Chapter 1

Linear Wave Digital Filters

In this chapter we will consider linear electrical circuits and the basics of Wave Digital Filter modeling. This comprises a tutorial review of linear Wave Digital Filter theory from the perspective of Virtual Analog circuit modeling, as well as some new contributions and a real-world case study on the Roland TR-808 Bass Drum's Output Filter.

The electrical elements we cover will run the gamut from standard to purely theoretical. Some of these elements (e.g., resistors, capacitors, inductors, transformers) are of importance to the Virtual Analog designer because they occur in standard audio circuits. Some are idealized but may reasonably stand in for parts of audio circuits—e.g., an ideal voltage source may represent the input to an audio circuit or a battery, ideal controlled sources may represent connections between subcircuits or amplifier two-ports, or a nullor may represent an operational amplifier in a negative feedback configuration, although the real-world circuits are not so idealized. Some are purely theoretical, although they serve an important role in filter design and/or theoretical purpose in managing aspects of a Wave Digital Filter design. These include ideal voltage/current/power converters, gyrators, dualizers, and circulators. Although these are not real circuit elements (or at least not ones that are used regularly in audio circuits), they are an important part of analog filter *design* and we will consider them for completeness.

We will focus on handling all of the standard one-, two-, and three-port linear electrical elements and their interconnection topologies rather than on the *computational cost* of doing so. The computational efficiency of Wave Digital Filters is a major advantage of the approach, but we will not dwell on it in this dissertation. Information on efficient realization of voltage-wave Wave Digital Filter adaptors is easily found in the literature [87, 91, 102].

The structure of this chapter is as follows. We'll begin with some preliminaries on ports in network theory, and realizability in digital structures (§1.1). These set up an overview and introduction to the Wave Digital Filter approach, after which we define the wave variables used in

Wave Digital Filter theory, including a new parametric wave definition. Following these preliminaries, we'll develop the basic building blocks of Wave Digital Filter theory, beginning with algebraic one-ports (§1.2): resistors, sources, open/short circuits, etc. After this, we'll develop Wave Digital Filter models of reactive one-ports (§1.3): capacitors and inductors. There we'll go beyond the standard Bilinear Transform approach to reactance discretization and put a special emphasis on alternative discretization methods, including Warped Bilinear Transform, Backward Euler, the α Transform, and the Möbius Transform. Following this we'll develop Wave Digital Filter models of two-ports (§1.4), including basic topological connections (series/parallel), two-port electrical devices (transformers, gyrators, etc.), abstractions of more complicated two-port amplifiers (y-, z-, h-, and g-parameter models, ideal converters, nullors), and more abstract two-ports like dualizers. Building up to three-ports (§1.5), we'll derive the foundational topological connections of Wave Digital Filter theory, the three-port series and parallel adaptors, and the three-port circulator. Finally rules for realizing N-port series and parallel adaptors and circulators from combinations of three-ports are reviewed (§1.6). Having derived the standard WDF building blocks, we conclude the chapter by demonstrating their use in a simulation of the TR-808 Bass Drum's Output Filter (§1.7) and reviewing the capabilities and limitations of the presented techniques (§1.8).

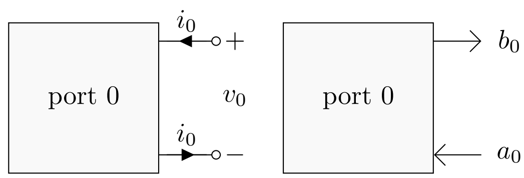
1.1 Preliminaries

In this section we'll review background material which forms the basis of Wave Digital Filter theory, including ports (§1.1.1), realizability of digital structures (§1.1.2), and the Connection Tree concept (§1.1.3) which allows the order of calculations in a Wave Digital Filter structure to by systematized. Following that we'll introduce wave variables (§1.1.4), the signal variables in the Wave Digital Filter context.

1.1.1 Port Definition

Many electrical elements can be characterized by the behavior at their ports. Ports are characterized by two terminals (one which is designated positive and one which is designated negative), a port voltage across these terminals, and a port current through these terminals [129]. The port voltage is defined as the voltage drop from the positive terminal to the negative terminal. The port current is defined as the current into the positive terminal or equivalently the current out of the negative terminal.

A one-port is shown in the Kirchhoff domain in Figure 1.1a and in the wave domain in Figure 1.1b. In the Kirchhoff domain, the port is characterized by its port voltage v_0 and its port current i_0. In the wave domain it is characterized by an incident wave a_0 and a reflected wave b_0, as well as a port resistance R_0 which parameterizes these waves.



(a) Kirchhoff Domain. (b) Wave Domain.

Figure 1.1: Kirchhoff- and Wave-Domain Representations of a Generic One-Port.

1.1.2 Realizability

For a digital filter structure to be "realizable" means that its computation can be structured explicitly into a series of adds, multiplies, reads and writes to delay registers, etc. Formal criteria defining realizability are given in the literature [138, 139, 5], but for our purposes realizability only means that there are no delay-free directed loops (or, "delay-free loops") in the structure. A delay-free loop is a signal flow path that forms a loop without passing through any delays.

1.1.3 Connection Tree

Typically, Wave Digital Filter structures are organized into tree structures. To ensure that these structures are realizable, we must identify and remove every delay-free loop in the structure. This is a daunting task, but luckily it has been systematized by the *Binary Connection Tree* (BCT) concept [6, 140, 132]. In this work, in recognition of adaptors with more than two downward-facing ports in a tree structure, we will always speak more broadly of (not necessarily binary) *Connection Trees* [141].

The Connection Tree concept involves designating one element of the tree as its root, and then letting the rest of the structure organize itself into a tree below the root. In these trees we have three types of elements: the aforementioned *root* which has no upward facing ports and one or more downward-facing port, *adaptors* which have one upward facing port and one or more downward facing ports, and *leaves* which have one upward-facing port. We notice that any delay-free loop in this computational structure can be seen as originating in an upward-facing port, following some path down the tree, connecting back to the same upward-facing port, and then closing the loop.

From this we conclude that breaking the delay-free loop at each upward-facing port in the Wave Digital Filter structure will suffice to ensure a realizable structure. Breaking this delay-free loop is called "adapting" the port, and is a central precept of Wave Digital Filter theory. We will find that some ports can be adapted and some cannot. The task of modeling a reference circuit as a Wave Digital Filter is simplest when the circuit happens to contain at most one non-adaptable element. Of course we can not count on this property holding in general, and the presence of multiple nonadaptable elements is a major source of problems in applying Wave Digital Filter theory to Virtual Analog modeling. Resolving this issue in the linear and nonlinear case will be a major focus later in this dissertation, in Chapters 2 and 4.

1.1.4 Wave Variable Definitions

In the two previous sections, we established that a good way to structure calculations in a digital simulation was to first decompose a circuit into a tree structure and then ensure that there can be no delay-free directed loops in the structure by ensuring that there can be no delay-free directed paths at the upward-facing port of each block in the simulation. However so far we have provided no way to satisfy this second criteria.

The Wave Digital Filter approach to avoiding delay-free directed paths at the upward-facing port of each block involves the introduction of *wave variables*. In the Wave Digital Filter literature, it is common to emphasize a physical interpretation of these wave variables. Indeed, the physical interpretation can help with developing an intuition about the scattering equations that parameterize the parts of a wave digital structure, provide a link to microwave engineering, and it also meshes philosophically with the energetic interpretation of Wave Digital Filters that helps to guarantee stability. However, this dissertation will emphasize an interpretation of the wave variable transformation as an arbitrary linear transformation.

In Wave Digital Filters, the use of voltage waves is common, although power and current waves are used as well [142]. We'll first define voltage, power, and current waves, then introduce a parametric wave definition that encompasses all these definitions. A discussion of alternative wave definitions follows.

Voltage Waves

At a port 0 characterized by port voltage v_0 and port current i_0, the incident and reflected voltage waves a_0 and b_0 are defined by linear combinations of v_0 and i_0 that are parameterized by a free parameter R_0 called port resistance[1]

$$\begin{aligned} a_0 &= v_0 + R_0 i_0 \\ b_0 &= v_0 - R_0 i_0 \,. \end{aligned} \tag{1.1}$$

An important property of wave variable definitions is that they are invertible. Consider (1.1) written in matrix form

$$\begin{bmatrix} a_0 \\ b_0 \end{bmatrix} = \begin{bmatrix} 1 & R_0 \\ 1 & -R_0 \end{bmatrix} \begin{bmatrix} v_0 \\ i_0 \end{bmatrix} . \tag{1.2}$$

[1] In this section and elsewhere, we'll usually drop both continuous (t) and discrete (n) time indices for compactness, expect when talking about reactances or time-varying signals.

The determinant of matrix $\left[\begin{smallmatrix} 1 & R_0 \\ 1 & -R_0 \end{smallmatrix}\right]$ is $-2R_0$. Hence, the wave variable definition is only invertible if $R_0 \neq 0$.[2] Assuming $R_0 \neq 0$ holds, the inverse of the voltage wave definition is

$$\begin{aligned} v_0 &= \frac{1}{2}a_0 + \frac{1}{2}b_0 \\ i_0 &= \frac{1}{2R_0}a_0 - \frac{1}{2R_0}b_0 \,. \end{aligned} \tag{1.3}$$

In fact, inverse definitions are what we will substitute into Kirchhoff-domain equations as the first step of deriving their behavior in the wave domain.

Finally, the voltage wave definition and its inverse for a collection of ports with port voltages $\boldsymbol{v}$, port currents $\boldsymbol{i}$, incident waves $\boldsymbol{a}$, reflected waves $\boldsymbol{b}$, and a diagonal matrix of port resistances $\mathbf{R}$ are written in matrix form as

$$\begin{aligned} \boldsymbol{a} &= \boldsymbol{v} + \mathbf{R}\boldsymbol{i} \\ \boldsymbol{b} &= \boldsymbol{v} - \mathbf{R}\boldsymbol{i} \end{aligned} \qquad \longleftrightarrow \qquad \begin{aligned} \boldsymbol{v} &= \frac{1}{2}\boldsymbol{a} + \frac{1}{2}\boldsymbol{b} \\ \boldsymbol{i} &= \frac{1}{2}\mathbf{R}^{-1}\boldsymbol{a} - \frac{1}{2}\mathbf{R}^{-1}\boldsymbol{b} \end{aligned} \,. \tag{1.4}$$

Power Waves

The use of power waves in Wave Digital Filters is also common. At a port 0, the power wave definition and its inverse are

$$\begin{aligned} a_0 &= R_0^{-1/2}v_0 + R_0^{1/2}i_0 \\ b_0 &= R_0^{-1/2}v_0 - R_0^{1/2}i_0 \end{aligned} \qquad \longleftrightarrow \qquad \begin{aligned} v_0 &= \frac{R_0^{1/2}}{2}a_0 + \frac{R_0^{1/2}}{2}b_0 \\ i_0 &= \frac{R_0^{-1/2}}{2}a_0 - \frac{R_0^{-1/2}}{2}b_0 \end{aligned} \,. \tag{1.5}$$

The power wave definition and its inverse for a collection of ports are written in matrix form as

$$\begin{aligned} \boldsymbol{a} &= \mathbf{R}^{-1/2}\boldsymbol{v} + \mathbf{R}^{1/2}\boldsymbol{i} \\ \boldsymbol{b} &= \mathbf{R}^{-1/2}\boldsymbol{v} - \mathbf{R}^{1/2}\boldsymbol{i} \end{aligned} \qquad \longleftrightarrow \qquad \begin{aligned} \boldsymbol{v} &= \frac{1}{2}\mathbf{R}^{1/2}\boldsymbol{a} + \frac{1}{2}\mathbf{R}^{1/2}\boldsymbol{b} \\ \boldsymbol{i} &= \frac{1}{2}\mathbf{R}^{-1/2}\boldsymbol{a} - \frac{1}{2}\mathbf{R}^{-1/2}\boldsymbol{b} \end{aligned} \,. \tag{1.6}$$

Power waves are preferred for time-varying reference circuits and structures involving nonlinear reactances [94].

[2] It is also common to require that R_0 is positive rather than negative. The requirement that R_0 is positive simplifies bookkeeping of energy metrics in the Wave Digital Filter, and will also follow as a consequence of adaptation rules, when all the elements in the wave digital filter are positive.

Current Waves

The use of Current waves in Wave Digital Filters is less common, although they are still occasionally used. At a port 0, the current wave definition and its inverse are

$$\begin{aligned} a_0 &= R_0^{-1} v_0 + i_0 \\ b_0 &= R_0^{-1} v_0 - i_0 \end{aligned} \qquad \longleftrightarrow \qquad \begin{aligned} v_0 &= \frac{R_0}{2} a_0 + \frac{R_0}{2} b_0 \\ i_0 &= \frac{1}{2} a_0 - \frac{1}{2} b_0 \end{aligned} \,. \tag{1.7}$$

The current wave definition and its inverse for a collection of ports are written in matrix form as

$$\begin{aligned} \boldsymbol{a} &= \mathbf{R}^{-1} \boldsymbol{v} + \boldsymbol{i} \\ \boldsymbol{b} &= \mathbf{R}^{-1} \boldsymbol{v} - \boldsymbol{i} \end{aligned} \qquad \longleftrightarrow \qquad \begin{aligned} \boldsymbol{v} &= \frac{1}{2} \mathbf{R} \boldsymbol{a} + \frac{1}{2} \mathbf{R} \boldsymbol{b} \\ \boldsymbol{i} &= \frac{1}{2} \boldsymbol{a} - \frac{1}{2} \boldsymbol{b} \end{aligned} \,. \tag{1.8}$$

Current waves are the dual of voltage waves [142].

Parametric Wave Definition

The previous voltage, power, and current wave definitions can all be considered specific instances of a more general wave definition that is parameterized by a parameter ρ

$$\begin{aligned} a_0 &= R_0^{\rho-1} v_0 + R_0^{\rho} i_0 \\ b_0 &= R_0^{\rho-1} v_0 - R_0^{\rho} i_0 \end{aligned} \qquad \longleftrightarrow \qquad \begin{aligned} v_0 &= \frac{1}{2} R_0^{1-\rho} a_0 + \frac{1}{2} R_0^{1-\rho} b_0 \\ i_0 &= \frac{1}{2} R_0^{-\rho} a_0 - \frac{1}{2} R_0^{-\rho} b_0 \end{aligned} \,, \tag{1.9}$$

where ρ defines the wave type. Standard wave definitions correspond to

$$\rho = \begin{cases} 1 & \text{for voltage waves} \\ 1/2 & \text{for power waves} \\ 0 & \text{for current waves} \end{cases} \,, \tag{1.10}$$

although other values of ρ are valid as well.

