SEMINAR 6 - MA

LANȚURI MARKOV

BREVIAR TEORETIC

1. Vector de probabilitate $p = (p_1, p_2, p_3, \dots, p_n) \in \mathbb{R}^n$ a.î. $p_i \geq 0$, $1 \leq i \leq n$ și $\sum_{i=1}^{n} p_i = 1$. Exemplu $p = \left(\frac{1}{4}, \frac{3}{8}, \frac{1}{4}, \frac{1}{8}\right) \in \mathbb{R}^4$.

2. (i) Matrice stochastică $\Pi = \begin{pmatrix} p_{11} & p_{12} & p_{13} \dots & p_{1n} \\ p_{21} & p_{22} & p_{23} \dots & p_{2n} \\ p_{n1} & p_{n2} & p_{n3} \dots & p_{nn} \end{pmatrix} = (p_{ij})_{1 \leq i,j \leq n} \in M_n(\mathbb{R})$

a.î. fiecare linie a matricei Π este vector de probabilitate, i.e. $\sum_{j=1}^{n} p_{ij} = 1$, $\forall i = \overline{1,n}$ și $p_{ij} \geq 0$, $1 \leq i, j \leq n$.

(ii) Matrice dublu stochastică: o matrice $\Pi \in M_n(\mathbb{R})$ a.î. Π și Π^T sunt matrice stochastice. Exemplu $\Pi = \begin{pmatrix} 1/5 & 2/5 & 2/5 \\ 1/6 & 1/3 & 1/2 \\ 1/4 & 3/8 & 3/8 \end{pmatrix} \in M_3(\mathbb{R})$

Π este matrice stochastică, dar NU dublu stochastică $\left(\frac{1}{5} + \frac{1}{6} + \frac{1}{4} \neq 1\right)$

3. $\boxed{\text{LANȚ MARKOV}}$ (L.M.)

Se consideră următoarele date:

(i) o mulțime $S = \{1, 2, 3, \dots, n\}$ ale cărei elemente se numesc stări

(ii) un vector de probabilitate $p^{(0)} = (p_1^{(0)}, p_2^{(0)}, p_3^{(0)}, \dots, p_n^{(0)})$, ale cărui componente se numesc probabilități inițiale

(iii) o matrice stochastică $\Pi = (p_{ij}) \in M_n(\mathbb{R})$ ale cărei elemente se numesc probabilități de trecere sau probabilități de tranziție; matricea Π se numește, ea însăși, matrice de trecere (matrice de pasaj)

Se numește lanț Markov omogen sau L.M. staționar în timp, asociat tripletului $(S, p^{(0)}, \Pi)$ un șir de v.a. discrete $(X_k)_{k \geq 0}$ având următoarele proprietăți:

$\underline{1^\circ}$ $P(X_0 = 1) = p_1^{(0)}$, $P(X_0 = 2) = p_2^{(0)}$, $P(X_0 = 3) = p_3^{(0)}, \dots, P(X_0 = n) = p_n^{(0)}$

$\underline{2^\circ}$ $\forall i, j \in S$, $\forall k \geq 0$, avem: $P(X_{k+1} = j / X_k = i) = p_{ij}$

$\underline{3^\circ}$ $\forall k \geq 0$, $\forall s_0, s_1, s_2, s_3, \dots, s_{k+1} \in S$ avem:

$$P(X_{k+1} = s_{k+1} / X_k = s_k, X_{k-1} = s_{k-1}, \dots, X_0 = s_0) = P(X_{k+1} = s_{k+1} / X_k = s_k)$$

4. ANALIZA UNUI LANȚ MARKOV

FORMULE PROBABILISTICE

(i) $P(A \cap B) = P(A) \cdot P(B/A) = P(B) \cdot P(A/B)$

(ii) $P(A \cap B \cap C) = P(A) \cdot P(B/A) \cdot P(C/A \cap B) = P(C) \cdot P(B/C) \cdot P(A/B \cap C)$

