SEMINAR 4 - MA

Variabile aleatoare (1D, 2D)

Breviar teoretic

1° Caracteristici numerice ale v.a. 1D $\longrightarrow$ vezi Seminar 3 MA

2° Caracteristici numerice ale v.a. 2D

$V = (X, Y)$

- Discret: $X = \begin{pmatrix} x_i \\ p_i \end{pmatrix}_{i \in I}$; $Y = \begin{pmatrix} y_j \\ q_j \end{pmatrix}_{j \in J} \Rightarrow (X, Y): \begin{pmatrix} (x_i, y_j) \\ p_{ij} \end{pmatrix}_{(i,j) \in I \times J}$

  $p_{ij} = P(X = x_i; Y = y_j)$

  X, Y indep $\Rightarrow p_{ij} = p_i q_j$

- Continua: $f_V(x,y) = f(x,y)$ p.d.f (X, Y)

  X, Y indep $\Rightarrow f_V(x,y) = f_X(x) \cdot f_Y(y)$

Moment de ordin (m,n), $m, n \in \mathbb{N}$

$$M_{m,n}(X,Y) = M(X^m Y^n) = \begin{cases} \sum\limits_{i \in I} \sum\limits_{j \in J} x_i^m y_j^n p_{ij} \text{; discret} \\ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y) x^m y^n \, dx\, dy \text{; continua} \end{cases}$$

Moment centrat de ordin (m,n) Fie $m_1 = M(X) = E(X)$; $m_2 = M(Y) = E(Y)$

$$\mu_{m,n}(X,Y) = M\left((X - m_1)^m (Y - m_2)^n\right) = \begin{cases} \sum\limits_{i \in I} \sum\limits_{j \in J} (x_i - m_1)^m (y_j - m_2)^n p_{ij} \text{; discret} \\ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x - m_1)^m (y - m_2)^n f(x,y)\, dx\, dy \text{; continua} \end{cases}$$

3° Covarianța

$$\begin{cases} cov(X,Y) \overset{def}{=} \mu_{1,1}(X,Y) = M\left((X - m_1)(Y - m_2)\right) \\ cov(X,Y) = M(XY) - M(X) M(Y) \end{cases}$$

$$\begin{cases} D^2(X+Y) = D^2(X) + D^2(Y) + 2 cov(X,Y) \\ cov(X,X) = D^2(X) = Var\, X \end{cases}$$

4° Coeficient de corelație

$$r(X,Y) = \frac{cov(X,Y)}{\sigma(X)\sigma(Y)} \text{; } \sigma(X) = \sqrt{D^2(X)} = \sqrt{Var(X)}$$

Discret $r(X,Y) = \dfrac{1}{\sigma(X)\sigma(Y)} \sum\limits_{i \in I} \sum\limits_{j \in J} (x - m_1)(y - m_2) p_{ij}$

Continua $r(X,Y) = \dfrac{1}{\sigma(X)\sigma(Y)} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x - m_1)(y - m_2) f(x,y)\, dx\, dy$

Proprietăți $r(X,Y)$ (i) X, Y indep $\Rightarrow r(X,Y)=0$ (și $cov(X,Y)=0$)

(ii) $|r(X,Y)| \leq 1 \Leftrightarrow -1 \leq r(X,Y) \leq 1$

(iii) $|r(X,Y)|=1 \Leftrightarrow \exists a \in \mathbb{R}^*, b \in \mathbb{R}$: $Y=aX+B$ (dependență liniară)

5° Dreapta de regresie a v.a. Y față de v.a. X

$$(d)\quad y-M(Y)=\frac{cov(X,Y)}{D^2(X)}(x-M(X)) \Leftrightarrow y-M(Y)=r(X,Y)\cdot\frac{\sigma(Y)}{\sigma(X)}(x-M(X))$$

Cazul discret (X, Y v.a simple) ~~$X:\begin{pmatrix} x_1 & x_2 & \dots & x_n \\ p_1 & p_2 & \dots & p_n\end{pmatrix}$ și Y~~

Se consideră seria statistică $\{(x_i, y_i): 1\leq i\leq n\}$:

X	x_1	x_2	$\dots$	x_n
Y	y_1	y_2	$\dots$	y_n

Fie $X:\begin{pmatrix} x_1 & x_2 & x_3 \dots & x_n \\ 1/n & 1/n & 1/n \dots & 1/n\end{pmatrix}$; $Y:\begin{pmatrix} y_1 & y_2 & y_3 \dots & y_n \\ 1/n & 1/n & 1/n \dots & 1/n\end{pmatrix}$ v.a. 1D uniforme asociate

Notăm $\bar{x}=M(X)=\frac{1}{n}\sum_{i=1}^{n}x_i$; $\bar{y}=M(Y)=\frac{1}{n}\sum_{i=1}^{n} \Rightarrow$

$\Rightarrow$ (d) $y-\bar{y}=\frac{cov(X,Y)}{D^2(X)}(x-\bar{x})$, unde $D^2(X)=M(X^2)-M^2(X)=\frac{1}{n}\sum_{i=1}^{n}x_i^2-\bar{x}^2$

și $cov(X,Y)=\frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x})(y_i-\bar{y})=\sum_{i=1}^{n}x_iy_i-\bar{x}\bar{y}$

Interpretare statistică

(d), numită și dreapta de regresie sau dreapta de estimare a seriei statistice $\{(x_i, y_i): i=\overline{1,n}\}$, este dreapta care se abate „cel mai puțin", în sensul metodei celor mai mici pătrate, de la „distribuția" punctelor $\{A_i(x_i, y_i): 1\leq i\leq n\}$.