This parametric wave definition and its inverse for a collection of ports are written in matrix form as

$$\begin{aligned} \boldsymbol{a} &= \mathbf{R}^{\rho-1} \boldsymbol{v} + \mathbf{R}^{\rho} \boldsymbol{i} \\ \boldsymbol{b} &= \mathbf{R}^{\rho-1} \boldsymbol{v} - \mathbf{R}^{\rho} \boldsymbol{i} \end{aligned} \qquad \longleftrightarrow \qquad \begin{aligned} \boldsymbol{v} &= \frac{1}{2} \mathbf{R}^{1-\rho} \boldsymbol{a} + \frac{1}{2} \mathbf{R}^{1-\rho} \boldsymbol{b} \\ \boldsymbol{i} &= \frac{1}{2} \mathbf{R}^{-\rho} \boldsymbol{a} - \frac{1}{2} \mathbf{R}^{-\rho} \boldsymbol{b} \end{aligned} \,. \tag{1.11}$$

Alternate Wave Definitions

Alternative wave definitions are sometimes used. A voltage wave formulation which differs by a factor of 1/2 is favored by Karjalainen [102] and Smith [97], and is identical to the variable definition used in digital waveguide modeling [143]. A version where one wave is not parameterized by port resistance was proposed in the 1970s [144, 121]. A fully generalized definition is proposed by Lawson [123] [124, pp. 61–63].

In this dissertation we will only consider the standard voltage wave definition and the parametric wave definition.

1.2 Algebraic One-Ports

One-port devices are the simplest electronic devices, whose behavior is defined as the relationship between voltage and current at a single port. For each of these one-port devices, a wave-domain equation is derived by substituting a wave definition into its Kirchhoff-domain v–i relationship and solving for the reflected wave b.

1.2.1 Resistor

The behavior of a resistor is defined by Ohm's law, which instantaneously relates port voltage v_0 to port current i_0 by the resistor's resistance R

$$v_0 = Ri_0\,. \tag{1.12}$$

The symbol and port definition for a resistor, adapted and unadapted Wave Digital Filter symbols, and adapted and unadapted Wave Digital Filter signal-flow diagrams are shown in Figure 1.2.

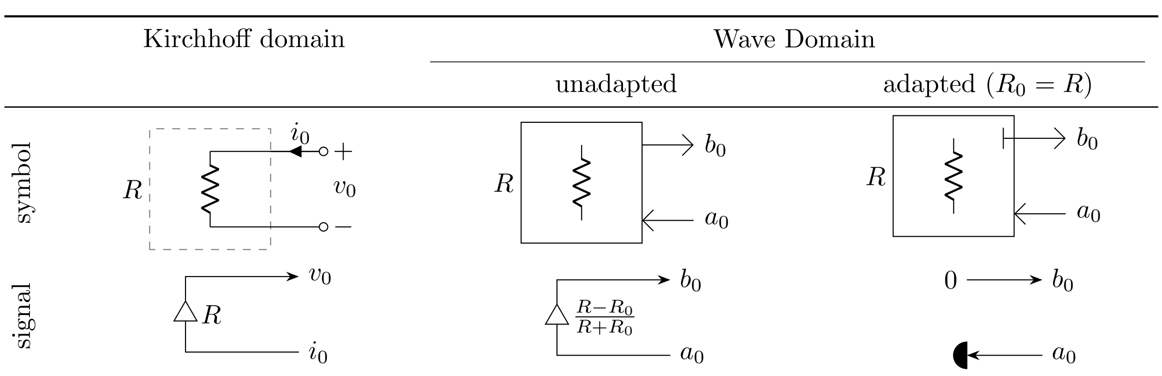


Figure 1.2: Kirchhoff- and Wave-Domain Representations of a Resistor One-Port.

Plugging the parametric wave definition (1.9) into the Kirchhoff-domain resistor equation (1.12)

$$\frac{1}{2}R_0^{1-\rho}a_0 + \frac{1}{2}R_0^{1-\rho}b_0 = \frac{1}{2}R\,R_0^{-\rho}a_0 - \frac{1}{2}R\,R_0^{-\rho}b_0 \tag{1.13}$$

and solving for the reflected wave b_0 yields the unadapted continuous-time wave-domain equation

$$b_0 = \frac{R - R_0}{R + R_0}a_0\,. \tag{1.14}$$

Since there are no derivatives in (1.14), this is also valid in discrete time. The unadapted wave-domain equation (1.14) is suitable for use at the root of a Wave Digital Filter connection tree. On the other hand, it contains a delay-free directed path from its input a_0 to its output b_0 so it is *not* yet suitable for use as a leaf in the connection tree.

To adapt the Wave Digital Filter resistor, hence rendering it suitable for inclusion in a Wave Digital Filter connection tree, we exploit the free port resistance parameter R_0 that we introduced when we plugged in the parametric wave variable definition. By setting $R_0 = R$, the coefficient $\frac{R-R_0}{R+R_0}$ is set to 0, yielding the adapted wave-domain resistor equation

$$b_0 = 0\,. \tag{1.15}$$

In Wave Digital Filter notation, the resistor's reflected wave is shown as a "wave source" that always produces $b_0 = 0$ and its incident wave is shown as a "wave sink" that just "throws out" whatever wave is incident upon it.

Under the physical interpretation of wave variables, the unadapted wave-domain equation for the resistor (1.14) simply expresses the scattering at an impedance mismatch, which is parameterized by an "impedance step over impedance sum" scattering coefficient. Under this interpretation, adapting the Wave Digital Filter resistor by setting $R_0 = R$ can be seen as an impedance matching operation. Matched impedances don't produce reflections, hence $b = 0$, and if no further impedance mismatch is seen, an incident wave will just keep travelling and never return (hence the incident wave sink).

Notice that the parametric wave factor ρ appears in neither of the unadapted resistor equation (1.14), nor the adaptation criteria $R_0 = R$, nor the adapted resistor equation (1.15). That means that the same Wave Digital Filter resistor model is valid for voltage, power, and current waves.

1.2.2 Ideal Voltage Source

An ideal voltage source produces a voltage $e(t)$ across its port

$$v_0(t) = e(t) \tag{1.16}$$

while drawing or sourcing as much port current as is necessary to produce this voltage. The symbol and port definition for an ideal voltage source, the unadapted Wave Digital Filter symbol, and the unadapted Wave Digital Filter signal-flow diagram are shown in Figure 1.3.

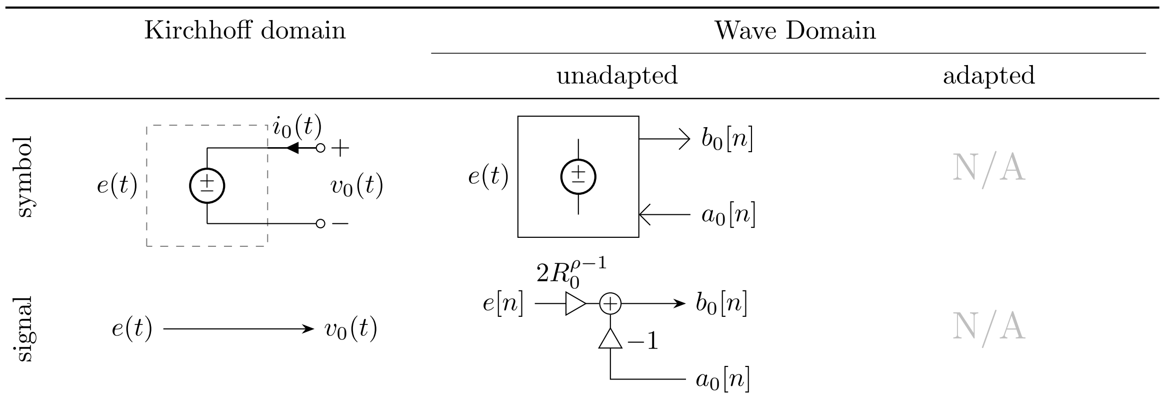


Figure 1.3: Kirchhoff- and Wave-Domain Representations of an Ideal Voltage Source One-Port.

Plugging the parametric wave definition (1.9) into the Kirchhoff-domain ideal voltage source equation (1.16)

$$\frac{1}{2}R_0^{1-\rho}a_0(t) + \frac{1}{2}R_0^{1-\rho}b_0(t) = e(t) \tag{1.17}$$

and solving for the reflected wave $b_0(t)$ yields the unadapted continuous-time wave-domain equation

$$b_0(t) = 2R_0^{\rho-1}e(t) - a_0(t)\,. \tag{1.18}$$

The discrete-time version is formed by replacing t with n

$$b_0[n] = 2R_0^{\rho-1}e[n] - a_0[n]\,. \tag{1.19}$$

The unadapted wave-domain equation (1.19) is suitable for use at the root of a Wave Digital Filter connection tree, but is not for use as a leaf in the connection tree on account of the delay-free directed path from its input $a_0[n]$ to its output $b_0[n]$. Unlike the case of the resistor, the ideal voltage source's wave domain equation does not depend on the port resistance and hence it cannot be adapted.

In Wave Digital Filter notation, the ideal voltage source's value $e[n]$ is again shown as a "wave source."

Notice that unlike the resistor, the parametric wave factor ρ *does* appears in the unadapted ideal voltage source equation (1.19). This means that the Wave Digital Filter model of an unadapted

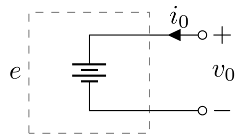


Figure 1.4: Battery Symbol and Port Definition.

ideal voltage source is different for voltage, power, and current waves

$$\underbrace{b_0[n] = 2e[n] - a_0[n]}_{\text{voltage waves}} \qquad \underbrace{b_0[n] = 2\sqrt{R_0}\, e[n] - a_0[n]}_{\text{power waves}} \qquad \underbrace{b_0[n] = 2R_0 e[n] - a_0[n]}_{\text{current waves}} . \tag{1.20}$$

Batteries are idealized as non-time-varying ideal voltage sources, as shown in Figure 1.4.

1.2.3 Ideal Current Source

An ideal current source produces a current $j(t)$ through its port

$$i_0(t) = -j(t) \tag{1.21}$$

while developing as much port voltage as is necessary to produce this current. Here we've chosen the polarity that respects the duality between Thévenin and Norton circuits; be aware that the port polarity is sometimes defined in the opposite way [145, p. 46]. The symbol and port definition for an ideal current source, unadapted Wave Digital Filter symbol, and unadapted Wave Digital Filter signal-flow diagram are shown in Figure 1.5.

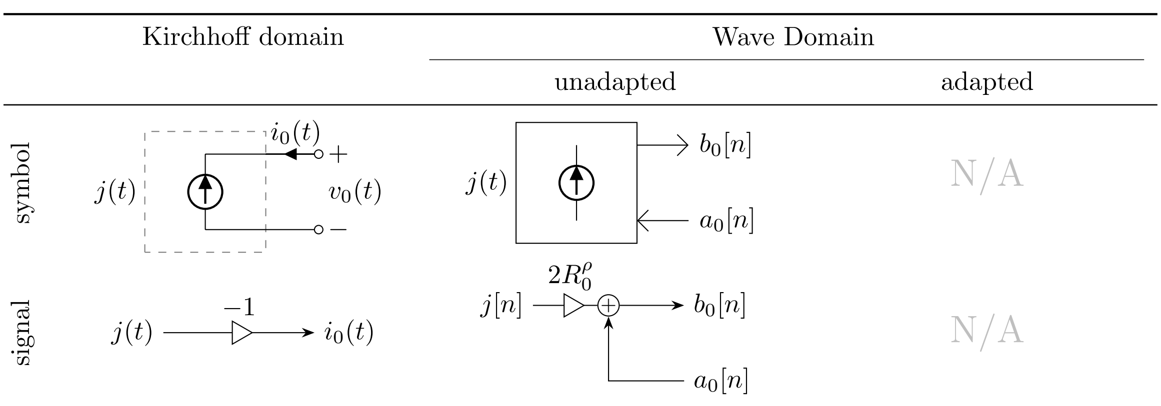


Figure 1.5: Kirchhoff- and Wave-Domain Representations of an Ideal Current Source One-Port.

Plugging the parametric wave definition (1.9) into the Kirchhoff-domain ideal current source equation (1.21)

$$\frac{1}{2}R_0^{-\rho}a_0(t) - \frac{1}{2}R_0^{-\rho}b_0(t) = -j_0(t) \tag{1.22}$$

and solving for the reflected wave $b_0(t)$ yields the unadapted continuous-time wave-domain equation

$$b_0(t) = 2R_0^{\rho}j(t) + a_0(t)\,. \tag{1.23}$$

Again the discrete-time version is formed by replacing t with n

$$b_0[n] = 2R_0^{\rho}j[n] + a_0[n]\,. \tag{1.24}$$

The unadapted wave-domain equation (1.24) is suitable for use at the root of a Wave Digital Filter connection tree, but is not for use as a leaf in the connection tree on account of the delay-free directed path from its input $a_0[n]$ to its output $b_0[n]$. As with the ideal voltage source, the ideal current source's wave domain equation does not depend on the port resistance and hence it cannot be adapted.

In Wave Digital Filter notation, the ideal current source's value $j[n]$ is again shown as a "wave source."

As with the ideal voltage source, the parametric wave factor ρ does appear in the unadapted ideal current source equation (1.24). This means that the Wave Digital Filter model of an unadapted ideal current source is different for voltage, power, and current waves

$$\underbrace{b_0[n] = 2R_0 j[n] + a_0[n]}_{\text{voltage waves}} \qquad \underbrace{b_0[n] = 2\sqrt{R_0}\,j[n] + a_0[n]}_{\text{power waves}} \qquad \underbrace{b_0[n] = 2j[n] + a_0[n]}_{\text{current waves}}\,. \tag{1.25}$$

1.2.4 Resistive Voltage Source

In §1.2.2 we derived the discrete, wave-domain equation for an ideal voltage source and found that it cannot be adapted. Hence, although it is suitable for use at the root of a Wave Digital Filter tree, it is unsuitable for inclusion as a leaf in a standard Wave Digital Filter tree. Since there may, in general, be more than one non-adaptable electrical element in a reference circuit and only one of them can be the root in a standard Wave Digital Filter, the nonadaptable ideal voltage can cause a problem.

This problem does not arise for resistive voltage sources. A resistive voltage source comprises a resistor R is series with an ideal voltage source $e(t)$. Using Kirchhoff's voltage law, this arrangement is described by

$$v_0(t) - e(t) = Ri_0(t)\,. \tag{1.26}$$

The symbol and port definition for a resistive voltage source, adapted and unadapted Wave Digital Filter symbols, and adapted and unadapted Wave Digital Filter signal-flow diagrams are shown in

Figure 1.6.

