A Passivity-Based Framework for Resilient Cyber Physical Systems

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***Abstract*—Resilient control systems play a special role in the area of cyber-physical systems, where the design must address the question how complex dynamic plants are to be controlled safely and reliably when a control system that is under a cyber attack. In this paper we describe a control theoretical framework based on the concept of passivity for designing a control network which can tolerate, for instance, denial-of-service attacks on networks used in the closed loop. In particular, we demonstrate how the *resilient power junction* structure could be applied, and show simulated results.**

I. Introduction

The design of resilient control systems necessitates novel developments at the intersection of computer science and control theory. The control of complex dynamic systems is a well-studied area, but much less is known about how to implement such control systems that are able to tolerate shortcomings of non-ideal software and network-based implementation platforms. Additionally, not only implementation side-effects have to be mitigated, but also potential issues related to security of the control system. For instance, if a network used in the control loop is under a denial-of-service attack, we still need to maintain the quality of control for the plant. If the controller itself is compromised, we need to detach it from a plant and an alternative controller must be brought online. In this paper we describe a control-theoretical framework based on *passivity* principles. Passivity-based controllers are ideally suited for high-confidence control systems that have infinite gain margins, thus possess a great deal of robustness to uncertainty.

Passivity is a mathematical property of the controller implementation, and could be realized in different ways. The approach described here applies to a large family of physical systems which can be described by both linear and nonlinear system models [1]–[3], including systems which can be described by cascades of passive systems such as quadrotor aircraft [4]. Furthermore, the theory can be applied to networked control design [5], [6] including over wireless networks [7].

For this paper we shall focus on the use of a structure called the *resilient power junction* (a special type of power junction [8], [9]) to demonstrate how a passive physical system (in which its dynamics are described by ordinary differential equations) can be interconnected to multiple-redundant-passive-digital controllers while maintaining L_2^m-stability. We shall discuss the conditions for the type of non-redundant controllers which can be tolerated if no detection scheme is used. In addition we demonstrate how potentially-destabilizing non-redundant controllers can be removed from the network when detection of the non-redundant controller occurs.

II. Resilient Control Architecture

Fig. 1 depicts a resilient digital control network which maintains L_2^m-stability, even when non-redundant controllers could be potentially introduced into the network. In particular, m redundant passive digital controllers (denoted $G_{\mathsf{cj}} : f_{\mathsf{cj}} \rightarrow e_{\mathsf{cj}}$ $\mathsf{j} \in \{1, \ldots, \mathsf{m}\}$) and $\mathsf{m_c} - \mathsf{m}$ non-redundant digital controllers $\mathsf{j} \in \{\mathsf{m}+1, \ldots, \mathsf{m_c}\}$ are interconnected to a *resilient power junction* in order to provide reliable control for a single continuous time plant (denoted $G_{\mathsf{pn}} : e_{\mathsf{pn}} \rightarrow f_{\mathsf{pn}}$ in which $\mathsf{n} = \mathsf{m_c} + 1$). The *resilient power junction* can be operated in a manner in which it does not explicitely detect non-redundant controllers or it can operate in a more-restricted environment in which it enforces that only redundant controllers can modify the behavior of the plant. In the more-restrictive mode, non-redundant controllers shall be isolated from the network so as not to potentially destabilize the rest of the network.

Section II-A reviews wave variables with which the power junction interacts with. Section II-B reviews the passive sampler and passive hold, which allow a continuous time plant to be interconnected to a digital control network. Section II-C introduces the resilient power junction (the main contribution to this paper). Section II-D provides the main stability result which shows that L_2^m-stability can be maintained in spite of non-redundant controllers being introduced to the network. Section III provides simulated results when various types of non-redundant controllers are connected to the network.

A. Wave Variables

Networks of a *passive* plant and controller are typically interconnected using *power variables*. *Power variables* are generally denoted with an *effort* and *flow* pair (e_*,f_*) whose product is power. They are typically used to show the exchange of energy between two systems using *bond graphs* [10], [11]. However, when these *power variables* are subject to communication delays the communication channel ceases to be *passive* which leads to network instabilities.

$$u_{\mathsf{pn}}(t) = \frac{1}{\sqrt{2b}}(b f_{\mathsf{pn}}(t) + e_{\mathsf{dcn}}(t)) \tag{1}$$

$$v_{\mathsf{pn}}(t) = \frac{1}{\sqrt{2b}}(b f_{\mathsf{pn}}(t) - e_{\mathsf{dcn}}(t)) \tag{2}$$