Notând $g(a,b)=\sum_{i=1}^{n}(A_iB_i)^2=\sum_{i=1}^{n}(ax_i+b-y_i)^2$, unde $A_i(x_i, y_i)$, $B_i(x_i, ax_i+b)$, ecuația dreptei (d) este:

(d) $y=ax+b$, unde a, b sunt soluții ale sistemului de ecuații

$\frac{\partial g}{\partial a}=0$, $\frac{\partial g}{\partial b}=0$

6° <u>Funcția de fiabilitate</u> (siguranța în funcționare); T v.a. 1D

$$R(t) = P(T>t) = 1 - F(t+0)$$

F continuă $\Rightarrow R(t) = 1 - F(t)$

$\begin{cases} F(t) \to \text{f. rep.}(T) \\ f(t) \to pdf(T) \\ F'(t) = f(t) \end{cases}$

7° <u>Rata de defectare (hazard)</u>

$$r(t) = \frac{f(t)}{R(t)} \Leftrightarrow r(t) = \frac{f(t)}{1-F(t)} \Leftrightarrow r(t) = -\frac{R'(t)}{R(t)} = \frac{F'(t)}{1-F(t)}$$

$$\Leftrightarrow r(t) = \frac{f(t)}{1-F(t)}$$

8° <u>Funcția caracteristică</u>

$$\varphi(t) = M(e^{jtX}) = E(\exp(jtX)) \,;\ t \in \mathbb{R}$$

<u>Discret</u> $\varphi(t) = \sum_{n \in I} p_n e^{jtx_n} \,;\ t \in \mathbb{R}$

<u>Continuu</u> $\varphi(t) = \int_{-\infty}^{\infty} f(x) e^{jtx} dx \,;\ t \in \mathbb{R}$

<u>Proprietăți</u> $\varphi(0) = 0;\ |\varphi(t)| \le 1;\ M_n(X) = j^{-n} \varphi^{(n)}(0)$

$$M(X) = E(X) = j^{-1}\varphi'(0);\ D^2(X) = Var(X) = -\varphi''(0) + (\varphi'(0))^2$$

$\boxed{\text{Ex. 1}}$ <u>Legea de probabilitate (repartiția) Gamma</u>

Fie $\alpha > 0$, $s > 0$ date. O v.a. 1D continuă, notată X are o <u>repartiție (distribuție) Gamma de parametri α și s</u> dacă p.d.f.(X) este:

$$f(x) = \begin{cases} 0; & x \le 0 \\ \frac{s^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-sx}; & x > 0 \end{cases} \qquad \text{Notație } X = \Gamma(\alpha; s)$$

Să se determine: <u>1°</u> Funcția caracteristică $\varphi(t)$; <u>2°</u> $M_n(X)$; <u>3°</u> $D^2(X)$

<u>4°</u> $M_o(X)$ <u>5°</u> Pentru legea $X = \Gamma(2;3)$ să se determine:

(i) funcția de fiabilitate și rata de hazard. ~~și ...~~

(ii) $k \in \mathbb{N}$ minim, pentru care $R(t) \ge 20 e^{-3t}$.

(iii) $t > 0$ a.i. $r(t) = 1$

<u>R.</u>

Rezolvare

1° $\varphi(t) = \int_{-\infty}^{\infty} f(x) e^{jtx} dx = \frac{s^\alpha}{\Gamma(\alpha)} \int_0^{\infty} x^{\alpha-1} e^{-sx} e^{jtx} dx =$

$= \frac{s^\alpha}{\Gamma(\alpha)} \int_0^{\infty} x^{\alpha-1} e^{-(s-jt)x} dx = \frac{s^\alpha}{\Gamma(\alpha)} \mathcal{L}\{x^{\alpha-1}\}(s-jt) =$

$= \frac{\cancel{\Gamma(\alpha-1+1)}}{(s-jt)^{\alpha-1+1}} \frac{s^\alpha}{\cancel{\Gamma(\alpha)}} = \frac{s^\alpha}{(s-jt)^\alpha} = \left(\frac{s}{s-jt}\right)^\alpha$

$\mathcal{L}\{f(x)\}(s) = \int_0^{\infty} f(x) e^{-sx} dx$

$\mathcal{L}\{x^\alpha\}(s) = \frac{\Gamma(\alpha+1)}{s^{\alpha+1}}$; $(\alpha > -1)$

$\mathcal{L}\{x^n\}(s) = \frac{n!}{s^{n+1}}$; $n \in \mathbb{N}$

$[(at+b)^\alpha]^{(n)} = \alpha(\alpha-1)\dots(\alpha-n+1) \cdot a^n (at+b)^{\alpha-n}$

$\Gamma(\alpha) = \int_0^{\infty} x^{\alpha-1} e^{-x} dx$; $\alpha > 0$

$\Gamma(\alpha+1) = \alpha \Gamma(\alpha)$; $\alpha > 0$

$\cancel{\Gamma(\alpha+1)} = \begin{cases} \Gamma(\alpha) = (\alpha-1)\Gamma(\alpha-1) \\ \alpha > 1 \end{cases}$