(iii) $P(A \cap B \cap C \cap D) = P(A) \cdot P(B/A) \cdot P(C/A \cap B) \cdot P(D/A \cap B \cap C) =$
$= P(D) \cdot P(C/D) \cdot P(B/C \cap D) \cdot P(A/B \cap C \cap D)$

(iv) $P(A/B) = \dfrac{P(A \cap B)}{P(B)}$; $P(B/A) = \dfrac{P(A \cap B)}{P(A)}$; $P(A/B) \underset{\text{Bayes}}{=} \dfrac{P(A)P(B/A)}{P(B)}$

4.1. Probabilitatea ca lanțul să evolueze pe traiectoria $s_0, s_1, s_2, s_3, \ldots, s_m$

$P(X_0 = s_0, X_1 = s_1, X_2 = s_2, X_3 = s_3, \ldots, X_m = s_m) =$

$= P(X_0 = s_0) \cdot P(X_1 = s_1 / X_0 = s_0) \cdot P(X_2 = s_2 / X_1 = s_1, \cancel{X_0 = s_0}) \cdot P(X_3 = s_3 / X_2 = s_2, \cancel{X_1 = s_1, X_0 = s_0})$

$\ldots P(X_m = s_m / X_{m-1} = s_{m-1}, \cancel{X_{m-2} = s_{m-2}, \ldots, X_1 = s_1, X_0 = s_0}) \overset{3^\circ}{=}$

$= P(X_0 = s_0) \cdot P(X_1 = s_1 / X_0 = s_0) \cdot P(X_2 = s_2 / X_1 = s_1) \ldots P(X_m = s_m / X_{m-1} = s_{m-1}) =$

$\overset{1^\circ}{\underset{2^\circ}{=}} p^{(0)}_{s_0} \cdot p_{s_0, s_1} \cdot p_{s_1, s_2} \cdots p_{s_{m-1}, s_m}$

4.2. Matricea probabilităților de trecere după m pași

Π $p_{ij} = P(X_{k+1} = j / X_k = i)$; $\Pi = (p_{ij}) = (p^{(1)}_{ij})$

$\Pi^2 = (p^{(2)}_{ij})_{1 \le i,j \le n}$: $p^{(2)}_{ij} = P(X_{k+2} = j / X_k = i) = P(X_k = j / X_l = i)$, dacă $k - l = 2$,

$\Pi^m = (p^{(m)}_{ij})_{1 \le i,j \le n}$: $p^{(m)}_{ij} = P(X_{k+m} = j / X_k = i) = P(X_k = j / X_l = i)$, dacă $k - l = m$,

4.3. Probabilități absolute

Def. (i) $p^{(k)}_i = P(X_k = i)$, $1 \le i \le n$; $k \ge 0$ $p^{(k)}_i \to$ probabilități absolute

(ii) $p^{(k)} = (p^{(k)}_1, p^{(k)}_2, p^{(k)}_3, \ldots, p^{(k)}_n)$; $k \ge 0$ $p^{(k)} \to$ repartiția de probabilitate a L.M. la momentul $k \ge 0$

Th. $p^{(k)} = p^{(0)} \Pi^k$, $\forall k \ge 0 \iff$

$\iff (p^{(k)}_1, p^{(k)}_2, \ldots, p^{(k)}_n) = (p^{(0)}_1, p^{(0)}_2, \ldots, p^{(0)}_n) \cdot \Pi^k$, $\forall k \ge 0$; $\Pi^0 = I_n$

Calcul recursiv $p^{(1)} = p^{(0)} \Pi$; $p^{(2)} = p_0 \Pi^2 = (p_0 \Pi) \Pi = p^{(1)} \Pi$;
$p^{(3)} = p^{(0)} \Pi^3 = (p_0 \Pi^2) \Pi = p^{(2)} \Pi$. În general $p^{(k)} = p^{(k-1)} \Pi$.

