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Command Filtered Adaptive Backstepping

W. Dong, J. A. Farrell, M. M. Polycarpou, M. Sharma

***Abstract*— As the order of the system increases, implementation of adaptive backstepping controllers becomes increasingly complex due to the necessity to calculate analytically the partial derivatives of certain stabilizing functions with respect to the system state. To remove the burden of computation, in this paper we propose a command filtered adaptive backstepping design method. In the proposed controller design method, analytic calculation of partial derivatives is not required. The control law and the update law become succinct. The stability properties of the controllers is analyzed through a sequence of theorems. Effectiveness of the proposed method is shown by simulation results.**

I. INTRODUCTION

Adaptive backstepping is a powerful tool for the design of controllers for nonlinear systems in or transformable to the parameter strict-feedback form [1], where $x \in \Re^n$ is the state, $u \in \Re$ is the control input, and $\theta \in R^p$ is an unknown constant vector. The adaptive backstepping approach, which will be reviewed in Section III, utilizes stabilizing functions $\bar{\alpha}_i$ and tuning functions τ_i for $i = 1, \ldots, n$. Calculation of these quantities utilizes the partial derivatives $\frac{\partial \bar{\alpha}_{i-1}}{\partial x_j}$ and $\frac{\partial \bar{\alpha}_i}{\partial \hat{\theta}}$. Theoretically, calculation of these partial derivatives is simple, but it can be quite complicated and tedious in applications when n is greater than three. See e.g., eqns. (44-45) in [2] or eqns. (3.8) and (3.10) in [3]. In certain applications, the analytic derivation is overly cumbersome, see e.g. [4].

For backstepping control of strict-feedback control systems without parametric uncertainty (i.e., $\theta = 0$), this issue has been addressed by a variety of methods [4], [5], [6], [7], [8], [9]. The authors in [4], [5] approximate the command derivatives using sliding mode filters [6]. Linear filters for derivative generation are considered in [7], [9]. Recently, the authors in [8] proposed a command filtered backstepping control design method to deal with the computation burden. In that article, singular perturbation theory is used to prove that the error between the trajectories of the command filtered and standard backstepping approaches are $\mathcal{O}\left(\frac{1}{\omega}\right)$, where ω is the natural frequency of the command filter.

The above articles only considered the non-adaptive case for systems without parametric uncertainty. The adaptive case is discussed in [10], [11], [12], [13], [14], [15], [16]. In [13], [14], the command derivatives are modeled as portions of unknown functions that are approximated during operation. The articles [15], [16] proposed control laws with the aid of dynamic surface control. Articles [10], [11], [12] use approaches similar to that described herein, but do not rigorously consider the effect of the command filtering on the stability of the adaptive closed loop system.

In this paper we consider adaptive control of generalized parameter strict-feedback systems and propose a command filtered adaptive backstepping control method. The proposed command filtered backstepping control design is a modified version of the adaptive backstepping in Ch. 4 of [1]. In the proposed command filtered backstepping control, analytic computation of partial derivatives in the stabilizing and tuning function is not necessary. Therefore, the control law and adaptive law are easily derived and implemented. To show effectiveness of the proposed design method, an example is included.

The organization of the remainder of this article is as follows. Section II formally states the control problem and its related technical assumptions. Section III derives a standard backstepping controller which is used as a point-of-reference throughout the paper. Command filtered adaptive backstepping (CFABS) is presented in Section IV, along with theorems concerning its stability properties. These are the main results of this article. In Section V, an example is given to show the effectiveness of the proposed control design method. The last section concludes this paper.

II. PROBLEM FORMULATION

Consider the following class of n-th order single-input-single-output nonlinear systems

$$\begin{aligned} \dot{x}_i &= f_i(z_i) + g_i(z_i)x_{i+1} + \psi_i^\top(z_i)\theta, \qquad (1) \\ &\quad \text{for } i = 1, \ldots, n-1 \\ \dot{x}_n &= f_n(x) + g_n(x)u + \psi_n^\top(x)\theta \qquad (2) \end{aligned}$$

where $x = [x_1, \cdots, x_n]^\top \in \Re^n$ is the state vector with initial condition $x(0) = x_0$, $z_i = [x_1, \cdots, x_i]^\top$, the first state x_1 is considered as the scalar output, and u is the scalar control signal. The vector $\theta \in \Re^p$ is unknown and constant. The functions $f_i : \Re^i \mapsto \Re$, $g_i : \Re^i \mapsto \Re$, and $\psi_i : \Re^i \mapsto \Re^p$ are assumed to be known.

