Basic Concepts of Interval Digital Signal Processing

Roque M. P. Trindade, Benjamín R. C. Bedregal, and Adrião D. Dória-Neto

***Abstract*—Interval mathematics has been used in Signal Processing as a tool for representing uncertainties that arise from finite numeric representation, limited precision sensors and the quantization process. In some control systems, such as soft computing or forecast systems, the uncertainties are a consequence of variable instability, signal variance, or the safety rate of some actuators. Papers with specific applications in this area have been published, but few have dealt with the theoretical foundation of interval mathematics applied to signal processing. This essay is a starting point on interval mathematics in the foundation of signal processing. It is an analytical approach for dealing with interval linear systems with an application perspective in signal processing. Interval linear systems will be used as mathematical models in real systems representation, where the intervals represent the uncertainty of the system. In this approach only linear and time invariant systems with single input and single output (SISO) are used. For this purpose the classical basic properties of real linear systems will be extended. These properties are: causality, stability, additivity and homogeneity. Finally, an interval convolution definition is proposed to represent uncertainty systems and signals more accurately; it is also an important tool for digital signal processing.

***Keywords*—Interval Linear Systems, Digital Signal Processing, Interval Mathematics.

I. Introduction

Nowadays the users of signal processing encounter problems in representing real hardware or software-dedicated systems in signal processing . Among these problems is the treatment of uncertainty systems. This can be inherent to the signal, to sensor bounds, the elected mathematic model, hardware bound implementation or due to lack of precision in some DSP operations. The use of interval mathematics in digital signal processing is proposed to solve these problems. Due to the complexity of the signal area, this work only deals with the linear system. Many signal processing methods are based on the divide-and-conquer principle when the superposition principle is used. This method is very efficient because it breaks a complex problem into many small ones. The **superposition** principle can be used only with linear systems. It offers many signal processing techniques. Linear systems provide the divide-and-conquer technique for the signal processing area. This can be seen in the parallel implementation of small systems to form a large and more complex system. It also occurs with cascade implementation.

The aforementioned information shows that interval methods are very important for the signal processing area because the results of interval algorithms contain a set of infinite-precision real numbers. The results preserve data uncertainties and computational errors. Other examples of signal processing applications are linear prediction, echo cancel, identification systems and channel equalization.

Moore in [6] proposed intervals to control errors caused by operations with finite representations of real numbers. Intervals can offer several interpretations, depending on the semantic field in which they are analyzed. In this work, the semantics used by Santiago et al. [9] will be considered, where intervals and interval functions are seen as representations of real numbers and real functions respectively.

Signal uncertainty is present in several applications such as space exploration, commercial applications, medicine, telecommunications, military operations, electric circuits, industry and scientific research. Thus, in recent years intervals have been used to model this type of uncertainty in signals. On the other hand, considering that linear systems play a fundamental role in signal processing, a well founded interval extension of linear system and properties, such as causality, linearity, invariance in time, stability, etc, would be needed for signal processing. However, studies lack mathematical foundations; particularly, for not presenting an interval extension of linear systems for signal processing.

II. Interval Mathematics

The foundations of interval mathematics in this work can be found in [6].

Let $\mathbb{IR}$ be the set of closed intervals with real end points. If $X,\ Y \in \mathbb{IR}$ then for each $\Box \in \{+, -, :, \times\}$, $X \Box Y = \{a \Box b \in \mathbb{R} \mid a \in X \wedge b \in Y\}$ with $0 \notin Y$ in the division case[1]. The main characteristic of the interval arithmetic defined by Moore is the property of monotonic inclusion, which guarantees that if $X\ \ and\ \ Y \in \mathbb{IR},\ \ x \in X$ and $y \in Y$ then $x \Box y \in X \Box Y$ ($0 \notin Y$ in the division case).

Two projections are associated to each interval, π_1 and π_2 defined by $\pi_1([a,b]) = a$ and $\pi_2([a,b]) = b$. For notational simplicity $\underline{X}$ is used to represent $\pi_1(X)$ and $\overline{X}$ to represent $\pi_2(X)$. Let $X : \mathbb{Z} \rightarrow \mathbb{IR}$, be a semi-interval sequence, the **lower limit** of $X[n]$ is the semi-interval function $\underline{X}[n] : \mathbb{IR} \rightarrow \mathbb{R}$, where $\underline{X}[n] = \pi_1(X[n]))$ and the **upper limit** of $X[n]$ is the semi-interval function $\overline{X}[n] : \mathbb{IR} \rightarrow \mathbb{R}$, where $\overline{X}[n] = \pi_2(X[n])$.

