Stability and Performance for Saturated Systems via Quadratic and Nonquadratic Lyapunov Functions

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***Abstract*—In this paper, we develop a systematic Lyapunov approach to the regional stability and performance analysis of saturated systems in a general feedback configuration. The only assumptions we make about the system are well-posedness of the algebraic loop and local stability. Problems to be considered include the estimation of the domain of attraction, the reachable set under a class of bounded energy disturbances and the nonlinear $\mathcal{L}_2$ gain. The regional analysis is established through an effective treatment of the algebraic loop and the saturation/deadzone function. This treatment yields two forms of differential inclusions, a polytopic differential inclusion (PDI) and a norm-bounded differential inclusion (NDI) that contain the original system. Adjustable parameters are incorporated into the differential inclusions to reflect the regional property. The main idea behind the regional analysis is to ensure that the state remain inside the level set of a certain Lyapunov function where the PDI or the NDI is valid. With quadratic Lyapunov functions, conditions for stability and performances are derived as linear matrix inequalities (LMIs). To obtain less conservative conditions, we use a pair of conjugate non-quadratic Lyapunov functions, the convex hull quadratic function and the max quadratic function. These functions yield bilinear matrix inequalities (BMIs) as conditions for stability and guaranteed performance level. The BMI conditions cover the corresponding LMI conditions as special cases, hence the BMI results are guaranteed to be as good as the LMI results. In most examples, the BMI results are significantly better than the LMI results.**

***Index Terms*—Deadzone, domain of attraction, Lyapunov functions, nonlinear $\mathcal{L}_2$ gain, reachable set, saturation.**

I. INTRODUCTION

A. Background

SATURATION is an ubiquitous nonlinearity in engineering systems and is the most studied in the literature as compared with other types of nonlinearities. Intensified efforts have been devoted to control systems with saturation since the earlier 1990s due to a few notable breakthroughs (see, e.g., [36], [46], and [48]). Saturation exists in different parts of a control system, such as the actuator, the sensor, the controller and components within the plant. Most research has been devoted to addressing actuator saturation, which involves fundamental control problems such as constrained controllability and global/semi-global stabilization. These problems have been discussed in great depth, e.g., in [22], [36], [45], [46], [48], and [49] (among which, [22] considers exponentially unstable systems). Another significant problem arising from actuator saturation is anti-windup compensation, which has attracted tremendous attention over the past decade (see, e.g., [4]–[6], [8]–[10], [12], [16]–[18], [28], [34], [35], [39]–[41], [47], [50], [52], and [54]).

The approach that is adopted in most of the recent literature to address saturated systems can be categorized as a Lyapunov approach. In this approach, some quantitative measures of stability and performance, such as the size of the domain of attraction, the convergence rate, and the $\mathcal{L}_2$ gain, are characterized by using Lyapunov functions or storage functions. Then the design parameters (e.g., of a controller or of an antiwindup compensator) are incorporated into an optimization problem to optimize these quantitative measures for the closed-loop system. This approach is mostly fueled by the numerical success in solving convex optimization problems with linear matrix inequalities (LMIs) (e.g., see [2]). This is a general approach which can be applied to deal with systems with saturation and deadzone occurring at different locations. The first papers that use LMI-based methods to deal with saturated systems include [21], [35], [42], where [21], [42] consider state feedback design and [35] analyzes antiwindup systems. Since then, extensive LMI-based algorithms have been developed for analysis and design of saturated systems (see, e.g., [4]–[6], [10], [13], [16]–[18], [22], [25], [26], [39], [40], [47], and [54].)

There are mainly two steps involved in the Lyapunov approach. The first step is to include the saturation function or the deadzone function in a sector so that the original system can be cast into the general framework of absolute stability, or can be described with a linear differential inclusion (LDI). The second step applies available tools from absolute stability theory or from general Lyapunov approaches for LDIs, such as the circle criterion or the LMI characterizations of stability and performance in [2]. Roughly speaking, all the analysis tools used in the aforementioned works are obtained by applying quadratic Lyapunov/storage functions to the LDIs except that [39] used a piecewise quadratic function.

Because of the two-step framework, the effectiveness of a particular method depends on how the original system is transformed into LDIs and what kind of analysis tools for LDIs are used. In many works involving anti-windup compensation, global sectors are used to describe saturation/deadzone functions. It is well known that a global sector can be very

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conservative for regional analysis and can only be applied when the closed-loop system is globally stable or to detect global stability. In some other works, regional LDI descriptions (some based on local sectors) are derived to reduce the conservatism (see, e.g., [4], [5], [10], [13], [21], [25], [26], [35], and [42]). Along this direction, the regional LDI description introduced in [25], [26] has proved very effective and easy to manipulate. It has been used successfully for different configurations or for different purposes in [4], [5], [10], [13], [27], [28].

With an effective regional LDI description, there is yet more potential to be explored in the second step about the analysis of LDIs. It is now generally accepted that quadratic Lyapunov functions can be very conservative even for stability analysis of LDIs (see, e.g., [7], [11], [32], and [55]). For this reason, considerable attention has been paid to the construction and development of non-quadratic Lyapunov functions (e.g., see [1], [3], [7], [32], [33], [38], [53], and [55]).

Recently, a pair of conjugate Lyapunov functions have demonstrated great potential in the analysis of LDIs and saturated linear systems [14], [15], [23], [27]. One is called the convex hull quadratic function since its level set is the convex hull of a family of ellipsoids. The other is called max quadratic function since it is obtained by taking pointwise maximum over a family of quadratic functions and its level set is the intersection of a family of ellipsoids. Some conjugate relationships about these two functions were established in [14], [15]. Since these functions are natural extensions of quadratic functions, they can also be used to perform quantitative performance analysis beyond stability, such as to estimate the $\mathcal{L}_2$ gain, and the reachable set, for LDIs. A handful of dual bilinear matrix inequalities (BMIs) have been derived for these purposes in [14]. As compared with the corresponding LMIs resulting from quadratic Lyapunov functions, these BMIs contain extra degrees of freedom in the bilinear terms, which are injected through the nonquadratic functions. Experience with low order systems shows that these BMIs can be solved effectively with the path-following method in [20]. Although it is possible that numerical difficulties may arise for higher order systems, the great potential of these nonquadratic Lyapunov functions has been demonstrated in [14], [15], [27] through a set of numerical examples.

B. Problem Formulation

With the recent developments and effective tools mentioned in the previous section, we are now able to address more effectively some stability and performance problems for systems with saturation/deadzone in the following general form:

$$\begin{cases} \dot{x} &= Ax + B_q q + B_w w \\ y &= C_y x + D_{yq} q + D_{yw} w \\ z &= C_z x + D_{zq} q + D_{zw} w \\ q &= \mathrm{dz}(y) \end{cases} \tag{1}$$

where $x \in \mathbb{R}^n$, $q, y \in \mathbb{R}^m$, $w \in \mathbb{R}^r$, $z \in \mathbb{R}^p$. The deadzone function $\mathrm{dz}(\cdot) : \mathbb{R}^m \to \mathbb{R}^m$ is defined as $\mathrm{dz}(y) := y - \mathrm{sat}(y)$, for all $y \in \mathbb{R}^m$, where $\mathrm{sat}(\cdot)$ is a vector saturation function

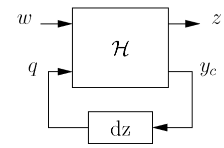

Fig. 1. Compact representation of a system with saturation/deadzone.

with the saturation levels given by a vector $\bar{u} \in \mathbb{R}^m$, $\bar{u}_i > 0$, $i = 1, 2, \ldots, m$. In particular

$$\mathrm{sat}(u_i) = \begin{cases} \bar{u}_i, & u_i \geq \bar{u}_i, \\ u_i, & u_i \in [-\bar{u}_i, \bar{u}_i], \\ -\bar{u}_i, & u_i \leq -\bar{u}_i. \end{cases} \quad \mathrm{sat}(u) = \begin{bmatrix} \mathrm{sat}(u_1) \\ \vdots \\ \mathrm{sat}(u_m) \end{bmatrix}.$$

In this paper, we consider symmetric saturation functions.[1] System (1) can be graphically depicted in block diagram form as in Fig. 1, where w is the exogenous input or disturbance and z is the output whose performance is under consideration. Many linear systems with saturation/deadzone components can be transformed into the aforementioned general form through a loop transformation. This general form has been used to study antiwindup systems in [16], [35], [40], [54]. When $D_{yq} = 0$, the system does not contain an algebraic loop, which can simplify the analysis and implementation. However, it was shown in [40] that the algebraic loop can be purposely introduced into the antiwindup configuration to reduce the global $\mathcal{L}_2$ gain. The importance of the parameter D_{yq} will also be illustrated in examples at the end of this paper.

We note that most of the previous works imposed various assumptions on the system, such as exponential stability of the original open-loop plant in an antiwindup configuration (e.g., [16], [40], and [54]). In these works, the global sector $[0, I]$ is used to describe the deadzone function. In some other works such as [4]–[6], [10], [13], [25]–[27], and [47] (among which [6] and [13] study the $\mathcal{L}_2$ gain), regional LDI descriptions are used to reduce the conservatism. In these works, the algebraic loop is absent ($D_{yq} = 0$) and the disturbance (in [6] and [13]) does not enter the deadzone function, i.e., $D_{yw} = 0$. In [31], the algebraic loop has a special structure, namely, D_{yq} is diagonal.

A recent attempt was made in [52] to perform regional analysis on the general form without the assumption on stability of the open-loop plant. The main idea, which had also been suggested in some other works, was to use a smaller sector $[0, K]$ with $K < I$ to bound the deadzone function. However, this idea would not work on the general form if $D_{yw} \neq 0$. As can be seen from the second equation in (1), y is not necessarily bounded in $\mathcal{L}_\infty$ norm when w is only bounded in the $\mathcal{L}_2$ norm. Hence, there exists no $K < I$ to bound the deadzone function even at $x = 0$. After all, as commented in [25] and [27], even in the absence of w, this kind of sector description is not only hard to manipulate, but also has a much restricted degree of freedom as compared with the regional LDI description initiated in [25] and [26].

In this paper, we will extend the regional LDI description in [25] and [26] to deal with the general situation where

[1] Asymmetric saturations can be treated with the methods developed here with some level of conservativeness by taking $\bar{u}_i$ as the minimum absolute value of the negative and positive saturation levels.

$D_{yq}, D_{yw} \neq 0$, and to address both stability and performance issues.

The only assumptions that we will make about the system (1) is its local stability (A is Hurwitz) and the well-posedness of the algebraic loop, which will be made precise in Section II. These were also the only assumptions made in [28] and they are clearly basic requirements for the system to be functional.

The objective of this paper is to carry out a systematic and comprehensive analysis of system (1) by using quadratic and nonquadratic Lyapunov functions. The following problems will be addressed.

1) Estimation of the domain of attraction (in the absence of w) by using invariant ellipsoids or invariant level sets of the nonquadratic Lyapunov functions.
2) With a given bound on the $\mathcal{L}_2$ norm of w, i.e, $\|w\|_2 \leq s$ for a given s, we would like to determine a set S as small as possible so that under the condition $x(0) = 0$, we have $x(t) \in S$ for all t. This set S will be considered as an estimate of the reachable set.
3) With $\|w\|_2 \leq s$ for a given s, we would like to determine a number $\gamma > 0$ as small as possible, so that under the condition $x(0) = 0$, we have $\|z\|_2 \leq \gamma\|w\|_2$. Performing this analysis for each $s \in (0, \infty)$, we obtain an estimate of the nonlinear $\mathcal{L}_2$ gain.

To address these problems systematically, we will first provide an effective treatment of the algebraic loop and the deadzone function in Section II. In particular, the necessary and sufficient condition for the well-posedness of the algebraic loop will be made explicit. Moreover, we will derive two forms of differential inclusions to describe the original system (1). The first one is a polytopic differential inclusion (PDI) involving a certain adjustable parameter or nonlinear function. This parameter or nonlinear function offers extra degrees of freedom associated with a local region under consideration. It will be optimized in conjunction with the Lyapunov functions in the final analysis problems. The second differential inclusion is a norm-bounded differential inclusion (NDI) which is derived from the PDI. The NDI is more conservative than the PDI but may be more numerically tractable for some cases.

In Section III, we will apply quadratic Lyapunov functions via the PDI and the NDI to characterize stability and performance of the original system (1). We note that quadratic functions have been used for these purposes in [4]–[6], [10], [13], [25], [26], [47] under the assumption that $D_{yq} = 0$ and $D_{yw} = 0$. In Section IV, we apply the convex hull quadratic function and the max quadratic function respectively via the PDI (It turns out that when these nonquadratics are applied to the NDI, they produce the same results as the quadratics). In Section V, we use a numerical example to demonstrate the effectiveness of this paper's results and the relationship between them. Section VI concludes this paper.

