\lambda_2} = 2000 \Rightarrow \lambda_2 = \frac{1}{2000}$

$\Rightarrow R_1(t) = e^{-(1/100)t}$; $R_2(t) = e^{-(1/2000)t}$; $t>0$

Fie $T = (T_1, T_2)$ v.a. 2D.

Avem $P(T \in (1500, 2500) \times (1500, 2500)) =$
$= P(T_1 \in (1500, 2500) ; T_2 \in (1500, 2500)) = P(T_1 \in (1500, 2500)) \cdot$
$\cdot P(T_2 \in (1500, 2500)) = (F_1(2500) - F_1(1500))(F_2(2500) - F_2(1500)) =$
$= (R_1(1500) - R_1(2500))(R_2(1500) - R_2(2500)) = \left(e^{-\frac{1500}{1000}} - e^{-\frac{2500}{1000}}\right) \cdot$
$\cdot \left(e^{-\frac{1500}{2000}} - e^{-\frac{25000}{2000}}\right) = (e^{-3/2} - e^{-5/2})(e^{-3/4} - e^{-5/4}) =$
$= e^{-5/2}(e-1) \cdot e^{-5/4}(e^{2/4} - 1) = e^{-5/2-5/4}(e-1)(\sqrt{e}-1) = e^{-\frac{15}{4}}(e-1)(\sqrt{e}-1) =$
$= \frac{(e-1)(\sqrt{e}-1)}{\sqrt[4]{e^{15}}} = \frac{\sqrt[4]{e}\,(e-1)(\sqrt{e}-1)}{e^4} \simeq 0.0258 = 2,58\%$

Ex. 1 În medie, 64% din semnalele recepționate într-un canal binar cu zgomot sunt simboluri „1", iar restul sunt simboluri „0". Se emit 5000 de simboluri „0" și „1" și admitem că variabila aleatoare care dă numărul de semnale „1" recepționate este de tip Bernoulli (binomială). Să se determine:

1° Probabilitatea ca numărul de simboluri „1" recepționate să fie situat între 3180 și 3248.

2° Un interval în care se află numărul de simboluri „0" recepționate, luând în considerare o eroare de 2% (probabilitate de 98%)

3° Un interval în care se află numărul de simboluri „1" recepționate, luând în calcul o eroare de 1% (probabilitate de 99%)

Rezolvare Fie X v.a. care reprezintă numărul de semnale „1" recepționate $\Rightarrow X = B(p; n)$; $n = 5000$; $p = 64\% = \frac{16}{25}$; $q = 1-p = \frac{9}{25}$ ...

Avem $m = np = 50\cancel{00} \cdot \frac{64}{1\cancel{00}} = 3200$; $\sigma = \sqrt{npq} = \sqrt{\cancel{5000} \cdot \frac{64}{\cancel{100}} \cdot \frac{36}{10\cancel{0}}} =$

$= \sqrt{32 \cdot 36} = \sqrt{16 \cdot 36 \cdot 2} = 24\sqrt{2} \simeq 33{,}94.$

1° $P(3180 \le X \le 3248) \simeq \phi\left(\frac{b-m}{\sigma}\right) - \phi\left(\frac{a-m}{\sigma}\right) = -\phi\left(\frac{3180-3200}{24\sqrt{2}}\right) +$

$\cancel{= \phi\left(\frac{-20}{24\sqrt{2}}\right)} + \phi\left(\frac{3248-3200}{24\sqrt{2}}\right) = -\phi\left(\underbrace{\frac{-20}{24\sqrt{2}}}_{-5/6\sqrt{2}}\right) + \phi\left(\frac{48^{2}}{24\sqrt{2}}\right) =$

$= \phi(\sqrt{2}) - \left(1 - \phi\left(\frac{5\sqrt{2}}{12}\right)\right) \simeq \phi(1.414) + \phi(0.589) - 1 =$

$\simeq 0.9214 + 0.7221 - 1 = \cancel{0.6435}\ 0.6435 \simeq 64{,}35\%$

2° $P(X \in [m-\alpha, m+\alpha]) = 2\phi\left(\frac{\alpha}{\sigma}\right) - 1 = 0.98 \Rightarrow$

$\Rightarrow \phi\left(\frac{\alpha}{\sigma}\right) = 0.99 \Rightarrow \frac{\alpha}{\sigma} = \cancel{0.99}\ \phi^{-1}(0.99) \Rightarrow$

$\Rightarrow \frac{\alpha}{\sigma} = 2{,}33 \Rightarrow \alpha = 2{,}33 \times 33{,}94 = 79.08$

$X \in [m-\alpha, m+\alpha] = [3200 - 79.08; 3200 + 79.08] \Rightarrow$

$\Rightarrow X \in [3120; 3280] \longrightarrow$ semnale „1"

$\bar{X} = 5000 - X \in [1720; 1880] \to$ semnale „1"

3° $P(X \in [m-\alpha, m+\alpha]) = 2\phi\left(\frac{\alpha}{\sigma}\right) - 1 = 0.99 \Rightarrow$

$\Rightarrow \phi\left(\frac{\alpha}{\sigma}\right) = 0.995 \Rightarrow \frac{\alpha}{\sigma} = \phi^{-1}(0.995) = 2,575 \Rightarrow$

$\alpha = 2,575 \cdot 33,94 \approx 87.39 \Rightarrow X \in [m-\alpha, m+\alpha] =$

$\approx [3200 - 87.39; 3200 + 87.39] \approx [3112; 3288]$

Ex. 3 Într-un sistem de așteptare, timpul de servire a unui client este descris de o v.a. $B(p;n)$, având media $m = 8$ (secunde), iar deviația standard este $\sigma = 2$.