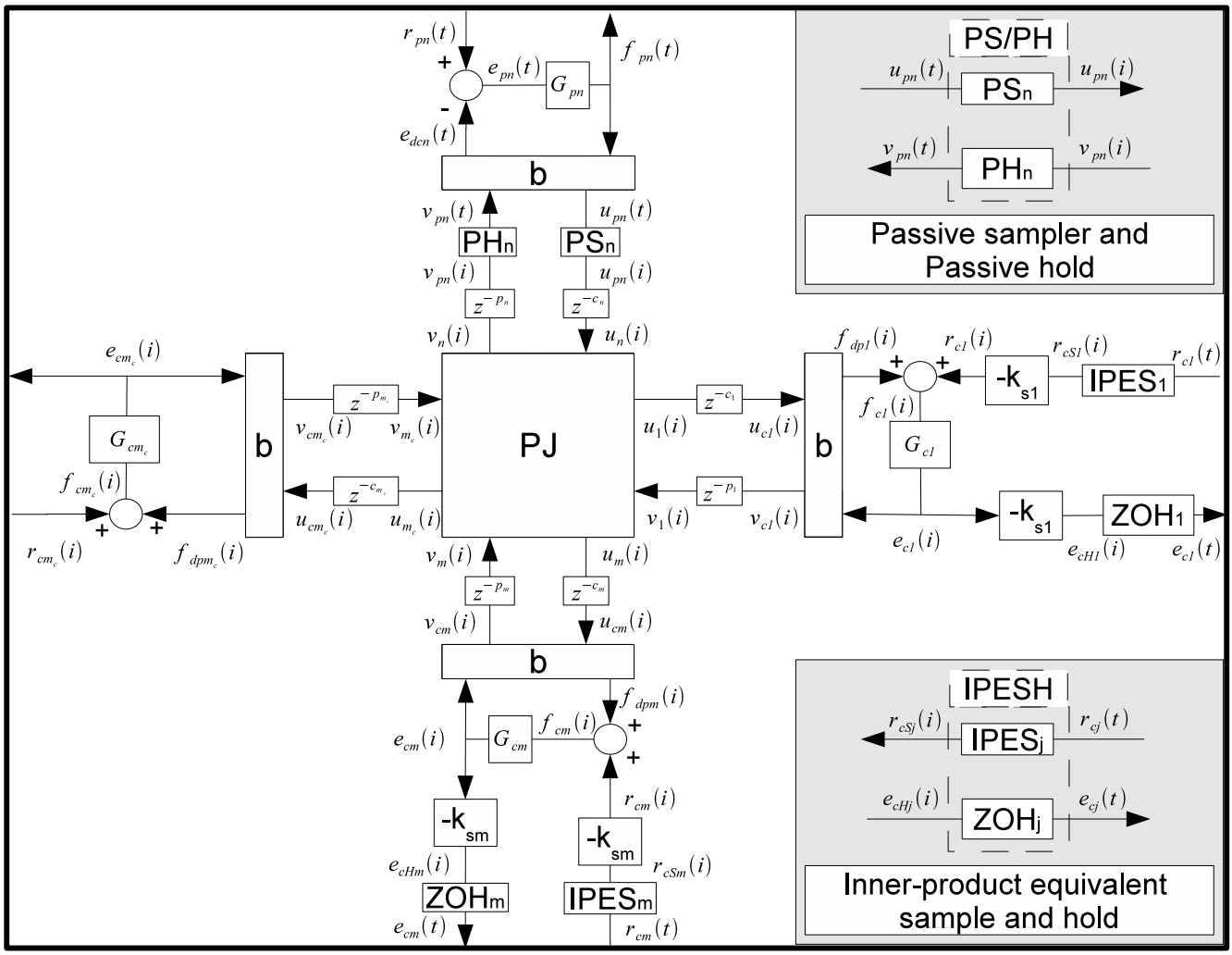

Fig. 1. An L_2^m-stable resilient *power junction* control network.

$$v_{\mathsf{cj}}(i) = \frac{1}{\sqrt{2b}}(bf_{\mathsf{dpj}}(i) - e_{\mathsf{cj}}(i)),\ \mathsf{j} \in \{1, \ldots, \mathsf{m_c}\} \quad (3)$$

$$u_{\mathsf{cj}}(i) = \frac{1}{\sqrt{2b}}(bf_{\mathsf{dpj}}(i) + e_{\mathsf{dpj}}(i)) \quad (4)$$

(1) can be thought of as the sensor output in a *wave variable* form for plant G_{pn} depicted in Fig. 1. Likewise, (3) can be thought of as each command output in a *wave variable* form for each controller $G_{\mathsf{cj}},\ \mathsf{j} \in \{1, \ldots, \mathsf{m_c}\}$ depicted in Fig. 1. The symbol $i \in \{0, 1, \ldots\}$ depicts discrete time for the controllers, and the symbol $t \in \mathbb{R}$ denotes continuous time and the two are related to the sample and hold time (T_s) such that $t = iT_s$. (1) and (2) respectively satisfy the following equality:

$$\frac{1}{2}(u_{\mathsf{pn}}^{\mathsf{T}}(t)u_{\mathsf{pn}}(t) - v_{\mathsf{pn}}^{\mathsf{T}}(t)v_{\mathsf{pn}}(t)) = f_{\mathsf{pn}}^{\mathsf{T}}(t)e_{\mathsf{dcn}}(t) \quad (5)$$

Similarly, (3) and (4) respectively satisfy the following equality $\forall\ \mathsf{j} \in \{1, \ldots, \mathsf{m_c}\}$:

$$\frac{1}{2}(u_{\mathsf{cj}}^{\mathsf{T}}(i)u_{\mathsf{cj}}(i) - v_{\mathsf{cj}}^{\mathsf{T}}(i)v_{\mathsf{cj}}(i)) = f_{\mathsf{dpj}}^{\mathsf{T}}(i)e_{\mathsf{cj}}(i). \quad (6)$$

Denote $I \in \mathbb{R}^{m_s \times m_s}$ as the identity matrix. When implementing the wave variable transformation the continuous time plant "outputs" $(u_{\mathsf{pn}}(t), e_{\mathsf{dcn}}(t))$ are related to the corresponding "inputs" $(v_{\mathsf{pn}}(t), f_{\mathsf{pn}}(t))$ as follows (Fig. 1):

$$\begin{bmatrix} u_{\mathsf{pn}}(t) \\ e_{\mathsf{dcn}}(t) \end{bmatrix} = \begin{bmatrix} -I & \sqrt{2b}I \\ -\sqrt{2b}I & bI \end{bmatrix} \begin{bmatrix} v_{\mathsf{pn}}(t) \\ f_{\mathsf{pn}}(t) \end{bmatrix} \quad (7)$$

Next, the discrete time controller "outputs" $(v_{\mathsf{cj}}(i), f_{\mathsf{dpj}}(i))$ are related to the corresponding "inputs" $(u_{\mathsf{cj}}(i), e_{\mathsf{cj}}(i))$ as follows (Fig. 1):

$$\begin{bmatrix} v_{\mathsf{cj}}(i) \\ f_{\mathsf{dpj}}(i) \end{bmatrix} = \begin{bmatrix} I & -\sqrt{\frac{2}{b}}I \\ \sqrt{\frac{2}{b}}I & -\frac{1}{b}I \end{bmatrix} \begin{bmatrix} u_{\mathsf{cj}}(i) \\ e_{\mathsf{cj}}(i) \end{bmatrix} \quad (8)$$

The *power junction* indicated in Fig. 1 by the symbol PJ has waves entering and leaving the power junction as indicated by the arrows. Waves leaving the controllers v_{cj} and entering the power junction v_{j} in which $\mathsf{j} \in \{1, \ldots, \mathsf{m_c}\}$ have the following relationship

$$v_{\mathsf{j}}(i) = v_{\mathsf{cj}}(i - p_{\mathsf{j}}(i))$$

in which $p_{\mathsf{j}}(i)$ denotes the time delay in transmitting the control wave from 'controller-j' to the power junction. Next, the input wave to the plant v_{pn} is a delayed version of the outgoing wave from the *power junction* v_{n} such that

$$v_{\mathsf{pn}}(i) = v_{\mathsf{n}}(i - p_{\mathsf{n}}(i))$$

in which $p_{\mathsf{n}}(i)$ denotes the discrete time delay in transmitting the outgoing wave to 'plant-n'. In Fig. 1 depicts the fixed time delays using the z-transform (i.e. $z^{-p_{\mathsf{n}}}$). Next, the outgoing wave from the plant u_{pn} is related to the wave entering the power junction u_{n} as follows:

$$u_{\mathsf{n}}(i) = u_{\mathsf{pn}}(i - c_{\mathsf{n}}(i))$$

in which $c_{\mathsf{n}}(i)$ denotes the discrete time delay in transmitting the wave from 'plant-n' to the power junction. Last, the input wave to the controller u_{cj} is a delayed version of the outgoing wave from the *power junction* $u_{\mathsf{j}},\ \mathsf{j} \in \{1, \ldots, \mathsf{m_c}\}$ such that

$$u_{\mathsf{cj}}(i) = u_{\mathsf{j}}(i - c_{\mathsf{j}}(i)),\ \mathsf{j} \in \{1, \ldots, \mathsf{m_c}\}$$

in which $c_{\mathsf{j}}(i)$ denotes the discrete time delay in transmitting the wave from the power junction to 'controller-j' (the delays are denoted as $z^{-c_{\mathsf{j}}}$ in Fig. 1).

