On-Line Approximation Control of Uncertain Nonlinear Systems: Issues with Control Input Saturation[1]

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Abstract

Various types and techniques of on-line approximation have been used in feedback control of uncertain nonlinear systems. In many practical applications, saturation of the control input influences significantly the performance of adaptive and learning control systems. This article addresses the issue of control input saturation in on-line approximation based control of nonlinear systems. A modified control design framework is presented for preventing the presence of input saturation from destroying the learning capabilities and memory of an on-line approximator in feedback control systems. The stability properties of the proposed feedback control law are obtained via Lyapunov analysis. Particular emphasis is given to aircraft longitudinal control, which extends the results to the backstepping feedback control procedure.

1 Introduction

A variety of feedback control approaches have been developed to deal with nonlinear systems, including feedback linearization [5], sliding mode control [9], and backstepping [11]. In their ideal form, both feedback linearization and backstepping rely on cancellation of known nonlinearities. To address the issue of uncertainty, several "robustifying" techniques have been developed: (i) *adaptive* methods deal with parametric uncertainty [7], where the nonlinearities are assumed to be known but some of the parameters that multiply these nonlinearities are unknown or uncertain; (ii) *robust* methods [2] deal with the case where known upper bounds on the unknown nonlinearities are available and therefore, they tend to be conservative, sometimes leading to high-gain feedback; (iii) *robust adaptive methods* [14] combine parametric uncertainty and unknown nonlinearities with partially known bounds.

The above control techniques are based on the assumption that the plant nonlinearities are either known or can be bounded by some known functions. In many applications, including control of high performance aircraft and uninhabited air vehicles (UAVs), some of the nonlinearities need to be approximated on-line. This may be due to modeling errors during the identification/modeling phase or, quite often, due to time-variations in the dynamics as a result of changes in the operating conditions or due to component wear or battle damage. To address the issue of unknown nonlinearities, various control system architectures have incorporated various network models as on-line approximators of unknown nonlinearities.

The application of on-line approximation methods to nonlinear systems in a feedback framework yields a complex nonlinear closed-loop system, which is analyzed using Lyapunov stability methods. Typically, the feedback control law and the adaptive law for updating the network weights are derived by utilizing a Lyapunov function, whose time derivative is forced to have some desirable stability properties (for example, negative semi-definiteness). Therefore, the stability of the closed-loop system is obtained during the synthesis of the adaptive control laws. Examples of this type of approach, which is referred to as Lyapunov synthesis method, include [3, 4, 10, 13, 15, 16].

From a practical perspective, one of the key problems in feedback control systems is that the signal $u(t)$ generated by the control law cannot be implemented due to physical constraints. A common example of such constraint is input saturation, which imposes limitations on the magnitude of the control input. In some applications this problem is crucial, especially in combination with nonlinear on-line approximation based control, which tends to be aggressive in seeking the desired tracking performance. In aircraft control applications, input saturation is caused by limitations in control surface deflections. For UAVs, the absence of humans in the air vehicle may allow more aggressive maneuvering, however the feedback control law has to deal both with unknown nonlinearities and input saturation.

[1]This research has been partially supported by the US Air Force (F33615-01-C-3108).

Input saturation in an adaptive linear control framework has been addressed in, for example, [1, 6, 8, 12]. One possible approach is to completely stop adaptation during saturation of the control input. While this ad-hoc method does prevent the tracking error induced by actuator constraints from corrupting parameter estimation, the stability properties of the closed-loop system cannot be established. Another approach that has been proposed, which we refer to as training signal hedging (TSH), see e.g. [1, 8], modifies the tracking error definition used in the parameter update laws. Finally, a third approach, referred to as pseudo-control hedging (PCH), alters the commanded input to the loop [6, 12]. The idea behind the PCH approach is to attenuate the command to the loop so that the generated control signal is implementable without saturation.

This article addresses the issue of control input saturation in on-line approximation based control systems. A modified control design framework is presented for preventing the presence of input saturation from destroying the learning capabilities and memory of an on-line approximator in feedback control systems. The design method is based on the TSH approach. The stability properties of the proposed feedback control law are obtained via Lyapunov analysis. Two design schemes are presented: the first is based on a direct learning control scheme and the second is based on an indirect learning control scheme. Particular emphasis is given to aircraft longitudinal control, which extends the results to the backstepping feedback control procedure.