Notation:

— $|\cdot|_\infty$: For $u \in \mathbb{R}^m$, $|u|_\infty := \max_i |u_i|$.

— $\|\cdot\|_2$: For $u \in \mathcal{L}_2$, $\|u\|_2 := \left(\int_0^\infty u^T(t)u(t)dt\right)^{1/2}$.

— $I[k_1, k_2]$: For two integers $k_1, k_2, k_1 < k_2$, $I[k_1, k_2] := \{k_1, k_1+1, \ldots, k_2\}$.

— $\mathrm{sat}(\cdot)$: The symmetric saturation function with implicit saturation level given by $\bar{u} \in \mathbb{R}^m$.

— $\bar{U} := \mathrm{diag}\{\bar{u}_1, \ldots, \bar{u}_m\}$ where $\bar{u}_i > 0$ is the saturation level for the ith component of $\mathrm{sat}(\cdot)$.

— $\mathrm{dz}(u)$: The deadzone function, $\mathrm{dz}(u) := u - \mathrm{sat}(u)$.

— $\mathrm{co}S$: The convex hull of a set S.

— $\mathcal{K}$: The set of diagonal matrices with 0 or 1 at each diagonal element.

— $\mathrm{He}X$: For a square matrix X, $\mathrm{He}X := X + X^T$.

— $\mathcal{E}(P)$: For $P \in \mathbb{R}^{n\times n}$, $P = P^T \geq 0$, $\mathcal{E}(P) := \{x \in \mathbb{R}^n : x^TPx \leq 1\}$.

— $\mathcal{L}(H)$: For $H \in \mathbb{R}^{m\times n}$, $\mathcal{L}(H) := \left\{x \in \mathbb{R}^n : |Hx|_\infty \leq 1\right\}$.

About the relationship between $\mathcal{E}(P)$ and $\mathcal{L}(\bar{U}^{-1}H)$, for a given $s > 0$, we have (see, e.g., [25]),

$$s\mathcal{E}(P) \subseteq \mathcal{L}(\bar{U}^{-1}H) \iff \begin{bmatrix} \frac{\bar{u}_\ell^2}{s^2} & H_\ell \\ H_\ell^T & P \end{bmatrix} \geq 0 \ \ \forall \ell \qquad (2)$$

where H_ℓ is the ℓth row of H and $\bar{u}_\ell$ is the ℓth diagonal element of $\bar{U}$.

II. Two Forms of Parameterized Differential Inclusions

Algebraic loops in linear systems can be easily solved (if they are well-posed). For system (1), the presence of the deadzone function makes the algebraic loop much harder to deal with. Theoretically, an explicit solution can be derived as a piecewise affine function, in terms of both x and w, by partitioning the vector space $\mathbb{R}^m$ into 3^m polytopic regions (see Remark 1). However, the complexity of the partition even for $m = 2$ or 3 makes the solution almost impossible to manipulate. In this paper, we would like to use convex sets to bound all the possible solutions. By doing that, we obtain differential inclusion descriptions for the original system (1) and make it more approachable with Lyapunov methods.

Recall that the deadzone function belongs to the $[0, I]$ sector, i.e., for each y there exists a diagonal $\Delta \in \mathbb{R}^{m\times m}$ satisfying $0 \leq \Delta \leq I$ and $\mathrm{dz}(y) = \Delta y$. Let $\mathcal{K}$ be the set of diagonal matrices whose diagonal elements are either 1 or 0. Then $\mathrm{co}\mathcal{K}$ is the set of diagonal Δ satisfying $0 \leq \Delta \leq I$. There are 2^m matrices in $\mathcal{K}$ and we number them as $K_i, i = 1, 2, \ldots, 2^m$. Then, we have $\mathcal{K} = \{K_i : i \in I[1, 2^m]\}$ and

$$\mathrm{dz}(y) \in \mathrm{co}\{K_i y : \ i \in I[1, 2^m]\}.$$

This relation holds for all $y \in \mathbb{R}^m$ but could be conservative over a local region where the system operates. In [25], [26], a flexible description was introduced for dealing with the saturated state feedback $\mathrm{sat}(Fx)$. This description can be easily adapted for the deadzone function. The main idea behind this description is the following simple fact.

Fact 1: Suppose $v_i \in [-\bar{u}_i, \bar{u}_i]$ (with $\bar{u}_i$ being the ith saturation level). For any $u_i \in \mathbb{R}$, we have $\mathrm{sat}(u_i) \in \mathrm{co}\{u_i, v_i\}$, i.e., $\mathrm{sat}(u_i) = \delta u_i + (1-\delta)v_i$ for some $\delta \in [0, 1]$, and $\mathrm{dz}(u_i) \in \mathrm{co}\{0, u_i - v_i\}$, i.e., $\mathrm{dz}(u_i) = \delta(u_i - v_i)$ for some $\delta \in [0, 1]$.

This simple fact has also been used in [13] to analyze the nonlinear $\mathcal{L}_2$ gain for a special case of (1), where D_{yq}, D_{yw},

D_{zq} and D_{zw} are all zero. For the general case where D_{yq} may be nonzero, we have the following algebraic loop:

$$y = C_y x + D_{yq}\mathrm{dz}(y) + D_{yw}w. \tag{3}$$

This algebraic loop is said to be well-posed if there exists a unique solution y for each $C_y x + D_{yw}w$. A sufficient condition for the algebraic loop to be well-posed is the existence of a diagonal matrix $W > 0$ such that $2W - D_{yq}W - WD_{yq}^T > 0$ (see, e.g., [16], [44]). In the following claim, whose proof is reported in [30], we give a precise characterization of the well-posedness of the algebraic loop.

Claim 1: Assume that ϕ is the deadzone function or the saturation function. Then $y = D\phi(y) + v$ has a unique solution for every $v \in \mathbb{R}^m$ if and only if $\det(I - D\Delta) \neq 0$ for all $\Delta \in \mathrm{co}\mathcal{K}$.

Remark 1: If the algebraic loop $y = D\phi(y) + v$ is well-posed, then the solution y is a piecewise affine function of v with 3^m polytopic regions. To understand this, consider the function g: $y \mapsto v = y - D\phi(y)$. It is piecewise affine with 3^m polytopic partitions. If there is a unique solution y for each v, then each polytope in the domain of g is uniquely and affinely mapped to a polytope in the range of g. Hence, the inverse function of g, i.e., the solution of the algebraic loop, is also piecewise affine, with partition corresponding to that of the original g. ∘

Based on Claim 1, we have the following criterion for the well-posedness of the algebraic loop (the proof can be found in [30]).

Claim 2: The algebraic loop (3) is well-posed if and only if the values of $\det(I - D_{yq}K_i), i \in I[1, 2^m]$, are all nonzero and have the same sign. In this case, we have

$$\begin{aligned}\{(I - \Delta D_{yq})^{-1}\Delta :\ \Delta \in \mathrm{co}\mathcal{K}\}& \\ \subseteq \mathrm{co}\{(I - K_i D_{yq})^{-1}K_i :\ i \in I[1, 2^m]\}.&\end{aligned} \tag{4}$$

The well-posedness condition in Claim 2 can be easily verified. The relation (4) will be used to bound the solution of the algebraic loop with a polytope.

Throughout this paper, we assume that this well-posedness condition is satisfied. For $i \in I[1, 2^m]$, denote

$$\begin{aligned}
T_i &= (I - K_i D_{yq})^{-1}K_i\\
A_i &= A + B_q T_i C_y \quad B_i = B_w + B_q T_i D_{yw}\\
C_i &= C_z + D_{zq}T_i C_y \quad D_i = D_{zw} + D_{zq}T_i D_{yw}.
\end{aligned} \tag{5}$$

Proposition 1: Let $h : \mathbb{R}^n \to \mathbb{R}^m$ be a given map and let h_ℓ be the ℓth component of h. Consider system (1). If $x \in \mathbb{R}^n$ satisfies $|h_\ell(x)| \leq \bar{u}_\ell$ for all $\ell \in I[1, m]$, then

$$\begin{bmatrix}\dot{x}\\ z\end{bmatrix} \in \mathrm{co}\left\{\begin{bmatrix}A_i x + B_i w - B_q T_i h(x)\\ C_i x + D_i w - D_{zq}T_i h(x)\end{bmatrix} :\ i \in I[1, 2^m]\right\}. \tag{6}$$

Proof: Since $|h_\ell(x)| \leq \bar{u}_\ell$ for all $\ell \in I[1, m]$, by Fact 1, we have

$$q = \mathrm{dz}(y) = \Delta(y - h(x))$$

for some $\Delta \in \mathrm{co}\mathcal{K}$. Recalling $y = C_y x + D_{yq}q + D_{yw}w$, we obtain $q = \Delta(C_y x + D_{yq}q + D_{yw}w - h(x))$. It follows that $q = (I - \Delta D_{yq})^{-1}\Delta(C_y x + D_{yw}w - h(x))$. By (4) and (5), we have

$$q \in \mathrm{co}\{T_i(C_y x + D_{yw}w - h(x)) :\ i \in I[1, 2^m]\}. \tag{7}$$

Applying this relation to the first and the third equations in (1), we obtain (6). □

By taking $h(x) = 0$ in (6), we obtain a polytopic linear differential inclusion (PLDI) representation which holds globally for the original system (1). A nonzero term $h(x)$ is used to inject additional degrees of freedom in some subset of the state space to reduce conservatism in regional analysis. When we use quadratic Lyapunov functions, we will choose $h(x) = Hx$ where H can be used as an optimizing parameter. When we use nonquadratic Lyapunov functions, a nonlinear $h(x)$ is more effective in general.

The PDI (6) involves 2^m vertices. This may present numerical difficulties when m is large (e.g., $m > 6$) and the order of the system is high. To reduce this computational burden, we may use a more conservative description; namely, to approximate the system (1) we may use an NDI, which is based on the following result, whose proof is in [30].

Claim 3: Let M be a positive diagonal matrix. Suppose that

$$2I - M^{-1}D_{yq}M - MD_{yq}^T M^{-1} = S^2$$

where S is symmetric and nonsingular. Then

$$\begin{aligned}&\mathrm{co}\{(I - K_i D_{yq})^{-1}K_i :\ i \in I[1, 2^m]\}\\ &\qquad \subseteq \{M(S^{-2} + S^{-1}\Omega S^{-1})M^{-1} : \|\Omega\| \leq 1\}\end{aligned} \tag{8}$$

where $\|\Omega\|$ is the spectral norm of Ω (namely its largest singular value). Furthermore, each vertex of the left-hand side is on the boundary of the righthand side.

Proposition 2: Assume that there exist a diagonal $M > 0$ and a symmetric nonsingular S such that

$$S^2 = 2I - M^{-1}D_{yq}M - MD_{yq}^T M^{-1}.$$

Let $H \in \mathbb{R}^{m\times n}$ be given. For $\Omega \in \mathbb{R}^{m\times m}$, define

$$\begin{aligned}\begin{bmatrix}A_\Omega & B_\Omega\\ C_\Omega & D_\Omega\end{bmatrix} &:= \begin{bmatrix}A & B_w\\ C_z & D_{zw}\end{bmatrix}\\ &\quad + \begin{bmatrix}B_q\\ D_{zq}\end{bmatrix} M(S^{-2} + S^{-1}\Omega S^{-1})M^{-1}\left[C_y - H \quad D_{yw}\right].\end{aligned}$$

Consider system (1). If $x \in \mathbb{R}^n$ satisfies $|\bar{U}^{-1}Hx|_\infty \leq 1$, then

$$\begin{bmatrix}\dot{x}\\ z\end{bmatrix} \in \left\{\begin{bmatrix}A_\Omega & B_\Omega\\ C_\Omega & D_\Omega\end{bmatrix}\begin{bmatrix}x\\ w\end{bmatrix} :\ \|\Omega\| \leq 1\right\}. \tag{9}$$

Proposition 2 can be proved like Proposition 1 by applying Claim 3 to (7) with $h(x) = Hx$ [note that $T_i = (I - K_i D_{yq})^{-1}K_i$)]. Then, we obtain

$$\begin{aligned}q \in \{&M(S^{-2} + S^{-1}\Omega S^{-1})\\ &\cdot M^{-1}((C_y - H)x + D_{yw}w) : \|\Omega\| \leq 1\}.\end{aligned}$$

Applying this to the original system (1), we obtain (9). We call (9) the NDI for (1). If $m = 1$, then the two sets in (8) are the same and the NDI is the same as the PDI. If $m > 1$, generally the NDI strictly contains the PDI. We also note that to obtain the NDI, there must exist a positive diagonal matrix M such that $2I - M^{-1}D_{yq}M - MD_{yq}^TM^{-1} > 0$, which is a stronger requirement than well-posedness.