Să se determine:

1° Probabilitatea ca durata de servire a primilor 50 de clienți să depășească 410 secunde

2° Probabilitatea ca durata de servire a primilor 200 clienți să fie situată în intervalul $(1590\,s; 1625\,s)$

3° Numărul de clienți serviți a.î. cu probabilitate de 95%, timpul de servire să depășească 500 secunde.

Rezolvare $X = B(p;n) \Longrightarrow P(a \le X \le b) \simeq \phi\left(\frac{b - M(X)}{\sigma(X)}\right) - \phi\left(\frac{a - M(X)}{\sigma(X)}\right)$

Rezolvare Fie X_k v.a. care descrie timpul de servire a clientului de rang k și $S_N = X_1 + X_2 + X_3 + \dots + X_N$; $N \ge 1$.

Avem: $M(S_N) = M(X_1) + M(X_2) + \dots + M(X_N) = mN = 8N$;

$\cancel{D^2(S_n) = Var(S_n) = D^2(X_1) + \dots + D^2(X_n) \ \text{v.a. indep}}$

$D^2(S_N) = Var(S_N) \overset{\text{v.a. indep}}{=} D^2(X_1) + D^2(X_2) + \dots + D^2(X_N) = N\sigma^2 = 4N$

Deoarece $X_1, X_2, X_3, \dots, X_N$ sunt v.a. de tip $B(p;n) \Longrightarrow$

$\Rightarrow S_N = \sum_{k=1}^{N} X_k$ este o v.a. de tip $B(p;mN)$, având

$M(S_N) = 8N$, și deviația standard (abaterea medie pătratică)

$\sigma(S_N) = \sqrt{D^2(S_N)} = 2\sqrt{N}$.

1° $P(S_{50} > 410) = P(410 < S_{50} < \infty) = \phi\left(\frac{\infty - M(S_{50})}{\sigma(S_{50})}\right) - \phi\left(\frac{410 - M(S_{50})}{\sigma(S_{50})}\right)$

$= \phi(\infty) - \phi\left(\frac{410 - 8 \cdot 50}{2\sqrt{50}}\right) = 1 - \phi\left(\frac{10}{10\sqrt{2}}\right) = 1 - \phi\left(\frac{\sqrt{2}}{2}\right) \simeq$

$\simeq 1 - \phi\left(\frac{1.414}{2}\right) = \cancel{\phi(1=} \ 1 - \phi(0.707) \simeq 1 - 0.7602 = 0.2398 = 23{,}98\%$.

2° $P(1590 < S_{200} < 1625) \simeq \phi\left(\frac{1625 - M(S_{200})}{\sigma(S_{200})}\right) - \phi\left(\frac{1590 - M(S_{200})}{\sigma(S_{200})}\right)$

$= \phi\left(\frac{1625 - 200 \cdot 8}{2\sqrt{200}}\right) - \phi\left(\frac{1590 - 200 \cdot 8}{2\sqrt{200}}\right) = \phi\left(\frac{25}{20\sqrt{2}}\right) - \phi\left(\frac{-10}{20\sqrt{2}}\right) =$

$= \phi\left(\frac{5\sqrt{2}}{8}\right) - \phi\left(-\frac{\sqrt{2}}{2}\right) \simeq \phi(0.884) - (1 - \phi(0.707)) =$

$\simeq 0.8117 + 0.7602 - 1 = 0.5719 = 57{,}19\%$

3° $P(S_N > 500) = 0.95 \Longleftrightarrow \cancel{1 - P(S_N \le 500) = 0.95} \Longleftrightarrow$

$\cancel{\Longleftrightarrow 1 - \phi} \ \Longleftrightarrow P(500 < S_N < \infty) = 0.95 \Longleftrightarrow$

$\Longleftrightarrow \phi\left(\frac{\infty - M(S_N)}{\sigma(S_N)}\right) - \phi\left(\frac{500 - M(S_N)}{\sigma(S_N)}\right) = 0.95 \Longleftrightarrow$

$\Longleftrightarrow 1 - \phi\left(\frac{500 - 8N}{2\sqrt{N}}\right) = 0.95 \Longleftrightarrow \phi\left(\frac{8N - 500}{2\sqrt{N}}\right) = 0.95 \Longleftrightarrow$

$1 - \phi(x) = \phi(-x)$.

$$\Leftrightarrow \frac{8N-500}{2\sqrt{N}} = \phi^{-1}(0.95) \Leftrightarrow \frac{4N-250}{\sqrt{N}} = 1.645$$

Fie $x = \sqrt{N} > 0 \Rightarrow 4x^2 - 1.645x - 250 = 0$

$$D = 1.645^2 + 2000 \Rightarrow x_{1,2} = \frac{1.645 \pm 44.752}{8} \Rightarrow$$

$\Rightarrow x \simeq 5.799 \Rightarrow x^2 \simeq 33,64$

Deci $N = x^2 = 34$.

4.10.3. Exemplu Legea exponențială-negativă T

$$f(x) = \begin{cases} 0; & x < 0 \\ \lambda e^{-\lambda x}; & x \geq 0 \end{cases} \quad ; \lambda > 0 \text{ dat}$$

Se cere (i) $\varphi_T(t)$; (ii) $M(T) = E(T)$; (iii) $D^2(T) = Var(T)$

(iv) $R_T(t)$; (v) $r_T(t)$

R: (i) $\varphi(t) = \int_{-\infty}^{\infty} f(x) e^{jtx} dx = \int_0^{\infty} \lambda e^{-\lambda x} e^{jtx} dx = \lambda \int_0^{\infty} e^{-(\lambda - jt)x} \cdot 1 \, dx =$

$= \lambda \mathcal{L}\{1\}(\lambda - jt) = \lambda \cdot \frac{1}{\lambda - jt} = \frac{\lambda}{\lambda - jt} = \frac{\lambda(\lambda + jt)}{\lambda^2 + t^2}$