2 Problem Formulation

To facilitate a more intuitive understanding of the issues of input saturation in adaptive and learning control systems, we start with a simple scalar system. Later on, the design and analysis techniques are extended to a class of higher-order systems. Consider the scalar nonlinear system

$$\dot{x} = f_0(x) + f(x) + g(x)u \tag{1}$$

where $x \in \mathcal{R}$ is the measured output, f_0 is a known function, $f(x)$ and $g(x)$ are unknown nonlinear functions to be approximated on-line, and u is the control signal. Suppose that the control objective is for $x(t)$ to track $x_r(t)$, which is the output of a reference model

$$\dot{x}_r = -\alpha x_r + \beta r, \tag{2}$$

where $\alpha > 0$, β are known parameters and $r(t)$ is a measurable command signal. Using on-line approximation methods, $\hat{f}(x, \hat{\theta}_f) = \hat{\theta}_f^\top \phi_f(x)$ and $\hat{g}(x, \hat{\theta}_g) = \hat{\theta}_g^\top \phi_g(x)$ are used as approximators[1] of $f(x)$ and $g(x)$ respectively, where $\hat{\theta}_f$, $\hat{\theta}_g$ are the adjustable parameters of the on-line approximator in vector form, and ϕ_f, ϕ_g, are the coresponding regressors or basis functions. Using standard techniques from adaptive and on-line approximation based control, a feedback control law of the form

$$\begin{aligned} u &= \frac{1}{\hat{g}(x, \hat{\theta}_g)} \left[-f_0(x) - \hat{f}(x, \hat{\theta}_f) - \alpha x + \beta r \right] && (3) \\ \dot{\hat{\theta}}_f &= \Gamma_f (x - x_r) \phi_f(x) && (4) \\ \dot{\hat{\theta}}_g &= \Gamma_g (x - x_r) \phi_g(x) u && (5) \end{aligned}$$

can be designed to achieve the desired tracking objective (where Γ_f, Γ_g are positive definite matrices representing the learning rate of the on-line approximation). The stability properties of the above feedback control law can be derived under certain conditions. Specifically, if: (i) the Minimum Functional Approximation Error (MFAE) between $f(x)$, $g(x)$ and $\hat{f}(x, \hat{\theta}_f)$, $\hat{g}(x, \hat{\theta}_g)$ respectively is zero; and (ii) the update law (5) is appropriately modified to guarantee that $\hat{g}(x, \hat{\theta}_g)$ does not approach zero (to avoid stabilizability problems), then it can be shown using Lyapunov analysis that all the variables remain bounded and the tracking error converges to zero (see, e.g., [4, 13, 15, 16]).

In many applications (including aircraft systems), the control law described by (3) may not always be realizable due to actuator constraints such as saturation. Assume that the control input u is constrained by the following known, constant upper/lower limit bounds: $u_L \le u(t) \le u_U$. In a more general setting, the limit bounds u_L, u_U may be functions of the state x.

Due to actuator saturation, the actual feedback learning control law being implemented is different from (3), (4), (5) as follows:

$$\begin{aligned} u &= \mathrm{sat}(u_0, u_L, u_U) && (6) \\ u_0 &= \frac{1}{\hat{g}(x, \hat{\theta}_g)} \left[-f_0(x) - \hat{f}(x, \hat{\theta}_f) - \alpha x + \beta r \right] && (7) \\ \dot{\hat{\theta}}_f &= \Gamma_f (x - x_r) \phi_f(x) && (8) \\ \dot{\hat{\theta}}_g &= \Gamma_g (x - x_r) \phi_g(x) u && (9) \end{aligned}$$

where the saturation function "sat" is linear with unity slope between its lower and upper limits; i.e.,

$$\mathrm{sat}(u_0, u_L, u_U) = \begin{cases} u_L & \text{if} \quad u_0 < u_L \\ u_0 & \text{if} \quad u_L \le u_0 \le u_U \\ u_U & \text{if} \quad u_0 > u_U \end{cases} .$$

Even though the problem formulation has been simplified to illustrate the key issues more clearly, it can be readily shown that the modified control law (6)-(9), with saturation limits, cannot guarantee the stability of the closed-loop system. In practice, the control design problem maybe more complicated due to the presence

[1] For simplicity, in this article we use linearly parametrized approximators. The case of nonlinearly parametrized approximators can also be considered by appropriate handling of the higher-order terms [15].

of several actuators that are available in the implementation of the control signal.

A key issue is what happens to learning during saturation. It is expected that during saturation the magnitude of the tracking error will increase, since the control signal is not being achieved. This tracking error is not the result of function approximation error, therefore we need to be careful so that the approximator does not cause "unlearning" during the period when the actuators are saturated.

How do we modify the adaptive laws (8), (9) for updating $\hat{\theta}_f$, $\hat{\theta}_g$ during saturation? Clearly, the adaptive laws (8), (9) depend on the tracking error $x - x_r$, therefore if the tracking error increases due to saturation, the adaptive law may cause a significant change in the weights in response to the increase in tracking error (the tracking error is driving the adaptation). A very simple solution may be to turn off adaptation in the event of saturation. However, this is not necessarily the best solution. Next, we develop two modified learning control schemes to address the input saturation problem.