$\cancel{\Gamma(n)} \begin{cases} \Gamma(n+\alpha) = (n+\alpha-1)(n+\alpha-2)\dots \\ \quad \dots \alpha \Gamma(\alpha) \\ \Gamma(n+\alpha) = \alpha(\alpha+1)(\alpha+2)\dots(\alpha+n-1) \cdot \Gamma(\alpha) \end{cases}$

$\begin{cases} \Gamma(n) = (n-1)!; \ n \in \mathbb{N}^* \\ \Gamma(1/2) = \sqrt{\pi} \end{cases}$

$\begin{cases} \Gamma(n+\frac{1}{2}) = \frac{(2n)!}{2^{2n} n!} \sqrt{\pi} \\ \Gamma(n+\frac{1}{2}) = \frac{(n+1)(n+2)\dots 2n}{2^{2n}} \sqrt{\pi} \end{cases}$

2° Metoda 1 $\boxed{M_n(X)} = j^{-n} \varphi^{(n)}(0) =$

$= j^{-n} \cdot s^\alpha \cdot [(s-jt)^{-\alpha}]^{(n)}(0) =$

$= j^{-n} s^\alpha \underbrace{(-\alpha)(-\alpha-1)\dots(-\alpha-n+1)}(-j)^n (s-jt)^{-\alpha-n} \Big|_{t=0} =$

$= j^{-n} s^\alpha (-1)^n \cdot \alpha(\alpha+1)\dots(\alpha+n-1) \cdot (-1)^n j^n s^{-\alpha-n} =$

$= \frac{1}{s^n} \alpha(\alpha+1)(\alpha+2)\dots(\alpha+n-1) = \boxed{\frac{\Gamma(n+\alpha)}{\Gamma(\alpha)} s^{-n}}$

Metoda 2 $M_n(X) = \int_{-\infty}^{\infty} f(x) \cdot x^n dx =$

$= \frac{s^\alpha}{\Gamma(\alpha)} \int_0^{\infty} x^{n+\alpha-1} e^{-sx} dx = \frac{s^\alpha}{\Gamma(\alpha)} \mathcal{L}\{x^{n+\alpha-1}\}(s) =$

$= \frac{s^\alpha}{\Gamma(\alpha)} \cdot \frac{\Gamma(n+\alpha-1+1)}{s^{n+\alpha-1+1}} = \frac{s^\alpha}{s^{n+\alpha}} \cdot \frac{\Gamma(n+\alpha)}{\Gamma(\alpha)} = \frac{1}{s^n} \frac{\Gamma(n+\alpha)}{\Gamma(\alpha)}$

3° $\boxed{D^2(X)} = \boxed{Var(X)} = M(X^2) - M^2(X) =$

$= M_2(X) - M_1^2(X) = \frac{1}{s^2} \cdot \frac{\Gamma(\alpha+2)}{\Gamma(\alpha)} - \left(\frac{1}{s} \cdot \frac{\Gamma(\alpha+1)}{\Gamma(\alpha)}\right)^2 =$

$= \frac{1}{s^2} \cdot \frac{(\alpha+1) \alpha \cancel{\Gamma(\alpha)}}{\cancel{\Gamma(\alpha)}} - \frac{1}{s^2} \left(\frac{\alpha \cancel{\Gamma(\alpha)}}{\cancel{\Gamma(\alpha)}}\right)^2 =$

$= \frac{1}{s^2}(\alpha^2 + \alpha - \alpha^2) = \boxed{\frac{\alpha}{s^2}}$

4° $x > 0 \Rightarrow f'(x) = \frac{s^\alpha}{\Gamma(\alpha)} (x^{\alpha-1} e^{-sx})' = \frac{s^\alpha}{\Gamma(\alpha)} e^{-sx} [(\alpha-1) x^{\alpha-2} - s \cdot x^{\alpha-1}] =$

$= \frac{s^\alpha}{\Gamma(\alpha)} e^{-sx} \cdot x^{\alpha-2} [(\alpha-1) - sx]$; $f'(x) = 0 \Leftrightarrow \underline{x = \frac{\alpha-1}{s} > 0} \Leftrightarrow \alpha > 1$

4° I) $0<\alpha<1 \Rightarrow f'(x)<0, \forall x>0$

x	0		∞
$f(x)$	∞	$\searrow$	0

$\Rightarrow f(x)$ nu are punct de maxim $\Rightarrow \nexists M_o(X)$.

II) $\alpha=1 \Rightarrow \begin{cases} f'(x) = \lambda e^{-\lambda x}(-\lambda) = -e^{-\lambda x}<0 \\ f(x) = \lambda e^{-\lambda x} \end{cases}$

x	$-\infty$	0	∞
$f(x)$		$0 \;\; \lambda$	$\searrow 0$

$\sup\{f(x) : x\in\mathbb{R}\} = \lambda \neq f(0)=0 \Rightarrow \nexists M_o(X)$

III) $\alpha>1$

x	0		$(\alpha-1)/\lambda$		∞
$f'(x)$	$\cancel{0}$	$+$	0	$-$	
$f(x)$	0	$\nearrow$	Max	$\searrow$	

$$\boxed{M_o(X) = \frac{\alpha-1}{\lambda}}$$

$$f_{max} = f(M_o(X)) = \cancel{\frac{\lambda^\alpha}{\Gamma(\alpha)}} = \frac{\lambda^\alpha}{\Gamma(\alpha)}\cdot\frac{(\alpha-1)^{\alpha-1}}{\lambda^{\alpha-1}}\cdot e^{1-\alpha} = \frac{\lambda}{\Gamma(\alpha)}\left(\frac{\alpha-1}{e}\right)^{\alpha-1}$$