Let $X = [r, s]$ and $Y = [t, u]$, then X is **less than or equal to** Y, denoted by $X \preceq Y$, if $r \leq t$ and $s \leq u$.

An interval X is called **positive** if $\underline{X} > 0$. And **negative** if $\overline{X} < 0$.

[1]Division by 0 is not defined in Interval Arithmetic

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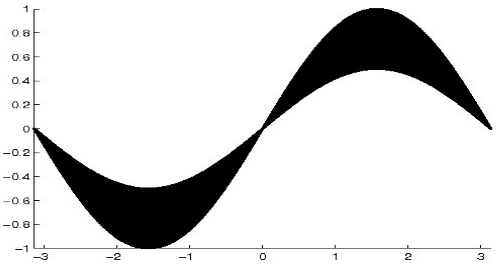

Fig. 1. Graphic representation of semi-interval function $[0.5, 1] \sin(x)$ for $-\pi \leq x \leq \pi$

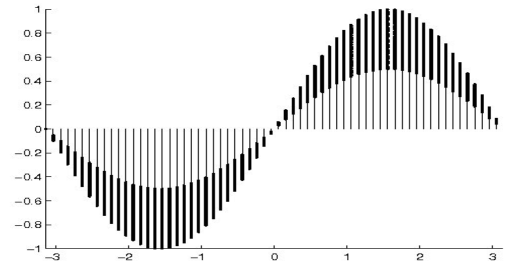

Fig. 2. An example of discretization using interval sequence on the semi-interval function $[0.5, 1] \sin(x)$ for $-\pi \leq x \leq \pi$

The arithmetic interval does not have a distributive property. This is a problem for constructing a strong theory for an interval linear system.

The arithmetic interval has **pseudo-distributivity** as shown in: Let be A, B an $C \in \mathbb{IR}$ then: $A(B + C) \subseteq AB + AC$.

Proposition 1: An interval A is the **infimum** of a set of intervals M w.r.t. $\preceq$, if $A = [\inf\{\underline{X} : X \in M\}, \inf\{\overline{X} : X \in M\}]$.

Proof: Straightforward from infimum definition of posets and classical properties for the infimum of lower bounded sets of real numbers.

Note that, analogous to the real case, all lower bounded intervals have infimum.

A. Interval Sequences

This section is presented because discrete-time signals are represented mathematically as number sequences [8].Thus, to construct a mathematical foundation of Interval Digital Signal Processing an interval version of sequences is required.

Definition 1 (Discrete Interval Sequence): A Discrete Interval Sequence is an application $X : \mathbb{Z} \rightarrow \mathbb{IR}$, represented by $\{X[n]\}$, where the n^{th} term is denoted by $X[n]^2$.
To simplify the notation, sequence $\{X[n]\}$ is referred to as $X[n]$. A practical discrete interval sequence can often arise from periodic sampling of an interval analog signal. In this case the numeric value of the n^{th} term of the sequence is equal to the value of the interval analog signal $X_a[t]$ at time nT; i.e.

$X[n] = X_a(nT)$, where $-\infty < n < \infty$. T is the **sampling period**; its reciprocal is the **sampling frequency**.

In this work all interval sequences are **semi-interval sequences**.

An example of signal discretization using an interval sequence can be seen in Figure 2, which represents an approximation of the semi-interval function $A \sin(x)$ when A is the interval $[0.5, 1]$ and $-\pi \leq x \leq \pi$ shown in Figure 1 by a discrete interval sequence.

1) Discrete interval sequence operations: In the treatment of the discrete-time signal, sequences are manipulated in several basic ways. To extend the discrete-time signal to its interval version, the discrete sequence operations must be extended to the interval version. In many cases this can be done by simply changing the real operations for interval operations.

[2][ ] is used to denote functions whose independent variable is an integer value and ( ) to denote the functions that have an independent variable in continuous space.

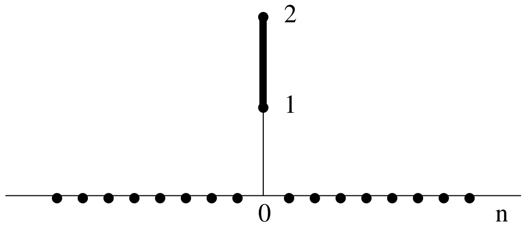

Fig. 3. The graphic representation of a product of sequences $\delta_i[n]$ and $X[n] = [1, 2] \; \forall \; n \in \mathbb{Z}$.