III. Analysis With Quadratic Lyapunov Functions

A. Some General Results for Linear Differential Inclusions

In [2], extensive results were established for stability and performance analysis of LDIs by using quadratic Lyapunov functions. Consider the LDI

$$\begin{bmatrix} \dot{x} \\ z \end{bmatrix} \in \left\{ \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} : \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \Phi \right\} \tag{10}$$

where Φ is a given convex set of matrices. The following lemma can be established like in the corresponding results in [2] by extending a polytopic Φ to a general Φ.

Lemma 1: Given $P = P^T > 0, \gamma > 0$, let $V(x) = x^TPx$ and denote by $\dot{V}(x,w)$ the derivative of V in any of the directions of the right-hand side of (10). The following holds.

1) $\dot{V}(x,w) < 0$ for all $x \in \mathbb{R}^n \setminus \{0\}$ and $w = 0$, if

$$A^TP + PA < 0 \quad \forall A \in \left\{ [I \quad 0]X \begin{bmatrix} I \\ 0 \end{bmatrix} : X \in \Phi \right\}.$$

2) $\dot{V}(x,w) \le w^Tw$ for all $x \in \mathbb{R}^n, w \in \mathbb{R}^r$, if

$$\text{He} \begin{bmatrix} PA & PB \\ 0 & -\frac{I}{2} \end{bmatrix} \le 0 \qquad \forall\, [A \quad B] \in \{[I \quad 0]X : X \in \Phi\}.$$

3) $\dot{V}(x,w) + (1/\gamma^2)z^Tz \le w^Tw$ for all $x \in \mathbb{R}^n, w \in \mathbb{R}^r$, if

$$\text{He} \begin{bmatrix} PA & PB & 0 \\ 0 & -\frac{I}{2} & 0 \\ C & D & -\frac{\gamma^2 I}{2} \end{bmatrix} \le 0 \;\; \forall \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \Phi. \tag{11}$$

The condition in item 1) guarantees that the ellipsoid $\mathcal{E}(P)$ is contractively invariant in the absence of w. It will be used for the estimation of the domain of attraction. The condition in item 2 guarantees that if $\|w\|_2 \le s$, then under the initial condition $x(0) = 0$, we will have $x(t) \in s\mathcal{E}(P)$ for all $t \ge 0$. This will be used to determine the reachable set under a class of bounded energy disturbances. Item 3) gives a condition for γ to be a bound for the $\mathcal{L}_2$ gain, i.e., $\|z\|_2 \le \gamma\|w\|_2$ for all w and $x(0) = 0$. The result in item 3) can also be found, e.g., in [19]. For the case where Φ is a polytope, we only need to verify the conditions at its vertices.

Combining Lemma 1 with the two differential inclusion descriptions, we will obtain different methods for the analysis of the original system (1). The crucial point is to guarantee that the PDI (6) [or the NDI (9)] is valid for all time under the class of disturbances and the set of initial $x(0)$'s under consideration. We are mainly concerned about the existence of a matrix H, such that $|\bar{U}^{-1}Hx(t)|_\infty \le 1$, i.e., $x(t) \in \mathcal{L}(\bar{U}^{-1}H)$, for all t. To ensure this property, we are going to construct a quadratic function $V(x) = x^TPx$, $P = P^T > 0$, and use Lemma 1 to guarantee that $x(t) \in s\mathcal{E}(P) \subseteq \mathcal{L}(\bar{U}^{-1}H)$ for all $t \ge 0$.

B. Analysis Based on the Polytopic Differential Inclusion

When $h(x) = Hx$, the PDI (6) can be written as

$$\begin{bmatrix} \dot{x} \\ z \end{bmatrix} \in \text{co}\left\{ \begin{bmatrix} A_i - B_qT_iH & B_i \\ C_i - D_{zq}T_iH & D_i \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} : i \in I[1,2^m] \right\}. \tag{12}$$

which corresponds to (10) with

$$\Phi = \text{co}\left\{ \begin{bmatrix} A_i - B_qT_iH & B_i \\ C_i - D_{zq}T_iH & D_i \end{bmatrix} : i \in I[1,2^m] \right\}.$$

We will restrict our attention to a certain ellipsoid $s\mathcal{E}(P)$. For the purpose of presenting the results in terms of LMIs, we state the results using $Q = P^{-1}$ and $Y = HQ$. To apply the PDI description within the ellipsoid $s\mathcal{E}(P) = s\mathcal{E}(Q^{-1})$, we need to ensure that $s\mathcal{E}(P) \subseteq \mathcal{L}(\bar{U}^{-1}H)$ so that $|\bar{U}^{-1}Hx|_\infty \le 1$ (i.e., $|h_\ell(x)| \le \bar{u}_\ell$ for all ℓ) for all $x \in s\mathcal{E}(P)$, which is equivalent to [recall from (2)]

$$\begin{bmatrix} \frac{\bar{u}_\ell^2}{s^2} & H_\ell \\ H_\ell^T & P \end{bmatrix} \ge 0 \quad \forall\, \ell \in I[1,m]$$

where H_ℓ is the ℓth row of H and $\bar{u}_\ell$ is the ℓth diagonal element of $\bar{U}$. Multiplying on the left and the right by $\text{diag}\{1,Q\}$, we obtain the equivalent condition

$$\begin{bmatrix} \frac{\bar{u}_\ell^2}{s^2} & Y_\ell \\ Y_\ell^T & Q \end{bmatrix} \ge 0 \quad \forall\, \ell \in I[1,m]. \tag{13}$$

Theorem 1: Given $Q \in \mathbb{R}^{n\times n}$, $Q = Q^T > 0$. Let $V(x) = x^TQ^{-1}x$. Consider system (1).

1) If there exists $Y \in \mathbb{R}^{m\times n}$ satisfying (13) with $s = 1$ and

$$QA_i^T + A_iQ - Y^TT_i^TB_q^T - B_qT_iY < 0 \;\; \forall i \in I[1,2^m] \tag{14}$$

then $\dot{V}(x,w) < 0$ for all $x \in \mathcal{E}(Q^{-1}) \setminus \{0\}$ and $w = 0$, i.e., $\mathcal{E}(Q^{-1})$ is a contractively invariant ellipsoid.

2) Let $s > 0$. If there exists $Y \in \mathbb{R}^{m\times n}$ satisfying (13) and

$$\text{He} \begin{bmatrix} A_iQ - B_qT_iY & B_i \\ 0 & -\frac{I}{2} \end{bmatrix} \le 0 \;\; \forall i \in I[1,2^m] \tag{15}$$

then $\dot{V}(x,w) \le w^Tw$ for all $x \in s\mathcal{E}(Q^{-1}), w \in \mathbb{R}^r$. If $x(0) = 0$ and $\|w\|_2 \le s$, then $x(t) \in s\mathcal{E}(Q^{-1})$ for all $t \ge 0$.

3) Let $\gamma, s > 0$. If there exists $Y \in \mathbb{R}^{m\times n}$ satisfying (13) and

$$\text{He} \begin{bmatrix} A_iQ - B_qT_iY & B_i & 0 \\ 0 & -\frac{I}{2} & 0 \\ C_iQ - D_{zq}T_iY & D_i & -\frac{\gamma^2 I}{2} \end{bmatrix} \le 0 \qquad \forall i \in I[1,2^m] \tag{16}$$

then $\dot{V}(x,w) + (1/\gamma^2)z^Tz \le w^Tw$ for all $x \in s\mathcal{E}(Q^{-1}), w \in \mathbb{R}^r$. If $x(0) = 0$ and $\|w\|_2 \le s$, then $\|z\|_2 \le \gamma\|w\|_2$.

Proof: Let $P = Q^{-1}$ and $H = YP$.

1) If we multiply (14) on the left and the right by P, we obtain $(A_i - B_qT_iH)^TP + P(A_i - B_qT_iH) < 0 \quad \forall i \in I[1,2^m]$.

Applying item 1) of Lemma 1 to the LDI (12), this guarantees that $\dot{V}(x,w) < 0$ for all $x \in \mathbb{R}^n \setminus \{0\}$ and $w = 0$ for (12). Because of (13) with $s = 1$, we have $\mathcal{E}(Q^{-1}) \subseteq \mathcal{L}(\bar{U}^{-1}H)$, i.e., $|\bar{U}^{-1}Hx|_\infty \le 1$ for all $x \in \mathcal{E}(P)$. By Proposition 1, system (1) satisfies (12) for all $x \in \mathcal{E}(Q^{-1})$. Hence, for system (1), we also have $\dot{V}(x,w) < 0$ for all $x \in \mathcal{E}(Q^{-1}) \setminus \{0\}$.

2) If we multiply (15) on the left and the right by $\mathrm{diag}\{P, I\}$, we obtain

$$\mathrm{He}\begin{bmatrix} PA_i - PB_qT_iH & PB_i \\ 0 & -\frac{I}{2} \end{bmatrix} \le 0 \quad \forall i \in I[1, 2^m].$$

By item 2) of Lemma 1, this ensures that $\dot{V}(x,w) \le w^Tw$ for all x and w for (12). Also, the condition (13) ensures that $s\mathcal{E}(Q^{-1}) \subseteq \mathcal{L}(\bar{U}^{-1}H)$ and hence (12) is valid within $s\mathcal{E}(Q^{-1})$. Therefore, we have $\dot{V}(x,w) \le w^Tw$ for all $x \in s\mathcal{E}(Q^{-1}), w \in \mathbb{R}^r$ for system (1). If $x(0) = 0$ and $\|w\|_2 \le s$, then by integrating both sides of $\dot{V} \le w^Tw$, we have $V(x(t)) \le s^2$, i.e., $x(t) \in s\mathcal{E}(Q^{-1})$ for all $t \ge 0$.

3) We note that (16) implies (15). So by item 2), it is ensured that $x(t) \in s\mathcal{E}(Q^{-1})$ for all $t \ge 0$ if $x(0) = 0$ and $\|w\|_2 \le s$. Hence, the LDI (12) is valid for system (1) for all $\|w\|_2 \le s$ and $x(0) = 0$. If we multiply (16) on the left and the right by $\mathrm{diag}\{P, I, I\}$, we obtain

$$\mathrm{He}\begin{bmatrix} PA_i - PB_qT_iH & PB_i & 0 \\ 0 & -\frac{I}{2} & 0 \\ C_i - D_{zq}T_iH & D_i & -\frac{\gamma^2 I}{2} \end{bmatrix} \le 0$$

for all $i \in I[1, 2^m]$. By Lemma 1, this ensures that $\dot{V}(x,w) + (1/\gamma^2)z^Tz \le w^Tw$ for all $x \in \mathbb{R}^n, w \in \mathbb{R}^r$ for system (12). For system (1), the inequality holds for all $x \in s\mathcal{E}(Q^{-1})$ and $w \in \mathbb{R}^r$. By integrating both sides of the inequality, we have $\|z\|_2 \le \gamma\|w\|_2$ as long as $\|w\|_2 \le s$ and $x(0) = 0$. □

It can be verified that for the special case where $D_{yq} = 0$, $D_{yw} = 0$, $D_{zq} = 0$ and $D_{zw} = 0$, items 1) and 3) reduce to the corresponding results in [25] and [13], respectively. The three parts in Theorem 1 can be respectively used to estimate the domain of attraction, the reachable set and the nonlinear $\mathcal{L}_2$ gain for system (1). For these purposes, we may formulate corresponding optimization problems with linear matrix inequality (LMI) constraints. For the estimation of the nonlinear $\mathcal{L}_2$ gain, we need to minimize γ for a selection of s over $[0, \infty)$.

1) Problem 1: Estimation of the Domain of Attraction: For the purpose of enlarging the estimation of the domain of attraction, we may choose a shape reference set X_R (see, e.g., [22], [25], and [26]) and maximize a scaling $\alpha > 0$ such that $\alpha X_R \subseteq \mathcal{E}(Q^{-1})$, with Q satisfying (13) and (14). The optimizing parameters are Q and Y. When X_R is a polygon or an ellipsoid, the resulting optimization problem has an LMI formulation.