5° $X=\Gamma(2;3) \Rightarrow f(x) = \begin{cases} 0; & x\le 0 \\ \frac{3^2}{\Gamma(2)}\cdot x^{2-1}e^{-3x}; & x>0 \end{cases}$

$$\boxed{f(x) = \begin{cases} 0; & x\le 0 \\ 9xe^{-3x}; & x>0 \end{cases}} \Rightarrow F(x) = \int_{-\infty}^{x} f(t)\,dt = \begin{cases} 0; & x\le 0 \\ \int_0^x 9te^{-3t}dt; & x>0 \end{cases}$$

Avem $\int_0^x te^{-3t}dt = \int_0^x t\cdot\left(\frac{e^{-3t}}{-3}\right)'dt = t\cdot\frac{e^{-3t}}{-3}\Big|_0^x + \frac{1}{3}\int_0^x t'\cdot e^{-3t}dt =$

$= -\frac{1}{3}xe^{-3x} - \frac{1}{9}e^{-3t}\Big|_0^x = -\frac{1}{3}xe^{-3x} - \frac{1}{9}e^{-3x} + \frac{1}{9} = \frac{1}{9}\left(1-3xe^{-3x}-e^{-3x}\right)$

Deci $F(x) = \begin{cases} 0; & x\le 0 \\ 1-e^{-3x}(3x+1); & x>0 \end{cases}$ sau $F(t) = \begin{cases} 0; & t\le 0 \\ 1-e^{-3t}(3t+1); & t>0 \end{cases}$

(i) Astfel, $R(t) = 1-F(t) = \begin{cases} 1; & t\le 0 \\ e^{-3t}(3t+1); & t>0 \end{cases}$; $f(t) = \begin{cases} 0; & t\le 0 \\ 9te^{-3t}; & t>0 \end{cases}$

$r(t) = \frac{f(t)}{R(t)} = \begin{cases} \frac{0}{1} = 0; & t\le 0 \\ \frac{9t}{3t+1}; & t>0 \end{cases}$

(ii) $R(t)\ge 20e^{-3t} \Rightarrow$

- $t\le 0$: $1\ge 20e^{-3t} \Rightarrow e^{-3t}\le\frac{1}{20}$ fals, deoarece $e^{-3t}\ge e^0=1$
- $t>0$: $e^{-3t}(3t+1)\ge 20e^{-3t} \Rightarrow (3t+1)\ge 20 \Rightarrow$

$\Rightarrow t\ge 19/3 = 6,(3) \Rightarrow t_{min} = 7,\ t\in\mathbb{N}$

(iii) $r(t) = 1 \Leftrightarrow 9t = 3t+1 \Rightarrow t = 1/6$.

OBS. $r'(t) = 9/(3t+1)^2 \Leftrightarrow 0\le r(t) < \lim_{t\to\infty} r(t) \Rightarrow 0\le r(t)<3$

Ex. 2 Funcția de fiabilitate asociată v. a. T, care reprezintă timpul de bună funcționare a unui tub catodic, este

$$R(t) = P(T > t) = \begin{cases} 1; & t \le 0 \\ e^{-\lambda t}; & t > 0 \end{cases} ; \lambda > 0 \text{ dat.}$$

1° Să se arate că T urmează o lege de probabilitate de tip exponențial-negativ, i.e. $T = \Gamma(1; \lambda)$.

2° Să se determine $M_n(T)$, $D^2(T)$, $\sigma(T)$, $Me(T)$, $As(T)$, $Ex(T)$.

3° Să se determine rata de hazard (defectare) $r(t)$ pentru T.

4° Dacă $\lambda = 0.001$, să se determine durata de bună funcționare a tubului catodic, ~~care~~ pentru care fiabilitatea este cel puțin 95%.

5° Știind că durata medie de viață a unui tub catodic este 1600 h, să se determine:

i) probabilitatea ca tubul să iasă din funcțiune în 1200 h

ii) probabilitatea ca tubul să iasă din funcțiune într-un interval de timp cuprins între 1500 h și 2000 h

6° Două tuburi catodice (care funcționează independent) au duratele medii de viață 1000 h, respectiv 2000 h. Să se determine probabilitatea ca tuburile să iasă din funcțiune în intervalul (1500 h, 2500 h).

Rezolvare

1° Observăm că $R(t)$ este o funcție continuă (în $t=0$: $l_s = l_d = R(0) = 1$)

Avem $F(t) = 1 - R(t) = \begin{cases} 0; & t \le 0 \\ 1 - e^{-\lambda t}; & t > 0 \end{cases} \Rightarrow f(t) = \begin{cases} 0; & t \le 0 \\ \lambda e^{-\lambda t}; & t > 0 \end{cases}$,

deoarece $f(t) = F'(t)$ și admitem $f(0) = 0$. Se observă că $f(t)$ este p.d.f. pentru legea $\Gamma(1; \lambda)$, deci $T = \Gamma(1; \lambda)$.

