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Fractional Factorial Experiments

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Introduction

- As the number of factors in a 2^k factorial design increases, the number of runs required for a complete replicate of the design rapidly outgrows the resources of most experimenters.
- For example, $2^2=4$; $2^3=8$; $2^4=16$; $2^5=32$; $2^6=64$;
- In this $2^6(=64)$ factorial experiment, only 6 d.f. out of 63 d.f. corresponds to main effects; 15 d.f corresponds to two factor interactions. Ie. Only 21 d.f. likely to be important where as remaining 42 d.f. corresponds to three or higher interactions.
- If the experimenter can reasonably assume that certain high-order interactions are negligible, information on the main effects and low-order interactions may be obtained by running only a fraction of the complete factorial experiment

Definitions and Basic Principles

- Consider 2^3 factorial experiment
- We have 8 treatment combinations and we can afford to run only 4
- Ie. We suggest a one-half fraction of 2^3 factorial experiment.
- Since the new design contains $2^{3-1} = 4$ treatment combinations, a one-half fraction of the 2^3 design is often called a **2^{3-1} design**.
- In general, we can write one half fraction of 2^k design as **2^{k-1} design**.
- In general, we can write one quarter fraction of 2^k design as **2^{k-2} design**.
- In general, we can write $1/2^p$ fractional factorial as **2^{k-p} design.**

Example: 2^{3-1} fractional factorial

- Consider the algebraic signs of the effect contrast

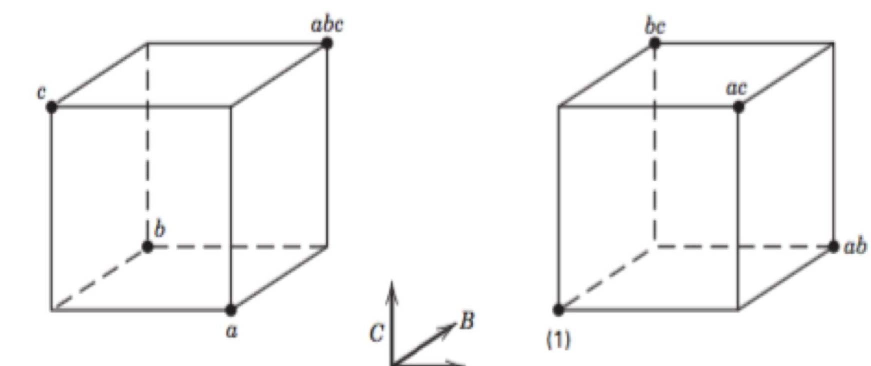
Plus and Minus Signs for the 2^3 Factorial Design

Treatment Combination	Factorial Effect							
	I	*A*	*B*	*C*	*AB*	*AC*	*BC*	*ABC*
a	+	+	−	−	−	−	+	+
b	+	−	+	−	−	+	−	+
c	+	−	−	+	+	−	−	+
abc	+	+	+	+	+	+	+	+
ab	+	+	+	−	+	−	−	−
ac	+	+	−	+	−	+	−	−
bc	+	−	+	+	−	−	+	−
(1)	+	−	−	−	+	+	+	−

Example: 2^{3-1} fractional factorial

- Suppose we select four treatment combination a, b, c & abc as our one-half fraction. From earlier table, we can see that all these treatment combinations has + sign under ABC effect. i.e. this fraction of treatment combination formed by +ABC.
- ABC is called the generator of this particular fraction. Generally, we will refer the generator such as ABC as a word.
- Note that column (1) (or I) always + sign, we call I = ABC as the defining relation of our design.
- If we use to generate a fraction using treatments with – sign under ABC, we call I = -ABC as the defining relation.

Example: 2^{3-1} fractional factorial



(a) The principal fraction, $I = +ABC$

(b) The alternate fraction, $I = -ABC$

Example: 2^{3-1} fractional factorial

- Consider the one-half of the treatment combinations generated by I=+ABC ie. The set of [a, b, c, abc]
- Four treatment combinations, so three d.f.
- Based on the algebraic signs of contrast (see earlier table), we can estimate the effect of A. B & C as

$$[A] = \frac{1}{2}(a - b - c + abc)$$
$$[B] = \frac{1}{2}(-a + b - c + abc)$$
$$[C] = \frac{1}{2}(-a - b + c + abc)$$