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4.4. Distribuția (repartiția) limită (de echilibru) a LM

Echivalent: Distribuția invariantă a LM.

(i) Def. Dacă $\exists\, p^* = \lim_{k\to\infty} p^{(k)}$, atunci p^* se numește repartiția limită (de echilibru; invariantă) a L.M. $(S, p^{(0)}, \Pi)$

(ii) Th. $p^{(k)} = p^{(k-1)}\Pi \xrightarrow{k\to\infty} \boxed{p^* = p^*\Pi}$

(iii) OBS. a) p^* este un vector de probabilitate

b) $p^* = p^*\Pi \Rightarrow (p^*)^T = (p^*\Pi)^T \Rightarrow (p^*)^T = \Pi^T\cdot(p^*)^T \Leftrightarrow \Pi^T(p^*)^T = 1\cdot(p^*)^T$, din

$(p^*)^T$ este unicul vector propriu al matricei Π^T, asociat valorii proprii $\lambda = 1$, având suma componentelor egală cu 1 ($Ax = \lambda x$)

(iv) LANȚ MARKOV REGULAT $\Leftrightarrow$ matricea Π este matrice regulată $\Leftrightarrow$

$\Leftrightarrow \exists\, m \geq 1$ a.î. Π^m are toate elementele > 0

OBS. Dacă $(X_k)_{k\geq 0}$ este L.M. regulat, atunci $\boxed{\exists\, p^*}$

$\boxed{\text{Ex. 1}}$ Se consideră lanțul Markov $\cancel{(S, p^{(0)}, \Pi)}$

$(X_k)_{k\geq 0}$ asociat tripletului $(S, p^{(0)}, \Pi)$ unde

$S = \{1, 2, 3\}$, $p^{(0)} = \left(a^2, a, \frac{5}{9}\right)$, $\Pi = \begin{pmatrix} a & b & 1/2 \\ \frac{1}{10} & \frac{1}{2} & \frac{2}{5} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{pmatrix}$.

Să se determine: 1) $a, b \in \mathbb{R}$

2) Probabilitatea ca lanțul să evolueze pe traiectoria $(2, 1, 3, 3)$

3) $P(X_{200} = 1, X_{199} = 2, X_{198} = 3 / X_{197} = 2)$

4) $P(X_{1000} = 1, X_{998} = 2 / X_{997} = 3)$

5) $P(X_0 = 2 / X_2 = 3)$

6) $P(X_1 = 1) + P(X_2 = 3)$

7) Repartiția limită a lanțului Markov

8) $P(X_{79} = 2 / X_{77} \neq 2, X_{77} \neq 1)$

9) $P(X_0 = 2 / X_1 = 1, X_3 = 2)$

10) a) $P(X_2 = 2 / X_4 = 1, X_5 = 2)$

b) $P(X_2 = 1 / X_3 = 2, X_1 = 3)$

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Rezolvare

(1) $\begin{cases} a^2+a+\frac{5}{9}=1 \\ a+b+\frac{1}{2}=1 \\ a,b\in[0,1] \end{cases} \Leftrightarrow \begin{cases} 9a^2+9a-4=0 \\ a+b=1/2 \\ a,b\in[0,1] \end{cases} \Rightarrow \begin{cases} a=\frac{-9\pm\sqrt{81+144}}{18} \quad \frac{-9-15}{18}<0 ;\ \frac{-9+15}{18}=\frac{1}{3} \\ \begin{cases} a=\frac{1}{3} \\ b=\frac{1}{6} \end{cases} \end{cases}$

$$p^{(0)}=\left(\frac{1}{9},\frac{1}{3},\frac{5}{9}\right);\quad \Pi=\begin{pmatrix} \frac{1}{3} & \frac{1}{6} & \frac{1}{2} \\ \frac{1}{10} & \frac{1}{2} & \frac{2}{5} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{pmatrix};\ p^{(0)}=(p_1^{(0)},p_2^{(0)},p_3^{(0)});\ \Pi=(p_{ij})_{1\le i,j\le 3}$$