The product and sum of two interval sequences $X[n]$ and $Y[n]$ are defined as the sample-by-sample product and sum, respectively. The multiplication of an interval sequence $X[n]$ by an interval constant C is defined as the interval multiplication of each interval sample by C. In the case of multiplication by a real number c, it is considered a degenerate interval $[c, c]$ and becomes peculiar to the case of multiplication by the interval constant.

An Interval sequence $Y[n]$ is said to be a **delayed or shifted** version of a sequence $X[n]$ if $Y[n]$ with values $Y[n] = X[n-n_0]$, where n_0 is an integer.

Definition 2: Let be $X_1[i]$ and $X_2[k]$ two discrete interval sequences. $X_1[n] \preceq X_2[n]$ if $X_1[i] \preceq X_2[k] \; \forall i = k$.

2) Basic Interval Sequences: In digital signal processing several basic sequences are of particular importance. These are discussed below.

The **Unit sample sequence**, denoted by $\delta[n]$, is defined as the sequence with values 0 if $n \neq 0$ and 1 if $n = 0$.

The **interval unit sample sequence**, denoted by $\delta_i[n]$, is defined as the sequence $\delta_i[n] = [\delta[n], \delta[n]]$.

Figure 3 shows an example of the product of two sequences, $\delta_i[n]$ and a constant sequence defined by $X[n] = [1, 2] \; \forall n \in \mathbb{Z}$.

We refer in this work to interval unit sample sequence as a *discrete-time impulse* or simply as an *impulse.*

Just as in real time discrete signal processing, the interval discrete sequence can be expressed as

$$X[n] = \sum_{k=-\infty}^{\infty} X[k] \delta_i[n-k].$$

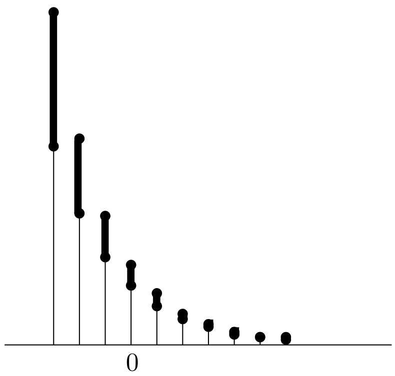

Fig. 4. An interval exponential sequence

The **unit step sequence,** denoted by $u[n]$, is defined as 0 if $n < 0$ and 1 if $n \geq 0$.

Just as the impulse sequence, the interval unit step sequence can be defined from the real unit step sequence, as shown below.

The **interval unit step sequence** , denoted by $u_i[n]$, is defined as $u_i[n] = [u[n], u[n]]$.

The interval unit step sequence can be represented by the interval impulse as $u_i[n] = \sum_{k=0}^{\infty} \delta_i[n-k]$.

Two important sequence classes in real signal processing and the analysis of linear time-invariant discrete-time systems are **exponential** and **sinusoidal sequences**. The interval version must then be done.

The **interval exponential sequences** have a general form $X[n] = A\alpha^n$, where A and α are intervals. An interval exponential sequence is positive and decreases when n increases, where A is a positive interval and $\alpha \subset (0, 1)$. When $\alpha \subset (-1, 0)$, the sequence alternates in sign but the magnitude decreases with an increase in n. If $|\alpha| > 1$, the magnitude of the sequence increases with n. The scope of this work is not sufficiently large to perform a more detailed analysis of interval sequence convergence.

An illustration of an exponential interval where A is positive and $\alpha \subset (0, 1)$ is shown in Figure 4.

An **interval sinusoidal sequence** can be defined as

$$X[n] = A\, cos(\omega_0 n + \phi) \text{ for all n, or} \tag{1}$$

$$X[n] = A \sin(\omega_0 n + \phi) \text{ for all n,} \tag{2}$$

Figure 2 shows an example of equation (2) where $A = [0.5, 1]$, $\omega_0 = \frac{\pi}{32}$, $\phi = 0$ and $-31 \leq n \leq 32$.