B. Passive Sampler and Passive Hold

In [12] it is shown how a passive sampler (PS) a passive hold (PH) in conjunction with a *inner-product equivalent sampler* (*IPES*) and zero-order-hold (*ZOH*) can be used to achieve a L_2^m-stable system consisting of (a) passive robot(s) and (a) digital controller(s). As can be seen in Fig. 1 we have connected the PS and PH to plant-n, while connecting the (*IPES*) and zero-order-hold (*ZOH*) block to each passive digital controller $G_{\mathsf{cj}},\ \mathsf{j} \in \{1, \ldots, \mathsf{m}\}$ in order to relate $r_{\mathsf{cj}}(i)$ to $r_{\mathsf{cj}}(t)$ and $e_{\mathsf{cj}}(i)$ to $e_{\mathsf{cj}}(t)$ in a passivity preserving manner. Therefore we recall the following set of definitions:

Definition 1: The passive sampler denoted ($\mathsf{PS_n}$) and the corresponding passive hold denoted ($\mathsf{PH_n}$) must be implemented such that the following inequality is satisfied $\forall N > 0$:

$$\begin{aligned}\int_0^{NT_s} (u_{\mathsf{pn}}^{\mathsf{T}}(t)u_{\mathsf{pn}}(t) - v_{\mathsf{pn}}^{\mathsf{T}}(t)v_{\mathsf{pn}}(t))dt - \\ \sum_{i=0}^{N-1} (u_{\mathsf{pn}}^{\mathsf{T}}(i)u_{\mathsf{pn}}(i) - v_{\mathsf{pn}}^{\mathsf{T}}(i)v_{\mathsf{pn}}(i)) \geq \quad 0. \end{aligned} \tag{9}$$

One way to implement the PS and PH is to use the *averaging passive sampler and hold.*

Definition 2: The *averaging passive sampler* denoted ($\mathsf{PS_n}$) and the corresponding *averaging passive hold* denoted ($\mathsf{PH_n}$) is implemented such that for each l^{th} component ($l \in \{1, \ldots, m_s\}$) of the discrete-time-sampled wave $u_{\mathsf{pn}}(i) \in \mathbb{R}^{m_s}$ (denoted $u_{\mathsf{pn}_l}(i)$) is determined from the respective l^{th} component of the continuous-time wave $u_{\mathsf{pn}}(t) \in \mathbb{R}^{m_s}$ (denoted $u_{\mathsf{pn}_l}(t)$) using $\mathsf{PS_n}$ as follows:

$$u_{\mathsf{pn}_l}(i) = \sqrt{\int_{(i-1)T_s}^{iT_s} u_{\mathsf{pn}_l}^2(t)dt}\ \mathtt{sgn}(\int_{(i-1)T_s}^{iT_s} u_{\mathsf{pn}_l}(t)dt) \tag{10}$$

and the continuous-time wave $v_{\mathsf{pn}}(t) \in \mathbb{R}^{m_s}$ is determined from the discrete-time wave $v_{\mathsf{pn}}(i) \in \mathbb{R}^{m_s}$ in terms of each of their respective l^{th} components using $\mathsf{PH_n}$ as follows:

$$v_{\mathsf{pn}_l}(t) = \frac{1}{\sqrt{T_s}} v_{\mathsf{pn}_l}(i),\ t \in\ [iT_s, (i+1)T_s). \tag{11}$$

Using a PS and PH such as the *averaging passive sampler and hold* we can now relate continuous time variables to discrete time wave variables associated with plant G_{pn}. Substituting (5) into (9) results in the following inequality for the plant

$$\int_0^{NT_s} f_{\mathsf{pn}}^{\mathsf{T}}(t)e_{\mathsf{dcn}}(t) \geq \sum_{i=0}^{N-1} (u_{\mathsf{pn}}^{\mathsf{T}}(i)u_{\mathsf{pn}}(i) - v_{\mathsf{pn}}^{\mathsf{T}}(i)v_{\mathsf{pn}}(i)). \tag{12}$$

If we assume that the networking time delays of the transmission and reception of the wave variables satisfy Proposition 1 (see Appendix A) then the following inequalities hold:

$$\|(u_{\mathsf{pn}})_N\|_2^2 - \|(v_{\mathsf{pn}})_N\|_2^2 \geq \|(u_{\mathsf{n}})_N\|_2^2 - \|(v_{\mathsf{n}})_N\|_2^2 \tag{13}$$

$$\|(u_{\mathsf{j}})_N\|_2^2 - \|(v_{\mathsf{j}})_N\|_2^2 \geq \|(u_{\mathsf{cj}})_N\|_2^2 - \|(v_{\mathsf{cj}})_N\|_2^2 \tag{14}$$

This leads us to the following corollary which relates (12) to the corresponding pair of waves entering and leaving the *power junction* $(u_{\mathsf{n}}(i), v_{\mathsf{n}}(i))$.

Corollary 1: The continuous time plant-n (flow $f_{\mathsf{pn}}(t)$ and effort $e_{\mathsf{dcn}}(t)$) pair depicted in Fig. 1 is related to their respective pair of waves entering and leaving the *power junction* $(u_{\mathsf{n}}(i), v_{\mathsf{n}}(i))$ such that

$$\begin{aligned}\int_0^{NT_s} f_{\mathsf{pn}}^{\mathsf{T}}(t)e_{\mathsf{dcn}}(t) &\geq \sum_{i=0}^{N-1} (u_{\mathsf{n}}^{\mathsf{T}}(i)u_{\mathsf{n}}(i) - v_{\mathsf{n}}^{\mathsf{T}}(i)v_{\mathsf{n}}(i)) \\ \langle f_{\mathsf{pn}}(t), e_{\mathsf{dcn}}(t)\rangle_{NT_s} &\geq \|(u_{\mathsf{n}}(i))_N\|_2^2 - \|(v_{\mathsf{n}}(i))_N\|_2^2 \\ \langle f_{\mathsf{pn}}, e_{\mathsf{dcn}}\rangle_{NT_s} &\geq \|(u_{\mathsf{n}})_N\|_2^2 - \|(v_{\mathsf{n}})_N\|_2^2 \end{aligned} \tag{15}$$

is satisfied if the wave variable communication time-delays satisfy any of the conditions listed in Proposition 1.

Since T_s is typically not an integer, we will typically drop the i or t symbol and use N to refer to extended discrete-time l_2^m norms and NT_s to refer to extended L_2^m norms. In an analogous manner we can relate the control effort and flow variables $(e_{\mathsf{cj}}(i), f_{\mathsf{dpj}}(i))$ to the power junction wave variables $(u_{\mathsf{j}}(i), v_{\mathsf{j}}(i)\ \forall \mathsf{j} \in \{1, \ldots, \mathsf{m_c}\}$ for the $\mathsf{m_c}$-digital controllers.