Our objective is trajectory tracking. Therefore, we assume there is a desired trajectory $x_d(t) : \Re^+ \mapsto \Re$. Specific assumptions related to this desired trajectory will be stated in subsequent sections. The objectives of the control design are to specify a control signal $u(t)$ to steer $x_1(t)$ from any initial conditions to track the reference input $x_d(t)$, to achieve boundedness of all signals and states defined in the control law, and to achieve boundedness for the system states $x_i(t)$ for $i = 2, \ldots, n$.

W. Dong is with the Dept. of EE, U. of Texas - Pan American, Edinburg, TX 78539, USA; J. A. Farrell is with the Dept. of EE, U. of California, Riverside, CA 92521, USA; M. Polycarpou is with the Dept. of ECE, U. of Cyprus, 75 Kallipoleos Ave., Nicosia 1678, Cyprus; M. Sharma is with the Boeing Co., El Segundo, CA 90245, USA

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III. Standard Adaptive Backstepping

For convenience, this section summarizes some results on adaptive backstepping. Detailed results can be found in Ch. 4 of [1]. Related to adaptive backstepping, we state the following assumption related to the desired trajectory $x_d(t)$.

Assumption 1: The desired bounded trajectory $x_d(t)$ and its derivatives $\frac{d^i}{dt^i}x_d(t)$ $(1 \le i \le n)$ are bounded, continuous, and known.

The tracking error vector is defined as $\bar{x} = [\bar{x}_1, \ldots, \bar{x}_n]$ with $\bar{x}_i = x_i - \bar{\alpha}_{i-1}$ for $i = 1, \ldots, n$ where $\bar{\alpha}_0 = x_d$. The vector of functions $\bar{\alpha} = [\bar{\alpha}_1, \ldots, \bar{\alpha}_n]^\top$, referred to as stabilizing functions, is defined as

$$\bar{\alpha}_1(z_1, t) = \frac{1}{g_1}\left(-k_1\bar{x}_1 + \dot{x}_d - f_1 - \psi_1^\top\hat{\theta}\right) \tag{3}$$

$$\begin{aligned}\bar{\alpha}_i(z_i, t) &= \frac{1}{g_i}\Big(- k_i\bar{x}_i - f_i - g_{i-1}\bar{x}_{i-1} - \varpi_i^\top\hat{\theta} \\ &+ \sum_{j=1}^{i-1}\frac{\partial\bar{\alpha}_{i-1}}{\partial x_j}(f_j + g_j x_{j+1}) + \sum_{j=0}^{i-1}\frac{\partial\bar{\alpha}_{i-1}}{\partial x_d^{[j]}}x_d^{[j+1]} \\ &+ \frac{\partial\bar{\alpha}_{i-1}}{\partial\hat{\theta}}\Gamma\tau_i + \sum_{j=2}^{i-1}\frac{\partial\bar{\alpha}_{j-1}}{\partial\hat{\theta}}\Gamma\varpi_i\bar{x}_j\Big),\end{aligned} \tag{4}$$

for $i = 2, \ldots, n$

where $k_i > 0$ for $i = 1, \ldots, n$, $x_d^{[j]}$ denotes the j-th derivative of x_d with respect to time, and

$$\tau_i = \tau_{i-1} + \varpi_i\bar{x}_i, \quad \tau_0 = 0 \tag{5}$$

$$\varpi_i = \psi_i - \sum_{j=1}^{i-1}\frac{\partial\bar{\alpha}_{i-1}}{\partial x_j}\psi_j. \tag{6}$$

The τ_i are called tuning functions. Substituting eqns. (3–6) into eqns. (1–2), with the control input and adaptive law defined as

$$u(t) = \bar{\alpha}_n(x(t), t) \tag{7}$$

$$\dot{\hat{\theta}} = \Gamma\tau_n, \tag{8}$$

the closed-loop tracking error differential equations are

$$\dot{\bar{x}}_1 = -k_1\bar{x}_1 + g_1\bar{x}_2 + \varpi_1^\top\tilde{\theta} \tag{9}$$

$$\begin{aligned}\dot{\bar{x}}_i &= -k_i\bar{x}_i + g_i\bar{x}_{i+1} - g_{i-1}\bar{x}_{i-1} + \sum_{j=2}^{i-1}\frac{\partial\bar{\alpha}_{j-1}}{\partial\hat{\theta}}\Gamma\varpi_i\bar{x}_j \\ &\quad - \sum_{j=i+1}^{n}\frac{\partial\bar{\alpha}_{i-1}}{\partial\hat{\theta}}\Gamma\varpi_j\bar{x}_j + \varpi_i^\top\tilde{\theta},\end{aligned} \tag{10}$$

$$\begin{aligned}\dot{\bar{x}}_n &= -k_n\bar{x}_n - g_{n-1}\bar{x}_{n-1} + \sum_{j=2}^{n-1}\frac{\partial\bar{\alpha}_{j-1}}{\partial\hat{\theta}}\Gamma\varpi_n\bar{x}_j \\ &\quad + \varpi_n^\top\tilde{\theta}\end{aligned} \tag{11}$$

$$\dot{\tilde{\theta}} = -\Gamma\tau_n \tag{12}$$

for $i = 2, \ldots, n-1$, where $\tilde{\theta} = \theta - \hat{\theta}$.