When the exponential interval sequence has A and α complex A and α can be rewritten as $|A|e^{j\phi}$ and $|\alpha|e^{j\omega_0}$, respectively. So equation 1can be rewritten as

$$\begin{aligned} X[n] &= A\alpha^n \\ &= |A|e^{j\phi}|\alpha|e^{j\omega_0 n} \\ &= |A||\alpha|^n e^{j(\omega_0 n+\phi)} \\ &= |A||\alpha|^n \cos(\omega_0 n + \phi) + j|A||\alpha|^n \sin(\omega_0 n + \phi). \end{aligned} \tag{3}$$

When $\alpha = 1$ in equation 3 the result is a **complex exponential sequence** as shown below.

$$\begin{aligned} X[n] &= |A|e^{j(\omega_0 n+\phi)} \\ &= |A|\cos(\omega_0 n + \phi) + |A|\sin(\omega_0 n + \phi). \end{aligned} \tag{4}$$

This paper does not explore the full interval complex sequence version because it is too complex for the scope of this article. This will be investigated in another study concerning Interval Complex Variables.

It is easy to see that when $|A| = [\min\{|a| : a \in A\}, \max\{|a| : a \in A\}]$, the equation (4) can be written by the sum of equations (1) and (2). As a result, a type of complex interval exponential sequence can be constructed. This is peculiar to a general complex interval sequence, where only the magnitude is interval. This sequence can be used to represent the particular systems that have uncertainty only in magnitude, not in phase. The reader can find more detailed interval complex theory studies in the following articles [2], [1], [4], [5], [7]. Boche pointed out the incompatibility of the polar form of interval complex representation and its rectangular form of representation[2]; this is still an open question.

Let $X : \mathbb{Z} \longrightarrow \mathbb{IR}$ be a semi-interval sequence, with a and b the first and the last term respectively. The **interval sum** of all $X[n]$ from $X[a]$ to $X[b]$ is defined by

$$\sum_{i=a}^{b} X[i] = \left[\sum_{i=a}^{b} \underline{X}[i], \sum_{i=a}^{b} \overline{X}[i] \right]. \tag{5}$$

An extension of equation (5) for undefined terms can be written as

$$\sum_{i=-\infty}^{+\infty} X[i] = \left[\sum_{i=-\infty}^{+\infty} \underline{X}[i], \sum_{i=-\infty}^{+\infty} \overline{X}[i] \right]. \tag{6}$$

Theorem 1: Let C be an interval constant. Then $C \sum_{i=a}^{b} X[i] \subseteq \sum_{i=a}^{b} CX[i]$.

Proof: A generalization of distributive property Due to pseudo-distributivity, interval linear systems can be applied only in positive systems or negative systems, because for systems in sets $\mathbb{IR}^+$ or $\mathbb{IR}^-$ the $\subseteq$ of therem 1can be replaced by $=$.

Theorem 2: Let $X_1[n]$ and $X_2[n]$ semi-interval sequences. Then

$$\sum_{i=a}^{b} (X_1[i] + X_2[i]) = \sum_{i=a}^{b} X_1[i] + \sum_{i=a}^{b} X_2[i].$$

Proof:

$$\sum_{i=a}^{b}(X_1[i]+X_2[i]) =$$

$$= \left[\sum_{i=a}^{b}\underline{(X_1[i]+X_2[i])}, \sum_{i=a}^{b}\overline{(X_1[i]+X_2[i])}\right]$$

$$= \left[\sum_{i=a}^{b}(\underline{X_1}[i]+\underline{X_2}[i]), \sum_{i=a}^{b}(\overline{X_1}[i]+\overline{X_2}[i])\right]$$

$$= \left[\sum_{i=a}^{b}\underline{X_1}[i], \sum_{i=a}^{b}\overline{X_1}[i]\right] + \left[\sum_{i=a}^{b}\underline{X_2}[i], \sum_{i=a}^{b}\overline{X_2}[i]\right]$$

$$= \sum_{i=a}^{b}X_1[i] + \sum_{i=a}^{b}X_2[i].$$

III. Basic properties of discrete interval systems

In this section the properties of discrete real systems are extended to discrete interval systems. A discrete interval system L is a **memoryless system** if $Y[n]$ depends only on its inputs $X[n]$ or on $x[n]$ (in the semi-interval case). An interval system is **time-invariant** when a variation in the time of the input sequence causes the same variation in the time of the output sequence, i.e. if $X_1(n) = X(n-n_0) \Rightarrow Y_1(n) = Y(n-n_0)$.

An interval system L is **additive** if the response of the addition of an input sequence is the addition of the respective response to individual inputs. Mathematically: $L(X_1+X_2) = L(X_1)+L(X_2)$.