Corollary 2: All $\mathsf{m_c}$ discrete time controller (flows $f_{\mathsf{dpj}}(i)$ and efforts $e_{\mathsf{cj}}(i)$) pairs depicted in Fig. 1 are related to their respective pair of waves leaving and entering the *power junction* $(u_{\mathsf{j}}(i), v_{\mathsf{j}}(i))$ such that $\forall \mathsf{j} \in \{1, \ldots, \mathsf{m_c}\}$

$$\|(u_{\mathsf{j}})_N\|_2^2 - \|(v_{\mathsf{j}})_N\|_2^2 \geq \langle e_{\mathsf{cj}}, f_{\mathsf{dpj}}\rangle_N \tag{16}$$

is satisfied if the wave variable communication time-delays satisfy any of the conditions listed in Proposition 1.

A properly implemented power junction will always satisfy the following inequality [8]:

$$\begin{aligned}u_{\mathsf{n}}^{\mathsf{T}}u_{\mathsf{n}} - v_{\mathsf{n}}^{\mathsf{T}}v_{\mathsf{n}} \geq &\sum_{\mathsf{j}=1}^{\mathsf{m}} (u_{\mathsf{j}}^{\mathsf{T}}u_{\mathsf{j}} - v_{\mathsf{j}}^{\mathsf{T}}v_{\mathsf{j}}) \\ &+ \sum_{\mathsf{j}=\mathsf{m}+1}^{\mathsf{m_c}} (u_{\mathsf{j}}^{\mathsf{T}}u_{\mathsf{j}} - v_{\mathsf{j}}^{\mathsf{T}}v_{\mathsf{j}}). \end{aligned} \tag{17}$$

Which leads us to the following lemma.

Lemma 1: The $\mathsf{m_c}$ discrete time controller (flows $f_{\mathsf{dpj}}(i)$ and efforts $e_{\mathsf{cj}}(i)$) pairs $\mathsf{j} \in \{1,\ldots,\mathsf{m_c}\}$ are related to the continuous time plant (flow $f_{\mathsf{pn}}(t)$ and effort $e_{\mathsf{dcn}}(t)$) pair depicted in Fig. 1 as follows

$$\begin{aligned}\langle f_{\mathsf{pn}}(t), e_{\mathsf{dcn}}\rangle_{NT_s} &\geq \sum_{j=1}^{\mathsf{m_c}} \langle e_{\mathsf{cj}}, f_{\mathsf{dpj}}\rangle_N \\ &\geq \sum_{\mathsf{j}=1}^{\mathsf{m}} \langle e_{\mathsf{cj}}, f_{\mathsf{dpj}}\rangle_N + \sum_{\mathsf{j}=\mathsf{m}+1}^{\mathsf{m_c}} \langle e_{\mathsf{cj}}, f_{\mathsf{dpj}}\rangle_N.\end{aligned} \tag{18}$$

if the wave variable communication time-delays satisfy any of the conditions listed in Proposition 1.
The proof for Lemma 1 is in Appendix A-A1.

C. The Resilient Power Junction

The resilient power junction is a special type of power junction which satisfies the following:

i) the general definition for the power junction [8, Definition 1], in particular the inequality (17) is satisfied
ii) may be implemented to *detect* non-redundant controllers during run-time, and *isolate* non-redundant controllers by simply setting $u_{\mathsf{j-detect}}(i) = v_{\mathsf{j-detect}}(i)\ \forall\ i\ \geq N_{\mathsf{j-detect}}$ in which $N_{\mathsf{j-detect}}$ indicates the point in time when controller-$\mathsf{j-detect}$'s $v_{\mathsf{j-detect}}(i) \neq v_1(i)$. In addition, the isolated non-redundant controllers will no longer add to the calculation of v_{n}.

For simplicity of discussion we assume two scenarios. The first scenario assumes that the resilient power junction runs assuming that all $\mathsf{m_c}$-controllers are redundant. The second scenario provides conditions for the resilient power junction to detect a non-redundant controller and isolate from the network.

Assumption 1: i) there are $\mathsf{m_c} - \mathsf{m}$ non-redundant controllers with indexes $\mathsf{j} \in \{\mathsf{m}+1,\ldots,\mathsf{m_c}\}$ and $\mathsf{m} \geq 1$ passive controllers with indexes $\mathsf{j} \in \{1,\ldots,\mathsf{m}\}$,
ii) at initial time $i = 0$, m is unknown and $\mathsf{m_c}$ is known,
iii) all power junction waves are vectors such that u_{n}, v_{n}, u_{j}, $v_{\mathsf{j}} \in \mathbb{R}^{m_s}$ and the l^{th} component ($l \in \{1,\ldots,m_s\}$) of each wave is denoted $u_{\mathsf{n}l},\ldots,v_{\mathsf{j}l}$ respectively,
iv) wave variable communication time-delays satisfy any of the conditions listed in Proposition 1.

Assumption 2: i) Assumption 1 holds *except* all wave-variable communication time-delays and data-dropouts between the power junction and the controllers are identical,
ii) the temporal order in which non-redundant controllers are detected will be such that

$$N_{\mathsf{m_c-detect}} \leq N_{(\mathsf{m_c}-1)-\mathsf{detect}} \leq \cdots \leq N_{(\mathsf{m}+1)-\mathsf{detect}}.$$

Definition 3: Given Assumption 1, the resilient power junction is implemented as follows:

i) initialize: $i = 0$, $\hat{\mathsf{m}}(i) = \mathsf{m_c}$, $\forall \mathsf{j} \in \{1,\ldots,\hat{\mathsf{m}}\}\ E_{\mathsf{j}}(i) = 0$
ii) compute $\hat{u}_1(i)$,
$$\hat{u}_1(i) = \frac{1}{\sqrt{\hat{\mathsf{m}}}} u_{\mathsf{n}} \tag{19}$$
iii) $N = i+1$, compute $\forall \mathsf{j} \in \{1,\ldots,\hat{\mathsf{m}}\}$
$$\begin{aligned}\hat{E}_{\mathsf{j}}(N) &= \|(u_1)_{N-1}\|_2^2 - \|(v_{\mathsf{j}})_{N-1}\|_2^2 \\ &\quad + (\hat{u}_1^{\mathsf{T}}(i)\hat{u}_1(i) - v_{\mathsf{j}}^{\mathsf{T}}(i)v_{\mathsf{j}}(i)) \\ &= E_{\mathsf{j}}(N-1) + (\hat{u}_1^{\mathsf{T}}(i)\hat{u}_1(i) - v_{\mathsf{j}}^{\mathsf{T}}(i)v_{\mathsf{j}}(i))\end{aligned}$$
iv) $\hat{\mathsf{m}}(N) = \hat{\mathsf{m}}(i)$
v) If in addition Assumption 2 holds then $\hat{\mathsf{m}}(N) = \max \mathsf{j} \in \{2,\ldots,\hat{\mathsf{m}}(i)\}$ in which $\hat{E}_{\mathsf{j}}(N) = \hat{E}_1(N)$.
vi) $\forall \mathsf{j} \in \{\hat{\mathsf{m}}(N)+1,\ldots,\mathsf{m_c}\}\ u_{\mathsf{j}}(i) = v_{\mathsf{j}}(i)$
vii) let $\hat{\mathsf{m}} = \hat{\mathsf{m}}(N)$ compute $u_1(i)$ using the right hand side of (19) and $\forall \mathsf{j} \in \{1,\ldots,\hat{\mathsf{m}}(N)\}$ set $u_{\mathsf{j}}(i) = u_1(i)$, and compute
$$E_{\mathsf{j}}(N) = E_{\mathsf{j}}(N-1) + (u_1^{\mathsf{T}}(i)u_1(i) - v_{\mathsf{j}}^{\mathsf{T}}(i)v_{\mathsf{j}}(i)).$$
viii) set $\hat{\mathsf{m}} = \hat{\mathsf{m}}(N)$ and compute $v_{\mathsf{n}}(i)$ by using the resilient power junction equation (20)
$$\begin{aligned}\mathsf{sf}_v &= \frac{|\sum_{\mathsf{j}=1}^{\hat{\mathsf{m}}} v_{\mathsf{j}l}|}{\sum_{\mathsf{j}=1}^{\hat{\mathsf{m}}} |v_{\mathsf{j}l}|} \\ v_{\mathsf{n}_l}(i) &= \mathsf{sf}_v \mathsf{sgn}(\sum_{\mathsf{j}=1}^{\hat{\mathsf{m}}} v_{\mathsf{j}l}(i)) \sqrt{\sum_{\mathsf{j}=1}^{\hat{\mathsf{m}}} v_{\mathsf{j}l}^2(i)}.\end{aligned} \tag{20}$$
ix) $i = N$ repeat ii)-viii)