The derivative of the Lyapunov function

$$V_o = \frac{1}{2}\sum_{i=1}^{n}\bar{x}_i^2 + \frac{1}{2}\tilde{\theta}^\top\Gamma^{-1}\tilde{\theta} \tag{13}$$

along the solutions of eqns. (9–12) satisfies

$$\dot{V}_o = -\sum_{i=1}^{n}k_i\bar{x}_i^2. \tag{14}$$

Using eqns. (13–14) it is standard to show that $\bar{x}$ converges to zero and that $\hat{\theta}$ is bounded. For the standard adaptive backstepping, we have the following results [1].

Theorem 1: For system (1)-(2), under Assumption 1, the control and adaptive laws (7-8) with $\bar{\alpha}_i$, τ_i, and ϖ_i defined by (3)-(6) cause $\bar{x}$ to converge to zero with $\hat{\theta}$ bounded.

Remark 1: In the stabilizing functions $\bar{\alpha}_i$ and the tuning functions τ_i in (3–6), partial derivatives $\frac{\partial\bar{\alpha}_{i-1}}{\partial x_j}$ and $\frac{\partial\bar{\alpha}_{i-1}}{\partial\hat{\theta}}$ are required. When n is large, computation of these partial derivatives may become tedious and quite complicated.

IV. CFABS

This section presents a modification of the adaptive backstepping approach in Section III. In this modified procedure, analytic computation of $\dot{\bar{\alpha}}_i$ is not necessary for $i = 1, \ldots, n-1$. Removal of partial differential terms of $\bar{\alpha}_i$ with respect to x_j and $\hat{\theta}$ in (4) and (6) results in a significant simplification of the implementation of the adaptive backstepping control law.

Our objective is again trajectory tracking. For the command filtered approach, instead of Assumption 1, we assume there is a desired trajectory $x_d(t) : \Re^+ \mapsto \Re$ that satisfies the following assumption.

Assumption 2: The desired trajectory x_d and its first derivative $\dot{x}_d$ are smooth and bounded. In addition, $x_d(t)$ and $\dot{x}_d(t)$ are known and available for use in the control law computations.

Remark 2: For $n \ge 2$, Assumption 2 is easier to satisfy than Assumption 1. The significance of this remark increases as n increases.

The objective of the control design is to specify a control signal $u(t)$ to steer $x_1(t)$ from any initial condition to track the reference input $x_d(t)$, to achieve boundedness of all signals and states defined in the control law, and to achieve boundedness for the system states $x_i(t)$ for $i = 2, \ldots, n$.

Throughout the discussion we will use the level sets defined for any $a > 0$ as

$$\Omega_a = \left\{x \in \Re^n \middle| \|x\| < a\right\} \text{ and } \bar{\Omega}_a = \left\{x \in \Re^n \middle| \|x\| \le a\right\}.$$

In these definitions, and through out the article, unless otherwise stated, all norms are the standard 2-norm. With this notation, the design of the control system will also utilize the following assumption.

Assumption 3: Let the domain $\mathcal{D} = \Omega_d \subset \Re^n$ be an open set that contains the origin and the trajectory $x_d(t)$. For the system (1–2), there exist positive constants ρ and μ such that for $i = 1, \ldots, n$

1) ψ_i is smooth with bounded derivatives on $\bar{\Omega}_d$;
2) f_i is smooth with bounded derivatives on $\bar{\Omega}_d$; and
3) g_i is smooth with bounded derivatives and with $\rho > |g_i| > \eta > 0$ on $\bar{\Omega}_d$.

By Assumption 3, the functions f_i and g_i are each locally Lipschitz [17] on $\mathcal{D} \subset \bar{\Omega}_d$.

A. Design Equations

Define the tracking error for the command filtered backstepping approach as

$$\tilde{x}_i = x_i - x_i^c \tag{15}$$

for $i = 1, \ldots, n$, where $x_1^c(t) = x_d(t)$ and x_i^c for $i = 2, \ldots, n$ are defined in eqns. (22).