An interval system is **homogeneous** if the output of the system for an input times an interval constant is always equal to this constant times the output of the system for the respective input. Mathematically: $L(CX) = CL(X)$. This is not a general property in interval linear systems. It occurs only in specific cases.

An interval system L is **linear** if for all interval sequences $X[n]$, $X_1(n)$ and $X_2(n)$ are interval constants A, $L\{X[n]\}$, $L\{X_1(n)\}$ and $L\{X_2(n)\}$ are interval sequences,

$$L\{X_1(n)+X_2(n)\} = L\{X_1(n)\}+L\{X_2(n)\} \quad (7)$$

and

$$L\{AX[n]\} = AL\{X[n]\}. \quad (8)$$

Equations (7) and (8) are called properties of additivity and homogeneity respectively and can be combined to form the overlapping principle:$L\{AX_1(n)+BX_2(n)\} = AL\{X_1(n)\} + BL\{X_2(n)\}$.

An analysis of the existence condition of interval linear systems will not be explored in this work, but a thorough discussion of this subject can be found in [3], [10].

Proposition 2: Let L be an interval system. If L is homogeneous, then $K \in \mathbb{IR}$ such that for each $X \in \mathbb{IR}$, $L[X] = KX$.

Proof: Let $K = L[[1,1]]$ then, by the homogeneity of L, $L[X] = L[X[1,1]] = XL[[1,1]] = KX$. The **impulse response** of a discrete interval linear system L denoted by H is:

$$H[n] = L[\delta_i[n]]. \quad (9)$$

Theorem 3: If $L : \mathbb{R} \to \mathbb{IR}$ is a discrete semi-interval linear system, then $\sum_{i=a}^{b} L[i] = L\left[\sum_{i=a}^{b} i\right]$.

Proof: If L is a discrete semi-interval linear system, then there is an interval K for each real x, such that $L[x] = Kx$. Therefore,

$$\sum_{i=a}^{b} L[x] = \sum_{i=a}^{b} Kx = K\sum_{i=a}^{b} x = L\left[\sum_{i=a}^{b} x\right].$$

Theorem 4: Let L be an interval time-invariant linear system, $X[n]$ a discrete interval signal represented by a semi-interval sequence, $H[n]$ the impulse response of L and $Y[n]$ the output of the system. Then $Y[n] = \sum_{i=-\infty}^{\infty} X[i]H[n-i]$.

Proof: For the interval system

$$Y[n] = L[X[n]]$$

$X[n]$ can be rewritten as an infinite sum of impulse responses and, therefore,

$$\begin{aligned} Y[n] &= L\left(\sum_{i=-\infty}^{\infty} X[i]\delta_i[n-i]\right) \\ &= \sum_{i=-\infty}^{\infty} L[X[i]\delta_i[n-i]] \text{ (linearity of L)} \\ &= \sum_{i=-\infty}^{\infty} X[i]L[\delta_i[n-i]] \text{ (by the equation (9))} \\ &= \sum_{i=-\infty}^{\infty} X[i]H[n-i]. \end{aligned}$$

The causality notion is basic for signal processing in $\mathbb{R}$ or $\mathbb{C}$. The interval extension of this property is trivial. An interval system is **causal** if for all values n_0 of the output sequence value at index $n = n_0$, it depends only on the input sequence values for $n \le n_0$.

An interval system is **stable**, (bounded-input bounded-output BIBO), if for each limited input a limited output is produced. The input $X[n]$ is limited if there is a fixed positive real value b_X such that:$\forall\ n,\ |X[n]| \le b_X < \infty$. An output $Y[n]$ is bounded if for each limited input there is a real positive fixed value, b_Y, such that: $\forall n,\ |Y[n]| \le b_Y < \infty$.

A. Convolution

Convolution is the most important operation of digital signal processing. It is a linear system. If a linear system is fully specified by impulse response then it satisfies all mathematical convolution conditions. It can also be used for average move filter implementation. The concept of convolution is strongly correlated with the concept of mobile average. The output of a linear system can be given by the convolution of input with the impulse response of the system. In statistics, the density function of the probability of the sum of two independent variables X and Y is given by the convolution of the respective probability density function. In the multiplication of polynomials, the coefficients of the product is given by the convolution of the coefficients of input polynomials.