Lemma 2: The resilient power junction has the following properties:

i) it satisfies [8, Definition 1] for the power junction as a result:
$$\begin{aligned}\|(u_{\mathsf{n}})_N\|_2^2 - \|(v_{\mathsf{n}})_N\|_2^2 &\geq \sum_{\mathsf{j}=1}^{\mathsf{m}} (\|(u_{\mathsf{j}})_N\|_2^2 - \|(v_{\mathsf{j}})_N\|_2^2) \\ &\quad + \sum_{\mathsf{j}=\mathsf{m}+1}^{\mathsf{m_c}} (\|(u_{\mathsf{j}})_N\|_2^2 - \|(v_{\mathsf{j}})_N\|_2^2)\end{aligned}$$
ii) in addition, when Assumption 2 holds and after the final non-redundant controller has been detected at time $N_{(\mathsf{m}+1)-\mathsf{detect}}$ and the corresponding finite-energy offset which will remain constant for all $\forall\ N \geq N_{(\mathsf{m}+1)-\mathsf{detect}}$ is assumed to equal zero, then (21) holds.
$$\|(u_{\mathsf{n}})_N\|_2^2 - \|(v_{\mathsf{n}})_N\|_2^2 \geq \sum_{\mathsf{j}=1}^{\mathsf{m}} (\|(u_{\mathsf{j}})_N\|_2^2 - \|(v_{\mathsf{j}})_N\|_2^2) \tag{21}$$

D. L_2^m-Stable Network

In order to show L_2^m stability of our digital control network depicted in Fig. 1 we need to relate $\forall \mathsf{j} \in \{1,\ldots,\mathsf{m}\}$ the discrete-time reference and effort variables associated with each passive digital controller G_{cj} (denoted by the respective tuple $(r_{\mathsf{cj}}(i), e_{\mathsf{cj}}(i))$) to a continuous-time reference and effort variable counterpart which we denote by the respective tuple $(r_{\mathsf{cj}}(t), e_{\mathsf{cj}}(t))$. In order to make this comparison we used the *inner-product equivalent sampler* (denoted $\mathsf{IPES_j}$ in Fig. 1)

and a zero-order-hold (denoted ZOH_j in Fig. 1). We will refer to the pair of these devices as the *inner-product equivalent sample and hold* (*IPESH*).

Definition 4: [12], [13] The m-*inner-product equivalent sample and hold*'s depicted in Fig. 1 by the pair of respective symbols (IPES_j,ZOH_j) $\mathsf{j} \in \{1, \ldots, \mathsf{m}\}$ in which the inputs are denoted by the pair $(r_{\mathsf{cj}}(t), e_{cHj}(i))$ and the outputs are denoted by the pair $(r_{\mathsf{cSj}}(i), e_{\mathsf{cj}}(t))$. The *inner-product equivalent sampler* (*IPES*) is implemented by sampling $r_{\mathsf{cj}}(t)$ at a rate (T_s) such that $\forall N > 0$:

$$\begin{aligned} x(t) =& \int_0^t r_{\mathsf{cj}}(\tau) d\tau \\ r_{\mathsf{cSj}}(i) =& x((i+1)T_s) - x(iT_s). \end{aligned} \tag{22}$$

The *ZOH* is implemented as follows:

$$e_{\mathsf{cj}}(t) = e_{\mathsf{cHj}}(i), \ t \ \in [iT_s, (i+1)T_s) \tag{23}$$

Corollary 3: Using the *IPESH* as stated in Definition 4 we have that

$$\langle e_{\mathsf{cj}}, r_{\mathsf{cj}} \rangle_{NT_s} = \langle e_{\mathsf{cHj}}, r_{\mathsf{cSj}} \rangle_N \text{ holds.} \tag{24}$$

Using the *ZOH* as stated in Definition 4 we also have the property that

$$\|(e_{\mathsf{cj}})_{NT_s}\|_2^2 = T_s \|(e_{\mathsf{cHj}})_N\|_2^2 \text{ holds.} \tag{25}$$

Finally Fig. 1 possesses some scalar scaling gains $k_\mathsf{s} \in \mathbf{R}^+$ to account for the using the power-junction, PS and PH and the *IPESH*, such that for all $\mathsf{j} \in \{1, \ldots, \mathsf{m}\}$:

$$r_{\mathsf{cj}}(i) = -\, k_{\mathsf{sj}} r_{\mathsf{cSj}}(i) \tag{26}$$

$$e_{\mathsf{cj}}(i) = -\, \frac{1}{k_{\mathsf{sj}}} e_{\mathsf{cHj}}(i). \tag{27}$$

Using Corollary 3, (26), and (27) we have the following relationships

$$\begin{aligned} \langle e_{\mathsf{cj}}, r_{\mathsf{cj}} \rangle_N =& \langle e_{\mathsf{cHj}}, r_{\mathsf{cSj}} \rangle_N \\ \langle e_{\mathsf{cj}}, r_{\mathsf{cj}} \rangle_N =& \langle e_{\mathsf{cj}}, r_{\mathsf{cj}} \rangle_{NT_s} \\ \|(e_{\mathsf{cj}})_N\|_2^2 =& \frac{1}{k_{\mathsf{sj}}^2} \|(e_{\mathsf{cHj}})_N\|_2^2 \end{aligned} \tag{28}$$

$$\|(e_{\mathsf{cj}})_N\|_2^2 = \frac{1}{T_s k_{\mathsf{sj}}^2} \|(e_{\mathsf{cj}})_{NT_s}\|_2^2. \tag{29}$$

Theorem 1: For the network controlled system depicted in Fig. 1, the resilient power junction (Definition 3) is used and Assumption 1 is satisfied, then the combined system in regards to the plant G_{pn}, and redundant and non-redundant controllers G_{cj} $\forall \mathsf{j} \in \{1, \ldots, \mathsf{m_c}\}$:

I. is L_2^m*-stable* if the plant $G_{\mathsf{pn}}(e_{\mathsf{pn}}(t))$ and all controllers G_{cj} $\forall \mathsf{j} \in \{1, \ldots, \mathsf{m_c}\}$ are *strictly-output passive*.