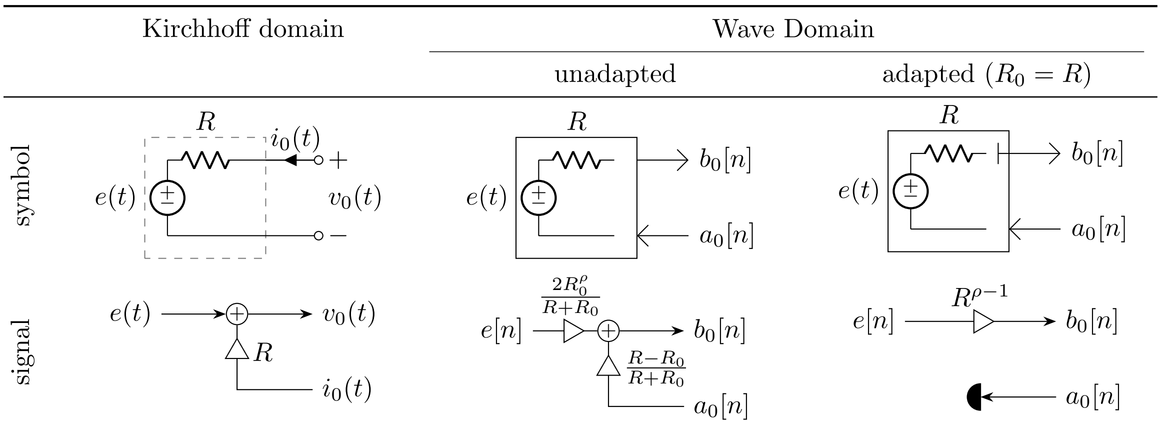


Figure 1.6: Kirchhoff- and Wave-Domain Representations of a Resistive Voltage Source One-Port.

Plugging the parametric wave definition (1.9) into the Kirchhoff-domain ideal current source equation (1.26)

$$\frac{1}{2}R_0^{1-\rho}a_0(t) + \frac{1}{2}R_0^{1-\rho}b_0(t) - e(t) = \frac{1}{2}R\,R_0^{-\rho}a_0(t) + \frac{1}{2}R\,R_0^{-\rho}b_0(t) \tag{1.27}$$

and solving for the reflected wave $b_0(t)$ yields the unadapted continuous-time wave-domain equation

$$b_0(t) = \frac{R - R_0}{R + R_0}a_0(t) + \frac{2R_0^\rho}{R + R_0}e(t)\,. \tag{1.28}$$

Again the discrete-time version is formed by replacing t with n

$$b_0[n] = \frac{R - R_0}{R + R_0}a_0[n] + \frac{2R_0^\rho}{R + R_0}e[n]\,. \tag{1.29}$$

The unadapted wave-domain equation (1.29) is suitable for use at the root of a Wave Digital Filter connection tree, but is not for use as a leaf in the connection tree on account of the delay-free directed path from its input $a_0[n]$ to its output $b_0[n]$.

To adapt the resistive voltage source, we exploit the free port resistance parameter R_0. By setting $R_0 = R$ (just as we did for the resistor), the coefficient $\frac{R-R_0}{R+R_0}$ is set to 0, yielding the adapted wave-domain resistive voltage source equation

$$b_0[n] = R^{\rho-1}e[n]\,. \tag{1.30}$$

Again, since the factor ρ appears in (1.30), the behavior depends on the wave definition.

If, in the reference circuit, the ideal voltage source is already in series with a resistor, we can (and should) combine the ideal voltage source and the resistor into a resistive voltage source without any loss of accuracy. If the ideal voltage source is not in series with a resistor, we can introduce a small fictitious resistor in series with the ideal voltage source. This changes the behavior of the electrical circuit, but if the resistor is small, it won't change by much. Also, no real-world voltage source is actually ideal since they have a limited ability to supply current. So arguably, even when a schematic contains an ideal voltage source, the actual reference circuit should have a series resistance. Ideally, this resistance value would be measured, predicted from first principles, or estimated by a rule of thumb. In practice, it is usually fine to just pick a small resistance like $1\,\Omega$; it is also worth mentioning that since resistors dissipate energy this is an incrementally passive modification.

1.2.5 Resistive Current Source

In §1.2.3 we derived the discrete, wave-domain equation for an ideal current source and found that it cannot be adapted. As with the voltage source, we now consider a resistive current source.

A resistive current source comprises a resistor R in parallel with an ideal current source $j(t)$. Using Kirchhoff's current law, this arrangement is described by

$$j(t) + i_0(t) = v_0(t)/R\,. \tag{1.31}$$

The symbol and port definition for a resistive current source, adapted and unadapted Wave Digital Filter symbols, and adapted and unadapted Wave Digital Filter signal-flow diagrams are shown in Figure 1.7.

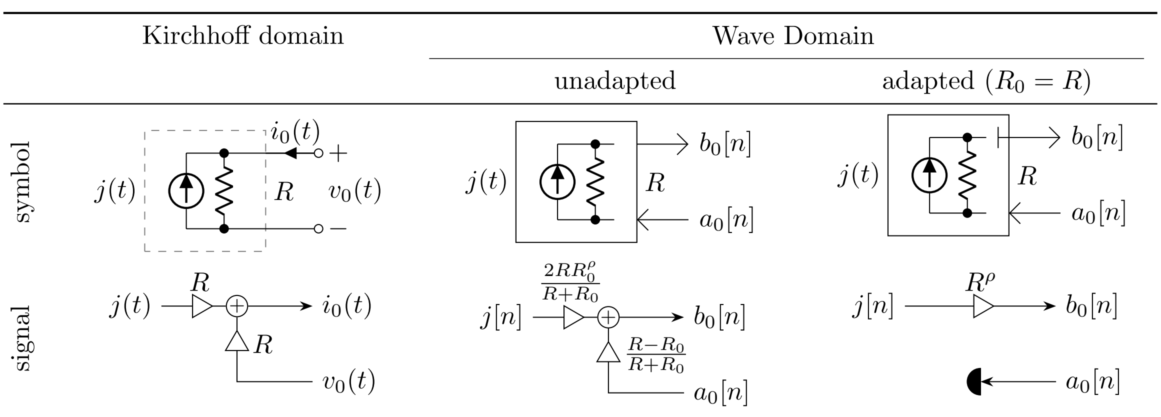


Figure 1.7: Kirchhoff- and Wave-Domain Representations of a Resistive Current Source One-Port.

Plugging the parametric wave definition (1.9) into the Kirchhoff-domain resistive current source equation (1.31)

$$j(t) + \frac{1}{2}R_0^{-\rho}a_0(t) - \frac{1}{2}R_0^{-\rho}b_0(t) = \frac{1}{2}\frac{R_0^{1-\rho}}{R}a_0(t) + \frac{1}{2}\frac{R_0^{1-\rho}}{R}b_0(t) \tag{1.32}$$

and solving for the reflected wave $b_0(t)$ yields the unadapted continuous-time wave-domain equation

$$b_0(t) = \frac{R - R_0}{R + R_0}a_0(t) + \frac{2R\,R_0^{\rho}}{R + R_0}j(t)\,. \tag{1.33}$$

Again the discrete-time version is formed by replacing t with n

$$b_0[n] = \frac{R - R_0}{R + R_0}a_0[n] + \frac{2R\,R_0^{\rho}}{R + R_0}j[n]\,. \tag{1.34}$$

The unadapted wave-domain equation (1.34) is suitable for use at the root of a Wave Digital Filter connection tree, but is not for use as a leaf in the connection tree on account of the delay-free directed path from its input $a_0[n]$ to its output $b_0[n]$.

To adapt the resistive current source, we exploit the free port resistance parameter R_0. By setting $R_0 = R$ (just as we did for the resistor), the coefficient $\frac{R-R_0}{R+R_0}$ is set to 0, yielding the adapted wave-domain resistive current source equation

$$b_0[n] = 2R^{\rho}j[n]\,. \tag{1.35}$$

Again, since the factor ρ appears in (1.35), the behavior depends on the wave definition.

1.2.6 Short Circuit

An ideal short circuit has zero port voltage

$$v_0 = 0 \tag{1.36}$$

while at the same time allowing an arbitrary amount of port current to flow. The symbol and port definition for an ideal short circuit, unadapted Wave Digital Filter symbol, and unadapted Wave Digital Filter signal-flow diagram are shown in Figure 1.8.

Plugging the parametric wave definition (1.9) into the Kirchhoff-domain short circuit equation (1.36)

$$\frac{1}{2}R_0^{1-\rho}a_0 + \frac{1}{2}R_0^{1-\rho}b_0 = 0 \tag{1.37}$$

and solving for the reflected wave b_0 yields the unadapted continuous-time wave-domain equation

$$b_0 = -a_0\,. \tag{1.38}$$

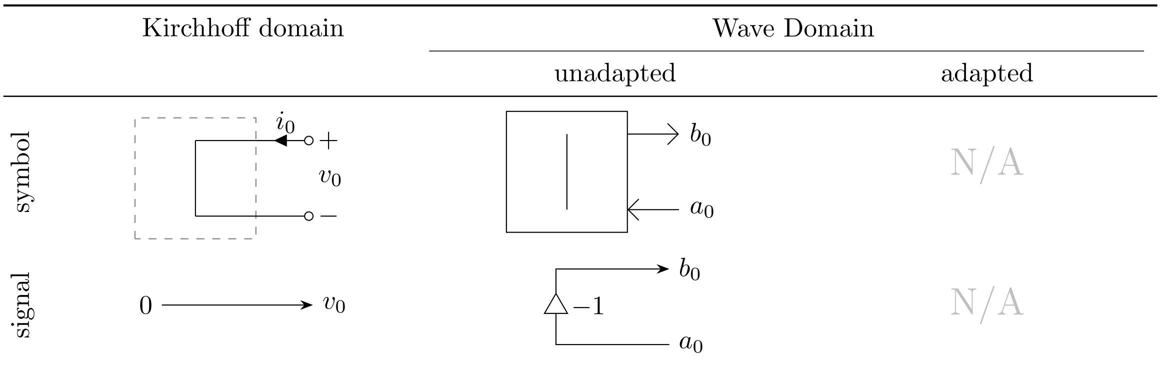


Figure 1.8: Kirchhoff- and Wave-Domain Representations of an Ideal Short Circuit One-Port.

The unadapted wave-domain equation (1.38) is suitable for use at the root of a Wave Digital Filter connection tree, but is not for use as a leaf in the connection tree on account of the delay-free directed path from its input a_0 to its output b_0. As with the ideal voltage and current sources, the short's wave domain equation does not depend on the port resistance and hence it cannot be adapted.

The unadapted short (1.38) is identical to an ideal voltage source (1.19) with $e = 0$ or an ideal resistor (1.14) with zero electrical resistance

$$b_0 = -a_0 = \left[2R_0^{\rho-1}e - a_0\right]_{e=0} = \left[\frac{R - R_0}{R + R_0}a_0\right]_{R=0} . \tag{1.39}$$

1.2.7 Open Circuit

An ideal open circuit has zero port current

$$i_0 = 0 \tag{1.40}$$

while placing no restrictions on port voltage. The symbol and port definition for an ideal open circuit, unadapted Wave Digital Filter symbol, and unadapted Wave Digital Filter signal-flow diagram are shown in Figure 1.9.

Plugging the parametric wave definition (1.9) into the Kirchhoff-domain short circuit equation (1.40)

$$\frac{1}{2}R_0^{-\rho}a_0 - \frac{1}{2}R_0^{-\rho}b_0 = 0 \tag{1.41}$$

and solving for the reflected wave $b(t)$ yields the unadapted continuous-time wave-domain equation

$$b_0 = a_0 \, . \tag{1.42}$$

The unadapted wave-domain equation (1.42) is suitable for use at the root of a Wave Digital Filter

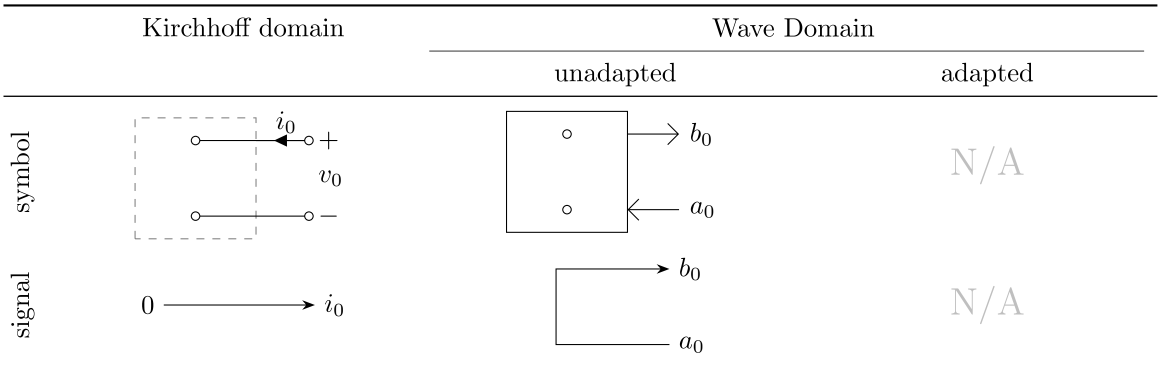


Figure 1.9: Kirchhoff- and Wave-Domain Representations of an Ideal Open Circuit One-Port.

connection tree, but is not for use as a leaf in the connection tree on account of the delay-free directed path from its input a_0 to its output b_0. As with the short, the open circuit's wave domain equation does not depend on the port resistance and hence it cannot be adapted.

The unadapted open circuit (1.42) is identical to an ideal resistor (1.14) with infinite electrical resistance

$$b_0 = a_0 = \lim_{R\to\infty} \frac{R - R_0}{R + R_0} a_0 \,. \tag{1.43}$$

1.2.8 Switch

The short circuit and open circuit one-ports can be considered the two different settings of an ideal single-pole, single-throw switch. From that perspective, the two one-ports are unified into a single wave-domain equation where λ accounts for whether the switch is closed or open

$$b_0[n] = \lambda\, a_0[n] \text{ , where } \lambda = \begin{cases} -1 & \text{closed switch} \\ +1 & \text{open switch} \end{cases} \tag{1.44}$$

Switches are an important part of analog audio circuits, where they are used to control settings. e.g., in a "bright switch/filter" [59, 7]. The symbol and port definition for a switch, unadapted Wave Digital Filter symbols, and unadapted Wave Digital Filter signal-flow diagram are shown in Figure 1.10.