(2) $P(X_0=2, X_1=1, X_2=3, X_3=3)=$

$=P\big((X_0=2)\cap(X_1=1)\cap(X_2=3)\cap(X_3=3)\big)\overset{(iii)}{=}$

$\overset{(iii)}{=}P(X_0)\cdot P(X_1=1/X_0=2)\cdot P(X_2=3/X_1=1,\cancel{X_0=2})\cdot P(X_3=3/X_2=3,\cancel{X_1=1,X_0=2})$

$\overset{3^\circ}{=}P(X_0)\cdot P(X_1=1/X_0=2)\cdot P(X_2=3/X_1=1)\cdot P(X_3=3/X_2=3)=$

$\overset{1^\circ}{\underset{2^\circ}{=}}p_1^{(0)}\cdot p_{21}\cdot p_{13}\cdot p_{33}=\frac{1}{9}\cdot\frac{1}{10}\cdot\frac{1}{2}\cdot\frac{1}{2}=\frac{1}{360}$

(3) $P(\underbrace{X_{200}=1, X_{199}=2, X_{198}=3}_{A}/\underbrace{X_{197}=2}_{B})=P(A/B)\overset{(iv)}{=}$

$\overset{(iv)}{=}\frac{P(A\cap B)}{P(B)}=\frac{P(X_{200}=1, X_{199}=2, X_{198}=3, X_{197}=2)}{P(X_{197}=2)}=$

$=\frac{P(X_{197}=2, X_{198}=3, X_{199}=2, X_{200}=1)}{P(X_{197}=2)}\overset{(iii)}{=}$

$\overset{(iii)}{=}\frac{\cancel{P(X_{197}=2)}\cdot P(X_{198}=3/X_{197}=2)\cdot P(X_{199}=2/X_{198}=3,\cancel{X_{197}=2})\cdot P(X_{200}=1/X_{199}=2,\ldots)}{\cancel{P(X_{197}=2)}}$

$P(X_{200}=1/X_{199}=2,\cancel{X_{198}=3},\cancel{X_{197}=2})$

$\overset{3^\circ}{=}P(X_{198}=3/X_{197}=2)\cdot P(X_{199}=2/X_{198}=3)\cdot P(X_{200}=1/X_{199}=2)\overset{2^\circ}{=}$

$\overset{2^\circ}{=}p_{23}\cdot p_{32}\cdot p_{21}=\frac{2}{5}\cdot\frac{1}{4}\cdot\frac{1}{10}=\cancel{\frac{1}{360}}\ \frac{1}{100}$

(4) $P(\underbrace{X_{1000}=1, X_{998}=2}_{A}/\underbrace{X_{997}=3}_{B})=P(A/B)\overset{(iv)}{=}\frac{P(A\cap B)}{P(B)}=$

$=\frac{P(X_{1000}=1, X_{998}=2, X_{997}=3)}{P(X_{997}=3)}=\frac{P(X_{997}=3, X_{998}=2, X_{1000}=1)}{P(X_{997}=3)}\overset{(iii)}{=}$

(iii) $\frac{\cancel{P(X_{997}=3)}\cdot P(X_{998}=2/X_{997}=3)\cdot P(X_{1000}=1/X_{998}=2,\cancel{X_{997}=3})}{\cancel{P(X_{997}=3)}}$

$\overset{3°}{=} P(X_{998}=2/X_{997}=3)\cdot P(X_{1000}=1/X_{998}=2) \underset{4.2;\, m=2}{\overset{2°}{=}}$

$= p_{32}\cdot p_{21}^{(1000-998)} = p_{32}\cdot \underbrace{p_{21}^{(2)}}_{\Pi^2} = \frac14\cdot(l_2\times c_1) =$