II. *passive* if the plant $G_{\mathsf{pn}}(e_{\mathsf{pn}}(t))$ and all controllers G_{cj} $\forall \mathsf{j} \in \{1, \ldots, \mathsf{m_c}\}$ are *passive*.

The proof of Theorem 1 is in Appendix A-A2. From Lemma 2 (21) is satisfied, therefore from Theorem 1 we can state the following corollary.

Corollary 4: For the network controlled system depicted in Fig. 1, the resilient power junction (Definition 3) is used and Assumption 2 is satisfied, then for $N \geq N_{(\mathsf{m}+1)-\mathsf{detect}}$ the combined system in regards to the passive plant G_{pn}, and the remaining m passive controllers G_{cj} $\forall \mathsf{j} \in \{1, \ldots, \mathsf{m}\}$ is:

I. L_2^m*-stable* if the plant $G_{\mathsf{pn}}(e_{\mathsf{pn}}(t))$ and all passive controllers G_{cj} $\forall \mathsf{j} \in \{1, \ldots, \mathsf{m}\}$ are *strictly-output passive*.

II. *passive* if the plant $G_{\mathsf{pn}}(e_{\mathsf{pn}}(t))$ is *passive*.

III. Simulations

In this section we shall control a single *strictly-output passive* continuous time plant with 3 *strictly-output passive* 'PID'-digital controllers, and 1 system destabilizing-digital controller if it is not properly detected and isolated. The plant is described by the following equation:

$$G_{\mathsf{pn}}(s) = \frac{k_{\mathsf{pn}}}{s + \omega_{\mathsf{pn}}} = \frac{2}{s+5},$$

The *strictly-output passive* 'PID'-digital controllers are of the following form:

$$G_{\mathsf{PID}}(z) = k_\mathsf{P} + G_\mathsf{I}(z) + G_\mathsf{D}(z)$$

in which $k_\mathsf{P} > 0$ is the proportional term, $G_\mathsf{I}(z)$ is the 'integral' term which is synthesized by applying the *IPESH-transform* [9, Definition 4] to the following continuous-time 'integrator' model (N.B. this is an integrator with finite-gain, such as seen when using a lag-compensator, in which $\epsilon > 0$ can be arbitrarily small in order to satisfy our *strictly-output passive* condition on the controller)

$$G_\mathsf{I}(s) = \frac{k_\mathsf{I}}{s + \epsilon k_\mathsf{I}}.$$

Similarly, $G_\mathsf{D}(z)$ is the 'derivative' term which is synthesized by applying the *IPESH-transform* to the following continuous-time 'derivative' model

$$G_\mathsf{D}(s) = k_\mathsf{D} \frac{\frac{NT_s}{\pi} s + 1}{\frac{T_s}{\pi} s + 1}.$$

Note that $N > 1$, is typically chosen to be around 10. With our nominal plant given, we use the following loop-shaping formulas to select the control gains in terms of the nyquist frequency $\omega_{\mathsf{nyquist}} = \frac{\pi}{T_s}$.

$$\begin{aligned} k_\mathsf{P} =& \alpha \frac{1}{3} \frac{\omega_{\mathsf{nyquist}} + \omega_{\mathsf{pn}}}{k_{\mathsf{pn}}} \\ k_\mathsf{I} =& \alpha \frac{1}{3} \frac{\omega_{\mathsf{nyquist}} (\omega_{\mathsf{nyquist}} + \omega_{\mathsf{pn}})}{k_{\mathsf{pn}}} \\ k_\mathsf{D} =& \alpha \frac{1}{3} \frac{2}{1+N} \frac{\omega_{\mathsf{nyquist}} + \omega_{\mathsf{pn}}}{k_{\mathsf{pn}}}. \end{aligned}$$

Other relevant parameters to the simulation are $b = 2$, $T_s = .1$, $\alpha = 1$, $\epsilon = .001$, $N = 10$. The unstable controller consisted of the discrete-time version of the *negative* 'PID'-digital controller with the integrator replaced with *three*-integrators.

Fig. 2 shows the nominal system response when all controllers are redundant, as we can see, the controller is able to reject periodic step-like disturbances. Fig. 3 shows the

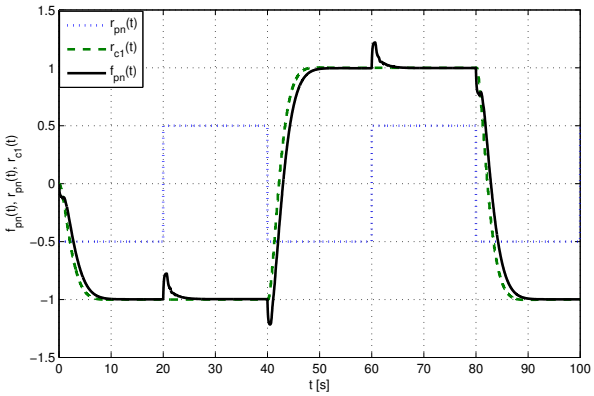

Fig. 2. Nominal system response when using the *resilient power junction* under Assumption 1 $\mathsf{m_c} = \mathsf{m}$.

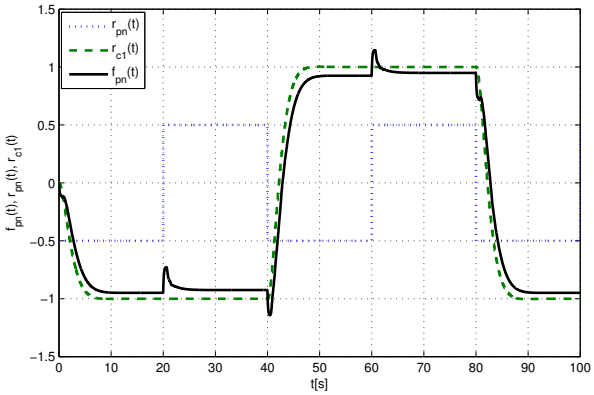

Fig. 3. System response when using the *resilient power junction* under Assumption 1 and integrator term of controller 4 is set to zero.

effect when one of the controllers is corrupted in a passive manner and looses its integrator-term, when controllers loose there proportional-term the overall degredation in performance is barely noticable. Fig. 4 shows that intermittent denial-of-service attacks lead to a graceful degradation and recovery of performance as a single controller is being attacked on the network. We should note that the denial of service attack can also be thought of as a single controller loosing, its integral, proportional term and derivative term set to zero. Fig. 5 shows that in a very short period of time, the introduction of the destabilizing controller with non-redundant-controller-detection disabled system instability will occur. Fig 6 indicates that when non-redundant-controller-detection is enabled the destabilizing-controller is isolated from the rest of the network and not only is stability preserved, but disturbances from $r_{\mathsf{pn}}(t)$ are completely eliminated.