An open circuit may be approximated as a resistance that is significantly larger than the other resistances in the circuit and a closed circuit may be approximated by a resistance that is significantly smaller than the other resistances in the circuit. Vladimirescu recommends a ratio of $R_{\text{off}}/R_{\text{on}} \leq 10^{12}$ [146, pp. 65–68].

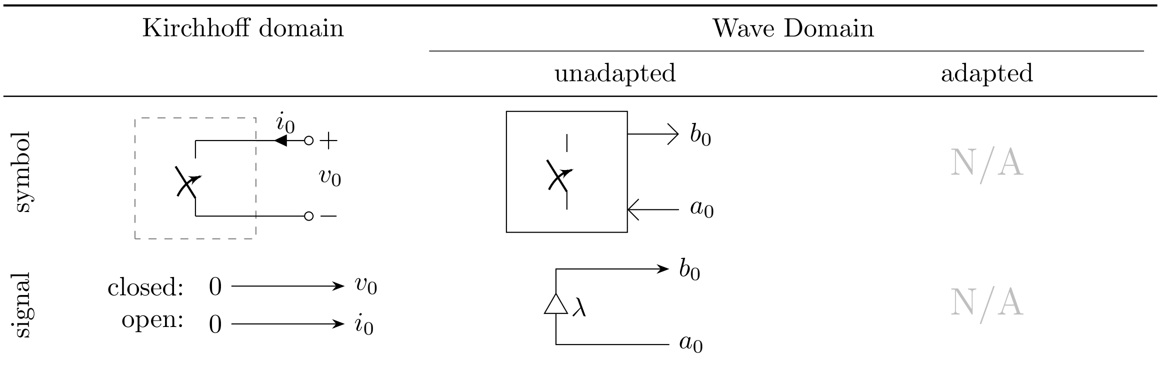


Figure 1.10: Kirchhoff- and Wave-Domain Representations of an Ideal Switch One-Port.

1.2.9 Singular Network Elements

We close our discussion of one-port algebraic devices by considering a class of special one-port network elements: the "nullator" and "norator." These theoretical network elements are singular [147] and sometimes called "degenerate" [148, p. 53].

Nullator

The *nullator* is a theoretical idealized network element that is characterized by zero port voltage and zero port current

$$v_0 = 0 \tag{1.45}$$

$$i_0 = 0\,. \tag{1.46}$$

The symbol and port definition for a nullator is shown in Figure 1.11.

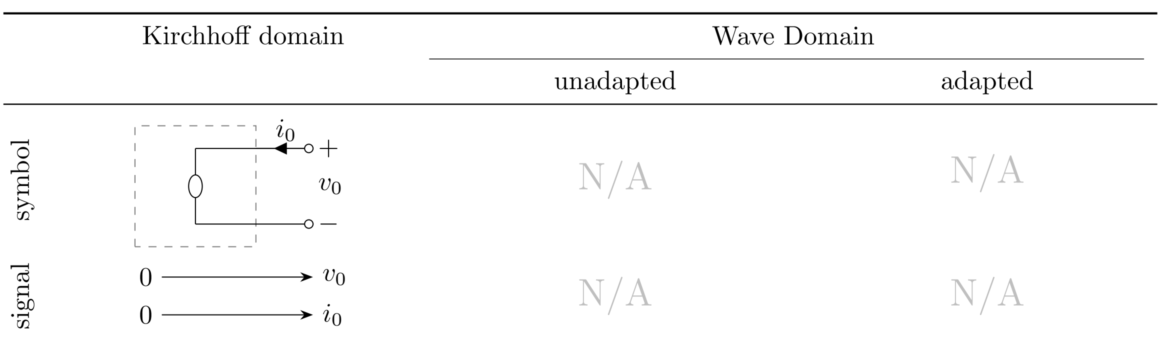


Figure 1.11: Kirchhoff- and Wave-Domain Representations of a Nullator One-Port.

A unique facet of this network element is that from the standpoint of voltage, the nullator looks like a short circuit, and from the standpoint of current, the nullator looks like an open circuit. Even theoretically, nullators cannot appear by themselves in a circuit—each one must by accompanied by another theoretical one-port device called a *norator* [147, 148].

In fact if we try to plug the parametric wave definition (1.9) into the nullator equations (1.45)–(1.46), we end up with the following two restrictions

$$a_0 = 0 \tag{1.47}$$

$$b_0 = 0\,. \tag{1.48}$$

Since it is not possible in the Wave Digital Filter context to put a restriction on the input wave to a port, but only to chose an output wave based on the input wave, this can be considered an overconstrained definition and thus is not suitable for inclusion in a Wave Digital Filter tree as is.

Norator

The *norator* is a theoretical idealized network element that is characterized by no restrictions on either port voltage or port current. The symbol and port definition for a norator is shown in Figure 1.2.

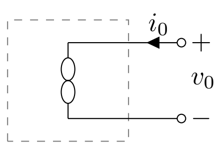
	Kirchhoff domain	Wave Domain	
		unadapted	adapted
symbol		N/A	N/A
signal	N/A	N/A	N/A

Figure 1.12: Kirchhoff- and Wave-Domain Representations of a Norator One-Port.

A unique facet of this network element is that from the standpoint of voltage, the nullator looks like an open circuit, and from the standpoint of current, the nullator looks like a short circuit. Even theoretically, norators cannot appear by themselves in a circuit—each one must by accompanied a nullator [147, 148].

Since the norator don't put any restrictions on the voltage or current of the port, they also put no restrictions on the reflected wave from the port. Since in the Wave Digital Filter context we mus chose an output wave based on the input wave, this can be considered an underconstrained definition and thus is not suitable for inclusion in a Wave Digital Filter tree as is.

1.3 Reactive One-Ports

So far, all the Wave Digital Filter one-ports that we have treated have been algebraic / memoryless. That is, their Kirchhoff-domain equations relate instantaneously a voltage, a current, and potentially an input. In deriving wave-domain equations, we have so far ended up with an instantaneous and linear relationship between an incident wave a, a reflected wave b, and potentially an input (e or j). In this section, we'll derive wave-domain equations for two *reactive* one-port elements: the capacitor (§1.3.1) and the inductor (§1.3.2). Reactive elements have a continuous-time derivative in their constitutive equations which must be approximated in discrete time by discretization.

Typically, Wave Digital Filters use the trapezoidal rule for integration to approximate these derivatives [5]. Thinking on the Laplace plane, the trapezoidal rule is identical to the Bilinear Transform, a conformal mapping from the s to z planes. In control engineering, the trapezoidal rule is also known as "Tustin's method" [149]. Here, we'll discuss alternative discretization strategies, including the Warped Bilinear Transform, α transform and the Möbius transform, which contains the α transform, forward Euler, backward Euler, and the warped and unwarped Bilinear Transforms as special cases.[3]

Here, we start with a continuous-time equation in the Kirchhoff domain, execute a change of variables to the wave domain, use the Laplace transform to write the constituent equation in terms of the Laplace differentiation operator s, discretize the Laplace expression using the Bilinear Transform of another $s \to z$ mapping, and use the inverse z-transform to find a discrete-time, wave-domain difference equation which can be adapted as with the algebraic one-ports.

1.3.1 Capacitor

The behavior of a linear capacitor with capacitance C is modeled by the differential equation

$$C\frac{\mathrm{d}v_0(t)}{\mathrm{d}t} = i_0(t) \tag{1.49}$$

which relates the time derivative of port voltage $\frac{\mathrm{d}v_0(t)}{\mathrm{d}t}$ to port current $i_0(t)$ by the capacitance C.

In the Laplace domain, this relationship is

$$CsV_0(s) = I_0(s)\,. \tag{1.50}$$

Plugging the parametric wave definition (1.9) into the Kirchhoff-domain capacitor equation (1.50)

$$\frac{1}{2}CR_0^{1-\rho}sA_0(s) + \frac{1}{2}CR_0^{1-\rho}sB_0(s) = \frac{1}{2}R_0^{-\rho}A_0(s) - \frac{1}{2}R_0^{-\rho}B_0(s) \tag{1.51}$$

[3] A completely different approach based on Runge–Kutta discretization has also been explored in the Wave Digital Filter context [150, 151, 152], but will not be considered further here since it profoundly alters the global simulation structure.

and solving for the reflected wave $B_0(s)$ yields the Laplace- and wave-domain transfer function

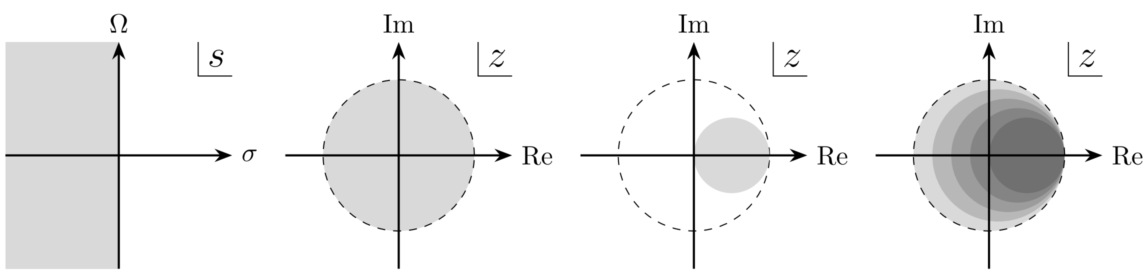
$$H_0(s) = \frac{B_0(s)}{A_0(s)} = \frac{1 - R_0Cs}{1 + R_0Cs}\,. \tag{1.52}$$

Bilinear Transform

Creating a discrete-time equation from (1.52) is not as simple as it is for the algebraic one-ports. To do so, we must choose a strategy for forming a z-plane transfer function $H_0(z^{-1}) = B_0(z^{-1})/A_0(z^{-1})$ that approximates (1.52) and then implement $H_0(z^{-1})$ as a difference equation using delays, multipliers, and adders. A standard way of approaching this is to use a mapping from the s-plane to the z-plane

$$H_0(z^{-1}) = H_0(s)\big|_{s=f(z^{-1})} \tag{1.53}$$

that corresponds to a numerical integration technique.



(a) s-Plane. (b) Bilinear Transform. (c) Backwards Euler. (d) α Transform.

Figure 1.13: Mappings from the s to z Planes, Including the Bilinear Transform, Backwards Euler, and the α Transform. On the z planes, the unit circle is shown as a dotted circle. For the α Transform, a family of transforms $\alpha \in [0.00, 0.25, 0.5, 0.75, 1.00]$ is shown.

The typical choice of mapping for a Wave Digital Filter reactance is the Bilinear Transform. The Bilinear Transform (BLT) forms $H(z^{-1})$ each s in some $H(s)$ by a function of z^{-1}

$$s = f_{\mathrm{BLT}}(z^{-1}) = \frac{2}{T}\frac{1 - z^{-1}}{1 + z^{-1}}\,. \tag{1.54}$$

Sometimes it is useful to look at the mapping in terms of z rather than z^{-1}

$$s = \frac{2}{T}\frac{z - 1}{z + 1}\,. \tag{1.55}$$

The Bilinear Transform maps the $j\Omega$ axis from the s plane exactly to the unit circle on the z plane, except for a well-known frequency warping. This warping is a consequence of the compression of the s-plane frequency range $\Omega \in [-\infty, +\infty]$ to the z-plane frequency range $\omega \in [-\pi, +\pi]$. This warping is described by

$$\omega = \frac{2}{T}\tan^{-1}\left(\Omega\frac{T}{2}\right). \tag{1.56}$$

As a conformal mapping, the BLT maps dc on the s plane ($s = 0$) to $z = 1$ on the z plane and the extended s-plane point $s \to \infty$ to the Nyquist location $z = -1$ on the z plane. The mapping of dc can be easily visualized by considering the numerator zero of (1.55) and the mapping of $s = \infty$ to Nyquist can be easily visualize by considering the denominator pole of (1.55). At the same time, the entire left-half plane on the s-plane is mapped to the inside of the unit circle on the z plane, as shown in Figures 1.13a–1.13b. The Bilinear Transform is identical to the trapezoidal rule for integration. Hence the trapezoidal rule reaches the highest order of consistency (consistent with order two) possible for A-stable[4] linear multi-step methods, a criteria known as the "first Dahlquist barrier" [153].

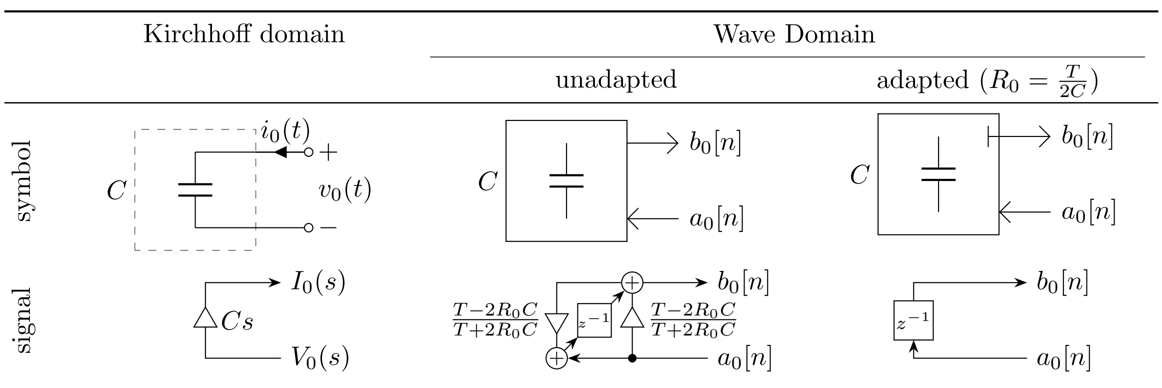


Figure 1.14: Kirchhoff- and Wave-Domain Representations of a Capacitor, Discretized Using the Bilinear Transform.

Applying the BLT to the continuous-time, wave-domain capacitor equation (1.52) and collecting terms

$$H_0(z^{-1}) = \frac{B_0(z^{-1})}{A_0(z^{-1})} = \frac{(T - 2R_0C) + (T + 2R_0C)z^{-1}}{(T + 2R_0C) + (T - 2R_0C)z^{-1}} \tag{1.57}$$

then taking the inverse z-transform and solving for the reflected wave $b_0[n]$ yields the unadapted discrete-time wave-domain equation

$$b_0[n] = -\frac{T - 2R_0C}{T + 2R_0C}b_0[n-1] + \frac{T - 2R_0C}{T + 2R_0C}a_0[n] + a_0[n-1]\,. \tag{1.58}$$

The unadapted wave-domain equation (1.58) is suitable for use at the root of a Wave Digital Filter connection tree. It can be adapted by setting $R_0 = \frac{T}{2C}$, which sets the coefficients $-\frac{T-2R_0C}{T+2R_0C}$ and

[4]not to be confused with "astable"

$\frac{T-2R_0C}{T+2R_0C}$ to zero and yields the adapted wave-domain capacitor equation

$$b_0[n] = a_0[n-1]\,. \tag{1.59}$$

Interestingly, the adapted Wave Digital Filter capacitor is simply a unit delay when it is discretized using the Trapezoidal Rule / BLT.