2) Problem 2: Estimation of the Reachable Set: Under the condition (13) and (15), an estimate of the reachable set is given by $s\mathcal{E}(Q^{-1})$. Since smaller (or tighter) estimates are desirable, we may formulate an optimization problem to minimize the size of $s\mathcal{E}(Q^{-1})$. There are different measures of size for ellipsoids, such as the trace of Q and the determinant of Q, among which the trace of Q is a convex measure and is much easier to handle. In a practical situation, we may be interested in knowing the size of a certain state or an output during the operation of the system. For instance, given a row vector $C \in \mathbb{R}^{1\times n}$, we would like to estimate the maximal value of $|Cx(t)|$ for all $t \ge 0$. Since $x(t) \in s\mathcal{E}(Q^{-1})$, the maximal value of $|Cx(t)|$ is less than

$$\bar{\alpha} := (\max\{x^TC^TCx : x^T(s^2Q)^{-1}x \le 1\})^{1/2}.$$

Given $\alpha > 0$. Consider the set $\mathcal{E}(C^TC/\alpha^2) = \{x : x^TC^TCx \le \alpha^2\} = \{x : |Cx| \le \alpha\}$. It is the region between the two hyperplanes $Cx = \alpha$ and $Cx = -\alpha$. It can also be considered as a degenerated ellipsoid corresponding to a positive semidefinite matrix C^TC. Hence we have $\alpha \ge \bar{\alpha}$ if and only if $\mathcal{E}((s^2Q)^{-1}) \subset \mathcal{E}(C^TC/\alpha^2)$, which is equivalent to $C^TC/\alpha^2 \le (s^2Q)^{-1}$. Thus, $\bar{\alpha} = \min\{\alpha : C^TC \le \alpha^2(s^2Q)^{-1}\}$. Note that $C^TC \le \alpha^2(s^2Q)^{-1}$ is equivalent to $Q^{1/2}C^TCQ^{1/2} \le \alpha^2/s^2I$ and to $CQC^T \le \alpha^2/s^2$, we have

$$\bar{\alpha} = \min\left\{\alpha : CQC^T \le \frac{\alpha^2}{s^2}\right\}.$$

To minimize $\bar{\alpha}$, we can minimize α^2 satisfying the linear (in Q and α^2) constraint $CQC^T \le \alpha^2/s^2$ with Q satisfying (13) and (15). With α determined this way, we have $|Cx(t)| \le \alpha$ for all $t \ge 0$. We may choose different C's, such as $C_i, i = 1, 2, \ldots, N$, and obtain a bound α_i on $|C_ix(t)|$ for each i. The polytope formed as $\{x \in \mathbb{R}^n : |C_ix| \le \alpha_i, i = 1, \ldots, N\}$ will also be an estimate of the reachable set.

3) Problem 3: Estimation of the Nonlinear $\mathcal{L}_2$ Gain: The problem of minimizing a bound on the $\mathcal{L}_2$ gain follows directly from item 3 of Theorem 1 by minimizing γ along with parameters Q and Y satisfying (13) and (16). For each $s > 0$, denote $\gamma^*(s)$ as the minimal γ, then we have

$$\|z\|_2 \le \gamma^*(\|w\|_2)\|w\|_2$$

for all w. In other words, $\gamma^*(s)$ serves as an estimate for the nonlinear $\mathcal{L}_2$ gain.

C. Analysis Based on the Norm-Bounded Differential Inclusion

For easy reference, the NDI description for (1) is repeated as follows. If $|\bar{U}^{-1}Hx|_\infty \le 1$, then

$$\begin{bmatrix} \dot{x} \\ z \end{bmatrix} \in \left\{ \begin{bmatrix} A_\Omega & B_\Omega \\ C_\Omega & D_\Omega \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} : \|\Omega\| \le 1 \right\} \tag{17}$$

where

$$\begin{bmatrix} A_\Omega & B_\Omega \\ C_\Omega & D_\Omega \end{bmatrix} = \begin{bmatrix} A & B_w \\ C_z & D_{zw} \end{bmatrix} + \begin{bmatrix} B_q \\ D_{zq} \end{bmatrix} MS^{-1}(I+\Omega)S^{-1}M^{-1}\,[C_y - H \quad D_{yw}] \tag{18}$$

and $M > 0$ is diagonal, S is symmetric and nonsingular such that $S^2 = 2I - M^{-1}D_{yq}M - MD_{yq}^TM^{-1}$.

The next lemma will be used to handle the norm-bounded differential inclusion (17).

Lemma 2: Given X, Y, Z, S of compatible dimensions, where S is symmetric and nonsingular. If

$$\mathrm{He}\begin{bmatrix} Z & X \\ Y & -\frac{S^2}{2} \end{bmatrix} \le 0 \tag{19}$$

then $\mathrm{He}(Z + XS^{-1}(I+\Omega)S^{-1}Y) \le 0 \quad \forall \|\Omega\| \le 1$.

This lemma follows directly using Schur complements and from $M\Omega N + N^T\Omega^T M^T \le MM^T + N^T N$ for all $\|\Omega\| \le 1$.

Theorem 2: Given $Q \in \mathbb{R}^{n\times n}$, $Q = Q^T > 0$. Let $V(x) = x^T Q^{-1} x$. Consider system (1).

1) If there exist $Y \in \mathbb{R}^{m\times n}$ and a diagonal $U > 0$ satisfying (13) with $s = 1$ and

$$\mathrm{He}\begin{bmatrix} AQ & B_qU \\ C_yQ - Y & -U + D_{yq}U \end{bmatrix} < 0 \tag{20}$$

then $\mathcal{E}(Q^{-1})$ is a contractively invariant ellipsoid.

2) Given $s > 0$, if there exist $Y \in \mathbb{R}^{m\times n}$ and a diagonal $U > 0$ satisfying (13) and

$$\mathrm{He}\begin{bmatrix} AQ & B_w & B_qU \\ 0 & -\frac{I}{2} & 0 \\ C_yQ - Y & D_{yw} & -U + D_{yq}U \end{bmatrix} \le 0 \tag{21}$$

then $\dot V(x,w) \le w^T w$ for all $x \in s\mathcal{E}(Q^{-1}), w \in \mathbb{R}^r$. If $x(0) = 0$ and $\|w\|_2 \le s$, then $x(t) \in s\mathcal{E}(Q^{-1})$ for all $t \ge 0$.

3) Given $\gamma, s > 0$, if there exist $Y \in \mathbb{R}^{m\times n}$ and a diagonal $U > 0$ satisfying (13) and

$$\mathrm{He}\begin{bmatrix} AQ & B_w & 0 & B_qU \\ 0 & -\frac{I}{2} & 0 & 0 \\ C_zQ & D_{zw} & -\frac{\gamma^2 I}{2} & D_{zq}U \\ C_yQ - Y & D_{yw} & 0 & -U + D_{yq}U \end{bmatrix} \le 0 \tag{22}$$

then $\dot V(x,w) + (1/\gamma^2)z^T z \le w^T w$ for all $x \in s\mathcal{E}(Q^{-1}), w \in \mathbb{R}^r$. If $x(0) = 0$ and $\|w\|_2 \le s$, then $\|z\|_2 \le \gamma\|w\|_2$.

Proof: The procedure is very similar to the proof of Theorem 1 except we need to establish that the conditions (20)–(22) imply the respective conditions in Lemma 1 for the NDI (17). This is a little more complicated than the counterpart for Theorem 1.

Here, we only show that (22) guarantees (11) when the differential inclusion (10) is specified to (17). The other correspondences in items 1) and 2) are similar and simpler. For system (17), the condition (11) in Lemma 1 can be written as

$$\mathrm{He}\begin{bmatrix} PA_\Omega & PB_\Omega & 0 \\ 0 & -\frac{I}{2} & 0 \\ C_\Omega & D_\Omega & -\frac{\gamma^2 I}{2} \end{bmatrix} \le 0 \qquad \forall \|\Omega\| \le 1. \tag{23}$$

From (18), we have

$$\begin{bmatrix} PA_\Omega & PB_\Omega & 0 \\ 0 & -\frac{I}{2} & 0 \\ C_\Omega & D_\Omega & -\frac{\gamma^2 I}{2} \end{bmatrix} = \begin{bmatrix} PA & PB_w & 0 \\ 0 & -\frac{I}{2} & 0 \\ C_z & D_{zw} & -\frac{\gamma^2 I}{2} \end{bmatrix} + \begin{bmatrix} PB_q \\ 0 \\ D_{zq} \end{bmatrix} MS^{-1}(I+\Omega)S^{-1}M^{-1}\begin{bmatrix} C_y - H & D_{yw} & 0 \end{bmatrix}.$$

By Lemma 2, to guarantee (23), it suffices to have

$$\mathrm{He}\begin{bmatrix} PA & PB_w & 0 & PB_qM \\ 0 & -\frac{I}{2} & 0 & 0 \\ C_z & D_{zw} & -\frac{\gamma^2 I}{2} & D_{zq}M \\ M^{-1}(C_y - H) & M^{-1}D_{yw} & 0 & -\frac{S^2}{2} \end{bmatrix} \le 0. \tag{24}$$

Multiplying on the left and the right by $\mathrm{diag}\{Q, I, I, M\}$, noticing that $\mathrm{He}(-S^2/2) = \mathrm{He}(-I + M^{-1}D_{yq}M)$, $Q = P^{-1}$, $Y = HQ$, (24) is equivalent to

$$\mathrm{He}\begin{bmatrix} AQ & B_w & 0 & B_qM^2 \\ 0 & -\frac{I}{2} & 0 & 0 \\ C_zQ & D_{zw} & -\frac{\gamma^2 I}{2} & D_{zq}M^2 \\ C_yQ - Y & D_{yw} & 0 & -M^2 + D_{yq}M^2 \end{bmatrix} \le 0$$

which is (22) with $U = M^2$. □

Remark 2: If we take $Y = 0$ in (22), then the inequality reduces to [16, eq. (10a)] (with some permutation). A nonzero parameter Y introduces additional degrees of freedom for regional analysis and makes the results applicable to the case where the system wrapped around the saturation is not globally exponentially stable. ○

As with Theorem 1, different optimization problems with LMI constraints can be formulated for stability and performance analysis of the original system (1) based on the three parts of Theorem 2. Since the NDI is a more conservative description than the PDI and since Theorems 1 and 2 are developed from the same framework, it is easy to see that the analysis results from using Theorem 2 are more conservative than those from using Theorem 1. Actually, even for the special case $m = 1$ for which the NDI and PDI descriptions are the same, Theorem 2 could still be more conservative than Theorem 1 because of using Lemma 2 to derive (24). The advantage of Theorem 2 is that the conditions involve fewer LMIs (but of larger size, i.e., $+m$ more than those in Theorem 1).

We should note that the results in Theorem 2 were established in [28] through the S-procedure. The approach taken in this paper helps us to understand the relationship between the results based on two different types of differential inclusions.

IV. Analysis With Nonquadratic Lyapunov Functions

In this section, we will use a pair of conjugate functions, the convex hull quadratic function and the max quadratic function to perform stability and performance analysis of system (1). For the PDI (6), significant improvement may be achieved with these nonquadratic functions. However, for the NDI (9), there is no advantage in using these nonquadratic functions over quadratic functions. As a matter of fact, this result also applies to any norm-bounded linear differential inclusion (NLDI) (see Remark 5). We first review some results about this pair of conjugate functions.

A. The Max Quadratic Function and the Convex Hull Quadratic Function

Given a family of positive–definite matrices $P_j \in \mathbb{R}^{n\times n}$, $P_j = P_j^T > 0, j \in I[1, J]$, the pointwise maximum quadratic function is defined as

$$V_{\max}(x) := \max\{x^T P_j x : \ j \in I[1,J]\}. \qquad (25)$$

Given $Q_j \in \mathbb{R}^{n\times n}$, $Q_j = Q_j^T > 0, j \in I[1,J]$. Let

$$\Gamma := \left\{\gamma \in \mathbb{R}^J : \ \gamma_1 + \gamma_2 + \cdots + \gamma_J = 1, \gamma_j \geq 0\right\}$$

the convex hull quadratic function is defined as

$$V_c(x) := \min_{\gamma\in\Gamma} x^T \left(\sum_{j=1}^{J} \gamma_j Q_j\right)^{-1} x. \qquad (26)$$

For simplicity, we say that V_c is composed from Q_j's. It was shown in [15] that $(1/2)V_{\max}$ is conjugate to $(1/2)V_c$ if $Q_j = P_j$ for each $j \in I[1,J]$. It is evident that V_c and $V_{\max}$ are homogeneous of degree 2, i.e., $V_c(\alpha x) = \alpha^2 V_c(x)$, $V_{\max}(\alpha x) = \alpha^2 V_{\max}(x)$. Also established in [15], [23] are that V_c is convex and continuously differentiable and that $V_{\max}$ is strictly convex.