(ii) $M(T) = +j^{-1}\varphi'(0) = +\frac{1}{j} \left. \frac{\lambda j}{(\lambda - jt)^2} \right|_{t=0} = \frac{\lambda}{\cancel{\lambda^2}} = \frac{1}{\lambda}$

(iii) $\varphi''(t) = \left(\frac{\lambda j}{(\lambda - jt)^2} \right)' = \frac{2\lambda j^2}{(\lambda - jt)^3} \Rightarrow D^2(T) = Var(T) = -\varphi''(0) + (\varphi'(0))^2 =$

$= \cancel{\frac{-2\lambda}{\lambda^3} = -\frac{2}{\lambda^2} \Rightarrow D^2} = +\frac{2\lambda}{\lambda^3} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2} \Rightarrow \sigma(T) = \frac{1}{\lambda} = M(T)$

(iv) $R(t) = 1 - F(t)$; $F(t) = \int_{-\infty}^{t} f(x) dx = \begin{cases} 0; & t < 0 \\ \lambda \int_0^t e^{-\lambda x} dx; & t \geq 0 \end{cases} =$

$= \begin{cases} 0; & t < 0 \\ \lambda \cdot \left. \frac{e^{-\lambda x}}{-\lambda} \right|_0^t; & t \geq 0 \end{cases} = \begin{cases} 0; & t < 0 \\ 1 - e^{-\lambda t}; & t \geq 0 \end{cases} \Rightarrow R(t) = \begin{cases} 1; & t < 0 \\ e^{-\lambda t}; & t \geq 0 \end{cases}$

(v) $r(t) = \frac{f(t)}{R(t)} = \begin{cases} 0; & t < 0 \\ \lambda; & t \geq 0 \end{cases} \Rightarrow r(t) = \lambda = ct, \forall t \geq 0$

OBS. Reciproc, dacă o v.a. 1D are rata de defectare constantă, atunci T urmează o lege de probabilitate exponențial-negativă

(p. 146-147)

9. Aplicație (Problemă de sondaj)

Din analize statistice, s-a constatat că 4% din calculatoarele produse de o firmă de profil prezintă defecțiuni. Un magazin comercializează 15000 de calculatoare de acest tip.

1° Să se determine probabilitatea ca în magazin să se afle:

(i) cel mult 630 calculatoare cu defecțiuni

(ii) un număr de calculatoare fără defecțiuni cuprins între 14340 și 14436

2° Să se determine cu o eroare de maximum 3% numărul de calculatoare cu defecțiuni din magazin.

3° Să se determine cu probabilitate de 98% (deci cu o eroare de 2%) numărul de calculatoare fără defecțiuni din magazin.

Rezolvare: Fie X v.a. ale cărei valori reprezintă numărul de calculatoare fără defecțiuni din magazin. $\Rightarrow$ X este de tip Bernoulli.

Avem $n = 15000$, $p = 96\% = \frac{96}{100} = \frac{24}{25}$; $q = 4\% = \frac{4}{100} = \frac{1}{25}$ $\Rightarrow$

$\Rightarrow m = np = 15000 \cdot \frac{96}{100} = 14400$; $\sigma = \sqrt{npq} = \sqrt{15000 \cdot \frac{96}{100} \cdot \frac{4}{100}} =$

$= \sqrt{\frac{60 \cdot 96}{10}} = \sqrt{6 \cdot 6 \cdot 16} = 6 \cdot 4 = 24 \Rightarrow m = 14400;\ \sigma = 24.$

1° (i) $P(\overline{X} \le 630) = P(X \ge 15000 - 630) = P(X \ge 14370) =$

$\boxed{X + \overline{X} = 15000}$ $= P(14370 \le X < \infty) \simeq \phi\left(\frac{\infty - m}{\sigma}\right) - \phi\left(\frac{14370 - m}{\sigma}\right)$

$= \phi(\infty) - \phi\left(\frac{14370 - 14400}{24}\right) = 1 - \phi\left(\frac{-30}{24}\right) = 1 - \phi(-1{,}25) =$

$= 1 - (1 - \phi(1{,}25)) = \phi(1{,}25) \simeq 0.8944 = 89{,}44\%$; $\phi(-x) = 1 - \phi(x)$

(ii) $P(13740 \le X \le 14036) = \phi\left(\frac{14436 - m}{\sigma}\right) - \phi\left(\frac{14340 - m}{\sigma}\right) =$

$= \phi\left(\frac{36}{60}\right) - \phi\left(\frac{-60}{60}\right) = \phi\left(\frac{3}{5}\right) - \phi(-1) = \phi(0.6) - (1 - \phi(1)) =$

$= \phi(0.6) + \phi(1) - 1 \simeq 0.6406 + 0.8413 - 1 = 0{,}4819 = 48{,}19\%$

2° $P(|X - m| < \alpha) = 2\phi\left(\frac{\alpha}{\sigma}\right) - 1 \Leftrightarrow P(X \in (m - \alpha, m + \alpha)) = 2\phi\left(\frac{\alpha}{\sigma}\right) - 1$

$P(X \in (m - \alpha, m + \alpha)) = 97\% = 0.97 \Longrightarrow 2\phi\left(\frac{\alpha}{\sigma}\right) - 1 = 0.97 \Rightarrow$

$\Rightarrow \phi\left(\frac{\alpha}{\sigma}\right) = 0.985 \Rightarrow \frac{\alpha}{24} = \phi^{-1}(0.985) \simeq 2{,}17 \Rightarrow$

$\Rightarrow \alpha \simeq 24 \cdot 2{,}17 = 52.08 \Longrightarrow X \in (m - \alpha, m + \alpha) \Rightarrow$

$\Rightarrow X \in (14400 - 52{,}08;\ 14400 + 52{,}08) \Rightarrow X \in [14347; 14453]$

3° $P(|X - m| < \alpha) = 2\phi\left(\frac{\alpha}{\sigma}\right) - 1 = 98\% = 0.98 \Rightarrow$