Define the stabilizing functions as

$$\alpha_1 = \frac{1}{g_1}\left(-k_1\tilde{x}_1 - f_1 + \dot{x}_1^c - \psi_1^\top \hat{\theta}\right) \tag{16}$$

$$\alpha_i = \frac{1}{g_i}\big(-k_i\tilde{x}_i - f_i + \dot{x}_i^c - g_{i-1}\tilde{x}_{i-1} - \varpi_i^\top \hat{\theta}\big), \tag{17}$$

for $i = 2, \ldots, n$ where $k_i > 0$. Using the stabilizing functions, we select the the control as

$$u(t) = \mathcal{S}_A(\alpha_n) \tag{18}$$

where $\mathcal{S}_A(\cdot)$ is any monotonic, odd and smooth function satisfying the following constraints

$$\mathcal{S}_A(v) = \begin{cases} v, & \text{for } |v| \leq A - \gamma \\ A, & \text{for } v \geq A + \gamma \\ -A, & \text{for } v \leq -(A+\gamma) \end{cases} \tag{19}$$

with $\gamma > 0$ being a small positive constant. In (19), for $v \in [A-\gamma, A+\gamma]$, $\mathcal{S}_A(v)$ can be any smooth and monotonic function; therefore, $\mathcal{S}_A(v)$ is a smooth function for $v \in \Re$. With this definition, $u(t) = \alpha_n$ for $|\alpha_n| < A - \gamma$.

Finally, for later use we define

$$\tau_i = \tau_{i-1} + \varpi_i\tilde{x}_i, \quad \tau_0 = 0 \tag{20}$$

$$\varpi_i = \psi_i. \tag{21}$$

To facilitate direct comparison with eqn. (6) of the backstepping approach, we have retained the variable ϖ_i, even though its definition is now trivial. Also, the definition of τ_i, in both approaches, can be rewritten as $\tau_i = \sum_{j=1}^{i} \varpi_j \tilde{x}_j$.

For $i = 2, \ldots, n$, the signal $x_i^c(t)$ is generated by the filter

$$\dot{x}_i^c = -\omega_i(x_i^c - \alpha_{i-1}) \tag{22}$$

where $\omega_i > 0$. For this filter, a natural choice for the initial condition is $x_i^c(0) = \alpha_{i-1}(z_{i-1}(0), x_d(0))$, which yields $\dot{x}_i^c(0) = 0$.

Eqn. (22) is a first-order, low-pass filter with unity DC gain and bandwidth parameterized by ω_i. Higher order filters can be useful. A second order filter is discussed in [8]. Herein, we consider a first order command filter to simplify the analysis. The purpose of this *command filter* is to generate x_i^c and its derivative $\dot{x}_c^c$ such that $|x_i^c(t) - \alpha_{i-1}|$ is small. If this is achieved successfully, then the adaptive backstepping controller can be implemented without analytic differentiation of the stabilizing functions, thus significantly simplifying the analytic portion of the controller derivation. The purpose of this article is to formalize an approach implementing this idea and to analyze its performance. An important aspect of the analysis is the effect of the errors $(x_i^c(t) - \alpha_{i-1}(t))$.

B. Analysis and Properties

The analysis will involve several steps in a manner similar to the analysis in [18], [19]. The underlying idea is to consider the controller as a system with slow dynamics defined by eqns. (1-2) and (16–21) together with a parameter adaptations law that is yet to be defined, and fast dynamics defined by eqns. (22). Assuming, without loss of generality, that the system starts at $t = 0$, Lemmas 1-3 address the performance of the combined system during a short time interval $t \in [0, T_1]$. We prove that for an arbitrary small positive constant number μ, there exist positive constants ω_i and T_1 such that $|x_i^c(t) - \alpha_{i-1}|$ converges to have magnitude less than μ prior to $t = T_1$, while maintaining boundedness of all signals within the system. Following this short start-up time interval (i.e., for $t > T_1$), we prove that $|x_i^c(t) - \alpha_{i-1}(t)| < \mu$, that the tracking error defined in eqn. (15) converges to less that a small positive bound ε, and that all system signals are bounded.

Due to space limitation, the proofs for the lemmas and theorems in this paper are omitted. Interested readers can find them in [20].

Lemma 1: Given Assumption 3, $A > 0$, and $x(0) \in \mathcal{D} = \Omega_d$ with $\|x(0)\| = b$, if $|u(t)| < A$, then there exists c with $b < c < d$ and $T_2 > 0$ such that the solution $x(t)$ of the initial value problem defined by eqns. (1–2) satisfies $x(t) \in \Omega_a$ for all $t < T_2$, where $a = \frac{b+c}{2}$.