3 Direct Learning Control Scheme

The first approach is based on a direct learning control framework. According to this approach, the standard feedback learning control law (6)-(9) is modified as follows:

$$u = \mathrm{sat}(u_0, u_L, u_U) \tag{10}$$

$$u_0 = \frac{1}{\hat{g}(x,\hat{\theta}_g)}\left[-f_0(x) - \hat{f}(x,\hat{\theta}_f) - \alpha x + \beta r + \eta\right] \tag{11}$$

$$\dot{\hat{\theta}}_f = \Gamma_f (x - x_r - \chi)\phi_f(x) \tag{12}$$

$$\dot{\hat{\theta}}_g = \Gamma_g (x - x_r - \chi)\phi_g(x) u \tag{13}$$

$$\dot{\chi} = -\alpha\chi + \hat{g}(x,\hat{\theta}_g)(u - u_0) \tag{14}$$

The above modified control law involves the utilization of the χ signal, which is a filtered version of the effect of input saturation on the variable being controlled. Note that in the case of no input saturation, then χ remains zero and the control law becomes the same as the standard learning control law described in the previous section. In the presence of input saturation, χ is non-zero, thus giving rise to a modified tracking error $\bar{x} = x - x_r - \chi$. The signal η in the specification of u_0 in equation (11) will be used later on to deal with the presence of functional approximation error.

Next, we proceed to investigate the stability properties of the closed-loop system, based on the learning control law described by (10)-(14). Consider the Lyapunov function $\mathcal{V} = \frac{1}{2}\bar{x}^2 + \frac{1}{2}\tilde{\theta}_f^\top \Gamma_f^{-1}\tilde{\theta}_f + \frac{1}{2}\tilde{\theta}_g^\top \Gamma_g^{-1}\tilde{\theta}_g$, where $\tilde{\theta}_z = \hat{\theta}_z - \theta_z^*$ and $z \in \{f,\ g\}$. The "optimal" parameter vector θ_f^* is defined as the weight vector of the on-line approximator that minimizes the functional approximator error between $f(x)$ and $\hat{f}(x,\theta_f)$ over a compact (i.e., closed and bounded) operating envelope $\mathcal{D}$ (correspondingly for $g(x)$):

$$\theta_z^* = \arg\min_{\theta_z}\left(\sup_{x\in\mathcal{D}} |z(x) - \hat{z}(x,\theta_z)|\right), \qquad z \in \{f,\ g\}$$

The minimum possible distance between $f(x)$ and $\hat{f}(x,\theta_f)$ over $\mathcal{D}$ is referred to as minimum functional approximation error (MFAE) over $\mathcal{D}$ and is denoted by $\varepsilon_z(x,\mathcal{D}) = z(x) - \hat{z}(x,\theta_z)|_{\theta_z=\theta_z^*}$ for $z \in \{f,\ g\}$. The bound on ε_z is denoted by $\bar{\varepsilon}_z = \sup_{x\in\mathcal{D}}|\varepsilon_z(x,\mathcal{D})|$. It is noted that ε_z, θ_f^*, θ_g^* are required only for the theoretical analysis, not for the design of the learning control law. On the other hand, the bound $\bar{\varepsilon}_z$ is required also in the design in order to robustify the learning scheme with respect to the possible presence of MFAEs. The requirement of assuming a known bound on the MFAE can be relaxed by using adaptive bounding techniques [13], where the bound is estimated on-line using adaptive methods.

By taking the time derivative of $\mathcal{V}$ along the solutions of (1), (2), (12)-(14) and taking into consideration the implemented control law (10), (11), we obtain

$$\begin{aligned}
\dot{\mathcal{V}} &= \bar{x}\left(f_0 + f + gu + \alpha x_r - \beta r + \alpha\chi - \hat{g}(u - u_0)\right) \\
&\quad + \left(\tilde{\theta}_f^\top \phi_f + \tilde{\theta}_g^\top \phi_g u\right) \\
&= -\alpha\bar{x}^2 + \bar{x}\left(\varepsilon_f + \varepsilon_g u + \eta\right) \\
&\le -\alpha\bar{x}^2 + |\bar{x}|\bar{\varepsilon} + \bar{x}\eta
\end{aligned} \tag{15}$$

where $\bar{\varepsilon}$ is a bound on the combined approximation error $\varepsilon_f + \varepsilon_g u$ over the operating envelope $\mathcal{D}$.

In the ideal case of zero approximation error, η can be set to zero, thus yielding $\dot{\mathcal{V}} = -\alpha\bar{x}^2$. If the approximation error is non-zero, but an upper bound $\bar{\varepsilon}$ is available (known), then the robustifying term η could be chosen as $\eta = -k\ \mathrm{sgn}(\bar{x})$, where sgn denotes the sign function and $k > 0$ is a constant gain function satisfying $k \ge \bar{\varepsilon}$. In this case, $|\bar{x}|\bar{\varepsilon} + \bar{x}\eta = -(k - \bar{\varepsilon})|\bar{x}| \le 0$. Alternatively, η could be chosen as $\eta = -k_0 q(\bar{x})$, where $k_0 > 0$ is a positive constant and q is an odd function of $\bar{x}$. In this case, it can be readily shown that $|\bar{x}|\bar{\varepsilon} + \bar{x}\eta \le 0$ for all $\bar{x}$ outside the region $\mathcal{Q} = \{\bar{x} \mid \mathrm{sgn}(\bar{x})q(x) \ge \bar{\varepsilon}/k_0\}$, which can be made arbitrarily small by increasing the value of the design constant k_0. This second approach has the advantage of avoiding the use of the discontinuous sign function, which can cause chattering in the feedback loop.