$\Rightarrow P(X \in (m - \alpha, m + \alpha)) = 2\phi\left(\frac{\alpha}{\sigma}\right) - 1 = 0{,}98 \Rightarrow \phi\left(\frac{\alpha}{\sigma}\right) = 0.99$

$\Rightarrow \phi\left(\frac{\alpha}{24}\right) = 0.99 \Rightarrow \frac{\alpha}{24} = \phi^{-1}(0.99) = 2.33 \Rightarrow \alpha = 24 \cdot 2{,}33 \Rightarrow$

$\Rightarrow \alpha = 55{,}92 \Longrightarrow X \in (14400 - 55{,}92,\ 14400 + 55{,}92) \Rightarrow$

$\Rightarrow X \in [14344; 14456) \Rightarrow \overline{X} = 15000 - X \in [544; 656]$

④ LANȚURI MARKOV

Ex. 1 Se consideră lanțul Markov ~~$(S, p^{(0)}, \Pi)$~~ $(X_k)_{k\geq 0}$ asociat tripletului $(S, p^{(0)}, \Pi)$ unde
$S = \{1, 2, 3\}$, $p^{(0)} = \left(a^2, a, \frac{5}{9}\right)$, $\Pi = \begin{pmatrix} a & b & 1/2 \\ \frac{1}{10} & \frac{1}{2} & \frac{2}{5} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{pmatrix}$.

Să se determine: ① $a, b \in \mathbb{R}$

② Probabilitatea ca lanțul să se deruleze pe traiectoria $(2, 1, 3, 3)$

③ $P(X_{200} = 1, X_{199} = 2, X_{198} = 3 / X_{197} = 2)$

④ $P(X_{1000} = 1, X_{998} = 2 / X_{997} = 3)$

⑤ $P(X_0 = 2 / X_2 = 3)$

⑥ $P(X_1 = 1) + P(X_2 = 3)$

⑦ Repartiția limită a lanțului Markov

⑧ $P(X_{79} = 2 / X_{77} \neq 2, X_{77} \neq 1)$

⑨ $P(X_0 = 2 / X_1 = 1, X_3 = 2)$

⑩ a) $P(X_2 = 2 / X_4 = 1, X_5 = 2)$
b) $P(X_2 = 1 / X_3 = 2, X_1 = 3)$

Rezolvare

① $\begin{cases} a^2 + a + \frac{5}{9} = 1 \\ a + b + \frac{1}{2} = 1 \\ a, b \in [0,1] \end{cases} \Leftrightarrow \begin{cases} 9a^2 + 9a - 4 = 0 \\ a + b = 1/2 \\ a, b \in [0,1] \end{cases} \Leftrightarrow \begin{cases} a = \frac{-9 \pm \sqrt{81+144}}{18} & \frac{-9-15}{18} < 0 ; \ \frac{-9+15}{18} = \frac{1}{3} \\ \begin{cases} a = \frac{1}{3} \\ b = \frac{1}{6} \end{cases} \end{cases}$

$p^{(0)} = \left(\frac{1}{9}, \frac{1}{3}, \frac{5}{9}\right)$; $\Pi = \begin{pmatrix} \frac{1}{3} & \frac{1}{6} & \frac{1}{2} \\ \frac{1}{10} & \frac{1}{2} & \frac{2}{5} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{pmatrix}$; $p^{(0)} = (p_1^{(0)}, p_2^{(0)}, p_3^{(0)})$; $\Pi = (p_{ij})_{1\leq i,j\leq 3}$

② $P(X_0 = 2, X_1 = 1, X_2 = 3, X_3 = 3) =$
$= P((X_0 = 2) \cap (X_1 = 1) \cap (X_2 = 3) \cap (X_3 = 3)) \stackrel{(iii)}{=}$
$\stackrel{(iii)}{=} P(X_0) \cdot P(X_1 = 1 / X_0 = 2) \cdot P(X_2 = 3 / X_1 = 1, \cancel{X_0 = 2}) \cdot P(X_3 = 3 / X_2 = 3, \cancel{X_1 = 1, X_0 = 2})$
$\stackrel{3^\circ}{=} P(X_0) \cdot P(X_1 = 1 / X_0 = 2) \cdot P(X_2 = 3 / X_1 = 1) \cdot P(X_3 = 3 / X_2 = 3) =$
$\stackrel{1^\circ, 2^\circ}{=} p_1^{(0)} \cdot p_{21} \cdot p_{13} \cdot p_{33} = \frac{1}{9} \cdot \frac{1}{10} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{360}$

③ $P(\underbrace{X_{200} = 1, X_{199} = 2, X_{198} = 3}_{A} / \underbrace{X_{197} = 2}_{B}) = P(A/B) \stackrel{(iv)}{=}$
$\stackrel{(iv)}{=} \frac{P(A \cap B)}{P(B)} = \frac{P(X_{200} = 1, X_{199} = 2, X_{198} = 3, X_{197} = 2)}{P(X_{197} = 2)} =$
$= \frac{P(X_{197} = 2, X_{198} = 3, X_{199} = 2, X_{200} = 1)}{P(X_{197} = 2)} \stackrel{(iii)}{=}$
$\stackrel{(iii)}{=} \frac{\cancel{P(X_{197} = 2)} \cdot P(X_{198} = 3 / X_{197} = 2) \cdot P(X_{199} = 2 / X_{198} = 3, \cancel{X_{197} = 2}) \cdot \cancel{P(X_{200} = 1 / X_{199} = 2 \ldots}}{\cancel{P(X_{197} = 2)}}$
$P(X_{200} = 1 / X_{199} = 2, \cancel{X_{198} = 3, X_{197} = 2})$
$\stackrel{3^\circ}{=} P(X_{198} = 3 / X_{197} = 2) \cdot P(X_{199} = 2 / X_{198} = 3) \cdot P(X_{200} = 1 / X_{199} = 2) \stackrel{2^\circ}{=}$
$\stackrel{2^\circ}{=} p_{23} \cdot p_{32} \cdot p_{21} = \frac{2}{5} \cdot \frac{1}{4} \cdot \frac{1}{10} = \cancel{\frac{1}{360}} \ \frac{1}{100}$