The parameter T_2 denotes a lower bound on the period of time over which the solution is defined to the initial value problem for the set of differential equations. Lemma 1 has proved that this period of time is not zero. Ultimately, our objective is to prove T_2 to be infinite, meaning that the solution to the initial value problem is defined for all positive time. We can prove that $T_2 = \infty$ by showing that $\|x(t)\|$ is bounded for all $t \in [0, \infty)$. This will be shown later in the article.

Because the time $T_2 > 0$ is independent of ω_i, the second step is to show that for any $\mu > 0$, there exists $\omega_i > 0$ and $0 < T_1 < T_2$ such that $|x_i^c(t) - \alpha_{i-1}(t)| \leq \mu,\ \forall t > T_1$. This will be accomplished in Lemma 3 using Lemma 2.

Lemma 2: Let $q(t)$ be a signal satisfying $|q(t)| < \beta$ and $|\dot{q}(t)| < \gamma$ with

$$\dot{z} = -\omega(z - q), \tag{23}$$

$\omega > 0$, and $z(0) = q(0)$, then there exists ω_o such that for $\omega > \omega_o$:

1) $|z(t) - q(t)| \leq \frac{\gamma}{\omega} < \frac{\gamma}{\omega_o}$,
2) $|z(t)| < \beta + \frac{\gamma}{\omega_o}$,
3) $|\dot{z}(t)| < \gamma$, and
4) $|\ddot{z}(t)| < 2\omega\gamma$

for any $t > 0$.

Lemma 3: Given Assumptions 2 and 3, on the interval $t \in [0, T_2]$ defined in Lemma 1, for the α_i defined in eqns. (16-17), and the command filter defined in eqn. (22), we have the following properties:

1) For any given $\mu > 0$, there exist sufficient large $\omega_i > 0$ such that the errors $(x_i^c(t) - \alpha_{i-1}(t))$ exponentially

converge to the μ-neighborhood of the origin within the time interval $(0, T_1]$.

2) The signal $|x_i^c(t) - \alpha_{i-1}(t)| \le \mu,\ \forall t \in [T_1, T_2]$.

By the definition of $\tilde{x}_i$ in eqn. (15) and the stabilizing functions in eqns. (16–17), the closed loop tracking error differential equations are

$$\dot{\tilde{x}}_1 = -k_1\tilde{x}_1 + g_1\tilde{x}_2 + g_1\left(x_2^c - \alpha_1\right) + \varpi_1^\top\tilde{\theta} \tag{24}$$

$$\dot{\tilde{x}}_i = -k_i\tilde{x}_i + g_i\tilde{x}_{i+1} - g_{i-1}\tilde{x}_{i-1} + g_i\left(x_{i+1}^c - \alpha_i\right) + \varpi_i^\top\tilde{\theta} \tag{25}$$

$$\dot{\tilde{x}}_n = -k_n\tilde{x}_n - g_{n-1}\tilde{x}_{n-1} + \varpi_n^\top\tilde{\theta} \tag{26}$$

where $\tilde{\theta} = \theta - \hat{\theta}$.

The system performance will depend on the nature of the rule that is selected for parameter adaptation. Two parameter adaptation laws will be considered in the subsequent theorems. We introduce the Lyapunov function and some preliminary analysis at this point of the presentation to motivate one remark and the subsequent theorems. Consider the Lyapunov function

$$V_1 = \frac{1}{2}\sum_{i=1}^{n}\tilde{x}_i^2 + \frac{1}{2}\tilde{\theta}^\top\Gamma^{-1}\tilde{\theta} \tag{27}$$

for $t \in [T_1, T_2)$. At $t = T_1$, $\|\tilde{x}(T_1)\|$ and $\|\tilde{\theta}(T_1)\|$ are bounded; hence, $V_1(T_1)$ is bounded. Differentiating V_1 along the solution of eqns. (24–26) for $t \in [T_1, T_2]$, we have