The symbol and port definition for a capacitor, adapted and unadapted Wave Digital Filter symbols, and adapted and unadapted Wave Digital Filter signal-flow diagrams for the Bilinear Transform discretization are shown in Figure 1.14.

Warped Bilinear Transform

A well-known variation on the Bilinear Transform is the *Warped* Bilinear Transform, which alters the Bilinear Transform $s \to z$ transformation to

$$s = f_{\text{WBLT}}(z^{-1}) = \frac{2}{T'}\frac{1-z^{-1}}{1+z^{-1}}\,. \tag{1.60}$$

where T' is the only term that is changed from the standard Bilinear Transform definition. T' can be chosen so that exactly one continuous time frequency $\Omega_0 \in]0,\pi[$ is mapped perfectly from the analog domain to the digital domain, according to

$$T' = 2\tan\left(\Omega_0 T/2\right)/\Omega_0\,. \tag{1.61}$$

The Warped Bilinear Transform is not discussed commonly in the Wave Digital Filter context; an example of its application is modeling the Hammond organ vibrato/chorus circuit [154].

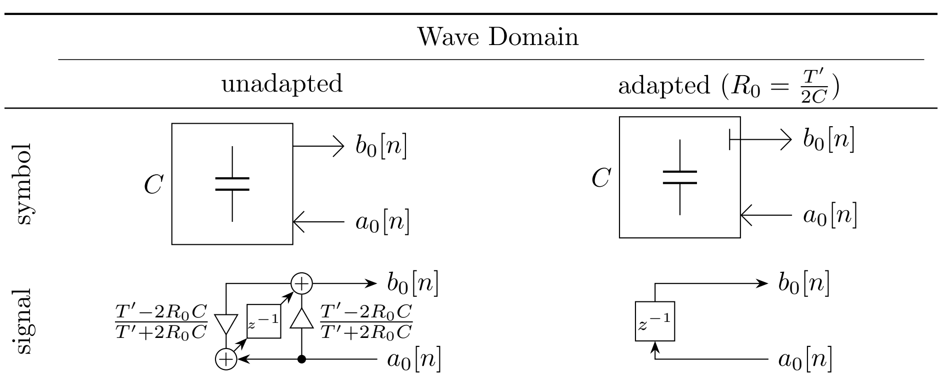


Figure 1.15: Wave-Domain Representations of a Capacitor, Discretized Using the *Warped* Bilinear Transform.

The derivation of the wave-domain z-plane transfer function of an unadapted capacitor is identical to the unwarped case, except that T is replaced by T'

$$H_0(z^{-1}) = \frac{B_0(z^{-1})}{A_0(z^{-1})} = \frac{(T' - 2R_0C) + (T' + 2R_0C)z^{-1}}{(T' + 2R_0C) + (T' - 2R_0C)z^{-1}}\,. \tag{1.62}$$

Again this results in a difference equation

$$b_0[n] = -\frac{T' - 2R_0C}{T' + 2R_0C}b_0[n-1] + \frac{T' - 2R_0C}{T' + 2R_0C}a_0[n] + a_0[n-1] \tag{1.63}$$

which is now adapted by $R_0 = \frac{T'}{2C}$, yielding the adapted difference equation

$$b_0[n] = a_0[n-1]\,. \tag{1.64}$$

Interestingly, the only difference between a wave-domain capacitor discretized with the standard Bilinear Transform and the Warped Bilinear Transform is the port resistance; in both cases the capacitor itself ends up as a simple delay.

The adapted and unadapted Wave Digital Filter symbols and adapted and unadapted Wave Digital Filter signal-flow diagrams for the capacitor discretized using the Warped Bilinear Transform are shown in Figure 1.15.

Backward Euler

Another standard discretization rule is Backwards Euler. Backwards Euler can be represented as an $s \to z$ mapping

$$s = f_{\mathrm{BE}}(z^{-1}) = \frac{1 - z^{-1}}{T}\,. \tag{1.65}$$

Although the Wave Digital Filter approach is usually associated with the Bilinear Transform, there is nothing preventing us from using Backwards Euler to discretize reactive elements. In fact it will sometimes be preferable to the Trapezoidal Rule, as we will see in the case study of Chapter 3.

As a conformal mapping, Backward Euler maps dc on the s plane $z = 1$ and $s \to \infty$ to the point $z = 0$. This is shown in Figure 1.13c. The mapping of dc can again be seen by looking at the transform numerator zero; since there is no explicit denominator pole, the mapping of $s \to \infty$ to $z = 0$ is due to the non-strictly-proper nature of the transform.

Applying Backwards Euler discretization to the continuous-time, wave-domain capacitor equation (1.52) and collecting terms

$$H_0(z^{-1}) = \frac{B_0(z^{-1})}{A_0(z^{-1})} = \frac{(T - R_0C) + (R_0C)\,z^{-1}}{(T + R_0C) + (R_0C)\,z^{-1}} \tag{1.66}$$

then taking the inverse z-transform and solving for the reflected wave $b_0[n]$ yields the unadapted

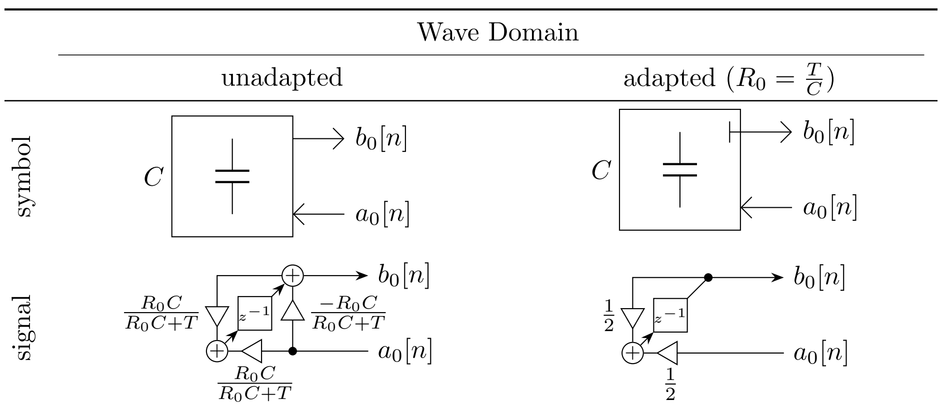


Figure 1.16: Wave-Domain Representations of a Capacitor, Discretized Using Backward Euler.

discrete-time wave domain equation

$$b_0[n] = \frac{R_0C}{T+R_0C} b_0[n-1] + \frac{T-R_0C}{T+R_0C} a_0[n] + \frac{R_0C}{T+R_0C} a_0[n-1]\,. \tag{1.67}$$

This discretization is adapted by $R_0 = T/C$, which yields the adapted difference equation

$$b_0[n] = \frac{1}{2} b_0[n-1] + \frac{1}{2} a_0[n-1]\,. \tag{1.68}$$

Notice that as with the Warped Bilinear Transform, the adaptation criteria is different than for the Bilinear Transform. Unlike with either version of the Bilinear Transform, the difference equation is now a first order IIR filter, albeit one with the special property that the delay-free feedforward gain is constrained to be 0.

The adapted and unadapted Wave Digital Filter symbols and adapted and unadapted Wave Digital Filter signal-flow diagrams for the capacitor discretized using Backward Euler are shown in Figure 1.16.

α Transform

Here we explore another discretization scheme, which we call the “α Transform”:

$$s = f_\alpha(z^{-1}) = \frac{\frac{1+\alpha}{T} - \frac{1+\alpha}{T} z^{-1}}{1+\alpha z^{-1}}\,. \tag{1.69}$$

This transform is parameterized by a parameter α. It was explored in the context of Kirchhoff-domain circuit modeling in [155] and can be considered a compromise between the Bilinear Transform and Backwards Euler. Indeed it includes Backward Euler ($\alpha = 0$), the Bilinear Transform ($\alpha = 1.0$), and even Forward Euler ($\alpha \to \infty$) as special cases. The concept of compromising between Backward

Euler and the Bilinear Transform has been explored in other contexts [156]; the α Transform defined in 1.69 is related by frequency scaling and a transformation of the parameterization to other known discretization schemes, including the "α-Approximation" [157] and the "Al-Alaoui Operator" [158].

As a conformal mapping, the α Transform maps dc on the s plane $z = 1$ and $s \to \infty$ to the point $z = -\alpha$. This is shown in Figure 1.13d [155]. This can be seen as a parameterization of the the transform's denominator pole as well as a frequency scaling by $\frac{1+\alpha}{T}$.

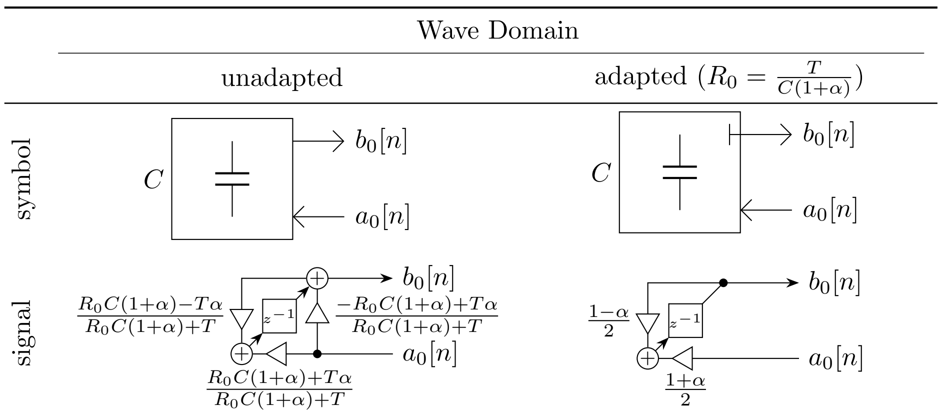


Figure 1.17: Wave-Domain Representations of a Capacitor, Discretized Using the α Transform.

Applying the α-Transform discretization to the continuous-time, wave-domain capacitor equation (1.52) and collecting terms

$$H_0(z^{-1}) = \frac{B_0(z^{-1})}{A_0(z^{-1})} = \frac{(T - R_0C\,(1+\alpha)) + (T\alpha + R_0C\,(1+\alpha))\,z^{-1}}{(T + R_0C\,(1+\alpha)) + (T\alpha + R_0C\,(1-\alpha))\,z^{-1}} \tag{1.70}$$

then taking the inverse z-transform and solving for the reflected wave $b_0[n]$ yields the unadapted discrete-time wave domain equation

$$b_0[n] = \frac{R_0C\,(1+\alpha) - T\alpha}{R_0C\,(1+\alpha) + T} b_0[n-1] + \frac{-R_0C\,(1+\alpha) + T}{R_0C\,(1+\alpha) + T} a_0[n] + \frac{R_0C\,(1+\alpha) + T\alpha}{R_0C\,(1+\alpha) + T} a_0[n-1]\,. \tag{1.71}$$

This discretization is adapted by $R_0 = \frac{T}{C(1+\alpha)}$, which yields the adapted difference equation

$$b_0[n] = \frac{1-\alpha}{2} b_0[n-1] + \frac{1+\alpha}{2} a_0[n-1]\,. \tag{1.72}$$

Again, the difference equation is a first-order IIR filter. The α transform is intended to operate in the range $\alpha \in [0,1]$ (bracketed by Backward Euler and the Bilinear Transform) and to a lesser extent the range $\alpha \in [1,\infty]$ (bracketed by Bilinear Transform and Forward Euler). Values of $\alpha < 0$ may be interesting as well, although they lack an interpretation as a compromise between known discretizations. The only condition that must be respected is that $\alpha \neq -1$; if $\alpha = -1$ were allowed it would correspond to an adaptation criteria of $R_0 = \infty$, which is disallowed for causing a non-invertible wave definition. At the same time it would correspond to a mapping from the entire left half s plane to the single point $z = 1$ on the z plane; a nonsensical situation that would colocate all poles and zeros of the system.

The adapted and unadapted Wave Digital Filter symbols and adapted and unadapted Wave Digital Filter signal-flow diagrams for the capacitor discretized using the α Transform are shown in Figure 1.17.

Möbius Transform

The α Transform contains the three most well-known discretization schemes–Backwards Euler, Bilinear Transform, and Forwards Euler—as special cases. Still, an even more general conformal mapping exists. This mapping is called the Möbius Transform and is defined by [155]

$$s = f_{\mathrm{M}}(z^{-1}) = \frac{a_M + b_M z^{-1}}{c_M + d_M z^{-1}} \,. \tag{1.73}$$

The Möbius Transform is parameterized by four parameters a_M, b_M, c_M, and d_M; it is simply an application of the Möbius Transform from the study of conformal mapping [159] to the problem of discretization.

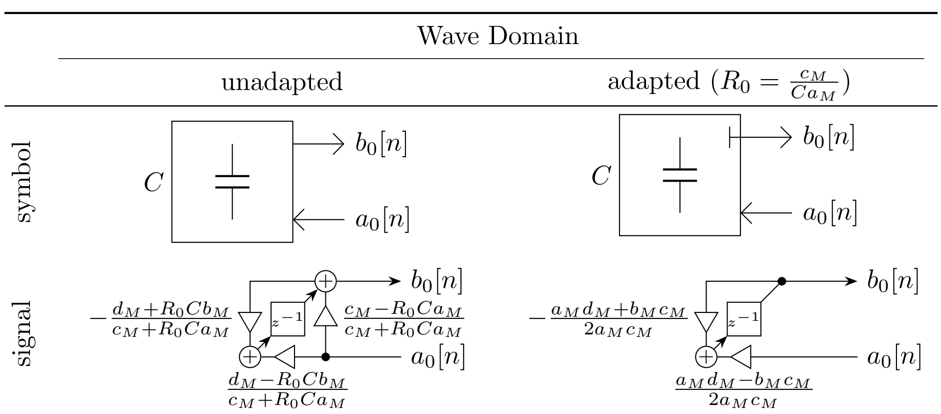


Figure 1.18: Wave-Domain Representations of a Capacitor, Discretized Using the Möbius Transform.