④ $P(\underbrace{X_{1000} = 1, X_{998} = 2}_{A} / \underbrace{X_{997} = 3}_{B}) = P(A/B) \stackrel{(iv)}{=} \frac{P(A \cap B)}{P(B)} =$
$= \frac{P(X_{1000} = 1, X_{998} = 2, X_{997} = 3)}{P(X_{997} = 3)} = \frac{P(X_{997} = 3, X_{998} = 2, X_{1000} = 1)}{P(X_{997} = 3)} \stackrel{(ii)}{=}$

$$\overset{(iii)}{=} \frac{\cancel{P(X_{997}=3)} \cdot P(X_{998}=2/X_{997}=3) \cdot P(X_{1000}=1/X_{998}=2, \cancel{X_{997}=3})}{\cancel{P(X_{997}=3)}}$$

$$\overset{3^\circ}{=} P(X_{998}=2/X_{997}=3) \cdot P(X_{1000}=1/X_{998}=2) \underset{4.2:\, m=2}{\overset{2^\circ}{=}}$$

$$= p_{32} \cdot p_{21}^{(1000-998)} = p_{32} \cdot \underbrace{p_{21}^{(2)}}_{\Pi^2} = \frac{1}{4} \cdot (l_2 \times c_1) =$$

$$= \frac{1}{4} \cdot \underbrace{\left(\frac{1}{10}, \frac{1}{2}, \frac{2}{5}\right) \times \begin{pmatrix} 1/3 \\ 1/10 \\ 1/4 \end{pmatrix}}_{} = \frac{1}{4}\left(\frac{1}{10} \cdot \frac{1}{3} + \frac{1}{2} \cdot \frac{1}{10} + \frac{2}{5} \cdot \frac{1}{4}\right) =$$

$$= \frac{1}{4}\left(\frac{1}{30} + \frac{1}{20} + \frac{1}{10}\right) = \frac{1}{4} \cdot \frac{2+3+6}{60} = \frac{11}{240}$$

$$\boxed{P(A/B) = \frac{P(A) \cdot P(B/A)}{P(B)} \quad \text{Bayes}}$$

(5) $P(\underbrace{X_0=2}_{A} / \underbrace{X_2=3}_{B}) \overset{(iv)}{=}$

$$\cancel{\overset{(iv)}{=} \frac{P(X_0=2) \cdot P(X_2=3/X_0)}{P(X_2)} \underset{4.2}{\overset{1^\circ}{=}}} \;\; \underset{4.2}{\overset{1^\circ}{=}} \; \cancel{p_2^{(0)} \cdot}$$

$$\overset{(iv)}{=} \frac{P(X_0=2) \cdot P(X_2=3/X_0=2)}{P(X_2=3)} \underset{4.2}{\overset{1^\circ}{=}} \frac{p_2^{(0)} \cdot p_{23}^{(2-0)}}{P(X_2=3)} = \frac{\frac{1}{3} \cdot p_{23}^{(2)}}{P(X_2=3)} \;(1)$$

$$p_{23}^{(2)} = l_2 \times c_3 = \left(\frac{1}{10}, \frac{1}{2}, \frac{1}{5}\right) \times \begin{pmatrix} 1/2 \\ 2/5 \\ 1/2 \end{pmatrix} = \frac{1}{20} + \frac{1}{5} + \frac{1}{10} = \frac{1+4+2}{20} = \frac{7}{20} \;(2)$$

$P(X_2=3) \underset{(i)}{\overset{4.3}{=}} p_3^{(2)}$, unde $p^{(2)} = (p_1^{(2)}, p_2^{(2)}, p_3^{(2)}) = (p^{0})\,\Pi^2 =$

$$= \left(\frac{1}{9}, \frac{1}{3}, \frac{5}{9}\right) \cdot \begin{pmatrix} 1/3 & 1/6 & 1/2 \\ 1/10 & 1/2 & 2/5 \\ 1/4 & 1/4 & 1/2 \end{pmatrix}^2 = \left(\frac{1}{9}, \frac{1}{3}, \frac{1}{9}\right) \begin{pmatrix} 1/3 & 1/6 & 1/2 \\ 1/10 & 1/2 & 2/5 \\ 1/4 & 1/4 & 1/2 \end{pmatrix} \begin{pmatrix} 1/3 & 1/6 & 1/2 \\ \frac{1}{10} & \frac{1}{2} & \frac{2}{5} \\ \frac{1}{9} & \frac{1}{4} & \frac{1}{2} \end{pmatrix}$$

$p_3^{(2)} =$ (col 3) din $p^{(0)} \cdot \Pi^2 = p^{(0)} \times$ (col 2 din Π^2) $= p^{(0)} \cdot \begin{pmatrix} \frac{1}{3} \cdot \frac{1}{6} + \frac{1}{6} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{4} \\ \frac{1}{10} \cdot \frac{1}{6} + \frac{1}{2} \cdot \frac{1}{2} + \frac{2}{5} \cdot \frac{1}{4} \\ \frac{1}{4} \cdot \frac{1}{6} + \frac{1}{4} \cdot \frac{1}{6} + \frac{1}{2} \cdot \frac{1}{4} \end{pmatrix} =$