$$\begin{aligned}
\dot{V}_1 &= -\sum_{i=1}^{n}k_i\tilde{x}_i^2 + \sum_{i=1}^{n-1}g_i(x_{i+1}^c - \alpha_i)\tilde{x}_i + \sum_{i=1}^{n}\varpi_i^\top\tilde{\theta}\tilde{x}_i \\
&\quad + \tilde{\theta}^\top\Gamma^{-1}\dot{\tilde{\theta}} \\
&= -\sum_{i=1}^{n}k_i\tilde{x}_i^2 + \sum_{i=1}^{n-1}g_i(x_{i+1}^c - \alpha_i)\tilde{x}_i \\
&\quad + \tilde{\theta}^\top\Gamma^{-1}(\dot{\tilde{\theta}} + \Gamma\tau_n) \\
&\le -\sum_{i=1}^{n}k_i|\tilde{x}_i|^2 + \mu\rho\sum_{i=1}^{n-1}|\tilde{x}_i| + \tilde{\theta}^\top\Gamma^{-1}(\dot{\tilde{\theta}} + \Gamma\tau_n) \\
&\le -k_o\|\tilde{x}\|^2 - k_o\sum_{i=1}^{n-1}\left(|\tilde{x}_i|^2 - \frac{\mu\rho}{k_o}|\tilde{x}_i|\right) \\
&\quad + \tilde{\theta}^\top\Gamma^{-1}\left(\dot{\tilde{\theta}} + \Gamma\tau_n\right) \\
&\le -k_o\|\tilde{x}\|^2 - k_o\sum_{i=1}^{n-1}\left(|\tilde{x}_i| - \frac{\mu\rho}{2k_o}\right)^2 + (n-1)\frac{\mu^2\rho^2}{4k_o} \\
&\quad + \tilde{\theta}^\top\Gamma^{-1}\left(\dot{\tilde{\theta}} + \Gamma\tau_n\right)
\end{aligned} \tag{28}$$

where $k_o = 0.5\min_i(k_i)$.

Remark 3: If the adaptive control law was selected as

$$\dot{\hat{\theta}} = \mathcal{P}(\Gamma\tau_n) \tag{29}$$

where Γ is a positive constant, $\mathcal{P}$ is a projection operator as described in Appendix E of [1]; then, $\dot{V}_1$ is negative for $\|\tilde{x}\| > \sqrt{n-1}\frac{\mu\rho}{2k_0}$. However, when $\|\tilde{x}\| < \sqrt{n-1}\frac{\mu\rho}{2k_0}$ it is possible for the Lyapunov function to increase. In fact it is possible that during some interval of time $t \in [t_1, t_2]$ for which $\|\tilde{x}(t)\| < \sqrt{n-1}\frac{\mu\rho}{2k_0}$ the magnitude of $V_1(t)$ may increase due to $\left\|\tilde{\theta}(t)\right\|$ increasing toward $\left\|\tilde{\theta}_M(t)\right\|$ which is the maximum parameter error allowed by the projection and is not necessarily small. Therefore, it is possible for $V_1(t_2) > V_1(t_1)$; in fact, it may be the case that $V_1(t_2) = \frac{1}{2}(n-1)\left(\frac{\mu\rho}{2k_0}\right)^2 + \frac{1}{2}\tilde{\theta}_M^\top\Gamma^{-1}\tilde{\theta}_M$. Let $\|\tilde{x}(t_2)\| = \sqrt{n-1}\frac{\mu\rho}{2k_0}$ and $\|\tilde{x}(t)\| \ge \sqrt{n-1}\frac{\mu\rho}{2k_0}$ for $t \in [t_2, t_3]$. During this time interval, the Lyapunov function is decreasing, but the only bound on $\|\tilde{x}\|$ is

$$\|\tilde{x}(t)\|^2 \le (n-1)\frac{\mu^2\rho^2}{4k_0^2} + \tilde{\theta}_M^\top\Gamma^{-1}\tilde{\theta}_M$$

which may be quite large. Such *bursting phenomenon* is not unique to the command filtered approach. It may occur in any adaptive application that is affected by noise or disturbances. Robustness to such effects can be addressed by various mechanisms: deadzones, persistence of excitation, σ-modification, and ϵ-modification [21]. The following theorem completes the presentation of the adaptive command filtered backstepping method for one possible parameter adaptation approach. ■

The following theorem will utilize a filtering method. Define the signals

$$\dot{\xi}_1 = -k_1\xi_1 + g_1\xi_2 + g_1(x_2^c - \alpha_1) \tag{30}$$

$$\dot{\xi}_i = -k_i\xi_i + g_i\xi_{i+1} - g_{i-1}\xi_{i-1} + g_i(x_{i+1}^c - \alpha_i) \tag{31}$$

$$\dot{\xi}_n = -k_n\xi_n - g_{n-1}\xi_{n-1} \tag{32}$$

for $i = 1, \ldots, n-1$ with $\xi(0) = 0$. This is a time-varying linear filter with bounded inputs $g_i(x_{i+1}^c - \alpha_i)$ for $i = 1, \ldots, n-1$.

Lemma 4: The system defined in eqns. (30-32) has states bounded by

$$\|\xi(t)\| \le \frac{\mu\rho}{2k_o}\left(1 - e^{-2k_o t}\right) \tag{33}$$

for bounded inputs satisfying $|g_i(x_{i+1}^c - \alpha_i)| \le \mu\rho$.