Applying the Möbius Transform discretization to the continuous-time, wave-domain capacitor equation (1.52) and collecting terms

$$H_0(z^{-1}) = \frac{B_0(z^{-1})}{A_0(z^{-1})} = \frac{(c_M - R_0Ca_M) + (d_M - R_0Cb_M)\, z^{-1}}{(c_M + R_0Ca_M) + (d_M + R_0Cb_M)\, z^{-1}} \tag{1.74}$$

then taking the inverse z-transform and solving for the reflected wave $b_0[n]$ yields the unadapted discrete-time wave domain equation

$$b_0[n] = -\frac{d_M + R_0 C b_M}{c_M + R_0 C a_M} b_0[n-1] + \frac{c_M - R_0 C a_M}{c_M + R_0 C a_M} a_0[n] + \frac{d_M - R_0 C b_M}{c_M + R_0 C a_M} a_0[n-1]\,. \tag{1.75}$$

This discretization is adapted by $R_0 = \frac{c_M}{C a_M}$, which yields the adapted difference equation

$$b_0[n] = -\frac{a_M d_M + b_M c_M}{2 a_M c_M} b_0[n-1] + \frac{a_M d_M - b_M c_M}{2 a_M c_M} a_0[n-1]\,. \tag{1.76}$$

This very general result contains all of the previously discussed discretization techniques as special cases as well as the fully class of transforms parameterized by a_M, b_M, c_M, and d_M. This class is constrained by a few special cases. The transform must satisfy $a_M \neq 0$ (to avoid the disallowed adaptation $R_0 = \infty$) and $c_M \neq 0$ (to avoid the disallowed adaptation $R_0 = 0$).

It is fascinating that the condition $a_M \neq 0$ can also be interpreted as a structural restriction that disqualifies all fully explicit discretization methods (such as Forward Euler). Ironically, although the central concept of the Wave Digital Filter approach is choosing port resistances to create fully local, fully explicit building blocks, the chosen discretization method must itself be implicit!

The adapted and unadapted Wave Digital Filter symbols and adapted and unadapted Wave Digital Filter signal-flow diagrams for the capacitor discretized using the Möbius Transform are shown in Figure 1.18.

1.3.2 Inductor

The behavior of a linear inductor with inductance L is modeled by the differential equation

$$v_0(t) = L\frac{\mathrm{d}i_0(t)}{\mathrm{d}t} \tag{1.77}$$

which relates the port voltage $v_0(t)$ to the time derivative of port current $\frac{\mathrm{d}i_0(t)}{\mathrm{d}t}$ by the inductance L. In the Laplace domain, this relationship is

$$V_0(s) = LsI_0(s)\,. \tag{1.78}$$

Plugging the parametric wave definition (1.9) into the Kirchhoff-domain inductor equation (1.78)

$$\frac{1}{2}R_0^{1-\rho}A_0(s) + \frac{1}{2}R_0^{1-\rho}B_0(s) = \frac{1}{2}R_0^{-\rho}LsA_0(s) - \frac{1}{2}R_0^{-\rho}LsB_0(s) \tag{1.79}$$

and solving for the reflected wave $B_0(s)$ yields the Laplace- and wave-domain transfer function

$$H_0(s) = \frac{B_0(s)}{A_0(s)} = \frac{Ls - R_0}{Ls + R_0}\,. \tag{1.80}$$

Bilinear Transform

Just like the capacitor, the inductor's continuous-time transfer function 1.80 is discretized using $s \to z$ mappings. Again we begin with the Bilinear Transform.

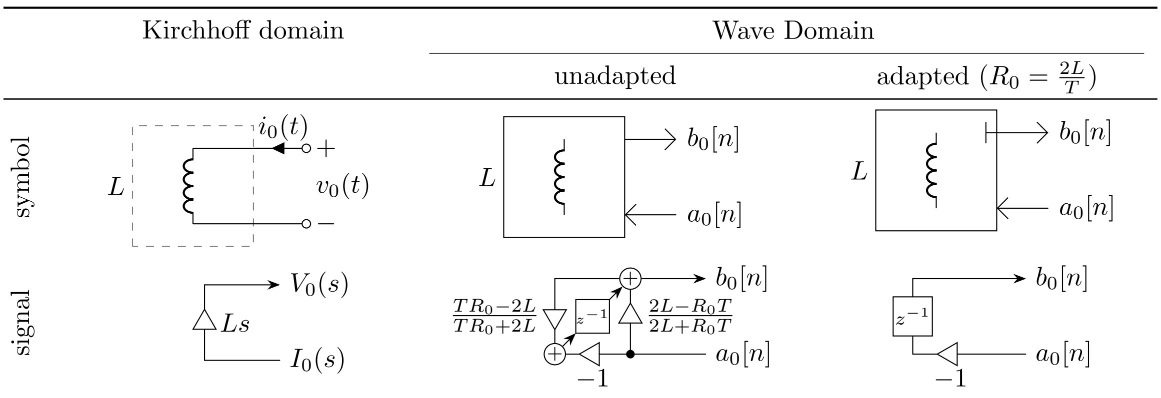


Figure 1.19: Kirchhoff- and Wave-Domain Representations of a Inductor One-Port, Discretized Using the Bilinear Transform.

Applying the BLT to the continuous-time, wave-domain inductor equation (1.80) and collecting terms

$$H_0(z^{-1}) = \frac{B_0(z^{-1})}{A_0(z^{-1})} = \frac{(2L - R_0T) + (-2L - 2R_0T)\, z^{-1}}{(2L + R_0T) + (-2L + R_0T)\, z^{-1}} \tag{1.81}$$

then taking the inverse z-transform and solving for the reflected wave $b_0[n]$ yields the unadapted discrete-time wave-domain equation

$$b_0[n] = \frac{2L - R_0T}{2L + R_0T} b_0[n-1] + \frac{2L - R_0T}{2L + R_0T} a_0[n] - a_0[n-1]\,. \tag{1.82}$$

This discretization is adapted by $R_0 = \frac{2L}{T}$, which yields the adapted difference equation

$$b_0[n] = -a_0[n-1]\,. \tag{1.83}$$

Similar to the capacitor, the adapted Wave Digital Filter inductor is simply a unit delay with a sign inversion when it is discretized using the trapezoidal rule / BLT.

The symbol and port definition for an inductor, adapted and unadapted Wave Digital Filter symbols, and adapted and unadapted Wave Digital Filter signal-flow diagrams for the Bilinear Transform discretization are shown in Figure 1.19.

Warped Bilinear Transform

Again, the Warped Bilinear Transform z-plane transfer function is identical to the unwarped case, except that T is replaced by T'

$$H_0(z^{-1}) = \frac{B_0(z^{-1})}{A_0(z^{-1})} = \frac{(2L - R_0T') + (-2L - 2R_0T')\,z^{-1}}{(2L + R_0T') + (-2L + R_0T')\,z^{-1}}\,. \tag{1.84}$$

Again this results in a difference equation

$$b_0[n] = \frac{2L - R_0T'}{2L + R_0T'} b_0[n-1] + \frac{2L - R_0T'}{2L + R_0T'} a_0[n] - a_0[n-1] \tag{1.85}$$

which is now adapted by $R_0 = \frac{2L}{T'}$, yielding the adapted difference equation

$$b_0[n] = -a_0[n-1]\,. \tag{1.86}$$

Again, only the adaptation is affected by the warping.

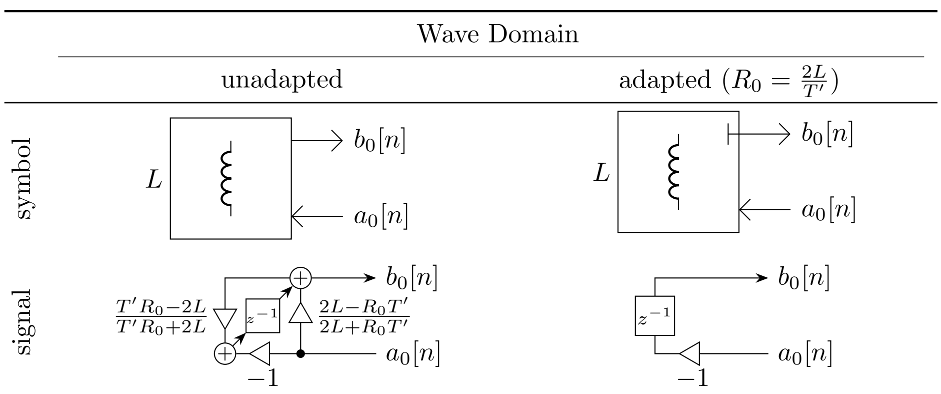


Figure 1.20: Wave-Domain Representations of an Inductor, Discretized Using the *Warped* Bilinear Transform.

The adapted and unadapted Wave Digital Filter symbols and adapted and unadapted Wave Digital Filter signal-flow diagrams for the inductor discretized using the Warped Bilinear Transform are shown in Figure 1.20.

Backwards Euler

Applying Backwards Euler to the continuous-time, wave-domain inductor equation (1.80) and collecting terms

$$H_0(z^{-1}) = \frac{B_0(z^{-1})}{A_0(z^{-1})} = \frac{(L - R_0T) + (-L)\,z^{-1}}{(L + R_0T) + (-L)\,z^{-1}} \tag{1.87}$$

then taking the inverse z-transform and solving for the reflected wave $b_0[n]$ yields the unadapted discrete-time wave domain equation

$$b_0[n] = \frac{L}{L + R_0 T} b_0[n-1] + \frac{L - R_0 T}{L + R_0 T} a_0[n] + \frac{L}{L + R_0 T} a_0[n-1] \,. \tag{1.88}$$

This discretization is adapted by $R_0 = \frac{L}{T}$, which yields that adapted difference equation

$$b_0[n] = \frac{1}{2} b_0[n-1] - \frac{1}{2} a_0[n-1] \,. \tag{1.89}$$

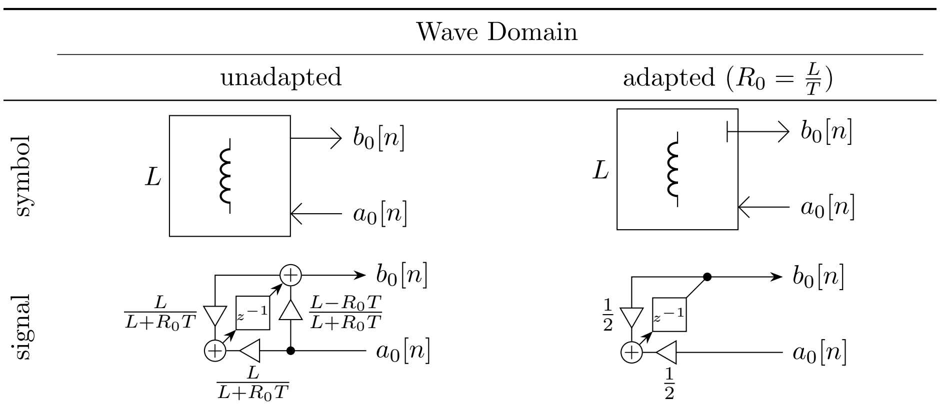


Figure 1.21: Wave-Domain Representations of an Inductor, Discretized Using Backward Euler.

The adapted and unadapted Wave Digital Filter symbols and adapted and unadapted Wave Digital Filter signal-flow diagrams for the inductor discretized using Backward Euler are shown in Figure 1.21.

α Transform

Applying the α Transform discretization to the continuous-time, wave-domain inductor equation (1.80) and collecting terms

$$H_0(z^{-1}) = \frac{B_0(z^{-1})}{A_0(z^{-1})} = \frac{(L\,(1+\alpha) - R_0 T) + (-L\,(1+\alpha) - R_0 T\alpha)\, z^{-1}}{(L\,(1+\alpha) + R_0 T) + (-L\,(1+\alpha) + R_0 T\alpha)\, z^{-1}} \tag{1.90}$$

then taking the inverse z-transform and solving for the reflected wave $b_0[n]$ yields the unadapted discrete-time wave domain equation

$$b_0[n] = \frac{L\,(1+\alpha) - R_0 T\alpha}{L\,(1+\alpha) + R_0 T} b_0[n-1] + \frac{L\,(1+\alpha) - R_0 T}{L\,(1+\alpha) + R_0 T} a_0[n] + \frac{L\,(1+\alpha) + R_0 T\alpha}{L\,(1+\alpha) + R_0 T} a_0[n-1] \,. \tag{1.91}$$

This discretization is adapted by $R_0 = \frac{L(1+\alpha)}{T}$, which yields the adapted difference equation

$$b_0[n] = \frac{1-\alpha}{2} b_0[n-1] - \frac{1+\alpha}{2} a_0[n-1]\,. \tag{1.92}$$

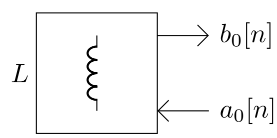
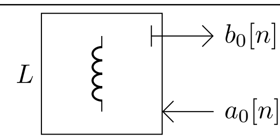
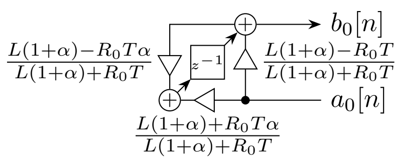
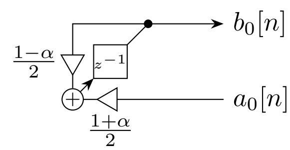
	Wave Domain	
	unadapted	adapted ($R_0 = \frac{L(1+\alpha)}{T}$)
symbol		
signal		

Figure 1.22: Wave-Domain Representations of an Inductor, Discretized Using the α Transform.

The adapted and unadapted Wave Digital Filter symbols and adapted and unadapted Wave Digital Filter signal-flow diagrams for the inductor discretized using the α Transform are shown in Figure 1.22.

Möbius Transform

Applying the Möbius Transform discretization to the continuous-time, wave-domain inductor equation (1.80) and collecting terms

$$H_0(z^{-1}) = \frac{B_0(z^{-1})}{A_0(z^{-1})} = \frac{(La_M - R_0c_M) + (Lb_M - R_0d_M)\,z^{-1}}{(La_M + R_0c_M) + (Lb_M + R_0d_M)\,z^{-1}} \tag{1.93}$$

then taking the inverse z-transform and solving for the reflected wave $b_0[n]$ yields the unadapted discrete-time wave domain equation

$$b_0[n] = -\frac{Lb_M + R_0d_M}{La_M + R_0c_M} b_0[n-1] + \frac{La_M - R_0c_M}{La_M + R_0c_M} a_0[n] + \frac{Lb_M - R_0d_M}{La_M + R_0c_M} a_0[n-1]\,. \tag{1.94}$$

This discretization is adapted by $R_0 = \frac{La_M}{c_M}$, which yields the adapted difference equation

$$b_0[n] = -\frac{b_Mc_M + a_Md_M}{2a_Mc_M} b_0[n-1] + \frac{b_Mc_M - a_Md_M}{2a_Mc_M} a_0[n-1]\,. \tag{1.95}$$

This time, the transform must satisfy $a_M \neq 0$ (to avoid the disallowed adaptation $R_0 = 0$) and $c_M \neq 0$ (to avoid the disallowed adaptation $R_0 = \infty$). Again, the condition $a_M \neq 0$ can also be interpreted as a structural restriction that disqualifies all fully explicit discretization methods (such as Forward Euler).