B. Interval Convolution

Let L be a discrete time-invariant interval linear system and $H[n]$ its impulse response. The interval **convolution** of a discrete signal is defined by the infinite sum of $H[n]$ times the sequence of interval inputs $X[n]$. Formally,

$$X[n] * H[n] = \sum_{i=-\infty}^{+\infty} X[i]H[n-i]. \quad (10)$$

In the following lines two properties of interval convolution are presented.

Proposition 3 (commutative): The interval convolution is commutative, that is, $H[n] * X[n] = X[n] * H[n]$.

Proof: Given that H and X are discrete interval time-invariant systems and owing to the commutative property of the interval product, we have

$$\begin{aligned} X[n] * H[n] &= \sum_{i=-\infty}^{+\infty} X[i]H[n-i] \\ &= \sum_{i=-\infty}^{+\infty} H[n-i]X[i]. \\ &= \sum_{i=-\infty}^{+\infty} H[v]X[n-v] \\ &= H[n] * X[n]. \end{aligned}$$

In this property the order of the signal is not important. Thus, a cascade system can be used.

Proposition 4 (Associative): The semi-interval convolution is associative, i.e.

$$[X[n] * H[n]] * G[n] = X[n] * [H[n] * G[n]].$$

Proof: Let $W[n] = X[n]*H[n]$ and $Z[n] = H[n]*G[n]$. Then,

$$\begin{aligned} [X[n] * H[n]] * G[n] &= W[n] * G[n] \\ &= \sum_{i=-\infty}^{+\infty} W[i]G[n-i] \\ &\quad \text{(by the equation (10))} \\ W[i] &= \sum_{v=-\infty}^{+\infty} X[v]H[i-v] \\ &\quad \text{(by the equation (10))} \end{aligned}$$

So,

$$\begin{aligned} &\sum_{i=-\infty}^{+\infty} W[i]G[n-i] = \\ &= \sum_{i=-\infty}^{+\infty} \left[\sum_{v=-\infty}^{+\infty} X[v]H[i-v]\right] G[n-i] \\ &= \sum_{i=-\infty}^{+\infty} \sum_{v=-\infty}^{+\infty} X[v]H[i-v]G[n-i] \end{aligned}$$

Changing the variable and putting $u = i - v$, we have

$$\begin{aligned} &\sum_{i=-\infty}^{+\infty} \sum_{v=-\infty}^{+\infty} X[v]H[i-v]G[n-i] = \\ &= \sum_{u=-\infty}^{+\infty} \sum_{v=-\infty}^{+\infty} X[v]H[u]G[[n-v]-u] \\ &= \sum_{v=-\infty}^{+\infty} \sum_{u=-\infty}^{+\infty} X[v]H[u]G[[n-v]-u] \\ &= \sum_{v=-\infty}^{+\infty} X[v] \sum_{u=-\infty}^{+\infty} H[u]G[[n-v]-u], \\ &\quad \text{(by the theorem 1),} \end{aligned}$$

$$\begin{aligned} &Z[n-v] = \sum_{u=-\infty}^{+\infty} H[u]G[[n-v]-u] \\ &\quad \text{(by the equation (10)).} \end{aligned}$$

So,

$$\begin{aligned} &\sum_{v=-\infty}^{+\infty} X[v] \sum_{u=-\infty}^{+\infty} H[u]G[[t-v]-u] = \\ &= \sum_{v=-\infty}^{+\infty} X[v] \left[\sum_{u=-\infty}^{+\infty} H[u]G[[t-v]-u]\right] \\ &= \sum_{v=-\infty}^{+\infty} X[v]Z[t-v] \\ &= X[n] * Z[t] \\ &= X[n] * [H[t] * G[t]]. \end{aligned}$$

The associative and distributive properties provide the parallel implementation of systems.

IV. Conclusion

The interval approach was confirmed as being almost intuitive, because all discrete systems that represent continuous systems can contain errors. These errors can not be only of data reading, but also of float point arithmetic, or inherent to the variance of the system. In this paper, several basic properties of digital signal processing in its interval version were presented as well as the discrete interval convolution. However, it should be pointed out that only discrete time-invariant linear systems were investigated. With the mathematical foundation for the discrete interval linear system described in this work, several signal processing tools can be defined, such as discrete interval filters and the analysis of system stability. This paper can provide the designers of digital signal processing with another way of dealing with the uncertainty of digital systems.

In future articles, the authors will investigate the complex version. But first, a study proposing equivalence between the polar form of interval complex numbers and their rectangular form will be carried out as well as work on the Z transform interval version.

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