Since the time derivative $\dot{\mathcal{V}}$ is negative semi-definite and $\dot{\mathcal{V}} = -\bar{x}^2$, we obtain that $\bar{x}$, $\tilde{\theta}_f$, $\tilde{\theta}_g$ remain bounded and, using Barbalat's Lemma, $\bar{x}(t)$ converges to zero as t goes to ∞. It is important to note that these results are based on the assumption that $\hat{\theta}_g^\top\phi(x)$ does not approach zero. In order to achieve that, some type of projection modification [15] is required in the update

Proceedings of the American Control Conference
Denver, Colorado June 4-6, 2003

of $\hat{\theta}_g(t)$. This required modification is independent of the saturation issue since it would have been needed even in the absence of any saturation constraints.

4 Indirect Learning Control Scheme

In the previous section we considered a direct type of learning scheme, in the sense that the tracking error is driving the adaptation. Another approach is to consider an indirect learning control scheme where the adaptation (learning) is driven by the identification error instead of the tracking error. Intuitively, it appears that such a scheme may be more robust to actuator saturation.

Based on the simple system (1), consider the identification model

$$\dot{\hat{x}} = -\alpha\hat{x} + \alpha x + f_0(x) + \hat{f}(x,\hat{\theta}_f) + \hat{g}(x,\hat{\theta}_g)u - \eta, \quad (16)$$

where η is to be designed to address the robustness issue in the presence of approximation errors. Let the identification error be defined as $e_I(t) = x(t) - \hat{x}(t)$. From (1), (16) we obtain

$$\dot{e}_I = -\alpha e_I + \left(f(x) - \hat{f}(x,\hat{\theta}_f)\right) + \left(g(x) - \hat{g}(x,\hat{\theta}_g)\right) u + \eta. \quad (17)$$

We proceed to design a feedback control law which incorporates input saturation, similar to the scheme developed in Section 3, and then we will analyze the stability properties of the indirect learning control scheme. The overall indirect scheme is described by

$$\begin{aligned}
u &= \operatorname{sat}(u_0, u_L, u_U) && (18)\\
u_0 &= \frac{1}{\hat{g}(x,\hat{\theta}_g)}\left[-f_0(x) - \hat{f}(x,\hat{\theta}_f) - \alpha x + \beta r + \eta\right] && (19)\\
\dot{\hat{\theta}}_f &= \Gamma_f (x - \hat{x})\phi_f(x) && (20)\\
\dot{\hat{\theta}}_g &= \Gamma_g (x - \hat{x})\phi_g(x)u && (21)\\
\dot{\hat{x}} &= \alpha x - \alpha\hat{x} + f_0(x) + \hat{f}(x,\hat{\theta}_f) + \hat{g}(x,\hat{\theta}_g)u - \eta && (22)
\end{aligned}$$

In the above indirect scheme, we note that the adaptation is driven by the identification error $e_I(t)$ instead of the modified tracking error used in the direct scheme. Furthermore, the χ-filter of the direct scheme has been replaced by the identification estimator (22). To investigate the stability properties if this scheme consider the Lyapunov function $\mathcal{V}_I = \frac{1}{2}e_I^2 + \frac{1}{2}\tilde{\theta}_f^\top\Gamma_f^{-1}\tilde{\theta}_f + \frac{1}{2}\tilde{\theta}_g^\top\Gamma_g^{-1}\tilde{\theta}_g$. The time derivative of $\mathcal{V}_I$ along the solutions of (17), (20), (21) satisfies

$$\begin{aligned}
\dot{\mathcal{V}} &= -\alpha e_I^2 + e_I\left(\varepsilon_f + \varepsilon_g u + \eta\right)\\
&\leq -\alpha e_I^2 + |e_I|\bar{\varepsilon} + e_I\eta && (23)
\end{aligned}$$

To obtain the desired stability properties η can be chosen as $\eta = k_0 q(e_I)$, where $k_0 > 0$ and q is an odd function. Therefore, $\dot{\mathcal{V}} \leq -\alpha e_I^2$, which implies that the parameter estimates remain bounded and the identification error $e_I(t)$ converges to zero. Note that, since this is an indirect scheme, the results are obtained independent of whether saturation is reached or not.