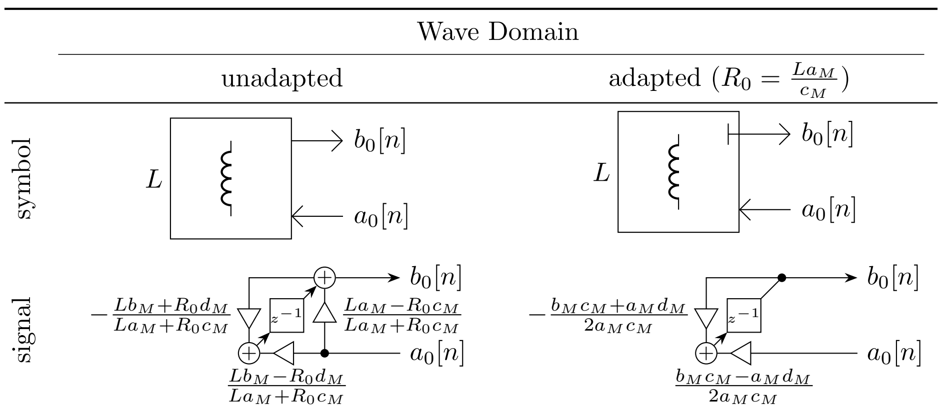


Figure 1.23: Wave-Domain Representations of an Inductor, Discretized Using the Möbius Transform.

The adapted and unadapted Wave Digital Filter symbols and adapted and unadapted Wave Digital Filter signal-flow diagrams for the inductor discretized using the Möbius are shown in Figure 1.23.

1.3.3 Choosing a Discretization Method

The choice of discretization method is an important one, and we've explored a number of options for Wave Digital Filter discretization of capacitors and inductors. These include Backwards Euler, the Bilinear Transform, the Warped Bilinear Transform, the α Transform, and the Möbius Transform.

The Bilinear Transform and the Warped Bilinear Transform are the most commonly used discretization methods. Indeed they are order-preserving, stability-preserving, and the Bilinear Transform is 2nd-order accurate. When a frequency response has an important feature, the warping control of the Warped Bilinear Transform is a useful way to map the frequency of that one feature exactly.

However, in nonlinear simulations the Bilinear Transform is known to suffer from spurious high-frequency oscillations [146, 160, 161, 162, 155]. These relate to the motion of instantaneous highly-damped poles, which "clump up" near the image of $s \to \infty$ on the z plane. Since that image is at Nyquist for the Bilinear Transform, it is unsurprising that it could create high-frequency oscillations.

Using Backwards Euler can often alleviate this issue—its image of $s \to \infty$ is at $z = 0$, the most damped possible location. Using the α Transform, it is possible to compromise between these two extremes, possibly with the goal of satisfying an instantaneous pole damping monotonicity condition [155]. In Chapter 3 we'll show a case study where this degree of control becomes important. The Möbius Transform is an even wider class of transforms that encompasses all these approaches and others, e.g., the recently-proposed application of the Fractional Bilinear Transform [163] (with the fractional delay realized with first-order Lagrange interpolation) and the AL-Alaoui operator to Wave Digital Filter modeling [164, 165]. Although not yet widely in use, the Möbius Transform has a high potential. In fact, since linear multistep methods above order 2 cannot be A-stable (a concept known as the "second Dahlquist Barrier" [153]), the Möbius transform is likely the most broad generalization of the highest order linear multistep methods that can achieve guaranteed passivity, an important Wave Digital Filter concept. Although we won't demonstrate it here, it is quite possible to use higher order linear multistep methods, such as the family of Gear's methods (also known as Backward Differentiation Formulas [166]), to discretize capacitors and inductors in the Wave Digital Filter context [167, 160, 168]. However, due to the Dahlquist barrier it appears they cannot be guaranteed passive. For this reason, discretizations that mesh philosophically with the Wave Digital Filter approach will likely all be special cases of the Möbius Transform approach.

Here, we've shown that although discretization schemes besides the Bilinear Transform are not often considered in the Wave Digital Filter context, all these transforms are viable. Another contribution of this section is the discovery of the fascinating condition that Wave Digital Filter discretization schemes, by constructions, must be implicit—fully explicit schemes like Forward Euler ruin the chance for adaptation and hence cannot be used.

1.4 Two-Ports

More complicated than one-ports are *two*-ports. Luckily, they are usually just algebraic (not reactive), so issues of discretization can be set aside.[5]

Typically, circuit two-ports are described in the Kirchhoff domain by two equations. Certain standard definitions based on amplifier y-, z-, h-, and g-parameters are often used, and indeed we will derive Wave Digital Filter adaptors based on those models later in this Section. However, for the purposes of derivation, we will simply use linear algebra on a very general Kirchhoff-domain matrix equation to solve for the scattering parameters of each two-port. This general Kirchhoff-domain matrix expression is

$$\begin{bmatrix} x_{00} & x_{01} \\ x_{10} & x_{11} \end{bmatrix} \begin{bmatrix} v_0 \\ v_1 \end{bmatrix} + \begin{bmatrix} y_{00} & y_{01} \\ y_{10} & y_{11} \end{bmatrix} \begin{bmatrix} i_0 \\ i_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tag{1.96}$$

or, in vector–matrix form

$$\mathbf{X}\boldsymbol{v} + \mathbf{Y}\boldsymbol{i} = \mathbf{0}. \tag{1.97}$$

[5]A two-port known as the Generalized Immittance Converter (GIC) [148, p. 37] which may involve embedded reactances is indeed involved in one thread of Wave Digital Filter research (e.g., [169, 170, 171, 165]), but is not important to audio circuitry and will not be considered further here.

Plugging the parametric wave definition (1.9) into this equation

$$\mathbf{X}\left(\frac{1}{2}\mathbf{R}^{1-\rho}\boldsymbol{a}+\frac{1}{2}\mathbf{R}^{1-\rho}\boldsymbol{b}\right)+\mathbf{Y}\left(\frac{1}{2}\mathbf{R}^{-\rho}\boldsymbol{a}-\frac{1}{2}\mathbf{R}^{-\rho}\boldsymbol{b}\right)=\mathbf{0} \tag{1.98}$$

and solving for $\boldsymbol{b}$ yields the unadapted wave-domain equation

$$\boldsymbol{b}=-\left(\mathbf{X}\mathbf{R}^{1-\rho}-\mathbf{Y}\mathbf{R}^{-\rho}\right)^{-1}\left(\mathbf{X}\mathbf{R}^{1-\rho}+\mathbf{Y}\mathbf{R}^{-\rho}\right)\boldsymbol{a} \tag{1.99}$$

which means the scattering matrix is defined

$$\mathbf{S}=-\left(\mathbf{X}\mathbf{R}^{1-\rho}-\mathbf{Y}\mathbf{R}^{-\rho}\right)^{-1}\left(\mathbf{X}\mathbf{R}^{1-\rho}+\mathbf{Y}\mathbf{R}^{-\rho}\right)\,. \tag{1.100}$$

In the rest of this section, we will discuss important electrical two-ports, including parallel adaptors, series adaptors (inverters), series and parallel adaptors containing absorbed sources, transformers, controlled sources (voltage-controlled voltage source, voltage-controlled current source, current-controlled voltage source, and current-controlled current source), ideal voltage, current, and power converters, unit elements, quasi-reciprocal lines (QUARLs), gyrators, dualizers, mutators, nullors, and y-, z-, h-, and h-parameter models.

1.4.1 Simple Two-Port Connections

The simplest two-ports are connections between ports. There are two kinds of connections, series and parallel. In the Wave Digital Filter context, a standard connection is a two-port parallel connection; hence the two-port series connection is also Wave Digital Filter *inverter*. We will find at the end of these derivations that the scattering matrix of an adapted parallel connection is simply an identity matrix—for this reason two-port parallel adaptors are almost never even notated, but rather just implied. Two-port series adaptors should be notated, since their adapted scattering matrix involves sign inversions.

Two-Port Parallel Adaptor

When two ports are in parallel, their voltages are equal and their currents are opposite

$$v_0 = v_1 \tag{1.101}$$

$$i_0 = -i_1\,. \tag{1.102}$$

The symbol and port definition for a two-port parallel adaptor, adapted and unadapted Wave Digital Filter symbols, and adapted and unadapted Wave Digital Filter signal-flow diagrams are shown in Figure 1.24.

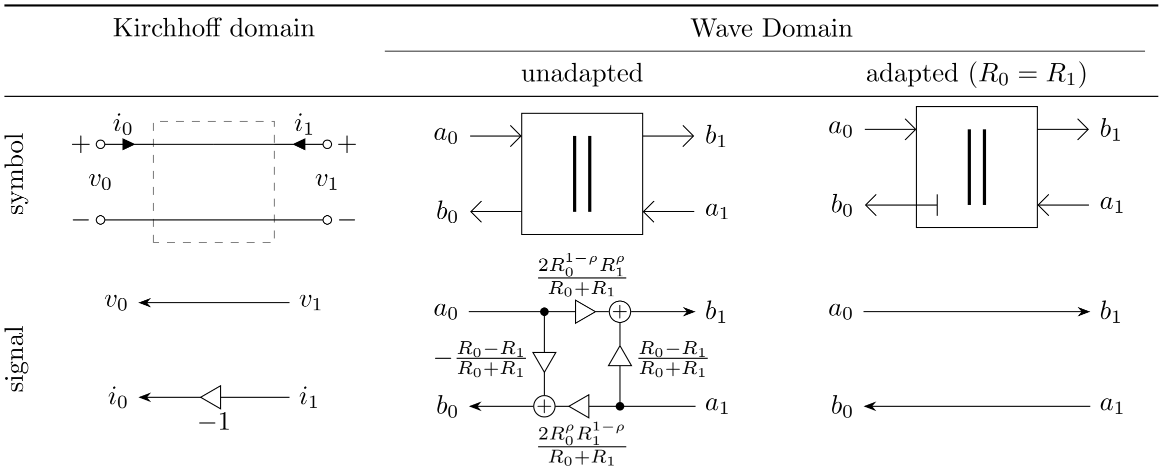


Figure 1.24: Kirchhoff- and Wave-Domain Representations of a Two-Port Parallel Connection.

Plugging the parametric wave definition (1.9) into the Kirchhoff-domain parallel adaptor equations (1.101)–(1.102) and solving for the reflected waves $\begin{bmatrix} b_0 & b_1 \end{bmatrix}^\top$ yields the unadapted wave-domain equation

$$\begin{bmatrix} b_0 \\ b_1 \end{bmatrix} = \begin{bmatrix} -\frac{R_0-R_1}{R_0+R_1} & \frac{2R_0^{\rho}R_1^{1-\rho}}{R_0+R_1} \\ \frac{2R_0^{1-\rho}R_1^{\rho}}{R_0+R_1} & \frac{R_0-R_1}{R_0+R_1} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} . \tag{1.103}$$

To adapt port 0 of the adaptor, we must find the value of R_0 that sets $-\frac{R_0-R_1}{R_0+R_1} = 0$. This is satisfied by $R_0 = R_1$ and produces the adapted wave-domain equation

$$\begin{bmatrix} b_0 \\ b_1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} . \tag{1.104}$$

This is identical to the criteria for adapting port 1.

The adapted two-port parallel connection is special because it is just a connection between ports. So, usually two-port parallel connections are not even drawn in an adaptor structure. Still, you may occasionally find a reason to use the two-port parallel adaptor as the root of a Wave Digital Filter tree.

Two-Port Series Adaptor / Inverter

The derivation of the two-port series adaptor proceeds in a very similar fashion to the two-port parallel adaptor. When two ports are in series, their voltages are opposite and their currents are equal

$$v_0 = -v_1 \tag{1.105}$$

$$i_0 = i_1\,. \tag{1.106}$$

The symbol and port definition for a two-port series adaptor, adapted and unadapted Wave Digital Filter symbols, and adapted and unadapted Wave Digital Filter signal-flow diagrams are shown in Figure 1.25.

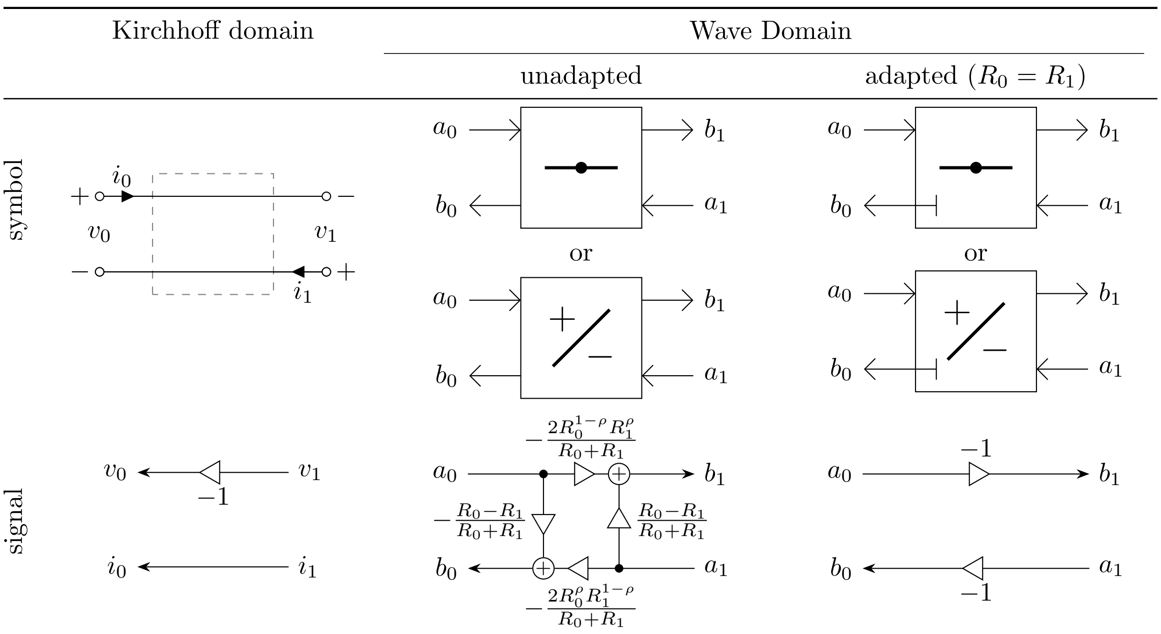


Figure 1.25: Kirchhoff- and Wave-Domain Representations of a Two-Port Series Connection (Inverter).