$= \frac14\cdot\underbrace{\left(\frac1{10},\frac12,\frac25\right)\times\begin{pmatrix}1/3\\1/10\\1/4\end{pmatrix}} = \frac14\left(\frac1{10}\cdot\frac13+\frac12\cdot\frac1{10}+\frac25\cdot\frac14\right) =$

$= \frac14\left(\frac1{30}+\frac1{20}+\frac1{10}\right) = \frac14\cdot\frac{2+3+6}{60} = \frac{11}{240}$

$\boxed{P(A/B)=\frac{P(A)\cdot P(B/A)}{P(B)} \quad \text{Bayes}}$

(5) $P(\underbrace{X_0=2}_{A}/\underbrace{X_2=3}_{B}) \overset{(iv)}{=}$

~~(iv)~~ $\cancel{\frac{P(X_0=2)\cdot P(X_2=3/X_0)}{P(X_2)} \underset{4.2}{\overset{1°}{=}}} \quad \cancel{\underset{4.2}{\overset{1°}{=}} p_2^{(0)}\cdot}$

(iv) $\frac{P(X_0=2)\cdot P(X_2=3/X_0=2)}{P(X_2=3)} \underset{4.2}{\overset{1°}{=}} \frac{p_2^{(0)}\cdot p_{23}^{(2-0)}}{P(X_2=3)} = \frac{\frac13\cdot p_{23}^{(2)}}{P(X_2=3)}$ (1)

$p_{23}^{(2)} = l_2\times c_3 = \left(\frac1{10},\frac12,\frac25\right)\times\begin{pmatrix}1/2\\2/5\\1/2\end{pmatrix} = \frac1{20}+\frac15+\frac1{10} = \frac{1+4+2}{20} = \frac7{20}$ (2)

$P(X_2=3) \underset{(i)}{\overset{4.3}{=}} p_3^{(2)}$, unde $p^{(2)} = (p_1^{(2)}, p_2^{(2)}, p_3^{(2)}) = (p^0)\,\Pi^2 =$

$= \left(\frac19,\frac13,\frac59\right)\cdot\begin{pmatrix}1/3 & 1/6 & 1/2\\ 1/10 & 1/2 & 2/5\\ 1/4 & 1/4 & 1/2\end{pmatrix}^2 = \left(\frac19,\frac13,\frac59\right)\begin{pmatrix}1/3 & 1/6 & 1/2\\ 1/10 & 1/2 & 2/5\\ 1/4 & 1/4 & 1/2\end{pmatrix}\begin{pmatrix}1/3 & 1/6 & 1/2\\ \frac1{10} & \frac12 & \frac25\\ \frac14 & \frac14 & \frac12\end{pmatrix}$

$p_3^{(2)} =$ (col 3) din $p^{(0)}\cdot\Pi^2 = p^{(0)}\times$(col 2 din Π^2) $= p^{(0)}\cdot\begin{pmatrix}\frac13\cdot\frac16+\frac16\cdot\frac12+\frac12\cdot\frac14\\ \frac1{10}\cdot\frac16+\frac12\cdot\frac12+\frac25\cdot\frac14\\ \frac14\cdot\frac16+\frac14\cdot\frac16+\frac12\cdot\frac14\end{pmatrix} =$

$= (p^0)\cdot\begin{pmatrix}\frac1{18}+\frac1{12}+\frac18\\ \frac1{60}+\frac14+\frac1{10}\\ \frac1{24}+\frac1{24}+\frac18\end{pmatrix} = \left(\frac19,\frac13,\frac59\right)\times\begin{pmatrix}\frac{19}{72}\\ \frac{22}{60}\\ \frac5{24}\end{pmatrix} = \left|\begin{matrix}9^2\cdot 8\\ 9\cdot 10 \to 9^2\cdot 3\cdot 40\\ 9\cdot 3\cdot 8\end{matrix}\right.$

$= \frac19\cdot\frac{19}{72}+\frac13\cdot\frac{11}{30}+\frac59\cdot\frac5{24} = \frac{19}{9\cdot 72}+\frac{11}{90}+\frac{25}{9\cdot 24} = \frac{15\cdot19+81\cdot55+45\cdot25}{9^2\cdot40\cdot3} =$