Now lets investigate what happens to the tracking error $\tilde{x} = x - x_r$. By combining (1), (2) and using the control law (18), (19) we obtain

$$\begin{aligned}
\dot{\tilde{x}} &= -\alpha\tilde{x} + \left(f(x) - \hat{f}(x,\hat{\theta}_f)\right) + \left(g(x) - \hat{g}(x,\hat{\theta}_g)\right) u\\
&\quad + \eta + \hat{g}(x,\hat{\theta}_g)(u - u_0) && (24)
\end{aligned}$$

Note that the tracking error dynamics for $\tilde{x}(t)$ are similar to the identification error dynamics $e_I(t)$, with the exception of the last term $\hat{g}(x,\hat{\theta}_g)(u - u_0)$. In fact, it can be readily shown that for appropriately chosen initial conditions $\hat{x}(0)$, we have $e_I(t) = \tilde{x}(t) - \chi(t)$, where $\chi(t)$ is generated (same as previously) by

$$\dot{\chi} = -\alpha\chi + \hat{g}(x,\hat{\theta}_g)(u - u_0), \qquad \chi(0) = 0. \quad (25)$$

To verify this relationship, let $z = e_I - \tilde{x}(t) + \chi(t)$. From (17), (24), (25) it can be easily shown that the z-dynamics satisfy $\dot{z} = -\alpha z$. Therefore if $\hat{x}(0) = x_r(0)$, then z will be zero, which implies that $e_I(t) = \tilde{x}(t) - \chi(t)$. Note that even if the initial conditions are not satisfied, the difference between $e_I(t)$ and $\tilde{x}(t) - \chi(t)$ will exponentially decay to zero.

Since, as shown earlier, $e_I(t)$ converges to zero, this implies that $\tilde{x}(t) - \chi(t)$ converges to zero, which is the same result as obtained in the previous section using the direct learning scheme method. In conclusion, the direct and indirect learning schemes are implemented differently but they yield the same stability properties.

5 Saturation in Backstepping

In this section, we extend the previous results to the case of a backstepping feedback control procedure. Specifically, we consider the backstepping control of the (γ, α, Q) loops of an aircraft model [17], where γ is the flight path angle, α is the angle of attack and Q is the pitch rate. The tracking error dynamic equations are described by

$$\begin{aligned}
\dot{\tilde{\gamma}} &= \frac{\bar{q}S\left(C_L + C_{L_\alpha}(\alpha_c + \tilde{\alpha})\right)}{mV_t\cos(\beta)} + f_\gamma - \dot{\gamma}_c && (26)\\
\dot{\tilde{\alpha}} &= -\frac{\bar{q}S\left(C_L + C_{L_\alpha}\alpha\right)}{mV_t\cos(\beta)} + Q_c + \tilde{Q} + f_\alpha - \dot{\alpha}_c && (27)\\
\dot{\tilde{Q}} &= c_5 P\, R - c_6(P^2 - R^2)\\
&\quad + c_7\, \bar{q}\, S\, \bar{c}(C_M + u_Q) - \dot{Q}_c && (28)
\end{aligned}$$

where γ_c, α_c, Q_c are the commanded flight path angle, commanded angle of attack and commanded pitch rate respectively, and $\tilde{\gamma}$, $\tilde{\alpha}$, $\tilde{Q}$ are the corresponding tracking errors. For exact definition of the variables and constants in the aircraft model dynamics, please refer to [17]. The objective in this section is to incorporate the modified learning control scheme of Section 3 for dealing with saturation of the actuator signals and investigate its effect on the learning algorithms.

The functions C_L, C_{L_α} and C_M are unknown and are approximated on-line by the following linearly-parameterized approximators: $\widehat{C}_L = \theta_{C_L}^T \phi(x)$; $\widehat{C}_{L_\alpha} = \theta_{C_{L_\alpha}}^T \phi(x)$; $\widehat{C}_M = \theta_{C_M}^T \phi(x)$, where x is the vehicle state vector that contains the most dominant elements.

Let $\bar{\gamma}(t) = \tilde{\gamma}(t) - \chi_\gamma(t)$ be the modified tracking error of the γ dynamics, where $\chi_\gamma(t)$ is generated by

$$\dot{\chi}_\gamma = -k_\gamma \chi_\gamma + \frac{\bar{q}S}{mV_t \cos(\beta)} \widehat{C}_{L_\alpha} \left(\alpha_c - \alpha_c^0\right).$$

where $\alpha_c(t) = \text{sat}(\alpha_c^0, \alpha_c^L, \alpha_c^U)$, and α_c^L, α_c^U are the lower and upper limits, respectively, of the control signal α_c. The dynamics of the modified tracking error $\bar{\gamma} = \tilde{\gamma} - \chi_\gamma$ satisfy

$$\begin{aligned} \dot{\bar{\gamma}} &= \frac{\bar{q}S\left(C_L + C_{L_\alpha}\left(\alpha_c + \tilde{\alpha}\right)\right)}{mV_t \cos(\beta)} + f_\gamma - \dot{\gamma}_c + k_\gamma \chi_\gamma \\ &\quad - \frac{\bar{q}S}{mV_t \cos(\beta)} \widehat{C}_{L_\alpha} \left(\alpha_c - \alpha_c^0\right) \end{aligned} \tag{29}$$