2° Din $\boxed{\text{Ex. 1}}$, luând $\alpha = 1$; $\beta = \lambda \Rightarrow M_n(X) = \frac{1}{\lambda^n} \cdot \frac{\Gamma(n+1)}{\Gamma(1)} = \frac{n!}{\lambda^n}$;

$M(X) = M_1(X) = \frac{1}{\lambda}$; $D^2(X) = Var(X) = \frac{1}{\lambda^2}$; $\sigma(X) = \sqrt{D^2(X)} = \frac{1}{\lambda}$.

Reținem $\underline{M(T) = E(T) = \sigma(T) = \sqrt{D^2(T)} = \sqrt{Var(T)} = \frac{1}{\lambda}}$

pentru o lege exponențial-negativă $T = \Gamma(1;\lambda);\ \lambda > 0$

$\underline{Me(X) = d} \iff F(d) = \frac{1}{2} \iff 1 - e^{-\lambda d} = \frac{1}{2} \iff e^{-\lambda d} = \frac{1}{2} \iff$

$\iff -\lambda d = \ln\frac{1}{2} \iff -\lambda d = -\ln 2 \iff \boxed{d = Me(T) = \frac{\ln 2}{\lambda}}$

$\underline{As(T) = \frac{\mu_3(T)}{\sigma^3(T)}};\ \mu_3(T) = M_3(T - m),\ m = M(T) = \frac{1}{\lambda} \Rightarrow$

$\Rightarrow \mu_3(T) = M\left(\left(T - \frac{1}{\lambda}\right)^3\right) = M\left(T^3 - 3\frac{1}{\lambda}T^2 + 3\frac{1}{\lambda^2}T - \frac{1}{\lambda^3}\right) =$

$= M(T^3) - \frac{3}{\lambda}M(T^2) + \frac{3}{\lambda^2}M(T) - \frac{1}{\lambda^3}M(1) = M_3(T) - \frac{3}{\lambda}M_2(T) +$

$+ \frac{3}{\lambda^2}M(T) - \frac{1}{\lambda^3}M(T^0) = \frac{3!}{\lambda^3} - \frac{3}{\lambda}\cdot\frac{2!}{\lambda^2} + \frac{3}{\lambda^2}\frac{1!}{\lambda} - \frac{1}{\lambda^3}\cdot 1 =$

$= \frac{1}{\lambda^3}(6 - 6 + 3 - 1) = \frac{2}{\lambda^3} \Rightarrow \underline{As(T) = \frac{2/\lambda^3}{1/\lambda^3} = 2}$

$\underline{Ex(T) = \frac{\mu_4(T)}{\sigma^4(T)} - 3} \qquad \mu_4(T) = M_3(T - m) = M\left((T-m)^4\right) =$

$= M_4(T) - 4m M_3(T) + 6m^2 M_2(T) - 4m^3 M(T) + m^4 \cdot M(T^0) =$

$= \frac{4!}{\lambda^4} - \frac{4}{\lambda}\cdot\frac{3!}{\lambda^3} + \frac{6}{\lambda^2}\frac{2!}{\lambda^2} - \frac{4}{\lambda^3}\cdot\frac{1!}{\lambda} + \frac{1}{\lambda^4}\cdot 1 = \frac{1}{\lambda^4}(24 - 24 + 12 - 4 + 1) =$

$= 9/\lambda^4 \Rightarrow \underline{Ex(X) = \frac{\mu_4(X)}{\sigma^4(X)} - 3 = \frac{9/\lambda^4}{1/\lambda^4} - 3 = 6}$

3° $r(t) = \frac{f(t)}{R(t)} = \begin{cases} \frac{0}{1};\ t \le 0 \\ \frac{\lambda e^{-\lambda t}}{e^{-\lambda t}};\ t > 0 \end{cases} = \begin{cases} 0;\ t \le 0 \\ \lambda;\ t > 0 \end{cases} \Rightarrow \underline{r(t) = const}$

OBS. Reciproc, dacă o v.a. 1D, notată T, are $r_T(t) = const \Rightarrow$
$\Rightarrow T$ urmează o lege de probabilitate exponențial-negativă.

4° $R(t) \ge \frac{95}{100} \iff t \le 0$ sau $\left(t > 0 \text{ și } e^{-\lambda t} \ge \frac{19}{20}\right) \iff$

$\iff t \le 0$ și $\left(-\lambda t \ge \ln\frac{19}{20},\ t > 0\right) \iff \left(t \le 0 \text{ și } \begin{cases} t \le \frac{1}{\lambda}\ln\frac{20}{19} \\ t > 0 \end{cases}\right) \xleftrightarrow{\lambda = 10^{-3}}$

$\Leftrightarrow t \le 0$ şi $(t>0,\ t \le 1000(\ln 20 - \ln 19) \simeq 1000 \cdot 0.0512 = 51,2)$

Deci $t \le 51.2$ h. Luăm t_{max} (întreg) $= 51$ h.