$= (285+4455+1125)/(27\cdot40) = 5865/27\cdot4\cdot0\cdot9 = \cancel{391/9720\ (3)}\ 5865/9720$ (3)

Din (1), (2), (3) $\Rightarrow p = P(X_1=2 / X_2=3) = \cancel{\frac{1}{5}} \cdot \frac{7}{\cancel{20}} \cdot \frac{\overset{162}{\cancel{9720}}}{5865} = \frac{1134}{5865} =$

$= \frac{388}{1955}$.

(6) $P(X_1=1) \overset{4.3}{\underset{(1)}{=}} p_1^{(1)}$, unde $p^{(1)} = p^{(0)}\Pi = (p_1^{(1)}, p_2^{(1)}, p_3^{(1)})$,

deci $P(X_1=1) = p^{(0)} \times \text{col}\,1(\Pi) = \left(\frac{1}{9}, \frac{1}{3}, \frac{5}{9}\right) \times \begin{pmatrix} 1/3 \\ 1/10 \\ 1/4 \end{pmatrix} =$

$= \frac{1}{9}\cdot\frac{1}{3} + \frac{1}{3}\cdot\frac{1}{10} + \frac{5}{9}\cdot\frac{1}{4} = \frac{\overset{20}{1}}{27} + \frac{\overset{18}{1}}{30} + \frac{\overset{15}{5}}{36} = \frac{20+18+75}{540} = \frac{113}{540}$ (1)

$P(X_2=3) \overset{4.3}{\underset{(1)}{=}} p_3^{(2)} \overset{(5)}{\underset{(3)}{=}} \frac{5865}{9720}$ (2)

Din (1) şi (2) $\Rightarrow P(X_1=1) + P(X_2=3) = \frac{\overset{18}{113}}{540} + \frac{5865}{9720} = \frac{2599}{9720}$

(7) $p^* = (p_1^*, p_2^*, p_3^*)$

$p^*\Pi = p^* \Leftrightarrow (p_1^*, p_2^*, p_3^*) \cdot \begin{pmatrix} \frac{1}{3} & \frac{1}{6} & \frac{1}{2} \\ \frac{1}{10} & \frac{1}{2} & \frac{2}{5} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{pmatrix} = (p_1^*, p_2^*, p_3^*) \Leftrightarrow$

$\Leftrightarrow \begin{cases} \frac{1}{3}p_1^* + \frac{1}{10}p_2^* + \frac{1}{4}p_3^* = p_1^* \;|\cdot 60 \\ \frac{1}{6}p_1^* + \frac{1}{2}p_2^* + \frac{1}{4}p_3^* = p_2^* \;|\cdot 60 \\ \frac{1}{4}p_1^* + \frac{2}{5}p_2^* + \frac{1}{2}p_3^* = p_3^* \;|\cdot 20 \\ p_1^* + p_2^* + p_3^* = 1 \end{cases} \Leftrightarrow \begin{cases} -40p_1^* + 6p_2^* + 15p_3^* = 0 \;|\cdot(-1) \\ 10p_1^* - 30p_2^* + 15p_3^* = 0 \\ 5p_1^* + 8p_2^* - 10p_3^* = 0 \\ p_1^* + p_2^* + p_3^* = 1 \;|\cdot 10 \end{cases} \overset{+}{\Rightarrow}$

$\Leftrightarrow \begin{cases} -40p_1^* + 6p_2^* + 15 = 0 \\ 5p_1^* - 15p_2^* + 4 = 0 \;|\cdot 8 \end{cases} \Leftrightarrow \begin{cases} -114p_2^* + 37 = 0 \\ 5p_1^* - 15p_2^* + 4 = 0 \end{cases} \Rightarrow \boxed{p_3^* = \frac{1}{3}}$