- Where the notation[A], [B] and [C] used to represent the linear combination to estimate the effect of A, B and C

Example: 2^{3-1} fractional factorial

- It is easy to verify that the effects of AB, AC and BC can also estimated as

$$[BC] = \frac{1}{2}(a - b - c + abc)$$
$$[AC] = \frac{1}{2}(-a + b - c + abc)$$
$$[AB] = \frac{1}{2}(-a - b + c + abc)$$

- Now, note that [A]=[BC]; [B]=[AC]; and [C]=AB];
- i.e. it is impossible to differentiate between A and BC; B and AC & C and AB.
- Ie. When we estimate A, B and C, we are really estimating A+BC; B+AC and C+AB. We can give a notation for this as

$$[A] \rightarrow A + BC, [B] \rightarrow B + AC, \text{ and } [C] \rightarrow C + AB.$$

- Two or more effects that has this property is called **aliases**

Example: 2^{3-1} fractional factorial

- In our example, A and BC are aliases, B and AC are aliases & C and AB are aliases.
- We can identify the alias structure using the defining relation
- Multiplying any effect word by the defining relation will yield the alias structure. For example, $A. I = A. ABC = A^2BC = BC$. i.e. $A=BC$
- Similarly, $B.I = B. ABC = AB^2C = AC$. i.e $B = AC$
- Similarly, $C.I = C. ABC = ABC^2 = AB$. i.e $C = AB$
- The half fraction with $I = +ABC$ is usually called **principal fraction**

Example: 2^{3-1} fractional factorial

- Suppose we had chosen the other fraction, associated with – sign in the ABC column.
- The treatment combinations are [(1), ab, ac and bc], which is generated using the defining relation, I = -ABC
- The linear combination of the observations, say [A]', [B]' and [C]' from this fraction is given as

$$[A]' \rightarrow A - BC$$
$$[B]' \rightarrow B - AC$$
$$[C]' \rightarrow C - AB$$

- i.e. When we estimate A, B and C with this fraction, we really estimating A-BC, B-AC and C-AB

Example: 2^{3-1} fractional factorial

- In practice, it does not matter which fraction you is actually used (we assume that higher order interactions (in this case AB, AC, BC and ABC) are negligible. Both fractions belong to the same family (i.e. two one-half fractions form the complete design).
- Suppose at after running the first half fraction, the other fraction also we run. Adding and subtracting the effects from each fraction we can get the actual effects.
- For example from first fraction, we really estimate [A] as A+BC and from the second fraction we estimate [A]' as A-BC. So

$$\frac{1}{2}([A] + [A]') = \frac{1}{2}(A + BC + A - BC) \rightarrow A$$

$$\frac{1}{2}([A] - [A]') = \frac{1}{2}(A + BC - A + BC) \rightarrow BC$$

Example: 2^{3-1} fractional factorial

- Similarly, we can obtain

i	**From** $\frac{1}{2}([i] + [i]')$	**From** $\frac{1}{2}([i] - [i]')$
A	A	BC
B	B	AC
C	C	AB

- By assembling the full 2^3 in this fashion with $I = +ABC$ in the first group of runs and $I = -ABC$ in the second, the 2^3 confounds ABC with blocks.

Design Resolution

- The fractional factorial design we considered is called a resolution III design. In such design, main effect are aliased with two factor interaction.
- A design is of resolution R if no p-factor effect is aliased with another effect containing less than R-p factors.
- The design we consider earlier, One half fraction of 2^3 design with defining relation I=ABC (or I=-ABC) is a 2_{III}^{3-1} design

Design Resolution

- **1. *Resolution III designs.*** These are designs in which no main effects are aliased with any other main effect, but main effects are aliased with two-factor interactions and some two-factor interactions may be aliased with each other. A 2^{3-1} design with *I=ABC* is a resolution III design (2_{III}^{3-1}).
- **2. *Resolution IV designs.*** These are designs in which no main effect is aliased with any other main effect *or* with any two-factor interaction, but two-factor interactions are aliased with each other. A 2^{4-1} design with *I=ABCD* is a resolution IV design (2_{IV}^{4-1}).
- **3. *Resolution V designs.*** These are designs in which no main effect or two-factor interaction is aliased with any other main effect or two-factor interaction, but two- factor interactions are aliased with three-factor interactions. A 2^{5-1} design with *I=ABCDE* is a resolution V design (2_{V}^{5-1}).