5° $M(T) = \frac{1}{\lambda} = 1600 \Rightarrow \lambda = 1/1600$

(i) $P(T<1200) = F(1200) = 1 - e^{-\frac{1200}{1600}} = 1 - e^{-3/4} = 1 - \frac{1}{\sqrt[4]{e^3}} =$

$= 1 - \frac{\sqrt[4]{e}}{e} \simeq 0,53$

(ii) $P(1500 < T < 2000) = F(2000) - F(1500) = R(1500) - R(2000) =$

$= e^{-1500/1600} - e^{-2000/1500} = e^{-15/16} - e^{-4/3} = \frac{\sqrt[16]{e}}{e} - \frac{1}{\sqrt[3]{e^4}} = \frac{\sqrt[16]{e}}{e} - \frac{\sqrt[3]{e^2}}{e^2} =$

$= \frac{e\sqrt[16]{e} - \sqrt[3]{e^2}}{e^2}$

6° Fie T_1, T_2 v.a. asociate celor două tuburi

$M(T_1) = \frac{1}{\lambda_1} = 1000 \Rightarrow \lambda_1 = \frac{1}{1000}$; $M(T_2) = \frac{1}{\lambda_2} = 2000 \Rightarrow \lambda_2 = \frac{1}{2000}$

$\Rightarrow R_1(t) = e^{-(1/100)t}$; $R_2(t) = e^{-(1/2000)t}$; $t>0$

Fie $T = (T_1, T_2)$ v.a. 2D.

Avem $P(T \in (1500, 2500) \times (1500, 2500)) =$

$= P(T_1 \in (1500, 2500);\ T_2 \in (1500, 2500)) = P(T_1 \in (1500, 2500)) \cdot$

$\cdot P(T_2 \in (1500, 2500)) = (F_1(2500) - F_1(1500))(F_2(2500) - F_2(1500)) =$

$= (R_1(1500) - R_1(2500))(R_2(1500) - R_2(2500)) = \left(e^{-\frac{1500}{1000}} - e^{-\frac{2500}{1000}}\right) \cdot$

$\cdot \left(e^{-\frac{1500}{2000}} - e^{-\frac{25000}{2000}}\right) = (e^{-3/2} - e^{-5/2})(e^{-3/4} - e^{-5/4}) =$

$= e^{-5/2}(e-1) \cdot e^{-5/4}(e^{2/4} - 1) = e^{-5/2 - 5/4}(e-1)(\sqrt{e}-1) = e^{-\frac{15}{4}}(e-1)(\sqrt{e}-1) =$

$= \frac{(e-1)(\sqrt{e}-1)}{\sqrt[4]{e^{15}}} = \frac{\sqrt[4]{e}(e-1)(\sqrt{e}-1)}{e^4} \simeq 0.0258 = 2,58\%$

$\boxed{\text{Ex. 3}}$ Funcţia $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = (8x^2 + 9y^3)\exp(-2x-3y)u(x)u(y)$,

i.e. $f(x,y) = \begin{cases} (8x^2 + 9y^3)e^{-2x-3y} & ; \text{ dacă } x \ge 0,\ y \ge 0 \\ 0 & ; \text{ în rest} \end{cases}$

este p.d.f. pentru o v.a. 2D, notată $V = (X, Y)$.

1° Să se determine momentele $M_{m,n}(X,Y)$; $m,n\in\mathbb{N}$ și să se verifice că f definită este medie, i.e. $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$.

2° Să se calculeze $\mathrm{cov}(X,Y)$ și $r(X,Y)$

3° Să se determine dreptele de regresie asociate v.a. (X,Y).

R. 1° $M_{m,n}(X,Y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^m y^n f(x,y)\,dx\,dy =$

$$= \int_0^\infty\int_0^\infty x^m y^n (8x^2+9y^3)e^{-2x}e^{-3y}\,dx\,dy =$$

$$= 8\int_0^\infty\int_0^\infty x^{m+2}y^n e^{-2x}e^{-3y}\,dx\,dy + 9\int_0^\infty\int_0^\infty x^m y^{n+3}e^{-2x}e^{-3y}\,dx\,dy =$$

$$= 8\int_0^\infty x^{m+2}e^{-2x}dx\cdot\int_0^\infty y^n e^{-3y}dy + 9\int_0^\infty x^m e^{-2x}dx\cdot\int_0^\infty y^{n+3}e^{-3y}dy =$$

$$= 8\cdot\mathcal{L}\{x^{m+2}\}(2)\cdot\mathcal{L}\{y^n\}(3) + 9\cdot\mathcal{L}\{x^m\}(2)\cdot\mathcal{L}\{y^{n+3}\}(3) =$$

$$= 8\cdot\frac{(m+2)!}{2^{m+3}}\cdot\frac{n!}{3^{n+1}} + 9\cdot\frac{m!}{2^{m+1}}\cdot\frac{(n+3)!}{3^{n+4}}\qquad 3^{n+4} = 3^{n+2}\cdot 3^2$$

$$\frac{(m+2)! = (m+2)(m+1)m!}{(n+3)! = (n+3)(n+2)(n+1)n!}$$

$$\cancel{= \frac{m!\,n!}{2^{m+1}\,3^{n+3}}\left[8(m+2)(m+1)\right]} = \frac{m!\,n!}{2^m\,3^{n+1}}\left[(m+2)(n+1) + \frac{(n+3)(n+2)(n+1)}{2\cdot 3}\right]$$

$$M_{m,n}(X,Y) = \frac{m!\,n!}{18\cdot 2^m\cdot 3^n}\left[6(m+1)(m+2) + (n+1)(n+2)(n+3)\right]$$

Verif. $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = M_{0,0}(X,Y) = \frac{1}{18}(6\cdot 2 + 1\cdot 2\cdot 3) = 1$.