The nominal control law is given by

$$\begin{aligned} \alpha_c^0 &= \frac{1}{\widehat{C}_{L_\alpha}} \left(\frac{m\ V_t\ \cos(\beta)}{\bar{q}\ S}\ \left(-f_\gamma + \dot{\gamma}_c - k_\gamma\ \tilde{\gamma}\right)\right) \\ &\quad - \frac{\widehat{C}_L}{\widehat{C}_{L_\alpha}} - \chi_\alpha. \end{aligned} \tag{30}$$

For simplicity, in this section we do not consider the robustness issue with respect to residual approximation errors, however, it can be directly addressed using the η term as in Section 3. Specifically, we assume that the residual approximation error is zero and the robustifying term $\nu_\gamma = 0$.

Note that the term χ_α, which appears in the control law (30) is the saturation filter for the α-dynamics, and it will appear in the next step of the backstepping procedure. By substituting (30) in (29) we obtain after some algebraic manipulation

$$\begin{aligned} \dot{\bar{\gamma}} &= -k_\gamma\ \bar{\gamma} + \frac{\bar{q}\ S\ \widehat{C}_{L_\alpha}(\bar{\alpha} - \chi_\alpha)}{m\ V_t\ \cos(\beta)} \\ &\quad - \frac{\bar{q}\ S}{m\ V_t\ \cos(\beta)} \left(\tilde{\theta}_{C_L}^T \phi(x,y) + \tilde{\theta}_{C_{L_\alpha}}^T \phi(x,y)\alpha\right). \end{aligned}$$

Define the γ-Lyapunov function as $\mathcal{V}_\gamma = \frac{1}{2}\bar{\gamma}^2 + \frac{1}{2}\left(\tilde{\theta}_{C_L}^T \Gamma_{C_L}^{-1} \tilde{\theta}_{C_L} + \tilde{\theta}_{C_{L_\alpha}}^T \Gamma_{C_{L_\alpha}}^{-1} \tilde{\theta}_{C_{L_\alpha}}\right)$. The derivative of $\mathcal{V}_\gamma$ along solutions of the $\bar{\gamma}$ dynamic equation is

$$\begin{aligned} \dot{\mathcal{V}}_\gamma &= -k_\gamma \bar{\gamma}^2 + \bar{\gamma} \frac{\bar{q}\ S\ \widehat{C}_{L_\alpha}(\bar{\alpha} - \chi_\alpha)}{m\ V_t\ \cos(\beta)} \\ &\quad + \tilde{\theta}_{C_L}^T \Gamma_{C_L}^{-1} \left(\dot{\theta}_{C_L} - \Gamma_{C_L} \frac{\bar{q}\ S}{m\ V_t\ \cos(\beta)} \phi \bar{\gamma}\right) \\ &\quad + \tilde{\theta}_{C_{L_\alpha}}^T \Gamma_{C_{L_\alpha}}^{-1} \left(\dot{\theta}_{C_{L_\alpha}} - \Gamma_{C_{L_\alpha}} \frac{\bar{q}\ S}{m\ V_t\ \cos(\beta)} \phi \alpha \bar{\gamma}\right) \end{aligned}$$

Since the unknown functions C_L and C_{L_α} also appear in the α dynamics, we delay the selection of the update laws for θ_{C_L} and $\theta_{C_{L_\alpha}}$ until the next step of the backstepping procedure.

Next we proceed to the α-dynamics. Let $\bar{\alpha}(t) = \tilde{\alpha}(t) - \chi_\alpha(t)$ be the modified tracking error of the α dynamics, where $\chi_\alpha(t)$ is generated by

$$\dot{\chi}_\alpha = -k_\alpha \chi_\alpha + \left(Q_c - Q_c^0\right).$$

where $Q_c = \text{sat}(Q_c^0, Q_c^L, Q_c^U)$ and Q_c^L, Q_c^U are the lower and upper limits, respectively, of the control signal Q_c.

The dynamics of the modified tracking error $\bar{\alpha} = \tilde{\alpha} - \chi_\alpha$ satisfy

$$\begin{aligned} \dot{\bar{\alpha}} &= -\frac{\bar{q}S\left(C_L + C_{L_\alpha}\alpha\right)}{mV_t \cos(\beta)} + Q_c + \tilde{Q} + f_\alpha - \dot{\alpha}_c + k_\alpha \chi_\alpha \\ &\quad - \left(Q_c - Q_c^0\right) \end{aligned} \tag{31}$$

The nominal (non-saturated) control law for the α-dynamics is given by

$$\begin{aligned} Q_c^0 &= \frac{\bar{q}S\left(\widehat{C}_L + \widehat{C}_{L_\alpha}\alpha\right)}{mV_t \cos(\beta)} - k_\alpha \tilde{\alpha} - f_\alpha + \dot{\alpha}_c \\ &\quad - \bar{\gamma} \frac{\bar{q}\ S\ \widehat{C}_{L_\alpha}}{m\ V_t\ \cos(\beta)} - \chi_Q \end{aligned} \tag{32}$$