$$= (p^{0}) \cdot \begin{pmatrix} \frac{1}{18} + \frac{1}{12} + \frac{1}{8} \\ \frac{1}{60} + \frac{1}{4} + \frac{1}{10} \\ \frac{1}{24} + \frac{1}{24} + \frac{1}{8} \end{pmatrix} = \left(\frac{1}{9}, \frac{1}{3}, \frac{5}{9}\right) \times \begin{pmatrix} \frac{19}{72} \\ \frac{22}{60} \\ \frac{5}{24} \end{pmatrix} = \left| \begin{matrix} 9^2 \cdot 8 \\ 9 \cdot 10 \\ 9 \cdot 3 \cdot 8 \end{matrix} \right. \to 9^2 \cdot 3 \cdot 40$$

$$= \frac{1}{9} \cdot \frac{19}{72} + \frac{1}{3} \cdot \frac{11}{30} + \frac{5}{9} \cdot \frac{5}{24} = \frac{19}{9 \cdot 72} + \frac{11}{90} + \frac{25}{9 \cdot 24} = \frac{15 \cdot 19 + 81 \cdot 55 + 45 \cdot 25}{9^2 \cdot 40 \cdot 3} =$$

$$= (285 + 4455 + 1125)/(27 \cdot 40) = 5865/27 \cdot 40 \cdot 9 = \cancel{391/9720} \; \cancel{(3)} \; 5865/9720 \;(3)$$

Din (1), (2), (3) $\Rightarrow p = P(X_1=2 / X_2=3) = \frac{\cancel{1}}{\cancel{3}}\cdot\frac{7}{\cancel{20}}\cdot\frac{\overset{162}{\cancel{9720}}}{5865} = \frac{1134}{5865} =$

$= \frac{388}{1955}.$

(6) $P(X_1=1) \overset{4.3}{\underset{(1)}{=}} p_1^{(1)}$, unde $p^{(1)} = p^{(0)}\Pi = (p_1^{(1)}, p_2^{(1)}, p_3^{(1)})$,

deci $P(X_1=1) = p^{(0)} \times \text{col } 1(\Pi) = \left(\frac{1}{9}, \frac{1}{3}, \frac{5}{9}\right) \times \begin{pmatrix} 1/3 \\ 1/10 \\ 1/4 \end{pmatrix} =$

$= \frac{1}{9}\cdot\frac{1}{3} + \frac{1}{3}\cdot\frac{1}{10} + \frac{5}{9}\cdot\frac{1}{4} = \overset{20}{\frac{1}{27}} + \overset{18}{\frac{1}{30}} + \overset{15}{\frac{5}{36}} = \frac{20+18+15}{540} = \frac{113}{540}$ (1)

$P(X_2=3) \overset{4.3}{\underset{(1)}{=}} p_3^{(2)} \overset{5}{\underset{(3)}{=}} \frac{5865}{9720}$ (2)

Din (1) și (2) $\Rightarrow P(X_1=1) + P(X_2=3) = \overset{18}{\frac{113}{540}} + \frac{5865}{9720} = \frac{2599}{9720}$

(7) $p^* = (p_1^*, p_2^*, p_3^*)$

$p^*\Pi = p^* \Leftrightarrow (p_1^*, p_2^*, p_3^*)\cdot\begin{pmatrix} \frac{1}{3} & \frac{1}{6} & \frac{1}{2} \\ \frac{1}{10} & \frac{1}{2} & \frac{2}{5} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{pmatrix} = (p_1^*, p_2^*, p_3^*) \Leftrightarrow$

$\Leftrightarrow \begin{cases} \frac{1}{3}p_1^* + \frac{1}{10}p_2^* + \frac{1}{4}p_3^* = p_1^* \mid \cdot 60 \\ \frac{1}{6}p_1^* + \frac{1}{2}p_2^* + \frac{1}{4}p_3^* = p_2^* \mid \cdot 60 \\ \frac{1}{4}p_1^* + \frac{2}{5}p_2^* + \frac{1}{2}p_3^* = p_3^* \mid \cdot 20 \\ p_1^* + p_2^* + p_3^* = 1 \end{cases} \Leftrightarrow \begin{cases} -40p_1^* + 6p_2^* + 15p_3^* = 0 \mid \cdot(-1) \\ 10p_1^* - 30p_2^* + 15p_3^* = 0 \\ 5p_1^* + 8p_2^* - 10p_3^* = 0 \\ p_1^* + p_2^* + p_3^* = 1 \mid \cdot 10 \end{cases}$

$\cancel{\Leftrightarrow \begin{cases} -40p_1^* + 6p_2^* + 5 = 0 \\ 5p_1^* - 11p_2^* + 4 = 0 \mid \cdot 8 \end{cases}} \Leftrightarrow \cancel{\begin{cases} -114p_2^* + 37 = 0 \\ 5p_1^* - 11p_2^* + 4 = 0 \end{cases}} \Rightarrow \cancel{\boxed{p_3^* = \frac{1}{3}}}$

$\cancel{\Rightarrow p_2^* = \frac{37}{114}, 5p_1^* = 11\cdot\frac{37}{114} - 4 = \frac{185-152}{38} \Rightarrow p_1^* = \frac{33}{38\cdot 5} = \frac{33}{190}}$

$\begin{cases} 50p_1^* - 36p_2^* = 0 \mid : 2 \\ 15p_1^* + 18p_2^* = 10 \end{cases} \xrightarrow{+} 40p_1^* = 10 \Rightarrow p_1^* = \frac{1}{4},\ p_2^* = \frac{25}{18}p_1^* =$

$= \frac{25}{18}\cdot\frac{1}{4} = \frac{25}{72}$

Din $p_1^* + p_2^* + p_3^* = 1 \Rightarrow p_3^* = 1 - \frac{1}{4} - \frac{25}{72} = \frac{29}{72}.$

Deci $p^* = \left(\frac{1}{4}, \frac{25}{72}, \frac{29}{72}\right).$

Seminar 6 - MA

(8) $P(X_{79}=2 / X_{77}\neq 2) = \cancel{P(X_{79}=2 / X_{77}}=$
$X_{77}\neq 1$

$= P(X_{79}=2 / X_{77}=3) \overset{4.2}{=} p_{32}^{(79-77)} = p_{32}^{(2)} = l_3 \times c_2 =$