The 1-level set of $V_{\max}$ and that of V_c are, respectively

$$L_{V_{\max}} := \{x \in \mathbb{R}^n : \ V_{\max}(x) \leq 1\}$$
$$L_{V_c} := \{x \in \mathbb{R}^n : \ V_c(x) \leq 1\}.$$

Since $V_{\max}$ and V_c are homogeneous of degree 2, we have

$$sL_{V_{\max}} = \left\{x \in \mathbb{R}^n : \ V_{\max}(x) \leq s^2\right\}$$
$$sL_{V_c} = \left\{x \in \mathbb{R}^n : \ V_c(x) \leq s^2\right\}.$$

It is easy to see that $L_{V_{\max}}$ is the intersection of the ellipsoids $\mathcal{E}(P_j)$'s. In [23], It was established that L_{V_c} is the convex hull of the ellipsoids $\mathcal{E}(Q_j^{-1})$'s, i.e.,

$$L_{V_c} = \left\{\sum_{j=1}^{J} \gamma_j x_j : \ x_j \in \mathcal{E}(Q_j^{-1}), \gamma \in \Gamma\right\}.$$

For a compact convex set S, a point $x \in S$ is called an extreme point if it cannot be represented as the convex combination of any other points in S. Clearly an extreme point must belong to the boundary of S (denoted as ∂S). For a strictly convex set, such as $L_{V_{\max}}$, every boundary point is an extreme point. In what follows, we characterize the set of extreme points of L_{V_c}. Since L_{V_c} is the convex hull of $\mathcal{E}(Q_j^{-1})$'s, an extreme point must be on the boundaries of both L_{V_c} and $\mathcal{E}(Q_j^{-1})$ for some $j \in I[1,J]$ (If $x \in \partial L_{V_c} \setminus \cup_{j=1}^{J}\mathcal{E}(Q_j^{-1})$, then x must be the convex combination of at least two points from $\cup_{j=1}^{J}\mathcal{E}(Q_j^{-1})$ and thus not an extreme point of L_{V_c}). Denote

$$E_j := \partial L_{V_c} \cap \partial\mathcal{E}(Q_j^{-1})$$
$$= \left\{x \in \mathbb{R}^n : V_c(x) = x^T Q_j^{-1} x = 1\right\}.$$

Then, $\bigcup_{j=1}^{J} E_j$ contains all the extreme points of L_{V_c}. The exact description of E_j is given as follows (see [30] for the proof).

Lemma 3: For each $j \in I[1,J]$, define $F_j = \{x \in \mathbb{R}^n : x^T Q_j^{-1}(Q_k - Q_j)Q_j^{-1}x \leq 0 \quad \forall k \in I[1,J]\}$. Then $E_j = \partial L_{V_c} \cap F_j$.

It is clear that $\alpha F_j = F_j$ for any $\alpha > 0$. Since L_{V_c} is convex and contains the origin in its interior, we have $L_{V_c} = \bigcup_{\delta\in[0,1]} \delta(\partial L_{V_c})$. It follows from Lemma 3 that $\bigcup_{\delta\in[0,1]} \delta E_j = L_{V_c} \cap F_j$.

The following lemma combines some results from [23] and [24].

Lemma 4: For a given $x_0 \in \mathbb{R}^n$, let $\gamma^* \in \Gamma$ be an optimal γ such that

$$x_0^T \left(\sum_{j=1}^{J} \gamma_j^* Q_j\right)^{-1} x_0 = \min_{\gamma\in\Gamma} x_0^T \left(\sum_{j=1}^{J} \gamma_j Q_j\right)^{-1} x_0 = V_c(x_0).$$

For simplicity and without loss of generality, assume that $\gamma_j^* > 0$ for $j \in I[1,J_0]$ and $\gamma_j^* = 0$ for $j \in I[J_0+1,J]$. Denote

$$Q_0 = \sum_{j=1}^{J_0} \gamma_j^* Q_j \quad x_j = Q_j Q_0^{-1} x_0, \qquad j \in I[1,J_0].$$

Then, $V_c(x_j) = V_c(x_0)$ and $x_j \in V_c(x_0)^{1/2} E_j$, $j \in I[1,J_0]$. Moreover, $x_0 = \sum_{j=1}^{J_0} \gamma_j^* x_j$, and

$$\nabla V_c(x_0) = \nabla V_c(x_j) = 2Q_j^{-1}x_j = 2Q_0^{-1}x_0, \ j \in I[1,J_0]$$

where $\nabla V_c(x)$ denotes the gradient of V_c at x.

The following lemma is adapted from a result of [27] to the slightly different definition of V_c and $V_{\max}$ (the two functions in [27] have the coefficient $1/2$ and the saturation levels in $\bar{U}$ are also included here).

Lemma 5: [27]Let $H \in \mathbb{R}^{m\times n}, \bar{U} \in \mathbb{R}^{m\times m}$ be positive–definite diagonal and denote the ℓth row of H by H_ℓ and the ℓ-th diagonal element of $\bar{U}$ by $\bar{u}_\ell$. We have

1) $L_{V_c} \subseteq \mathcal{L}(\bar{U}^{-1}H)$ if and only if $(1/\bar{u}_\ell)H_\ell^T \in L_{V_{\max}}$ for all $\ell \in I[1,m]$;
2) $L_{V_{\max}} \subseteq \mathcal{L}(\bar{U}^{-1}H)$ if and only if $(1/\bar{u}_\ell)H_\ell^T \in L_{V_c}$ for all $\ell \in I[1,m]$.

B. Analysis With Convex Hull Quadratic Functions

In this section, we apply the convex hull quadratic function to the analysis of system (1) through the polytopic differential inclusion (6), which is repeated as follows for easy reference:

$$\begin{bmatrix} \dot{x} \\ z \end{bmatrix} \in \mathrm{co}\left\{\begin{bmatrix} A_i x + B_i w - B_q T_i h(x) \\ C_i x + D_i w - D_{zq} T_i h(x) \end{bmatrix} : \ i \in I[1,2^m]\right\}. \qquad (27)$$

This PDI is a valid description for (1) as long as $|\bar{U}^{-1}h(x)|_\infty \leq 1$. We will restrict our attention to a level set sL_{V_c}, where $|\bar{U}^{-1}h(x)|_\infty \leq 1$ for all $x \in sL_{V_c}$. As with the case of using quadratic functions, the crucial point is to guarantee that $x(t) \in sL_{V_c}$ under the class of norm-bounded w and the set of initial states under consideration.

It may appear that choosing $h(x)$ as a linear function Hx within sL_{V_c} should lead to simpler results than choosing it as a nonlinear function. However, it turns out that a nonlinear $h(x)$

not only reduces conservatism but also leads to cleaner and numerically more tractable results. As expected, the derivation of the results is more involved than the former cases in Section III because of the nonquadratic Lyapunov function and the nonlinear function $h(x)$. For this reason, we present the results separately for the estimation of the domain of attraction, the reachable set and the $\mathcal{L}_2$ gain. Based on technical considerations, we first present the result about the reachable set.

Theorem 3: (Reachable set by $\mathcal{L}_2$-norm-bounded inputs) Given $Q_j = Q_j^T > 0$, $j \in I[1, J]$, let V_c be composed from Q_j's as in (26). Given $s > 0$. System (1) with $x(0) = 0$ satisfies $x(t) \in sL_{V_c}$ for all $t \geq 0$ and for all w such that $\|w\|_2 \leq s$ if there exist $Y_j \in \mathbb{R}^{m\times n}$ and $\lambda_{ijk} \geq 0$, $i \in I[1, 2^m]$, $j, k \in I[1, J]$ such that

$$\mathrm{He}\begin{bmatrix} A_iQ_j - B_qT_iY_j + \sum_{k=1}^{J} \lambda_{ijk}(Q_j - Q_k) & B_i \\ 0 & -\frac{I}{2} \end{bmatrix} \leq 0 \quad \forall\, i \in I[1, 2^m], j \in I[1, J] \tag{28}$$

$$\begin{bmatrix} \frac{\bar{u}_\ell^2}{s^2} & Y_{j,\ell} \\ Y_{j,\ell}^T & Q_j \end{bmatrix} \geq 0 \quad \forall\, \ell \in I[1, m], j \in I[1, J] \tag{29}$$

where $Y_{j,\ell}$ is the ℓth row of Y_j.

Proof: We will prove the theorem by showing that for all $x \in sL_{V_c}$ and $w \in \mathbb{R}^r$, we have $\dot{V}_c(x, w) \leq w^Tw$, where $\dot{V}_c(x, w)$ is the time derivative of V_c in the direction of the right-hand side of (1), which depends on x and w.

Let $P_j = Q_j^{-1}$, $H_j = Y_jQ_j^{-1}$. Multiplying (28) on the left and the right by $\mathrm{diag}\{P_j, I\}$, we have

$$\mathrm{He}\begin{bmatrix} P_j(A_i - B_qT_iH_j) + \sum_{k=1}^{J} \lambda_{ijk}P_j(Q_j - Q_k)P_j & P_jB_i \\ 0 & -\frac{I}{2} \end{bmatrix} \leq 0.$$

This implies that for all $i \in I[1, 2^m]$, $j \in I[1, J]$

$$2x^TP_j(A_ix + B_iw - B_qT_iH_jx) - w^Tw \leq 2\sum_{k=1}^{J} \lambda_{ijk}x^TP_j(Q_k - Q_j)P_jx \quad \forall\, x \in \mathbb{R}^n, w \in \mathbb{R}^r. \tag{30}$$

Given $j \in I[1, J]$ and any $\delta > 0$. Consider $x \in \delta E_j$. By Lemma 3, we have

$$\sum_{k=1}^{J} \lambda_{ijk}x^TP_j(Q_k - Q_j)P_jx \leq 0.$$

It follows from (30) that

$$2x^TP_j(A_ix + B_iw - B_qT_iH_jx) - w^Tw \leq 0 \quad \forall\, x \in \delta E_j, w \in \mathbb{R}^r,\ \delta > 0. \tag{31}$$

(In view of (27) and (29), this actually shows that $\dot{V}_c(x, w) \leq w^Tw$ for all $x \in s(L_{V_c} \cap E_j)$, recalling from Lemma 4 that $\nabla V_c(x) = 2P_jx$ for $x \in E_j$. More explanation can be seen later). We proceed to show that $\dot{V}_c(x, w) \leq w^Tw$ holds for all $x \in sL_{V_c}$ by exploiting the properties of V_c.

Now, consider $x_0 \in sL_{V_c}$. Then $V_c(x_0) = \delta^2$ for some $\delta \in (0, s]$. By Lemma 4, there exist $x_j \in \delta E_j$, $\gamma_j > 0$, $j \in I[1, J_0]$ with $J_0 \leq J$ such that $\sum_{j=1}^{J_0} \gamma_j = 1$ and $x_0 = \sum_{j=1}^{J_0} \gamma_jx_j$ (we note that the indexes j can always be reordered to make this true for each x_0). Let

$$Q_0 = \sum_{j=1}^{J_0} \gamma_jQ_j \quad Y_0 = \sum_{j=1}^{J_0} \gamma_jY_j \quad H_0 = Y_0Q_0^{-1}. \tag{32}$$

Then, we also have $x_0^TQ_0^{-1}x_0 = V_c(x_0) = \delta^2$ and

$$\nabla V_c(x_0) = 2Q_0^{-1}x_0 = 2Q_j^{-1}x_j, \qquad j \in I[1, J_0]. \tag{33}$$

Applying convex combination to the inequalities in (29), we have

$$\begin{bmatrix} \frac{\bar{u}_\ell^2}{s^2} & Y_{0,\ell} \\ Y_{0,\ell}^T & Q_0 \end{bmatrix} \geq 0 \Leftrightarrow \begin{bmatrix} \frac{\bar{u}_\ell^2}{s^2} & H_{0,\ell} \\ H_{0,\ell}^T & Q_0^{-1} \end{bmatrix} \geq 0\ \forall\, \ell \in I[1, m].$$

By (2), this implies that $s\mathcal{E}(Q_0^{-1}) \subseteq \mathcal{L}(\bar{U}^{-1}H_0)$. Since $x_0^TQ_0^{-1}x_0 = \delta^2 \leq s^2$, we have $|\bar{U}^{-1}H_0x_0| \leq 1$. Thus, (27) is valid at x_0 with $h(x_0) = H_0x_0$. Hence, we have

$$\dot{x}|_{x=x_0} \in \mathrm{co}\{A_ix_0 + B_iw - B_qT_iH_0x_0 :\ i \in I[1, 2^m]\}. \tag{34}$$

and

$$\dot{V}_c(x_0, w) \in \mathrm{co}\{(\nabla V_c(x_0))^T(A_ix_0 + B_iw - B_qT_iH_0x_0) : i \in I[1, 2^m]\}. \tag{35}$$

Recalling that

$$x_0 = \sum_{j=1}^{J_0} \gamma_jx_j, \quad x_j \in \delta E_j$$
$$\nabla V_c(x_0) = 2Q_0^{-1}x_0 = 2Q_j^{-1}x_j = 2P_jx_j. \tag{36}$$

Applying (31) to x_j and replacing $2x_j^TP_j$ with $(\nabla V_c(x_0))^T$, we obtain

$$(\nabla V_c(x_0))^T(A_ix_j + B_iw - B_qT_iH_jx_j) - w^Tw \leq 0\ \forall\, w \in \mathbb{R}^r. \tag{37}$$

By the definition of Q_0, H_0 and Y_0 in (32),

$$H_0x_0 = Y_0Q_0^{-1}x_0 = \left(\sum_{j=1}^{J_0} \gamma_jY_j\right) Q_0^{-1}x_0 \tag{38}$$

and, from (33), we have

$$H_jx_j = Y_jQ_j^{-1}x_j = Y_jQ_0^{-1}x_0, \qquad j \in I[1, J_0]. \tag{39}$$

Combining (36), (38), and (39), and noting that $\gamma_1+\gamma_2+\cdots+\gamma_{J_0}=1$, we have

$$\begin{aligned}&A_i x_0 + B_i w - B_q T_i H_0 x_0\\&\quad=\sum_{j=1}^{J_0}\gamma_j A_i x_j+\sum_{j=1}^{J_0}\gamma_j B_i w - B_q T_i\sum_{j=1}^{J_0}\gamma_j Y_j Q_0^{-1}x_0\\&\quad=\sum_{j=1}^{J_0}\gamma_j(A_i x_j + B_i w - B_q T_i H_j x_j)\qquad\forall\, w\in\mathbb{R}^r\end{aligned}\tag{40}$$

Note that this is satisfied for all $i\in I[1,2^m]$. It follows from (37) that for each $i\in I[1,2^m]$ and $w\in\mathbb{R}^r$,

$$\begin{aligned}&(\nabla V_c(x_0))^T(A_i x_0 + B_i w - B_q T_i H_0 x_0) - w^T w\\&\quad=\sum_{j=1}^{J_0}\gamma_j[(\nabla V_c(x_0))^T(A_i x_j + B_i w - B_q T_i H_j x_j) - w^T w]\\&\quad\le 0.\end{aligned}$$

By (35), we obtain $\dot V_c(x_0,w)-w^Tw\le 0$ for all $w\in\mathbb{R}^r$. Note that x_0 is an arbitrary point in sL_{V_c}.