$\Rightarrow p_2^* = \frac{37}{114},\; 5p_1^* = \overset{5}{\cancel{15}} \cdot \frac{37}{\underset{38}{\cancel{114}}} - 4 = \frac{185-152}{38} \Rightarrow p_1^* = \frac{33}{38\cdot 5} = \frac{33}{190}$

$\begin{cases} 50p_1^* - 36p_2^* = 0 \;|:2 \\ 15p_1^* + 18p_2^* = 10 \end{cases} \overset{+}{\Rightarrow} 40p_1^* = 10 \Rightarrow p_1^* = \frac{1}{4},\; p_2^* = \frac{25}{18}p_1^* =$

$= \frac{25}{18}\cdot\frac{1}{4} = \frac{25}{72}$

Din $p_1^* + p_2^* + p_3^* = 1 \Rightarrow p_3^* = 1 - \frac{1}{4} - \frac{25}{72} = \frac{29}{72}$.

Deci $p^* = \left(\frac{1}{4}, \frac{25}{72}, \frac{29}{72}\right)$.

(8) $P(X_{79}=2 / X_{77}\neq 2) = \cancel{P(X_{79}=2 / X_{77})} =$
$X_{77}\neq 1$

$= P(X_{79}=2 / X_{77}=3) \overset{4.2}{=} p_{32}^{(79-77)} = p_{32}^{(2)} = l_3 \times c_2 =$

$= \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{2}\right) \times \left(\frac{1}{6}, \frac{1}{2}, \frac{1}{4}\right)^T = \frac{1}{24} + \frac{1}{8} + \frac{1}{8} = \frac{7}{24}$

(9) $P(\underbrace{X_0=2}_{A} / \underbrace{X_1=1, X_3=2}_{B}) \overset{4}{\underset{(iv)}{=}} \frac{P(X_0=2, X_1=1, X_3=2)}{P(X_0=2)} \overset{4}{\underset{(ii)}{=}}$

$= \frac{\cancel{P(X_0=2)} \cdot P(X_1=1/X_0=2) \cdot P(X_3=2/X_1=1, \cancel{X_0=2})}{\cancel{P(X_0=2)}} \overset{3°}{=}$

$= P(X_1=1/X_0=2) \cdot P(X_3=2/X_1=1) \overset{2°}{\underset{4.2}{=}} p_{21} \cdot p_{12}^{(3-1)} = p_{21} \cdot p_{12}^{(2)} =$

$= \frac{1}{10} \cdot (l_1 \times c_2) = \frac{1}{10} \cdot \left(\frac{1}{3}, \frac{1}{6}, \frac{1}{2}\right) \times \left(\frac{1}{6}, \frac{1}{2}, \frac{1}{4}\right)^T = \left(\frac{1}{18} + \frac{1}{12} + \frac{1}{8}\right) \cdot \frac{1}{10}$

$= \frac{8+12+18}{144} \cdot \frac{1}{10} = \frac{38}{144 \cdot 10} = \frac{19}{720}$

(10) a) $P(\underbrace{X_2=2}_{A} / \underbrace{X_4=1, X_5=2}) \overset{4}{\underset{(iv)}{=}} \frac{P(X_2=2, X_4=1, X_5=2)}{P(X_4=1, X_5=2)} =$

$\overset{4}{\underset{(i),(ii)}{=}} \frac{P(X_2=2) \cdot P(X_4=1/X_2=2) \cdot P(X_5=2/X_4=1, \cancel{X_2=2})}{P(X_4=1) \cdot P(X_5=2/X_4=1)} =$

$\overset{1°,2°,3°}{\underset{4.2}{=}} \frac{p_2^{(2)} \cdot p_{21}^{(2)} \cdot \cancel{p_{12}}}{p_1^{(4)} \cdot \cancel{p_{12}}} = \frac{(p^{(0)} \cdot \Pi^2)_{col\,2}}{(p^{(0)} \cdot \Pi^4)_{col\,1}}$.