In practice, $\dot{\alpha}_c$ is not available for measurement and hence is approximated by a signal of the form $\dot{\hat{\alpha}}_c = \frac{s}{\tau_\alpha s + 1}\alpha_c$. For notational simplicity we avoid this approximation in the current analysis and assume that $\dot{\alpha}_c$ is available. By substituting (32) in (31), the modified tracking error dynamics for $\bar{\alpha}$ satisfy

$$\begin{aligned} \dot{\bar{\alpha}} &= -k_\alpha \bar{\alpha} + \tilde{Q} - \chi_Q - \bar{\gamma} \frac{\bar{q}\ S\ \widehat{C}_{L_\alpha}}{m\ V_t\ \cos(\beta)} \\ &\quad - \frac{\bar{q}\ S}{m\ V_t\ \cos(\beta)} \left(\left(C_L - \widehat{C}_L\right) + \left(C_{L_\alpha} - \widehat{C}_{L_\alpha}\right)\alpha\right). \end{aligned}$$

Define the (γ, α) Lyapunov function as $\mathcal{V}_{(\gamma,\alpha)} = \mathcal{V}_\gamma + \frac{1}{2}\bar{\alpha}^2$. After some algebra, it can be shown that the derivative of $\mathcal{V}_{(\gamma,\alpha)}$ along solutions of the (γ, α) dynamic equations is

$$\begin{aligned} \dot{\mathcal{V}}_{(\gamma,\alpha)} &= -\left(k_\gamma \bar{\gamma}^2 + k_\alpha \bar{\alpha}^2\right) + (\tilde{Q} - \chi_\alpha)\bar{\alpha} \\ &\quad + \tilde{\theta}_{C_L}^T \Gamma_{C_L}^{-1} \left(\dot{\theta}_{C_L} - \Gamma_{C_L} \frac{\bar{q}\ S}{m\ V_t\ \cos(\beta)} \phi\left(\bar{\gamma} - \bar{\alpha}\right)\right) \\ &\quad + \tilde{\theta}_{C_{L_\alpha}}^T \Gamma_{C_{L_\alpha}}^{-1} \left(\dot{\theta}_{C_{L_\alpha}} - \Gamma_{C_{L_\alpha}} \frac{\bar{q}\ S\ \bar{\alpha}}{m\ V_t\ \cos(\beta)} \phi \alpha \left(\bar{\gamma} - \bar{\alpha}\right)\right) \end{aligned}$$

Based on the above Lyapunov function derivative, we now select the adaptive laws for θ_{C_L} and $\theta_{C_{L_\alpha}}$:

$$\dot{\theta}_{C_L} = \Gamma_{C_L} \frac{\bar{q}\ S}{m\ V_t\ \cos(\beta)} \phi\left(\bar{\gamma} - \bar{\alpha}\right) \tag{33}$$

$$\dot{\theta}_{C_{L_\alpha}} = \Gamma_{C_{L_\alpha}} \frac{\bar{q}\ S}{m\ V_t\ \cos(\beta)} \phi\alpha\left(\bar{\gamma} - \bar{\alpha}\right) \tag{34}$$

In the third and final step we proceed to the Q-dynamics. Let $\bar{Q}(t) = \tilde{Q}(t) - \chi_Q(t)$ be the modified tracking error of the Q dynamics, where $\chi_Q(t)$ is generated by

$$\dot{\chi}_Q = -k_Q \chi_Q + c_7\ \bar{q}\ S\ \bar{c}\left(u_Q - u_Q^0\right),$$

and $u_Q(t) = \mathrm{sat}(u_Q^0, u_Q^L, u_Q^U)$. The dynamics of the modified tracking error $\bar{Q} = \tilde{Q} - \chi_Q$ satisfy

$$\begin{aligned} \dot{\bar{Q}} &= c_5 P\ R - c_6(P^2 - R^2) + c_7\ \bar{q}\ S\ \bar{c}(C_M + u_Q) \\ &\quad - \dot{Q}_c + k_Q \chi_Q - c_7\ \bar{q}\ S\ \bar{c}\left(u_Q - u_Q^0\right) \end{aligned} \tag{35}$$

The nominal (non-saturated) control law for the Q-dynamics is given by

$$u_Q^0 = -\widehat{C}_M + \frac{-k_Q\tilde{Q} - \bar{\alpha} + \dot{Q}_c - \left(c_5 P\ R - c_6(P^2 - R^2)\right)}{c_7\ \bar{q}\ S\ \bar{c}}. \tag{36}$$

By substituting (36) into (35) the modified tracking error dynamics for $\bar{Q}$ satisfy

$$\dot{\bar{Q}} = -k_Q\bar{Q} - \bar{\alpha} + c_7\ \bar{q}\ S\ \bar{c}\left(C_M - \widehat{C}_M\right).$$