2° $\mathrm{cov}(X,Y) = M(XY) - M(X)M(Y) = M_{1,1}(X,Y) - M_{1,0}(X)M_{0,1}(Y)$

$$= \frac{1}{18\cdot 6}(6\cdot 2\cdot 3 + 2\cdot 3\cdot 4) - \frac{1}{18\cdot 2}(6\cdot 2\cdot 3 + 1\cdot 2\cdot 3)\cdot\frac{1}{18\cdot 3}(6\cdot 1\cdot 2 + 2\cdot 3\cdot 4) =$$

$$= \frac{60}{18\cdot 6} - \frac{1}{18^2\cdot 6}\cdot 42\cdot 36 = \frac{10}{18} - \frac{1}{\underbrace{\cancel{18\cdot 18\cdot 6}}_{3\cdot 3}}\cdot 7\cdot 6\cdot 6 =$$

$$= \frac{5}{9} - \frac{7}{9} = -\frac{2}{9} = -0,(2)$$

$r(X,Y) = \frac{cov(X,Y)}{\sigma(X)\sigma(Y)}$; $D^2(X) = M(X^2) - M^2(X) = M_{2,0}(X,Y) - M_{1,0}^2(X,Y) =$

$= \frac{2\cdot 1}{18\cdot 2^2\cdot 1}(6\cdot 3\cdot 4 + 1\cdot 2\cdot 3) - \left(\frac{1}{18\cdot 2}(6\cdot 2\cdot 3 + 1\cdot 2\cdot 3)\right)^2 = \frac{78}{36} - \left(\frac{42}{36}\right)^2 =$

$= \frac{13}{6} - \frac{49}{36} = \frac{29}{36}$; $D^2(Y) = M(Y^2) - M^2(Y) = M_{0,2}(X,Y) - M_{0,1}^2(X,Y) =$

$= \frac{2}{18\cdot 9}(6\cdot 1\cdot 2 + 3\cdot 4\cdot 5) - \left(\frac{1}{54}(6\cdot 1\cdot 2 + 2\cdot 3\cdot 4)\right)^2 = \frac{1}{81}\cdot 72 - \left(\frac{36}{54}\right)^2 =$

$= \frac{8}{9} - \left(\frac{2}{3}\right)^2 = \frac{4}{9} \Rightarrow \sigma(X) = \frac{\sqrt{29}}{6}$; $\sigma(Y) = \frac{2}{3}$

Deci $r(X,Y) = -\frac{\cancel{2}}{9}\cdot\frac{6}{\sqrt{29}}\cdot\frac{3}{\cancel{2}} = -\frac{2}{\sqrt{29}} \simeq -0.371$

3°
$$\begin{cases} y - M(Y) = \frac{cov(X,Y)}{D^2(X)}(x - M(X)) & (d_1) \quad Y/X \\ x - M(X) = \frac{cov(X,Y)}{D^2(Y)}(y - M(Y)) & (d_2) \quad X/Y \end{cases}$$

Avem $M(X) = M_{1,0}(X,Y) = \frac{1}{18\cdot 2}(6\cdot 2\cdot 3 + 1\cdot 2\cdot 3) = \frac{1}{36}\cdot 42 = \frac{7}{6}$

$M(Y) = M_{0,1}(X,Y) = \frac{1}{18\cdot 3}(6\cdot 1\cdot 2 + 2\cdot 3\cdot 4) = \frac{1}{54}\cdot 36 = \frac{2}{3}$.

Deci
$$\begin{cases} (d_1): y - \frac{2}{3} = -\frac{2}{\cancel{9}}\cdot\frac{\cancel{36}\,4}{29}\left(x - \frac{7}{6}\right) \Leftrightarrow \frac{1}{3}(3y-2) = -\frac{8}{29}\cdot\frac{6x-7}{6} \\ (d_2): x - \frac{7}{6} = -\frac{\cancel{2}}{\cancel{9}}\cdot\frac{\cancel{9}}{\cancel{4}\,2}\left(y - \frac{2}{3}\right) \Leftrightarrow \frac{6x-7}{6} = -\frac{3y-2}{6} \end{cases}$$

$$\begin{cases} (d_1)\ 24x + 87y = 86 \\ (d_2)\ 2x + y = 3 \end{cases}$$

$\boxed{\text{Ex. 4}}$ Să se determine dreapta de regresie a seriei statistice

x	-1	1	2	3
y	0	-1	3	5

, i.e. aproximarea liniară prin metoda celor mai mici pătrate pentru seria statistică (~~de~~ setul de date experimentale) de mai sus.

R.

Metoda 1. Se consideră v.a. $X:\begin{pmatrix}-1 & 1 & 2 & 3\\ 1/4 & 1/4 & 1/4 & 1/4\end{pmatrix}$ și

$Y:\begin{pmatrix}0 & -1 & 3 & 5\\ 1/4 & 1/4 & 1/4 & 1/4\end{pmatrix}$, i.e. v.a. uniforme

$$(d)\quad y-\bar{y}=\frac{cov(X,Y)}{D^2(X)}(x-\bar{x}),$$

unde $\bar{x}=M(X)=\frac{1}{4}(-1+1+2+3)=5/4$; $\bar{y}=M(Y)=\frac{7}{4}$;

$D^2(X)=M(X^2)-M^2(X)=\frac{1}{4}(1+1+4+9)-\bar{x}^2=\frac{15}{4}-\frac{25}{16}=\frac{35}{16}$

$cov(X,Y)=\frac{1}{4}\sum_{i=1}^{4}(x_i-\bar{x})(y_i-\bar{y})=M((X-\bar{x})(Y-\bar{y}))=$