b) $P(\underbrace{X_2=1}_{A} / \underbrace{X_3=2, X_1=3}_{B}) \overset{4}{\underset{iv}{=}} \frac{P(X_2=1, X_3=2, X_1=3)}{P(X_3=2, X_1=3)} =$

$= \frac{P(X_1=3, X_2=1, X_3=2)}{P(X_1=3, X_3=2)} \overset{4(ii)}{\underset{4(i)}{=}} \frac{\cancel{P(X_1=3)} \cdot P(X_2=1/X_1=3) \cdot P(X_3=2/X_2=1, \cancel{X_1=3})}{\cancel{P(X_1=3)} \cdot P(X_3=2/X_1=3)}$

$\overset{2°,3°}{\underset{4.2}{=}} \frac{p_{31} \cdot p_{12}}{p_{32}^{(3-1)}} = \frac{p_{31} \cdot p_{12}}{p_{32}^{(2)}} = \frac{\frac{1}{4} \cdot \frac{1}{6}}{l_3 \times c_2} = \frac{\frac{1}{24}}{\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{2}\right) \times \left(\frac{1}{6}, \frac{1}{2}, \frac{1}{4}\right)^T} =$

$= \frac{1/24}{\frac{1}{24} + \frac{1}{8} + \frac{1}{8}} = \frac{1}{24} \cdot \frac{24}{7} = \frac{1}{7}$

Temă

Ex. 2. Se consideră LM $(X_k)_{k\geq 0}$ asociat tripletului $(S, p^{(0)}, \Pi)$, unde $S = \{1, 2\}$, $p^{(0)} = (a, b)$, $\Pi = \begin{pmatrix} 4a^2 & 3/4 \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix}$. Să se determine:

1. a, b; 2. Probabilitatea ca lanțul să evolueze pe traiectoria $(2, 1, 2, 1)$
3. $P(X_{150} = 1, X_{149} = 2 / X_{148} = 2)$; 4. $P(X_{80} = 1, X_{79} = 2, X_{77} = 2 / X_{76} = 1)$
5. $P(X_0 = 1 / X_2 = 2)$; 6. $P(X_1 = 2) + P(X_2 = 1)$
7. Repartiția limită a L.M.; 8. $P(X_{729} = 1 / X_{727} \neq 1)$;
9. $P(X_0 = 2 / X_1 = 2, X_2 = 2)$; 10. a) $P(X_2 = 1 / X_3 = 1, X_0 = 2)$
   b) $P(X_2 = 2 / X_3 = 1, X_0 = 1)$

Temă

Ex. 3 Se consideră LM $(X_k)_{k\geq 0}$ asociat tripletului $(S, p^{(0)}, \Pi)$, unde $S = \{1, 2, 3\}$, $p^{(0)} = (a, 2a, 3a)$, $\Pi = \begin{pmatrix} b & b^2 & \frac{1}{4} \\ 1/3 & 0 & 2/3 \\ 2/5 & 1/5 & 2/5 \end{pmatrix}$.

Să se determine: 1. a, b 2. Probabilitatea ca lanțul să evolueze pe traiectoria $(3, 1, 3, 2)$; 3. $P(X_{500} = 1, X_{499} = 2, X_{498} = 3 / X_{497} = 3)$
4. $P(X_{251} = 1, X_{250} = 2 / X_{248} = 3, X_{246} = 2)$
5. $P(X_2 = 1 / X_0 = 2)$; 6. $P(X_2 = 1) + P(X_3 = 3, X_4 = 2 / X_2 = 1)$
7. Repartiția limită a L.M.; 8. $P(X_{91} \neq 1 / X_{89} = 1)$
9. $P(X_1 = 1 / X_0 = 2, X_3 = 2)$
10. a) $P(X_2 = 3 / X_0 = 1, X_1 = 2)$
    b) $P(X_2 = 2 / X_3 = 1, X_0 = 2, X_1 = 3)$