$= \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{2}\right) \times \left(\frac{1}{6}, \frac{1}{2}, \frac{1}{4}\right)^T = \frac{1}{24} + \frac{1}{8} + \frac{1}{8} = \frac{7}{24}$

(9) $P(\underbrace{X_0=2}_{A} / \underbrace{X_1=1, X_3=2}_{B}) \overset{4}{\underset{(iv)}{=}} \frac{P(X_0=2, X_1=1, X_3=2)}{P(X_0=2)} \overset{4}{\underset{(ii)}{=}}$

$= \frac{\cancel{P(X_0=2)} \cdot P(X_1=1/X_0=2) \cdot P(X_3=2/X_1=1, \cancel{X_0=2})}{\cancel{P(X_0=2)}} \overset{3^\circ}{=}$

$= P(X_1=1/X_0=2) \cdot P(X_3=2/X_1=1) \overset{2^\circ}{\underset{4.2}{=}} p_{21} \cdot p_{12}^{(3-1)} = p_{21} \cdot p_{12}^{(2)} =$

$= \frac{1}{10} \cdot (l_1 \times c_2) = \frac{1}{10} \cdot \left(\frac{1}{3}, \frac{1}{6}, \frac{1}{2}\right) \times \left(\frac{1}{6}, \frac{1}{2}, \frac{1}{4}\right)^T = \left(\frac{1}{18} + \frac{1}{12} + \frac{1}{8}\right) \cdot \frac{1}{10}$

$= \frac{8+12+18}{144} \cdot \frac{1}{10} = \frac{38}{144 \cdot 10} = \frac{19}{720}$

(10) a) $P(\underbrace{X_2=2}_{A} / X_4=1, X_5=2) \overset{4}{\underset{(iv)}{=}} \frac{P(X_2=2, X_4=1, X_5=2)}{P(X_4=1, X_5=2)} =$

$\overset{4}{\underset{(i),(ii)}{=}} \frac{P(X_2=2) \cdot P(X_4=1/X_2=2) \cdot P(X_5=2/X_4=1, \cancel{X_2=2})}{P(X_4=1) \cdot P(X_5=2/X_4=1)} =$

$\overset{1^\circ,2^\circ,3^\circ}{\underset{4.\ldots}{=}} \frac{p_2^{(2)} \cdot p_{21}^{(2)} \cdot \cancel{p_{12}}}{p_1^{(4)} \cdot \cancel{p_{12}}} = \frac{(p^{(0)} \cdot \Pi^2)_{col\,2}}{(p^{(0)} \cdot \Pi^4)_{col\,1}}$

b) $P(\underbrace{X_2=1}_{A} / \underbrace{X_3=2, X_1=3}_{B}) \overset{4}{\underset{iv}{=}} \frac{P(X_2=1, X_3=2, X_1=3)}{P(X_3=2, X_1=3)} =$

$= \frac{P(X_1=3, X_2=1, X_3=2)}{P(X_1=3, X_3=2)} \overset{4(ii)}{\underset{4(i)}{=}} \frac{\cancel{P(X_1=3)} \cdot P(X_2=1/X_1=3) \cdot P(X_3=2/X_2=1, \cancel{X_1=3})}{\cancel{P(X_1=3)} \cdot P(X_3=2/X_1=3)}$

$\overset{2^\circ,3^\circ}{\underset{4.2}{=}} \frac{p_{31} \cdot p_{12}}{p_{32}^{(3-1)}} = \frac{p_{31} \cdot p_{12}}{p_{32}^{(2)}} = \frac{\frac{1}{4} \cdot \frac{1}{6}}{l_3 \times c_2} = \frac{\frac{1}{24}}{\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{2}\right) \times \left(\frac{1}{6}, \frac{1}{2}, \frac{1}{4}\right)^T} =$

$= \frac{1/24}{\frac{1}{24} + \frac{1}{8} + \frac{1}{8}} = \frac{1}{24} \cdot \frac{24}{7} = \frac{1}{7}$

VEZI ȘI SEMINAR 7 EX. 1

FILTRE NUMERICE (TRANSFORMATA Z)

② Ex. 1. Utilizând metoda transformării „Z", să se calculeze suma seriei numerice $s=\sum_{n=-1}^{\infty}\frac{3n^2(-1)^n+2n\cos^2\left(\frac{n\pi}{3}\right)}{2^n}$

R. Avem: $Z\{x(n)\}(z)=\sum_{n=0}^{\infty}\frac{x(n)}{z^n}$; $x\in S_d^+$

$$s=\sum_{n=0}^{\infty}\frac{3n^2(-1)^n+2n\cos^2\left(\frac{n\pi}{3}\right)}{2^n}+\left.\frac{3n^2(-1)^n+2n\cos^2\left(\frac{n\pi}{3}\right)}{2^n}\right|_{n=-1}$$

$$s=3\sum_{n=0}^{\infty}\frac{n^2(-1)^n}{2^n}+2\sum_{n=0}^{\infty}\frac{n\cos^2\left(\frac{n\pi}{3}\right)}{2^n}+\frac{3\cdot1(-1)-2\cos^2(-\pi/3)}{2^{-1}}$$

$$s=3\sum_{n=0}^{\infty}\frac{n^2}{(-2)^n}+2\sum_{n=0}^{\infty}\frac{n\cos^2\left(\frac{n\pi}{3}\right)}{2^n}-7=3s_1+2s_2-7\quad(1)$$

$$3s_1=\sum_{n=0}^{\infty}\frac{\overset{x(n)}{n^2}}{\underset{z}{(-2)}^n}=3Z\{n^2\}(-2)=3\cdot\left.\frac{z(z+1)}{(z-1)^3}\right|_{z=-2}=3\frac{(-2)(-1)}{(-3)^3}=-\frac{2}{9}\quad(2)$$