Hence, we have that $\dot V_c(x,w)\le w^Tw$ for all $x\in sL_{V_c}$ and $w\in\mathbb{R}^r$. Now, suppose $x(0)=0$ and $\|w\|_2^2\le s^2$. Then, for any $t_0>0$, as long as $x(t)\in sL_{V_c}$ for all $t\in(0,t_0)$, we have $V_c(x(t_0))\le\int_0^{t_0}w^T(\tau)w(\tau)d\tau\le s^2$, i.e., $x(t_0)\in sL_{V_c}$. On the other hand, if there exists $t_0>0$ such that $V_c(x(t))\le s^2$ for all $t\in(0,t_0)$ and $V_c(x(t_0))=s^2$ then we must have $\int_{t_0}^{\infty}w^T(\tau)w(\tau)d\tau=0$ and $\dot V_c(x(t),w(t))\le 0$ for almost all $t\ge t_0$. Hence, $V_c(x(t))\le s^2$ for all $t\ge t_0$. Therefore, we conclude that $x(t)\in sL_{V_c}$ for all $t\ge 0$. □

Remark 3: (Optimization issues) With conditions (28) and (29), we may formulate an optimization problem to minimize the estimate of the reachable set as with the quadratic function case. We observe that (28) is a bilinear matrix inequality (BMI) which contains some bilinear terms as the product of a full matrix and a scalar at the (1,1) block of the left-hand side matrix. Similar bilinear terms are contained in the matrix inequalities in [14], [15], and [27] for stability and performance analysis of linear differential/difference inclusions. A direct method to solve BMI problems is to alternatively fix one set of parameters and optimize the other set. In [14], [15], and [27], we adopted the path-following method from [20] and our experience with a set of numerical examples shows that the path-following method is much more effective than the straightforward iterative method. We actually implemented a two-step algorithm which combines the path-following method and the direct iterative method. The first step uses the path-following method to update all the parameters at the same time. The second step fixes λ_{ijk}'s and solves the resulting LMI problem which includes Q_j's and Y_j's as variables. This two-step method proves very effective on the BMI problems in [14], [15], and [27], and also works well on the example in Section V. We also see that if we take $Q_j=Q$ and $Y_j=Y$ for all j, then the bilinear terms vanish and the conditions reduce to the LMIs in (13) and (15). In our computation, we first solve the resulting optimization problem with LMI constraints and then use the optimal Q^* and Y^* to start the two-step algorithm, with $Q_j=Q^*$ and $Y_j=Y^*$ for all j and $\lambda_{ijk}\ge 0$ randomly chosen. This approach also proves effective for the problems of estimating the $\mathcal{L}_2$ gain and the domain of attraction, which will be addressed in Theorems 4 and 5.

Although there is no guarantee that the global optimal solution can be located, the convergence of the algorithms is satisfactory. Furthermore, since the initial value of the optimizing parameters can be inherited from the optimal solution obtained with quadratic functions, the algorithms ensure that the results are at least as good as those from using quadratic functions in Theorem 1. The aforementioned discussion also applies to the optimization problems resulting from Theorems 4 and 5. ∘

Remark 4: (About the nonlinear function $h(x)$): From the proof of Theorem 3, we see that a nonlinear function $h(x_0)=H_0(x_0)x_0$ is constructed from Q_j's and Y_j's so that $|\bar U^{-1}H_0(x_0)x_0|\le 1$ for all $x_0\in sL_{V_c}$ [see (32) where H_0 is constructed and the subsequent discussion up to (34)]. This makes the proof more complicated than with a linear function Hx but the result turns out to be cleaner and more easily tractable numerically. If we attempt to use a linear function $h(x)=Hx$ such that $|\bar U^{-1}Hx|_\infty\le 1$ for all $x\in sL_{V_c}$, we would have Y_j in (28) replaced with HQ_j and $Y_{j,\ell}$ in (29) replaced with $H_\ell Q_j$. When we formulate an optimization problem to estimate the reachable set by taking H and Q_j's as optimizing parameters, this would result in more complex BMI terms including HQ_j which may cause difficulties in the algorithms, such as slow convergence or getting stuck easily at a local solution. ∘

Remark 5: (Discussion about results based on NDIs): With similar developments as in the proof of Theorem 3, we can obtain a corresponding condition by using the norm-bounded differential inclusion (9) instead of using the PDI (6). The resulting condition involves the existence of Y_j's, $\lambda_{jk}\ge 0$, and a diagonal $U>0$ satisfying (29) and

$$\mathrm{He}\begin{bmatrix}AQ_j+\sum_{k=1}^{J}\lambda_{jk}(Q_j-Q_k) & B_w & B_qU\\ 0 & -\frac{I}{2} & 0\\ C_yQ_j-Y_j & D_{yw} & -U+D_{yq}U\end{bmatrix}\le 0\tag{41}$$

for all $j\in I[1,J]$. The bilinear terms in the first block seem to inject extra degrees of freedom as compared with (21) in Theorem 2 but they actually wouldn't help to reduce the conservatism. In other words, (41) implies the existence of a Q satisfying (21). To see this, we form a matrix

$$\Psi=\begin{bmatrix}-\sum_{k=2}^{J}\lambda_{1k} & \lambda_{21} & \cdots & \lambda_{J1}\\ \lambda_{12} & -\sum_{k=1,k\ne 2}^{J}\lambda_{2k} & \cdots & \lambda_{J2}\\ \vdots & \vdots & \vdots & \vdots\\ \lambda_{1J} & \lambda_{2J} & \cdots & -\sum_{k=1}^{J-1}\lambda_{Jk}\end{bmatrix}.$$

Then Ψ is a Metzler matrix. Since the sum of each column of Ψ is 0, the eigenvalue with the maximal real part is 0. Hence there exists a vector $c\ne 0$ with $c_i\ge 0$ such that $\Psi c=0$ (e.g., see [37]) and in particular we assume $\sum_{j=1}^{J}c_j=1$ (i.e., $c\in\Gamma$). If we let $Q=\sum_{j=1}^{J}c_jQ_j$, and $Y=\sum_{j=1}^{J}c_jY_j$, then Q and Y will satisfy (21) and (13). Furthermore, $s\mathcal{E}(Q)\subseteq sL_{V_c}$ is a smaller estimate of the reachable set. This means that with the NDI description, using the convex hull quadratic Lyapunov function offers no advantage to using the quadratic Lyapunov function. The same situation occurs for the estimation of the $\mathcal{L}_2$ gain or the domain of attraction, or, when applying a max quadratic function to NDIs.

For the special case where $H = 0$, the regional NDI (9) becomes a global NLDI. Thus we can conclude that for any NLDI, the convex hull quadratic function or the max quadratic function offers no advantage over quadratic functions when these stability and performance issues are concerned. $\circ$

We next address the problems of estimating the $\mathcal{L}_2$ gain and the domain of attraction.

Theorem 4: ($\mathcal{L}_2$ gain for norm-bounded w): Given $Q_j = Q_j^T > 0$, $j \in I[1, J]$, let V_c be composed from Q_j's as in (26). Consider system (1). Given $s, \gamma > 0$. If there exist $Y_j \in \mathbb{R}^{m\times n}$ and $\lambda_{ijk} \geq 0$, $i \in I[1, 2^m]$, $j, k \in I[1, J]$ such that

$$\mathrm{He}\begin{bmatrix} A_iQ_j - B_qT_iY_j + \sum_{k=1}^{J}\lambda_{ijk}(Q_j - Q_k) & B_i & 0 \\ 0 & -\frac{I}{2} & 0 \\ C_iQ_j - D_{zq}T_iY_j & D_i & -\frac{\gamma^2 I}{2} \end{bmatrix} \leq 0 \quad \forall\, i \in I[1, 2^m], j \in I[1, J] \tag{42}$$

$$\begin{bmatrix} \frac{\bar{u}_\ell^2}{s^2} & Y_{j,\ell} \\ Y_{j,\ell}^T & Q_j \end{bmatrix} \geq 0 \qquad \forall\, \ell \in I[1, m], j \in I[1, J] \tag{43}$$

then for all w such that $\|w\|_2 \leq s$ and $x(0) = 0$, we have $\|z\|_2 \leq \gamma\|w\|_2$.

Proof: We will prove the theorem by showing that for all $x \in sL_{V_c}$ and $w \in \mathbb{R}^r$, $\dot{V}_c(x, w) + (1/\gamma^2)z^Tz \leq w^Tw$. Since (42) implies (28), by Theorem 3, we have $x(t) \in sL_{V_c}$ for all t and for all $\|w\|_2 \leq s$, $x(0) = 0$. Also, all the relationships established in the proof of Theorem 3 are true under the conditions of the current theorem.

Let $P_j = Q_j^{-1}$, $H_j = Y_jQ_j^{-1}$ and

$$W_{ij} = P_jA_i - P_jB_qT_iH_j + \sum_{k=1}^{J}\lambda_{ijk}P_j(Q_j - Q_k)P_j.$$

Multiplying (42) on the left and the right by $\mathrm{diag}\{P_j, I, I\}$, we have

$$\mathrm{He}\begin{bmatrix} W_{ij} & P_jB_i & 0 \\ 0 & -\frac{I}{2} & 0 \\ C_i - D_{zq}T_iH_j & D_i & -\frac{\gamma^2 I}{2} \end{bmatrix} \leq 0.$$

By Schur complements, this is equivalent to

$$\mathrm{He}\begin{bmatrix} W_{ij} & P_jB_i \\ 0 & -\frac{I}{2} \end{bmatrix} + \frac{1}{\gamma^2}\begin{bmatrix} (C_i - D_{zq}T_iH_j)^T \\ D_i^T \end{bmatrix}[C_i - D_{zq}T_iH_j \;\; D_i] \leq 0. \tag{44}$$

Denote

$$\begin{aligned} f_{ij}(x, w) &= A_ix + B_iw - B_qT_iH_jx \\ g_{ij}(x, w) &= C_ix + D_iw - D_{zq}T_iH_jx. \end{aligned}$$

Then (44) implies that for all $x \in \mathbb{R}^n$, $w \in \mathbb{R}^r$,

$$2x^TP_jf_{ij}(x, w) + \frac{1}{\gamma^2}g_{ij}^T(x, w)g_{ij}(x, w) - w^Tw \leq 2\sum_{k=1}^{J}\lambda_{ijk}x^TP_j(Q_k - Q_j)P_jx. \tag{45}$$

Consider $x \in \delta E_j$ for $\delta > 0$. Like in the proof of Theorem 3, we have

$$\sum_{k=1}^{J}\lambda_{ijk}x^TP_j(Q_k - Q_j)P_jx \leq 0 \; \forall x \in \delta E_j, w \in \mathbb{R}^r, \delta > 0.$$

It follows from (45) that

$$2x^TP_jf_{ij}(x, w) + \frac{1}{\gamma^2}g_{ij}^T(x, w)g_{ij}(x, w) - w^Tw \leq 0, \quad \forall\, x \in \delta E_j, w \in \mathbb{R}^r, \delta > 0. \tag{46}$$

We note that this is true for all $i \in I[1, 2^m]$ and $j \in I[1, J]$.