IV. Conclusions

In this paper we have described how a general technique: passivity, and a particular controller structure involving the resilient power junction can be used. The resilient power junction operating under Assumption 1 when interconnected

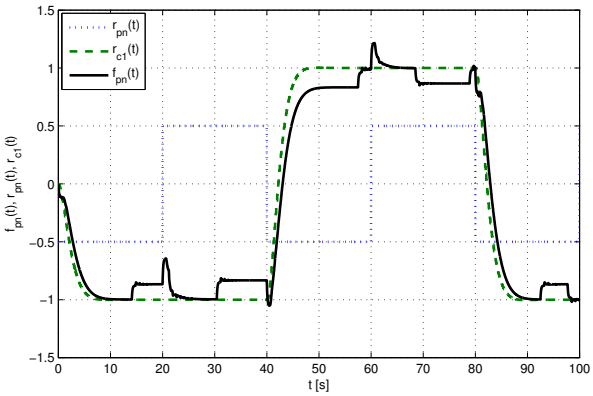

Fig. 4. System response when using the *resilient power junction* under Assumption 1 and introducing a intermitent denial of service attack to controller 4.

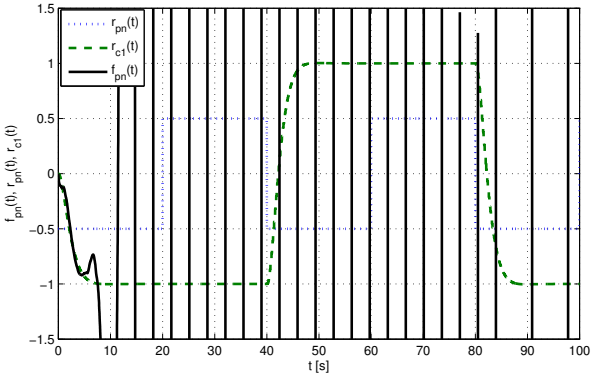

Fig. 5. Unstable system response when using the *resilient power junction* under Assumption 1 and introducing a higly unstable (and non-passive) digital-controller to the network.

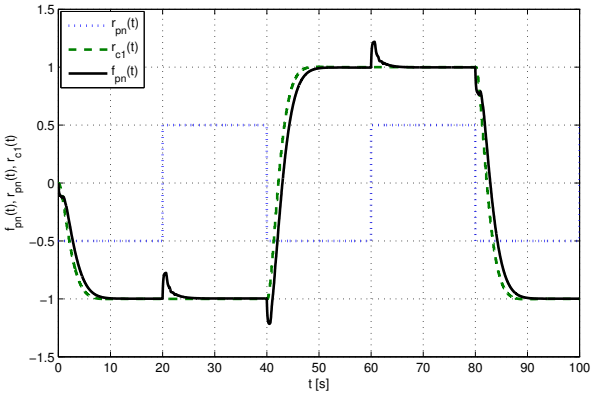

Fig. 6. Stable system response when using the *resilient power junction* under Assumption 2.

to m_c-redundant controllers and a single plant will always perform well under both denial-of-service attacks on individual controllers and degrade gracefully as additional *strictly-output passive* 'corrupted' digital controllers are introduced into the network. However, when introducing a highly-unstable controller into the network great care must be taken in order to identify and isolate the digital controller. Assumption 1 had to be made quite a bit-stricter in order to isolate these unstable controllers, in particular the time-delays and data-dropouts needed to be identical when transmitting controller wave variables to and from the power-junction. This can be fairly easily satisfied on a real-time-operating system but more difficult over a network. We did provide the important result, however, that controllers can be removed without either destabilizing the system and showed that they can still maintain uninterrupted performance.

The theoretical framework presented gives a tool to the control engineer for building digital control systems that can survive, and even 'operate through' attacks, while maintaining the quality of control. Naturally, there are critical points in the implementation (e.g. the realization of the resilient power junction) that needs to be created with great care. In any case, passivity-based approaches to controller design provide a promising direction for designing controllers that are significantly more robust than other techniques. As illustrated, mathematical proofs exist for their properties, and they could be widely applied to linear and non-linear plants alike.

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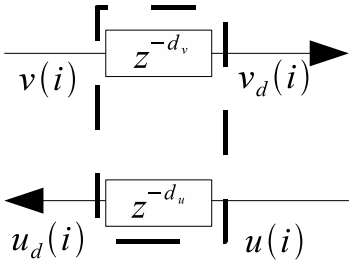

Fig. 7. Generalized wave variable delay figure.

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APPENDIX A
WAVE VARIABLES RESILIENCE TO TIME VARYING DELAYS

It is well established for the continuous-time plant and controller framework that wave variables allow both *effort* and *flow* variables to be transmitted over a network in a *passive* manner when subject to arbitrary fixed time delays and data dropouts [14], [15]. More recently the conditions required on the time-delay characteristics of discrete-time wave variables has been established. From some of the work involving discrete-time wave variables the engineer may be led to believed that any arbitrary discrete-time delay can be tolerated. This indeed is not the case, Proposition 1 makes this explicitly clear by summarizing recent observations made in [7], [13], [16], [17].

Proposition 1: More generally, given the two pairs of wave variables $(u(i), v_d(i))$, $(u_d(i), v(i))$ depicted in Fig. 7 in which the received-waves with the d-subscript are related to their corresponding non-delayed transmitted-counterparts such that

$$u_d(i) = \begin{cases} u(i - d_u(i)), & \text{if } d_u(i) \leq i \\ 0, & \text{otherwise.} \end{cases}$$

$$v_d(i) = \begin{cases} v(i - d_v(i)), & \text{if } d_v(i) \leq i \\ 0, & \text{otherwise.} \end{cases}$$

where $d_u(i),\ d_v(i) \in \{1, 2, \ldots, \}$ is the respective delay at

time i. A necessary condition for

$$\sum_{i=0}^{N-1} u^{\mathsf{T}}(i)u(i) - v_d^{\mathsf{T}}(i)v_d(i) \geq \sum_{i=0}^{N-1} u_d^{\mathsf{T}}(i)u_d(i) - v^{\mathsf{T}}(i)v(i) \tag{30}$$

or equivalently

$$\sum_{i=0}^{N-1} u^{\mathsf{T}}(i)u(i) - u_d^{\mathsf{T}}(i)u_d(i) + \sum_{i=0}^{N-1} v^{\mathsf{T}}(i)v(i) - v_d^{\mathsf{T}}(i)v_d(i) \geq 0$$

to be satisfied for all $N > 0$ is that both

$$\sum_{i=0}^{N-1} u^{\mathsf{T}}(i)u(i) - u^{\mathsf{T}}(i - d_u(i))u(i - d_u(i)) \geq 0 \text{ and}$$

$$\sum_{i=0}^{N-1} v^{\mathsf{T}}(i)v(i) - v^{\mathsf{T}}(i - d_v(i))v(i - d_v(i)) \geq 0$$

are satisfied for all $N > 0$. Therefore:

I. if delays are fixed ($d_u(i) = d_u, d_v(i) = d_v$) then (30) is always satisfied,

II. if the delays are such that data is always dropped ($d_u(i) = d_v(i) = (i+1)$) then (30) is always satisfied,

III. if the delays are switched arbitrarily between a constant delay or a drop-out delay ($d_u(i) \in \{d_u, (i+1)\}$ and ($d_v(i) \in \{(i+1), d_v\}$)) then (30) is always satisfied,

IV. if the delays are such that no duplicate wave-transmissions are processed then (30) is always satisfied, more precisely if we denote the set of received indexes up to time $N-1$ for u_d and v_d as $\mathcal{D}_u = \{0 - d_u(0), 1 - d_u(1), \ldots, (N-1) - d_u(N-1)\}$ and $\mathcal{D}_v = \{0 - d_v(0), 1 - d_v(1), \ldots, (N-1) - d_v(N-1)\}$ respectively and
   - each index $i \in \{0, 1, \ldots, N-1\}$ appears in $\mathcal{D}_u$ no more than once and
   - each index $i \in \{0, 1, \ldots, N-1\}$ appears in $\mathcal{D}_v$ no more than once.

   An example of a delay which violates this final condition is when $d_u(i) = i$ in which $\mathcal{D}_u = \{0, 0, \ldots, 0\}$ and the index 0 appears N times.

TCP/IP is a transmission protocol which will satisfy (30) however the UDP protocol could replicate packets and violate (30). Applications which choose to use UDP can be easily modified to satisfy Propositions 1-IV.

A. Additional Proofs

1) Lemma 1: *Proof:* Summing the both sides of (17) with respect to index $i \in \{0, 1, \ldots, N\}$ we have:

$$\|(u_{\mathsf{n}})_N\|_2^2 - \|(v_{\mathsf{n}})_N\|_2^2 \geq \sum_{\mathsf{j}=1}^{\mathsf{m}} (\|(u_{\mathsf{j}})_N\|_2^2 - \|(v_{\mathsf{j}})_N\|_2^2) + \sum_{\mathsf{j}=\mathsf{m}+1}^{\mathsf{m_c}} (\|(u_{\mathsf{j}})_N\|_2^2 - \|(v_{\mathsf{j}})_N\|_2^2), \tag{31}$$

take the left-hand-side (LHS) of (15) into the LHS of (31), likewise substitute the right-hand-side (RHS) of (16) into the RHS of (31) which yields (18). ■

2) Theorem 1: *Proof:* We recall from Lemma 1 that if any of the conditions listed in Proposition 1 are met for the wave variable communication time-delays $c_{\mathsf{j}}(i) = c_{\mathsf{n}}(i) = d_u(i)$, $p_{\mathsf{j}}(i) = p_{\mathsf{n}}(i) = d_v(i)$ that

$$\langle f_{\mathsf{pn}}, e_{\mathsf{dcn}} \rangle_{NT_s} \geq \sum_{\mathsf{j}=1}^{\mathsf{m_c}} \langle e_{\mathsf{cj}}, f_{\mathsf{dpj}} \rangle_N \tag{32}$$

holds for all $N \geq 1$. We recall, that the *strictly-output passive* plant satisfies

$$\langle f_{\mathsf{pn}}, e_{\mathsf{pn}} \rangle_{NT_s} \geq \epsilon_{\mathsf{pn}} \|(f_{\mathsf{pn}})_{NT_s}\|_2^2 - \beta_{\mathsf{pn}} \tag{33}$$

while each *strictly-output passive* controller for $\mathsf{j} \in \{1, \ldots, \mathsf{m_c}\}$ satisfies (34).

$$\langle e_{\mathsf{cj}}, f_{\mathsf{cj}} \rangle_N \geq \epsilon_{\mathsf{cj}} \|(e_{\mathsf{cj}})_N\|_2^2 - \beta_{\mathsf{cj}} \tag{34}$$

In addition, we can substitute (29) into (34) which yields

$$\langle e_{\mathsf{cj}}, f_{\mathsf{cj}} \rangle_N \geq \frac{\epsilon_{\mathsf{cj}}}{T_s k_s^2} \|(e_{\mathsf{cj}})_{NT_s}\|_2^2 - \beta_{\mathsf{cj}}. \tag{35}$$

Substituting, $e_{\mathsf{dcn}} = r_{\mathsf{pn}} - e_{\mathsf{pn}}$ and $f_{\mathsf{dpj}} = f_{\mathsf{cj}} - r_{\mathsf{cj}}$ into (32) yields

$$\langle f_{\mathsf{pn}}, r_{\mathsf{pn}} - e_{\mathsf{pn}} \rangle_{NT_s} \geq \sum_{\mathsf{j}=1}^{\mathsf{m_c}} \langle e_{\mathsf{cj}}, f_{\mathsf{cj}} - r_{\mathsf{cj}} \rangle_N$$

which can be rewritten as

$$\langle f_{\mathsf{pn}}, r_{\mathsf{pn}} \rangle_{NT_s} + \sum_{\mathsf{j}=1}^{\mathsf{m_c}} \langle e_{\mathsf{cj}}, r_{\mathsf{cj}} \rangle_N \geq \langle f_{\mathsf{pn}}, e_{\mathsf{pn}} \rangle_{NT_s} + \sum_{\mathsf{j}=1}^{\mathsf{m_c}} \langle e_{\mathsf{cj}}, f_{\mathsf{cj}} \rangle_N \tag{36}$$

so that we can then substitute (33), (35), and (28) into (36) to yield

$$\langle f_{\mathsf{pn}}, r_{\mathsf{pn}} \rangle_{NT_s} + \sum_{\mathsf{j}=1}^{\mathsf{m_c}} \langle e_{\mathsf{cj}}, r_{\mathsf{cj}} \rangle_{NT_s} \geq \epsilon [\|(f_{\mathsf{pn}})_{NT_s}\|_2^2 + \sum_{\mathsf{j}=1}^{\mathsf{m_c}} \|(e_{\mathsf{cj}})_{NT_s}\|_2^2] - \beta \tag{37}$$

in which $\epsilon = \min(\epsilon_{\mathsf{pn}}, \frac{\epsilon_{\mathsf{cj}}}{T_s k_s^2})$, $\mathsf{j} \in \{1, \ldots, \mathsf{m_c}\}$ and $\beta = \beta_{\mathsf{pn}} + \sum_{\mathsf{j}=1}^{\mathsf{m_c}} \beta_{\mathsf{cj}}$. Thus (37) satisfies [8, Definition 4-iii)] for *strictly-output passivity* in which the input is the row vector of all controller and plant inputs $[r_{\mathsf{c1}}, \ldots, r_{\mathsf{cm_c}}, r_{\mathsf{pn}}]$, and the output is the row vector of all controller and plant outputs $[e_{\mathsf{c1}}, \ldots, e_{\mathsf{cm_c}}, f_{\mathsf{pn}}]$. When we let $\epsilon_{\mathsf{pn}} = \epsilon_{\mathsf{cj}} = 0$ we see that all the plants and controllers are *passive*, therefore the system depicted in Fig. 1 is *passive*. ■