Define $v = [v_1, \ldots, v_n]^\top$ where

$$v_i = \tilde{x}_i - \xi_i \quad \text{for} \quad i = 1, \ldots, n. \tag{34}$$

Differentiating both sides of eqn. (34) and simplifying by subtracting eqns. (30-32) from (24)-(26) yields the dynamic equation for the v_i variables as

$$\dot{v}_1 = -k_1v_1 + g_1v_2 + \varpi_1^\top\tilde{\theta} \tag{35}$$

$$\dot{v}_i = -k_iv_i + g_iv_{i+1} - g_{i-1}v_{i-1} + \varpi_i^\top\tilde{\theta} \tag{36}$$

$$\dot{v}_n = -k_nv_n - g_{n-1}v_{n-1} + \varpi_n^\top\tilde{\theta}. \tag{37}$$

Note that eqns. (30–32) are implemented within the controller. Eqns. (35–37) describe the dynamics of the v_i variables; however, these equations are not implemented by the controller directly. Instead, the compensated tracking errors v_i for $i = 1, \ldots, n$ are computed directly from eqn. (34), since $\tilde{x}_i$ and ξ_i are available. Nonetheless, eqns. (35–37) will useful for the analysis of the following theorem.

Theorem 2: For system (1)-(2), under Assumption 1, for any $\mu > 0$, there exist $\omega_i > 0$ such that for the command filter based control laws of eqns. (16-21), (22), (30–32) and the adaptive law defined as

$$\dot{\hat{\theta}} \;=\; \mathcal{P}\left(\Gamma \sum_{i=1}^{n} v_i \varpi_i\right) \qquad (38)$$

we have the following results:

1) The signal $\tilde{x}(t)$ is ultimately bounded by $\epsilon \left(\geq \frac{\mu\rho}{2k_o}\right)$.
2) The signal $v(t) = [v_1(t), \ldots, v_n(t)]^\top$ is asymptotically converges to the origin.
3) For the system of differential equations defined in this theorem, solution trajectories corresponding to initial value problems are defined for $t \in [0, \infty)$.

Note that Lemmas 1 and 3 are still applicable as their proofs did not depend on the form of the adaptive law.

C. Parameter Selection

Remark 4: The parameter A in the definition of eqn. (19) can be selected as

$$A = \begin{cases} A_1, & \text{for } t \in [0, T_1) \\ A_2, & \text{for } t \in [T_1, \infty) \end{cases}$$

where A_1 is a small positive number, A_2 is a large positive number, and T_1 the time at which $|x_i^c(t) - \alpha_i(t)| \leq \mu$ for $i = 1, \ldots, n$. By this choice of A, the control signal magnitude is constrained to be less than A_1 in magnitude during the start-up transient during which the filtered command signals converge to the value of the stabilizing functions. Once this convergence is achieved, the value A_2 is set large enough so that the control signal is effectively unconstrained.

Remark 5: Increasing the ω_i will decrease the tracking error by decreasing μ, which allows the ultimate bound ϵ to be decreased correspondingly.

Remark 6: In this article, we choose to implement first order command filters mainly to simplify the presentation and bounds in Lemmas 1 and 3. A second order filter is used in the analysis of [8]. Higher order filters are also feasible as long as the filter outputs are $x_i^c(t)$ and $\dot{x}_i^c(t)$, and a filter parameter $\omega > 0$ can be adjusted to cause the absolute error between the filter input and output (i.e., $|\alpha_{i-1} - x_i^c|$) to converge to be less than μ within time T_2 where both μ and T_2 can be selected to be arbitrarily small. Second and higher order filters have the desirable property that $\dot{x}_i^c$ is computed from α_{i-1} by integration (not differentiation), which has useful noise reduction properties.

Remark 7: In Assumption 3, there is no restriction on the size of d. Ideally, d would be large. The results herein are referred to as *semi-global* [22], [23].

V. Simulation

Consider the wing-rock example in Section 4.6 in [1]. The model of the system is

$$\begin{aligned} \dot{\phi} &= p \\ \dot{p} &= b\delta_A + \psi(\phi, p)^\top \theta \\ \dot{\delta}_A &= -\tfrac{1}{\tau}\delta_A + \tfrac{1}{\tau}u \end{aligned} \qquad (39)$$

where $\psi(\phi, p) = [1, \phi, p, |\phi|p, |p|p]^\top$ and $\theta = [\theta_1, \theta_2, \theta_3, \theta_4, \theta_5]^\top$ is unknown, and $b = 1.5$ and $\tau = 1/15$ are known constants. The physical interpretation of the example is described in detail in [1]. Our objective is to choose the control signal u to cause the roll angle ϕ to track a given reference trajectory $\phi_r(t)$. According to the controller design procedure in Section IV, we define