Plugging the parametric wave definition (1.9) into the Kirchhoff-domain series adaptor equation (1.105)–(1.106) and solving for the reflected waves $\begin{bmatrix} b_0 & b_1 \end{bmatrix}^\top$ yields the unadapted wave-domain equation

$$\begin{bmatrix} b_0 \\ b_1 \end{bmatrix} = \begin{bmatrix} -\frac{R_0-R_1}{R_0+R_1} & -\frac{2R_0^{\rho}R_1^{1-\rho}}{R_0+R_1} \\ -\frac{2R_0^{1-\rho}R_1^{\rho}}{R_0+R_1} & \frac{R_0-R_1}{R_0+R_1} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix}. \tag{1.107}$$

To adapt port 0 of the series adaptor, we must find the value of R_0 that sets $-\frac{R_0-R_1}{R_0+R_1} = 0$. This is satisfied by $R_0 = R_1$ and produces the adapted wave-domain equation

$$\begin{bmatrix} b_0 \\ b_1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix}. \tag{1.108}$$

Again this is identical to the criteria for adapting port 1.

Notice that the two-port series adaptor is very similar to the two-port parallel adaptor, differing only by a sign inversion on the antidiagonal scattering matrix entries. In fact series adaptors are the dual of parallel adaptors and can be realized through certain transpositional properties by a series adaptor and sign inversions (and vice versa) [172]. Although the two-port series adaptor may appear as a root element, its normal use is as a *polarity inverter* [173]. Many audio circuits have a single input and a single output, and are ac-coupled; for these circuits getting the polarity of ports correct is often not important and at worst could case a sign flip in the output signal. However, in more complicated circuits, especially those involving nonlinearities, incorrect polarities can be disasterous and completely change the behavior of a circuit [174, 175, 173, 176]. The systematic use of the two-port series adaptor / inverter will be illustrated later in the case study of this Chapter.

1.4.2 Two-Port Series/Parallel Adaptors Containing Sources

Here we consider two-port series adaptors with absorbed ideal current sources and two-port parallel adaptors with absorbed ideal voltage sources [6]. These two-ports are essential for creating realizable Wave Digital Filter trees out of reference circuits with multiple sources without altering the reference circuit by adding extra resistances. This hints at broader issues of realizability due to multiple nonadaptable linear electrical one-ports. An alternate and more comprehensive strategy for handling multiple nonadaptable linear elements will be presented at the end of Chapter 2.

Two-Port Parallel Adaptor + Ideal Current Source

When two ports are in parallel with an ideal current source j, the two port current and the source current sum to zero and the voltages currents are equal

$$v_0 = v_1 \tag{1.109}$$

$$i_0 + i_1 + j = 0\,. \tag{1.110}$$

The symbol and port definition for a two-port parallel adaptor with an absorbed current source, adapted and unadapted Wave Digital Filter symbols, and adapted and unadapted Wave Digital Filter signal-flow diagrams are shown in Figure 1.26.

Plugging the parametric wave definition (1.9) into the Kirchhoff-domain equations (1.109)–(1.110) and solving for the reflected waves $\begin{bmatrix} b_0 & b_1 \end{bmatrix}^\top$ yields the unadapted wave-domain equation

$$\begin{bmatrix} b_0 \\ b_1 \end{bmatrix} = \begin{bmatrix} -\frac{R_0-R_1}{R_0+R_1} & \frac{2R_0^{\rho}R_1^{1-\rho}}{R_0+R_1} \\ \frac{2R_0^{1-\rho}R_1^{\rho}}{R_0+R_1} & \frac{R_0-R_1}{R_0+R_1} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} + \begin{bmatrix} 2\frac{R_0^{\rho}R_1}{R_0+R_1} \\ 2\frac{R_0R_1^{\rho}}{R_0+R_1} \end{bmatrix} j\,. \tag{1.111}$$

Exactly like in the normal two-port parallel adaptor, to adapt port 0 of the adaptor, we must find

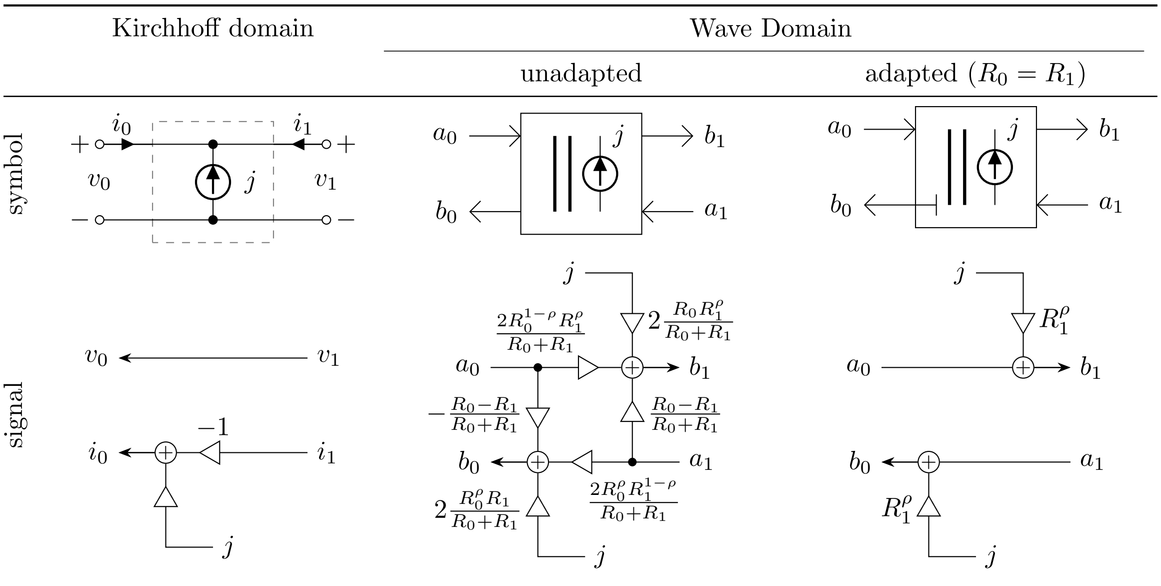
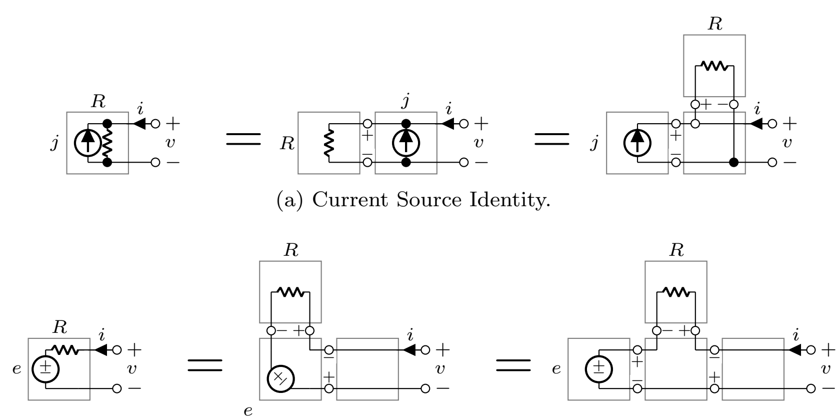


Figure 1.26: Kirchhoff- and Wave-Domain Representations of a Two-Port Parallel Connection Including an Ideal Current Source.

the value of R_0 that sets $-\frac{R_0-R_1}{R_0+R_1} = 0$. This is satisfied by $R_0 = R_1$ and produces the adapted wave-domain equation

$$\begin{bmatrix} b_0 \\ b_1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} + \begin{bmatrix} R_1^\rho \\ R_1^\rho \end{bmatrix} j\,. \tag{1.112}$$

Again this is identical to the criteria for adapting port 1.



(a) Current Source Identity.

(b) Voltage Source Identity.

Figure 1.27: Current and Voltage Source Identities.

Arguably, as shown in Figure 1.27a, we can view resistive current sources as a combination of a resistor with a parallel adaptor with an absorbed current source.

Two-Port Series Adaptor + Ideal Voltage Source

When two ports are in series with an ideal voltage source e, the two port voltages and the source voltage sum to zero and the port currents are equal

$$v_0 + v_1 + e = 0 \tag{1.113}$$
$$i_0 = i_1\,. \tag{1.114}$$

Take note that we diverge slightly from how this adaptor is defined in [6], arguing that their port polarity actually reflects a parallel connection. The symbol and port definition for a two-port series adaptor with absorbed voltage source, adapted and unadapted Wave Digital Filter symbols, and adapted and unadapted Wave Digital Filter signal-flow diagrams are shown in Figure 1.28.

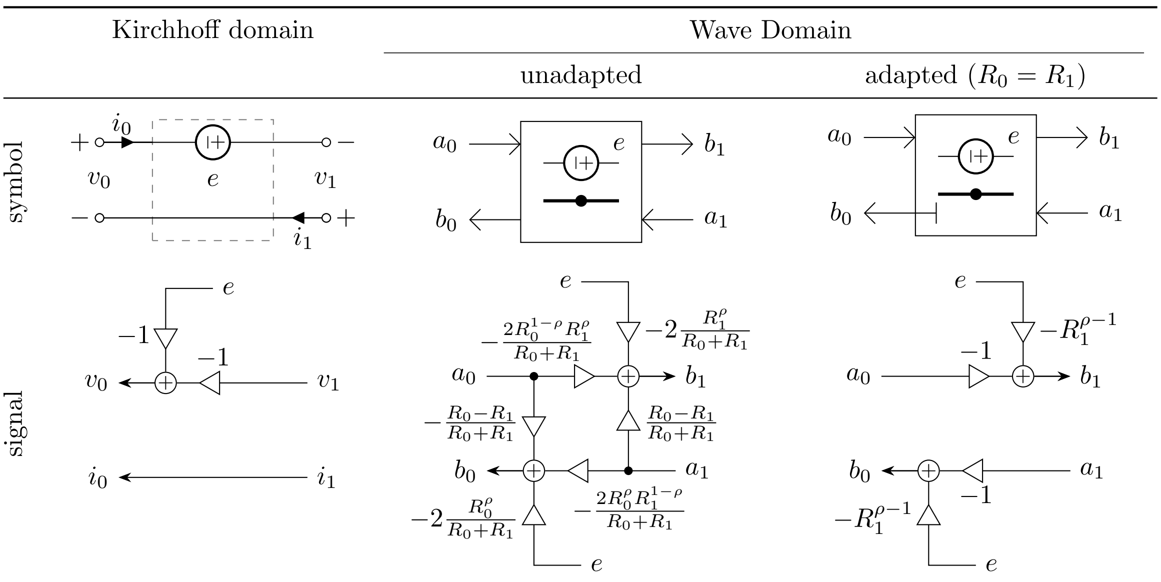


Figure 1.28: Kirchhoff- and Wave-Domain Representations of a Two-Port Series Connection Including an Ideal Voltage Source.

Plugging the parametric wave definition (1.9) into the Kirchhoff-domain parallel adaptor equations (1.113)–(1.114) and solving for the reflected waves $\begin{bmatrix} b_0 & b_1 \end{bmatrix}^\top$ yields the unadapted wave-domain equation

$$\begin{bmatrix} b_0 \\ b_1 \end{bmatrix} = \begin{bmatrix} -\frac{R_0-R_1}{R_0+R_1} & -\frac{2R_0^\rho R_1^{1-\rho}}{R_0+R_1} \\ -\frac{2R_0^{1-\rho}R_1^\rho}{R_0+R_1} & \frac{R_0-R_1}{R_0+R_1} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} + \begin{bmatrix} -2\frac{R_0^\rho}{R_0+R_1} \\ -2\frac{R_1^\rho}{R_0+R_1} \end{bmatrix} e\,. \tag{1.115}$$

Exactly like in the normal two-port series adaptor, to adapt port 0 of the adaptor, we must find the value of R_0 that sets $-\frac{R_0-R_1}{R_0+R_1}=0$. This is satisfied by $R_0=R_1$ and produces the adapted wave-domain equation

$$\begin{bmatrix} b_0 \\ b_1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} + \begin{bmatrix} -R_1^{\rho-1} \\ -R_1^{\rho-1} \end{bmatrix} e\,. \tag{1.116}$$

Again this is identical to the criteria for adapting port 1.

Arguably, as shown in Figure 1.27b, we can view resistive voltage sources as a combination of a resistor with a series adaptor with an absorbed voltage source, and an inverter.

1.4.3 Ideal Transformer

We define the ideal transformer (sometimes more specifically called the "ideal real transformer" [148]) as the linear two-port device that relates the port voltages and currents at two ports according to

$$v_0 = n\,v_1 \tag{1.117}$$

$$i_1 = -n\,i_0\,, \tag{1.118}$$

where n is the "turns ratio," the ratio between windings on the primary and secondary coils. This definition uses sign conventions that conform with the standard port definition (currents pointing inwards on the positive terminals). Sometimes a cascade definition is used [177], which emphasizes how a transformer transforms an impedance that it is attached to.

$$v_0 = n\,v_1 \tag{1.119}$$

$$i_1 = n\,i_0\,. \tag{1.120}$$

In this work, we will always use the first transformer definition (1.117)–(1.118). To use this derivation for a transformer using the sign conventions of (1.119)–(1.120), it suffices to put our derived Wave Digital Filter transformer in series with a Wave Digital Filter inverter, i.e., a two-port series adaptor §1.4.1. The symbol and port definition for an ideal transformer, adapted and unadapted Wave Digital Filter symbols, and adapted and unadapted Wave Digital Filter signal-flow diagrams are shown in Figure 1.29.

Plugging the parametric wave definition (1.9) into the Kirchhoff-domain transformer equations (1.117)–(1.118) and solving for the reflected waves $\begin{bmatrix} b_0 & b_1 \end{bmatrix}^\top$ yields the unadapted wave-domain equation

$$\begin{bmatrix} b_0 \\ b_1 \end{bmatrix} = \begin{bmatrix} -\frac{R_0-n^2R_1}{R_0+n^2R_1} & \frac{2nR_0^{\rho}R_1^{1-\rho}}{R_0+n^2R_1} \\ \frac{2nR_0^{1-\rho}R_1^{\rho}}{R_0+n^2R_1} & \frac{R_0-n^2R_1}{R_0+n^2R_1} \end{bmatrix} = \begin{bmatrix} a_0 \\ a_1 \end{bmatrix}\,. \tag{1.121}$$

To adapt port 0 of the transformer adaptor, we must find the value of R_0 that sets $-\frac{R_0+n^2R_1}{R_0+n^2R_1}=0$.