Utilizând egalitatea $\cos^2\alpha=\frac{1}{2}(1+\cos 2\alpha)$, avem:

$$2s_2=\cancel{2}\sum_{n=0}^{\infty}\frac{n}{2^n}\cdot\frac{1+\cos\frac{2n\pi}{3}}{\cancel{2}}=\sum_{n=0}^{\infty}\frac{n}{2^n}+\sum_{n=0}^{\infty}\frac{\overset{x(n)}{n\cdot\cos\frac{2n\pi}{3}}}{\underset{z}{2}^n}=s_3+s_4\quad(3)$$

$$s_3=\sum_{n=0}^{\infty}\frac{\overset{x(n)}{n}}{\underset{z}{2}^n}=Z\{n\}(2)=\left.\frac{z}{(z-1)^2}\right|_{z=2}=2\quad(4)$$

Pentru s_4, utilizăm Teorema derivării ①, (7), cu $x(n)=\left(\cos\frac{2n\pi}{3}\right)u(n)$ și $z=2$ $\Rightarrow s_4=Z\{n\cos\frac{2n\pi}{3}\}(2)=-z\cdot Z'\{\cos\frac{2n\pi}{3}\}(z)\Big|_{z=2}$

Deoarece $\boxed{Z\{\cos(an)u(n)\}(z)=\frac{z(z-\cos a)}{z^2-2z\cos a+1}}$, $a=\frac{2\pi}{3}$ (p.248),

primim: $s_4=-2\cdot\left(\frac{z(z-\cos\frac{2\pi}{3})}{z^2-2z\cos\frac{2\pi}{3}+1}\right)'_{z=2}=-2\cdot\left(\frac{z^2+\frac{z}{2}}{z^2+z+1}\right)'_{z=2}=$

$$= -\left(\frac{2z^2+z}{z^2+z+1}\right)'_{z=2} = -\frac{(4z+1)(z^2+z+1)-(2z+1)(2z^2+z)}{(z^2+z+1)^2}\Big|_{z=2} =$$

$$= -\frac{9\cdot 7 - 5\cdot 10}{7^2} = -\frac{13}{49} \quad (5)$$

Din (1) - (5) $\Rightarrow s = -\frac{2}{9} + 2 - \frac{13}{49} - 7 = -5 - \frac{98+117}{49\cdot 9} = -5 - \frac{215}{441} \Rightarrow$

$$\Rightarrow s = -\frac{2420}{441}$$

3 Ex. 2 Utilizând metoda (tehnica) transformării „z", să se rezolve ecuația cu diferențe finite:

$$x(n+2) - 4x(n+1) + 4x(n) = 3^n u(n) \quad ; \; x \in S_d^+$$
$$x(0) = 0; \; x(1) = 2$$

R. Formulare echivalentă $x(n) = x_n$.

Să se determine șirul $(x_n)_{n\geq 0}$ care verifică relația de recurență liniară $x_{n+2} - 4x_{n+1} + 4x_n = 3^n$; $n\geq 0$; $x_0 = 0, x_1 = 2$

Etapa 1 Determinăm $X(z) = \mathcal{Z}\{x(n)\}(z)$ (1)

Din Teorema translației la stânga, avem:

$$(2)\begin{cases} \mathcal{Z}\{x(n+1)\} = zX(z) - zx(0) \\ \mathcal{Z}\{x(n+2)\} = z^2X(z) - z^2x(0) - zx(1) \end{cases}$$

Aplicând operatorul (transformata) z ecuației date, obținem

$$\mathcal{Z}\{x(n+2)\}(z) - 4\mathcal{Z}\{x(n+1)\}(z) + 4\mathcal{Z}\{x(n)\}(z) = \mathcal{Z}\{3^n u(n)\}(z)$$

Din (1) și (2), rezultă

$$z^2X(z) - \cancel{z^2 0} - z\cdot 2 - 4(zX(z) - \cancel{z 0}) + 4X(z) = \frac{z}{z-3}$$

$$X(z)\cdot(z^2-4z+4) = 2z + \frac{z}{z-3} \Leftrightarrow X(z) = \frac{2z^2-5z}{(z-3)(z^2-4z+4)} \Leftrightarrow$$

$$X(z) = \frac{z(2z-5)}{(z-3)(z-2)^2} \quad (3)$$

Etapa 2 Determinăm $x(n) = \mathcal{Z}^{-1}\{X(z)\}(n)$

Din (3) $\Rightarrow \dfrac{X(z)}{z} = \dfrac{2z-5}{(z-3)(z-2)^2} = \dfrac{A}{z-3} + \dfrac{B}{z-2} + \dfrac{C}{(z-2)^2} \Leftrightarrow$

$\Leftrightarrow 2z-5 = A(z-2)^2 + B(z-2)(z-3) + C(z-3)$

$z=3 \Rightarrow A=1$; $z=2 \Rightarrow C=1$; z^2: $0 = A+B \Rightarrow B=-1$

Deci: $\dfrac{X(z)}{z} = \dfrac{1}{z-3} - \dfrac{1}{z-2} + \dfrac{1}{(z-2)^2}$

$$X(z) = \frac{z}{z-3} - \frac{z}{z-2} + \frac{z}{(z-2)^2} \Big/ \mathcal{Z}^{-1}$$

$$x(n) = \mathcal{Z}^{-1}\left\{\frac{z}{z-3}\right\}(n) - \mathcal{Z}^{-1}\left\{\frac{z}{z-2}\right\}(n) + \mathcal{Z}^{-1}\left\{\frac{z}{(z-2)^2}\right\}(n)$$

$$\boxed{x(n) = \left(3^n - 2^n + n\cdot 2^{n-1}\right) u(n)}$$

VEZI și SEMINAR 7 EX.1 / EX2