$=\frac{1}{4}\left[\left(-1-\frac{5}{4}\right)\left(0-\frac{7}{4}\right)+\left(1-\frac{5}{4}\right)\left(-1-\frac{7}{4}\right)+\left(2-\frac{5}{4}\right)\left(3-\frac{7}{4}\right)+\left(3-\frac{5}{4}\right)\left(5-\frac{7}{4}\right)\right]=$

$=\frac{1}{4}\left(\frac{63}{16}+\frac{11}{16}+\frac{15}{16}+\frac{91}{16}\right)=\frac{1}{4}\cdot\frac{180}{16}=\frac{45}{16}$

$$(d)\quad y-\frac{7}{4}=\frac{45/16}{35/16}\left(x-\frac{5}{4}\right)\Leftrightarrow \underbrace{y-\frac{7}{4}=\frac{9}{7}\left(x-\frac{5}{4}\right)\Leftrightarrow y=\frac{9}{7}x+\frac{1}{7}}$$

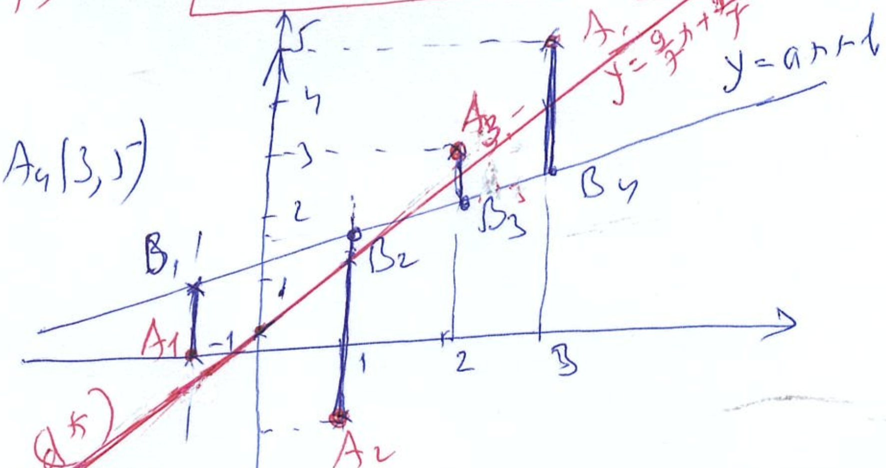
Metoda 2

$A_1(-1,0), A_2(1,-1), A_3(2,3), A_4(3,5)$

$(d):\ \boxed{y=ax+b}$

$B_1(-1,-a+b), B_2(1,a+b)$

$B_3(2,2a+b), B_4(3,3a+b)$



$g(a,b)=A_1B_1^2+A_2B_2^2+A_3B_3^2+A_4B_4^2\longrightarrow \min$

$g(a,b)=(-a+b)^2+(a+b+1)^2+(2a+b-3)^2+(3a+b-5)^2\longrightarrow \min$

$$\begin{cases}\frac{\partial g}{\partial a}=0\\ \frac{\partial g}{\partial b}=0\end{cases}\Leftrightarrow \begin{cases}\frac{1}{2}\frac{\partial g}{\partial a}=(-a+b)(-1)+(a+b+1)\cdot 1+(2a+b-3)\cdot 2+(3a+b-5)\cdot 3=0\\ \frac{1}{2}\frac{\partial g}{\partial b}=(-a+b)\cdot 1+(a+b+1)\cdot 1+(2a+b-3)\cdot 1+(3a+b-5)\cdot 1=0\end{cases}$$

$$\Leftrightarrow\begin{cases}15a+5b=20\ |:5\\ 5a+4b=7\end{cases}\Leftrightarrow\begin{cases}3a+b=4\ |\cdot(-4)\\ 5a+4b=7\end{cases}\Leftrightarrow\begin{cases}3a+b=4\\ -7a=-9\end{cases}\Leftrightarrow\begin{cases}a=\frac{9}{7}\\ b=4-\frac{27}{7}=\frac{1}{7}\end{cases}$$

$(d^*):\ y=\frac{9}{7}x+\frac{1}{7}$

$\boxed{\text{Ex. 5}}$ Temă

$\boxed{5.1}$ Vezi Ex. 1. Pentru legea $X = \Gamma(2;2)$, i.e.

$$f(t) = \begin{cases} 0; & x \leq 0 \\ \frac{4}{\Gamma(2)} t e^{-2t}; & x > 0 \end{cases}$$ să se determine

(i) $Me(X)$; (ii) $\sigma(X)$; (iii) $Mo(X)$;

(iv) Funcția ~~caracteristică~~ de repartiție; (v) $P(|X| \leq \ln 3)$

(vi) funcția de fiabilitate $R(t)$; (vii) Funcția de hazard $r(t)$

(viii) t minim pentru care $R(t) \geq 19 e^{-2t}$; (ix) $\sup\{r(t) : t \in \mathbb{R}\}$

(x) $t \in \mathbb{R}$ a.î. $r(t) \geq 1$.

$\boxed{5.2}$ Similar cu Ex. 3 pentru $f(x,y) = (x^2+2y)^2 e^{-x-4y} u(x) u(y)$

$\boxed{5.3}$ Similar cu Ex. 4 pentru

x	0	2	3	5
y	-1	1	2	0