Now, consider $x_0 \in sL_{V_c}$. Then $V_c(x_0) = \delta^2$ for some $\delta \in (0, s]$. Like in the proof of Theorem 3, there exist $x_j \in \delta E_j$, $\gamma_j > 0, j \in I[1, J_0]$ such that $\sum_{j=1}^{J_0}\gamma_j = 1$ and $x_0 = \sum_{j=1}^{J_0}\gamma_jx_j$. Let H_0, Q_0, Y_0 be defined as in (32). Then, we also have $|\bar{U}^{-1}H_0x_0| \leq 1$. Applying Proposition 1 at x_0, we have

$$\begin{bmatrix} \dot{x} \\ z \end{bmatrix} \in \mathrm{co}\left\{\begin{bmatrix} A_ix_0 + B_iw - B_qT_iH_0x_0 \\ C_ix_0 + D_iw - D_{zq}T_iH_0x_0 \end{bmatrix} : i \in I[1, 2^m]\right\}.$$

Let

$$\begin{aligned} f_{i0}(x_0, w) &= A_ix_0 + B_iw - B_qT_iH_0x_0 \\ g_{i0}(x_0, w) &= C_ix_0 + D_iw - D_{zq}T_iH_0x_0. \end{aligned}$$

Then, see (47), as shown at the bottom of the page.

Since $2x_j^TP_j = 2x_0^TQ_0^{-1} = (\nabla V_c(x_0))^T$ [see (33)], applying (46) at x_j, we obtain

$$(\nabla V_c(x_0))^Tf_{ij}(x_j, w) + \frac{1}{\gamma^2}|g_{ij}(x_j, w)|^2 - w^Tw \leq 0 \quad \forall\, w \in \mathbb{R}^r,\; i \in I[1, 2^m].$$

$$\dot{V}_c(x_0, w) + \frac{1}{\gamma^2}z^Tz - w^Tw \leq \max\{(\nabla V_c(x_0))^Tf_{i0}(x_0, w) + \frac{1}{\gamma^2}|g_{i0}(x_0, w)|^2 - w^Tw : i \in I[1, 2^m]\} \tag{47}$$

Like in (40), we have

$$f_{i0}(x_0,w)=\sum_{j=1}^{J_0}\gamma_j f_{ij}(x_j,w)$$
$$g_{i0}(x_0,w)=\sum_{j=1}^{J_0}\gamma_j g_{ij}(x_j,w).$$

It follows that

$$(\nabla V_c(x_0))^T f_{i0}(x_0,w)+\frac{1}{\gamma^2}|g_{i0}(x_0,w)|^2-w^Tw\le 0$$

and from (47)

$$\dot V_c(x_0,w)+\frac{1}{\gamma^2}z^Tz-w^Tw\le 0 \tag{48}$$

which is satisfied for all $x_0\in sL_{V_c}$ and $w\in\mathbb{R}^r$. Since $x(0)=0$, $x(t)\in sL_{V_c}$ for all t and for all $\|w\|_2\le s$, integrating both sides of (48), we have $\|z\|_2^2\le\gamma^2\|w\|_2^2$. This completes the proof. □

Theorem 5: (Estimation of the domain of attraction): Given $Q_j=Q_j^T>0, j\in I[1,J]$, let V_c be composed from Q_j's as in (26). Consider system (1) with $w\equiv 0$. We have $\dot V_c(x)<0$ for all $x\in L_{V_c}\setminus\{0\}$ if there exist $\lambda_{ijk}\ge 0$, $Y_j\in\mathbb{R}^{m\times n}, i\in I[1,2^m], j,k\in I[1,J]$ such that

$$\mathrm{He}(A_iQ_j-B_qT_iY_j+\sum_{k=1}^{J}\lambda_{ijk}(Q_j-Q_k))<0 \quad \forall i\in I[1,2^m], j\in I[1,J]$$
$$\begin{bmatrix}1 & Y_{j,\ell}\\ Y_{j,\ell}^T & Q_j\end{bmatrix}\ge 0 \quad \forall \ell\in I[1,m], j\in I[1,J].$$

Proof: The proof can be adapted from the proof of Theorem 3 by assuming that $B_i=0$. Then, with the same procedure, it can be shown that $\dot V_c(x)<0$ for all $x\in L_{V_c}\setminus\{0\}$. □

Remark 6: Note that the condition in Theorem 5 is similar to (but less conservative than) that of [27, Th. 4], which is developed for a special case without algebraic loops. Similar numerical complexity can be expected. ∘

C. Analysis With Max Quadratic Functions

The max quadratic function is not differentiable everywhere. Following the definition of [43, p. 215], a subgradient of a convex function $f:\mathbb{R}^n\to\mathbb{R}$ at x_0 is a vector $v\in\mathbb{R}^n$ such that

$$f(x)-f(x_0)\ge v^T(x-x_0)\ \ \forall x\in\mathbb{R}^n \tag{49}$$

and the subdifferential, denoted as $\partial f(x_0)$ (not to be confused as the boundary of a set) , is the set of all subgradient at x_0. The function $f(x)$ is differentiable at x_0 if and only if $\partial f(x_0)$ is single valued. We use $\partial V_{\max}(x)$ to denote the sub-differential of $V_{\max}$ at x.

Lemma 6: Consider $x_0\in\mathbb{R}^n$. Suppose that there exists $J_0\in I[1,J]$ such that $V_{\max}(x_0)=x_0^TP_jx_0$ for $j\in I[1,J_0]$ and $V_{\max}(x_0)>x_0^TP_jx_0$ for $j>J_0$. Then

1) $\partial V_{\max}(x_0)=\mathrm{co}\{2P_jx_0 : j\in I[1,J_0]\}$;
2) for a vector $\zeta\in\mathbb{R}^n$, the directional derivative of $V_{\max}$ at x_0 along ζ is

$$\lim_{t\to 0^+}\frac{V_{\max}(x_0+t\zeta)-V_{\max}(x_0)}{t}=\max_{\xi\in\partial V_{\max}(x_0)}\{\xi^T\zeta\}.$$

Proof: See [30]. □

For simplicity and with some abuse of notation, for $\dot x$ given by (1), denote

$$\dot V_{\max}(x,w):=\max_{\xi\in\partial V_{\max}(x)}\{\xi^T\dot x\}$$
$$=\max_{\xi\in\partial V_{\max}(x)}\{\xi^T(Ax+B_qq+B_ww)\}.$$

Then, by Lemma 6, with $\zeta=\dot x$, $V_{\max}$ is decreasing along $\dot x$ if and only if $\dot V_{\max}(x,w)<0$.

Theorem 6: (Reachable set by bounded inputs) Given $P_j=P_j^T>0, j\in I[1,J]$, let $V_{\max}$ be the max quadratic function formed by P_j's as in (25). Given $s>0$. System (1) with $x(0)=0$ satisfies $x(t)\in sL_{V_{\max}}$ for all $t\ge 0$ and for all w such that $\|w\|_2\le s$ if there exist $H\in\mathbb{R}^{m\times n}$, $\lambda_{ijk}\ge 0$, $\alpha_{\ell j}\ge 0$, $j,k\in I[1,J]$, $i\in I[1,2^m]$, $\ell\in I[1,m]$, such that $\sum_{j=1}^{J}\alpha_{\ell j}=1$, and

$$\mathrm{He}\begin{bmatrix}P_jA_i-P_jB_qT_iH+\sum_{k=1}^{J}\lambda_{ijk}(P_j-P_k) & P_jB_i\\ 0 & -\frac{I}{2}\end{bmatrix}\le 0 \quad \forall i\in I[1,2^m], j\in I[1,J] \tag{50}$$

$$\begin{bmatrix}\frac{\bar u_\ell^2}{s^2} & H_\ell\\ H_\ell^T & \sum_{j=1}^{J}\alpha_{\ell j}P_j\end{bmatrix}\ge 0\ \forall \ell\in I[1,m]. \tag{51}$$

Proof: By the definition of V_c, condition (51) implies that $V_c((s/\bar u_\ell)H_\ell)\le 1$ for all $\ell\in I[1,m]$. By Lemma 5, this implies that $L_{V_{\max}}\subseteq\mathcal{L}(s\bar U^{-1}H)=(1/s)\mathcal{L}(\bar U^{-1}H)$, i.e., $sL_{V_{\max}}\subseteq\mathcal{L}(\bar U^{-1}H)$. Hence, $|\bar U^{-1}Hx|_\infty\le 1$ for all $x\in sL_{V_{\max}}$. By Proposition 1, we have

$$\dot x\in\mathrm{co}\{A_ix+B_iw-B_qT_iHx : i\in I[1,2^m]\}\ \forall x\in sL_{V_{\max}}.$$

On the other hand, it can be verified that (50) implies that

$$2x^TP_j(A_ix+B_iw-B_qT_iHx)-w^Tw \le 2\sum_{k=1}^{J}\lambda_{ijk}x^T(P_k-P_j)x \tag{52}$$

for all $j\in I[1,J], i\in I[1,2^m]$. The state–space of x can be partitioned as the following subsets:

$$S_j=\{x\in\mathbb{R}^n : x^T(P_k-P_j)x\le 0, k\in I[1,J]\},\ j\in I[1,J].$$

If $x \in S_j \setminus \cup_{k\neq j} S_k$, then $V_{\max}(x) = x^T P_j x$ and $\partial V_{\max}(x) = 2P_j x$. If $x \in \cap_{j=1}^{J_0} S_j \setminus \cup_{j=J_0+1}^{J} S_j$, then $V_{\max}(x) = x^T P_j x, j \in I[1, J_0]$ and $\partial V_{\max}(x) = \mathrm{co}\{2P_j x : j \in I[1, J_0]\}$.

We first consider $x \in S_j \setminus \cup_{k\neq j} S_k$. Then

$$\sum_{k=1}^{J} \lambda_{ijk} x^T (P_k - P_j) x \leq 0 \tag{53}$$

and

$$\dot{V}_{\max}(x, w) - w^T w \leq \max_{i\in I[1,2^m]} (2x^T P_j (A_i x + B_i w - B_q T_i H x) - w^T w).$$

If $x \in \cap_{j=1}^{J_0} S_j \setminus \cup_{j=J_0+1}^{J} S_j$, then (53) is satisfied for all $j \in I[1, J_0]$ and we have

$$\dot{V}_{\max}(x, w) - w^T w \leq \max_{j\in I[1,J_0]} \max_{i\in I[1,2^m]} (2x^T P_j (A_i x + B_i w - B_q T_i H x) - w^T w).$$

It follows from (52) and (53) that $\dot{V}_{\max}(x, w) - w^T w \leq 0$. The remaining part of the proof is similar to the proof of Theorem 3. □

The following results can be proven similarly to Theorem 6 (see [30] for proofs).

Theorem 7: ($\mathcal{L}_2$ gain for norm-bounded w) Given $P_j = P_j^T > 0, j \in I[1, J]$. Consider system (1) and $s, \gamma > 0$. If there exist $H \in \mathbb{R}^{m\times n}$, $\lambda_{ijk} \geq 0$, $\alpha_{\ell j} \geq 0$, $j, k \in I[1, J]$, $i \in I[1, 2^m]$, $\ell \in I[1, m]$, such that $\sum_{j=1}^{J} \alpha_{\ell j} = 1$ and

$$\mathrm{He}\begin{bmatrix} P_j(A_i - B_q T_i H) + \sum_{k=1}^{J} \lambda_{ijk}(P_j - P_k) & P_j B_i & 0 \\ 0 & -\frac{I}{2} & 0 \\ C_i - D_{zq} T_i H & D_i & -\frac{\gamma^2 I}{2} \end{bmatrix} \leq 0 \quad \forall\, i \in I[1, 2^m], j \in I[1, J] \tag{54}$$

$$\begin{bmatrix} \frac{\bar{u}_\ell^2}{s^2} & H_\ell \\ H_\ell^T & \sum_{j=1}^{J} \alpha_{\ell j} P_j \end{bmatrix} \geq 0 \ \forall\, \ell \in I[1, m] \tag{55}$$

then for all w such that $\|w\|_2 \leq s$ and $x(0) = 0$, we have $\|z\|_2 \leq \gamma \|w\|_2$.

Theorem 8: (Estimation of the domain of attraction): Given $P_j = P_j^T > 0, j \in I[1, J]$. Consider system (1) with $w \equiv 0$. We have $\dot{V}_{\max}(x, 0) < 0$ for all $x \in L_{V_{\max}} \setminus \{0\}$ if there exist $H \in \mathbb{R}^{m\times n}$, $\lambda_{ijk} \geq 0$, $\alpha_{\ell j} \geq 0$, $j, k \in I[1, J]$, $i \in I[1, 2^m]$, $\ell \in I[1, m]$, such that $\sum_{j=1}^{J} \alpha_{\ell j} = 1$ and

$$\mathrm{He}\left(P_j A_i - P_j B_q T_i H + \sum_{k=1}^{J} \lambda_{ijk}(P_j - P_k)\right) < 0 \quad \forall\, i \in I[1, 2^m], j \in, I[1, J] \tag{56}$$

$$\begin{bmatrix} \bar{u}_\ell^2 & H_\ell \\ H_\ell^T & \sum_{j=1}^{J} \alpha_{\ell j} P_j \end{bmatrix} \geq 0 \quad \forall\, \ell \in I[1, m]. \tag{57}$$

As compared with the counterpart results from using convex hull quadratic functions, the conditions (50), (54), and (56) in Theorems 6–8 appear to be less tractable because of the bilinear term $P_j B_q T_i H$ in the first blocks of the matrices. Also, the same H for all P_j's seems to offer fewer degrees of freedom as compared with different Y_j's for different Q_j's in Theorems 3–5. However, numerical examples show that Theorems 6–8 may produce better results in some cases.