$$\begin{aligned} \alpha_1 &= -k_1\tilde{x}_1 + \dot{\phi}_d \\ \alpha_2 &= \frac{1}{b}\left(-k_2\tilde{x}_2 - \tilde{x}_1 - \psi^\top\hat{\theta} + \dot{x}_2^c\right) \\ \alpha_3 &= \tau\left(-k_3\tilde{x}_3 + \frac{1}{\tau}\delta_A - b\tilde{x}_2 + \dot{x}_3^c\right) \end{aligned}$$

where the tracking errors are

$$\tilde{x}_1 = \phi - \phi_d \qquad \tilde{x}_2 = p - x_2^c \qquad \tilde{x}_3 = \delta_A - x_3^c,$$

the reference states x_i^c and $\dot{x}_i^c$ are generated by eqn. (22), the control signal is $u = S_A(\alpha_3)$, and the adaptive law defined by eqn. (38).

For the simulation, the correct value of the unknown parameter vector is $\theta = [0, -26.67, 0.76485, -2.9225, 0]^\top$ [1]. With these parameters, the open loop system has a stable limit cycle with amplitude 0.6 rads and period 1.2 s.

For the adaptive controller implementation, the parameter estimates are initialized as $\hat{\theta} = 1.35\theta$. We choose $k_1 = 1$, $k_2 = 5$, and $k_3 = 10$, $\epsilon = 0.01$, $\gamma = 1$, $\omega_2 = 50$, $\omega_3 = \omega_2^2$, and

$$A = \begin{cases} 1.1, & \text{if } t < T_1 \\ \infty, & \text{otherwise.} \end{cases}$$

The value of T_1 was determined by monitoring $x_i^c - \alpha_{i-1}$ for $i = 2, 3$. Fig. 1 shows a phase-plane plot of the first two states ϕ and p for two simulated trajectories. Both simulations start from $x(0) = [0.0, 0.1, 0.0]^\top$.

The objective of the first simulation is the same as that of Section 4.6 in [1] where the goal is to regulate the state, thereby eliminating the effect of wing-rock. The simulated trajectory is the dashed line starting at $[0.0, 0.1]$ and ending at the origin. The simulation shows that the controller is able to effectively remove the wing-rock phenomenon when that is the objective.

The objective of the second simulation is to force ϕ to follow the signal $\phi_r(t) = 0.1\sin(t/2)$. The simulated trajectory is shown as the narrow dotted curve in Fig. 1. The results show that the adaptive control law allows tracking of a reference input trajectory.

In a true phase-plane plot for a time-invariant system, trajectories may only intersect at equilibrium points. In Fig. 1, the trajectories may cross for two reasons, even for the regulation example. First, the third state is not depicted. Second, due to adaptation of θ, the closed loop system is not time-invariant.

VI. Conclusion

This paper considered adaptive backstepping control for parameter strict-feedback systems. A command filtered method is proposed which introduces $(n-1)$ linear filters. The purpose of each command filter is to compute a signal and its derivative. By adjustment of a bandwidth parameter ω_i, the i-th command filter output signal x_i^c can

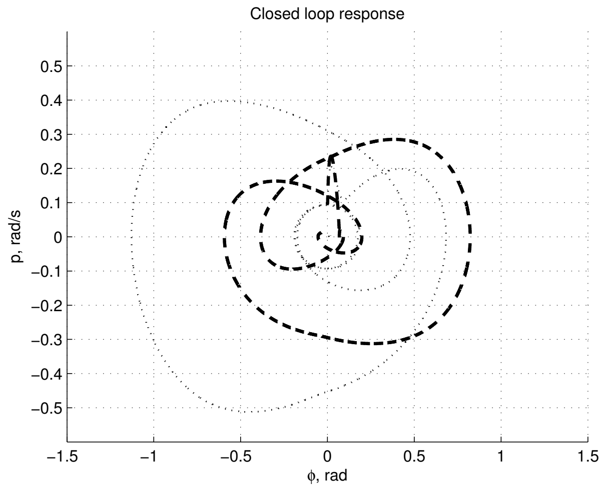

Fig. 1. Response of ϕ vs $\dot{\phi}$ (dashed: Regulation, dotted: Tracking).

be made arbitrarily close to the desired stabilizing signal α_{i-1}, as expressed in eqns. (16 -17). This article analyzes the stability and performance of the approach, proving in Theorem 2 that certain compensated tracking errors are asymptotically convergent to the origin. The proposed control method considerably simplifies the adaptive backstepping implementation and allows the backstepping approach to be applied to a broader category of systems.

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