Define the (γ, α, Q) Lyapunov function as

$$\mathcal{V}_{(\gamma,\alpha,Q)} = \mathcal{V}_{(\gamma,\alpha)} + \frac{1}{2}\bar{Q}^2 + \frac{1}{2}\left(\tilde{\theta}_{C_M}^T \Gamma_{C_M}^{-1} \tilde{\theta}_{C_M}\right).$$

The derivative of $\mathcal{V}_{(\gamma,\alpha,Q)}$ along solutions of the (γ, α, Q) dynamic equations and update laws for the weights of the on-line approximators is

$$\begin{aligned} \dot{\mathcal{V}}_{(\gamma,\alpha,Q)} &= -\left(k_\gamma\bar{\gamma}^2 + k_\alpha\bar{\alpha}^2 + k_Q\bar{Q}^2\right) \\ &\quad + \tilde{\theta}_{C_M}^T \Gamma_{C_M}^{-1}\left(\dot{\theta}_{C_M} - \Gamma_{C_M} c_7\ \bar{q}\ S\ \bar{c}\phi\bar{Q}\right) \end{aligned}$$

Therefore, we select the adaptive law for θ_{C_M} as follows:

$$\dot{\theta}_{C_M} = \Gamma_{C_M} c_7\ \bar{q}\ S\ \bar{c}\phi\bar{Q}.$$

If $(k_\gamma, k_\alpha, k_Q)$ are all positive then $\dot{\mathcal{V}}_{(\gamma,\alpha,Q)}$ is negative semi-definite. Therefore, using Barbalat's Lemma the modified tracking errors $\bar{\gamma}$, $\bar{\alpha}$, $\bar{Q}$ converge to zero asymptotically.

6 Conclusions

This article has presented an approach for dealing with control input saturation in uncertain nonlinear system, where the uncertainty is approximated on-line with learning techniques. A key issue has been the modification of the update laws associated with the weights of the on-line approximator, such that the presence of saturation limits do not cause instability in the closed-loop system nor "unlearning" in the on-line approximation process. The presented design and analysis procedure first considered the case of a first-order plant, and later was extended to a backstepping control scheme based on an aircraft vehicle model.

References

[1] A. M. Annaswamy and J. E. Wong. Adaptive control in the presence of saturation nonlinearity. *International Journal of Adaptive Control and Signal Processing*, 11:3–19, 1997.

[2] B. R. Barmish, M. Corless, and G. Leitmann. A new class of stabilizing controllers for uncertain dynamical systems. *SIAM Journal on Control and Optimization*, 21:246–255, 1983.

[3] F.-C. Chen and H. K. Khalil. Adaptive control of a class of nonlinear discrete-time systems using neural networks. *IEEE Trans. on Autom. Control*, 40:791–801, May 1995.

[4] J. Farrell. Stability and approximator convergence in nonparametric nonlinear adaptive control. *IEEE Trans. on Neural Net.*, 9:1008–1020, September 1998.

[5] A. Isidori. *Nonlinear Control Systems.* Springer-Verlag, second edition, 1995.

[6] E. Johnson and A. Calise. Neural network adaptive control systems with input saturation. In Proc.*American Control Conference*, Arlington VA, June 2001.

[7] I. Kanellakopoulos, P. Kokotovic, and A. S. Morse. Systematic design of adaptive controllers for feedback linearizable systems. *IEEE Trans. on Autom. Control*, 36:1241–1253, 1991.

[8] S. P. Karason and A. M. Annaswamy. Adaptive control in the presence of input constraints. *IEEE Transactions on Automatic Control*, 39(11):2325–2330, 1994.

[9] H. Khalil. *Nonlinear Systems.* Prentice Hall, 2nd edition, 1996.

[10] B. Kim and A. Calise. Nonlinear flight control using neural networks. *Journal of Guidance Control and Dynamics*, 20(1):26–33, 1997.

[11] M. Krstic, I. Kanellakopoulos, and P. Kokotovic. *Nonlinear and Adaptive Control Design.* Wiley, 1995.

[12] R. Monopoli. Adaptive control for systems with hard saturation. In Proc.*IEEE Conference on Decision and Control*, 1975.

[13] M. M. Polycarpou. Stable adaptive neural control scheme for nonlinear systems. *IEEE Transactions on Automatic Control*, 41(3):447–451, 1996.

[14] M. M. Polycarpou and P. Ioannou. A robust adaptive nonlinear control design. *Automatica*, 32(3):423–427, 1996.

[15] M. M. Polycarpou and M. J. Mears. Stable adaptive tracking of uncertain systems using nonlinearly parameterized on-line approximators. *Inter. Journ. Control*, 70(3):363–384, 1998.

[16] R. M. Sanner and J. E. Slotine. Gaussian networks for direct adaptive control. *IEEE Transactions on Neural Networks*, 3(6):837–863, November 1992.

[17] B. Stevens and F. Lewis. *Aircraft Control and Simulation.* Jihn Wiley, 1992.