V. Examples

1) Example 1: Consider system (1) with the following parameters:

$$\begin{bmatrix} A & B_q & B_w \\ C_y & D_{yq} & D_{yw} \\ C_z & D_{zq} & D_{zw} \end{bmatrix} = \left[\begin{array}{ccc|cc|cc} 0 & 0 & -1 & 1 & 0 & 0 & 1 \\ 1 & 0 & -2 & 0 & 1 & 1 & 0 \\ 0 & 1 & -3 & 1 & -1 & 1 & 1 \\ \hline 1 & 0 & 1 & -3 & -1 & 1 & -1 \\ 0 & 1 & 0 & -2 & -4 & 0 & 1 \\ \hline 0 & 1 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & -1 \end{array}\right].$$

The well-posedness of the system is easily verified through Claim 2. We use the four methods in Theorems 1, 2, 4, and 7 to estimate the nonlinear $\mathcal{L}_2$ gain. The resulting estimates are plotted in Fig. 2, where the dotted curve is from applying quadratics via NDI (Theorem 2), the dashed–dotted one is from applying quadratics via PDI (Theorem 1), the dashed one is from applying max quadratics (with $J = 2$) via PDI (Theorem 7) and the solid one is from applying convex hull quadratics ($J = 2$) via PDI (Theorem 4). Each of the four curves tends to a constant value as $\|w\|_2$ goes to infinity. This constant value will be an estimate of the global $\mathcal{L}_2$ gain. As expected, applying quadratics via PDI always leads to better results than applying quadratics via NDI, and applying one of the two nonquadratics always leads to better results than applying quadratics. However, the relationship between the results from applying the two nonquadratic functions is not definite. The situation exhibited in Fig. 2 can be reversed if we change the parameters of the system. In what follows, we present several scenarios through some adjustments of the parameters.

2) Case 2: If we change D_{yq} to $D_{yq} = \begin{bmatrix} -3 & -1.3 \\ -2.3 & -4 \end{bmatrix}$ (well-posedness ensured), then the global $\mathcal{L}_2$ gain by using quadratics via NDI is unbounded (or, global stability is not confirmed), while that by using quadratics via PDI is 170.1473. By using max quadratics and convex hull quadratics, the global $\mathcal{L}_2$ gains are, respectively, 20.7833 and 19.3307.

3) Case 3: If we change D_{yq} to $D_{yq} = \begin{bmatrix} -3 & -2 \\ -2 & -4 \end{bmatrix}$ (well-posedness ensured), then the global $\mathcal{L}_2$ gain by using quadratics via either NDI or PDI is unbounded. By using max quadratics and convex hull quadratics, the global $\mathcal{L}_2$ gains are, respectively, 42.3354 and 31.6731.

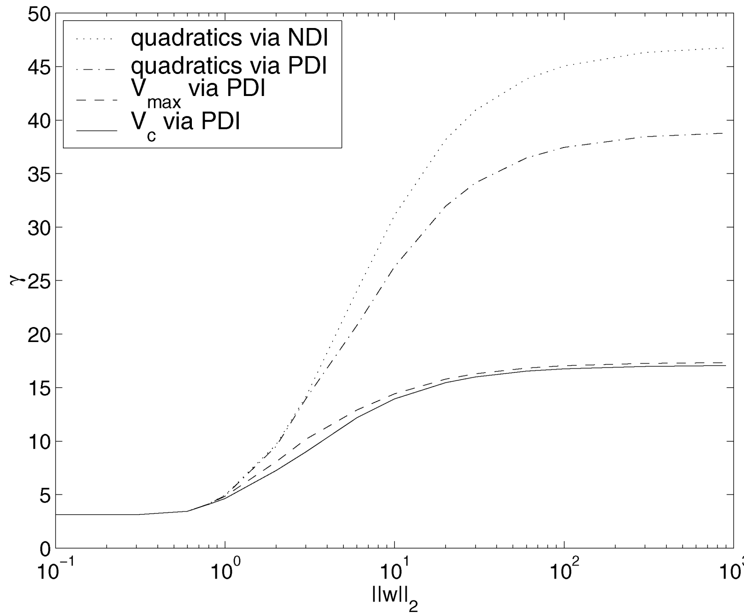

Fig. 2. Different estimates of the nonlinear $\mathcal{L}_2$ gain: Case 1.

The above two situations also show how the stability and performance results by the same method can be affected by the parameter D_{yq} which describes the algebraic loop. As discussed in [40], this parameter is one of the two key design parameters in static anti-windup synthesis and can have a dramatic impact on antiwindup performance.

Due to space limitation, we will not present computational results about the estimation of the domain of attraction or the estimation of the reachable set. Interested readers are referred to [27] for some numerical results. From the different situations exhibited through the $\mathcal{L}_2$ gain, it is not hard to infer that the difference among the estimations by using quadratics/non-quadratics via NDI/PDI can be made arbitrarily large through adjusting the four elements of D_{yq}. For instance, Case 2 suggests that the estimate of the domain of attraction by using quadratics via NDI is bounded while that by using quadratics via PDI is the whole state space. Case 3 suggests that the domain of attraction estimated by nonquadratic functions is the whole state space while that by quadratics (via PDI or NDI) is bounded. On the other hand, the estimate of the reachable set by nonquadratics can be bounded while that by quadratics is not.

We should remark that for this particular example, the algorithm for applying convex hull quadratics converges very well for all the values of s that we considered in our numerical computation, even under different parameter changes. The algorithm for applying max quadratics generally converges well but for some values of s it showed some difficulties where we needed to stop the algorithm and restart it from different initial values of λ_{ijk} which are randomly generated. In any case, improvement is expected from the nonquadratic functions.

4) Example 2: We adopt Example 2 from [16]. The plant is a cart-spring-pendulum system with one control input, one disturbance input, four states and one measurement output. The plant and controller parameters can be found in [16]. For this example, the closed-loop system without antiwindup compensation is not globally stable. Also, there exists no static antiwindup compensation to make the global $\mathcal{L}_2$ gain bounded. With dynamic anti-windup augmentation, an upper bound for the achievable global $\mathcal{L}_2$ gain is found to be 181.1424 (by using quadratic Lyapunov functions). When this achievable gain is approached, some parameters of the antiwindup compensator will approach infinity. To make the parameters within a reasonable range, we have to allow a slightly larger global $\mathcal{L}_2$ gain. A particular dynamic antiwindup compensator is given as follows, with notation adopted from [16]:

$$\left[\begin{array}{c|c}\Lambda_1 & \Lambda_2 \\ \hline \Lambda_3 & \Lambda_4\end{array}\right] = \left[\begin{array}{cccc|c} -10.04 & -8.67 & 5.95 & -34.81 & 625.3 \\ 16.78 & -0.0077 & -50.52 & 33.4 & 214.6 \\ 27.26 & 12.9 & -176.84 & -20.2 & 1534 \\ 6.8 & 9.56 & -54.1 & -35 & 410 \\ \hline 157 & -10 & -148 & 105 & 1467 \\ 3209 & -1315 & 1458 & 6281 & 3452 \\ 972 & -763 & 1102 & -196 & -6949 \\ 74719 & -50878 & 27569 & -10528 & 24840 \\ -1152 & -367 & 5992 & 387 & -54618 \end{array}\right].$$

When quadratic Lyapunov functions are used via the PDI, the estimated global $\mathcal{L}_2$ gain is 182.3080. When V_c (with $J = 2$) is used via the PDI, a slightly smaller estimate is given as 181. 2326. For other values of bound on $\|w\|_2$, the improvement by using V_c is also small. However, if we change some parameters of the system, the difference between estimates by quadratics and nonquadratics can be arbitrarily large.

For this particular system, we have $D_{yq} = \Lambda_4(5)$. Hence, the algebraic loop is directly affected by $\Lambda_4(5)$. Suppose that we change $\Lambda_4(5)$ from -54618 to -52618. Two estimates of the nonlinear $\mathcal{L}_2$ gain are plotted in Fig. 3, where the dashed curve corresponds to the estimate obtained by applying quadratic functions and the solid one to that obtained by applying V_c (with $J = 2$), both via PDI description. Also plotted as a

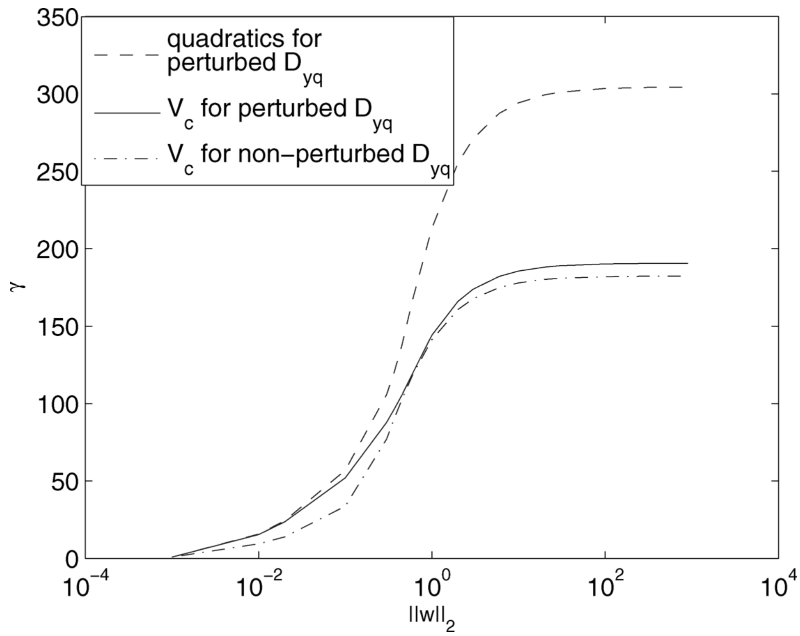

Fig. 3. Different estimates of $\mathcal{L}_2$ gain under parameter perturbation.

dashed–dotted curve is the estimate obtained by using V_c when $\Lambda_4(5) = -54618$. The above computational results suggest that nonquadratic functions may also have advantage for analyzing robust performance under parameter perturbations. This will motivate further research problems.

The order of the closed-loop system for this example is 12, including the state of the plant, the controller and the dynamic antiwindup compensator. The BMI problem for V_c with $J = 2$ involves 189 variables (the two matrices Q_1 and Q_2 for V_c contain 156 variables). It takes about 2 h to generate the solid curve (a connection of 18 points). The smoothness of the curve suggests the uniformity of the convergence to some optimal or suboptimal solutions, considering that the algorithm was run only once for each value of $\|w\|_2$ and the initial values of λ_{ijk}'s were chosen randomly.

VI. Conclusion

For a general system with saturation or deadzone components, regional stability and performance analysis relies on an effective regional treatment of the algebraic loop and the deadzone function. This paper provides such a treatment which yields two forms of parameterized differential inclusions. Applying available tools based on quadratic Lyapunov functions to these differential inclusions, we obtained conditions for stability and performance in the form of LMIs. These conditions are easily tractable but could be conservative in view of the quadratic Lyapunov functions applied. Further improvement relies on using non-quadratic Lyapunov functions. We explored a pair of conjugate Lyapunov functions in this paper and reduced the conservatism of the conditions with a series of BMI conditions. Numerical experience shows that these BMI conditions can be effectively solved with the path following method. Although there is no guarantee that the global optimal solutions will be obtained, the great potential of these nonquadratic Lyapunov functions has been revealed by numerical examples. The effectiveness demonstrated through these examples motivates further investigation on these nonquadratic Lyapunov functions and the development of more efficient algorithms to handle them for more complicated situations. This paper's results lay foundations for the design of saturated controllers and for the design of anti-windup compensators. Preliminary results have been obtained in [29] for regional dynamic anti-windup design which is based on the analysis result by applying quadratic functions via NDI. The analysis results based on PDI and nonquadratic functions can be applied for design purposes by incorporating controller design parameters into the existing optimization problem. In this regard, main efforts will be devoted to making the optimization problems more tractable through careful algebraic manipulation and appropriate parameter transformations.

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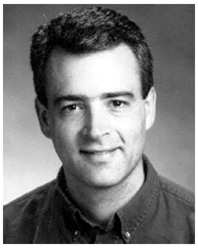

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