$\dot{x} = A_i x$ (i =1, 2), *viz.* $U(x)$ is a Lyapunov function for the system $\dot{x}(t) = A_2 x(t)$; obviously no such function can exist as A_2 is not Hurwitz.

VI. Conclusions

In this technical note, incremental stability theory gets extended in several respects. First of all a *continuous* converse Lyapunov Theorem for incremental iISS is derived. Interestingly, this allows to show that incremental iISS implies incremental ISS (rather surprisingly, as the opposite implication holds in the non-incremental case), thanks to a construction which also shows how local and global incremental stability notions are equivalent. Then, a counter-example is provided showing that candidate Lyapunov functions of the form $V(x_1 - x_2)$ need not be general enough to study incremental stability properties of nonlinear systems.

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Command Filtered Backstepping

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***Abstract*—Implementation of backstepping becomes increasingly complex as the order of the system increases. This increasing complexity is mainly driven by the need to compute command derivatives at each step of the design, with the ultimate step requiring derivatives of the same order as the plant. This article addresses a modification that obviates the need to compute analytic derivatives by introducing command filters in the backstepping design. While the concept of the command filter has previously been introduced in the literature, the main contribution of this technical note is the rigorous analysis of the effect of the command filter on closed-loop stability and performance, and a proof of stability based on Tikhonov's theorem. The implementation approach includes a compensated tracking error that retains the standard stability properties of backstepping approaches.**

***Index Terms*—Backstepping, nonlinear control, singular perturbation.**

I. Introduction

A typical requirement, in tracking control for an n-th order system, is that the desired output and its first n derivatives must be available for use in the control law implementation. In many applications, the user input device or trajectory planner only specifies a desired output signal $x_d^o(t)$. The signal $x_d^o(t)$ is constrained to be bounded, but may contain discontinuities or other features that may not be achievable by the physical system. A standard practice in applications is to treat $x_d^o(t)$ as the input to a prefilter with state space representation

$$\begin{aligned} \dot{z}_i &= z_{i-1} \quad \text{for } i = \{1, 2, \ldots, n-1\} \\ \dot{z}_n &= -a_1 (z_1 - x_d^o) - a_2 z_2 - \ldots - a_n z_n \end{aligned} \tag{1}$$

where the characteristic equation $s^n + a_n s^{n-1} + \ldots + a_1 = 0$ is selected to be stable and to specify the desired bandwidth and transient response of the system, see e.g. [14]–[16]. Then the desired system output is defined as $x_d(t) = z_1(t)$. The designer of the system ensures that the user input device and the prefilter are compatible in the sense that the error $x_d(t) - x_d^o(t)$ is small. The first $n-1$ derivatives of x_d are the states $z_2, \ldots, z_n$, which are continuous and bounded if $x_d^o(t)$ is bounded. The n-th derivative of x_d is specified by (1). The signal $x_d(t)$ and its first n derivatives are used in the implementation of the nonlinear control law, for example, by feedback linearization [6] or backstepping [8]. Alternative means for differentiating a signal are discussed for example in [1], [10], [21].

In the backstepping control approach [7], [8], the control law design uses states as virtual control signals. At each step of the design, virtual control signals, denoted by $\bar{\alpha}_i$ in Section III, and their derivatives are required. Theoretically, calculation of the virtual control signal derivatives is simple, but it can be quite complicated and tedious in applications when n is greater than three because the control signal u will include the derivative of $\bar{\alpha}_n$, which requires the second derivative

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of $\bar{\alpha}_{n-1}$, which requires the third derivative of $\bar{\alpha}_{n-2}$, and so on. See e.g., (44)–(45) in [12] or (3.8) and (3.10) in [9]. In certain applications, the analytic derivation is overly cumbersome, see e.g. [11]. This issue has been addressed by a variety of methods. The authors in [11], [20] approximate the command derivatives using sliding mode filters [10]. Linear filters for derivative generation are considered in [22]. In [17], [18], the command derivatives are modeled as portions of unknown functions that are approximated during operation.

The method described herein requires that only the signals x_d and $\dot{x}_d$ be available as inputs to the control system. If necessary, these signals can be the outputs of a *command filter* of order at least one, as described in (1). This article uses the command filtering idea to derive and analyze a practical extension of the backstepping approach. A main motivation for this extension is simplification of the process of determining the command derivatives required for backstepping implementation. Preliminary versions of the method presented herein were applied to aircraft control in [2]–[5], [19] without the formal proof presented herein.

The benefits of the presented Command Filtered Backstepping (CFBS) approach include: (1) Decoupling of the design of the controllers for the backstepping iterations; and, (2) Obviating the need for analytic computation of command signal derivatives. Note that even if the designer were to derive exact analytical expressions for the command derivatives relative to the design model, these are still approximations because that model is an approximate representation of the plant. Therefore, the choice is not between a correct analytic expression or a filtered estimate of the command derivatives; instead, the choice is between two estimates of the command derivatives.

The organization of the remainder of this article is as follows. Section II formally states the control problem and its related technical assumptions. Section III derives a standard backstepping controller which is used as a point-of-reference throughout the technical note. Command filtered backstepping (CFBS) is presented in Section IV, along with theorems concerning its stability properties. These are the main results of this article. Section V derives dynamic equations that are required to prove these theorems; the proofs themselves are presented in Section VI.

II. Problem Formulation

Consider the following class of n-th order single-input-single-output nonlinear systems

$$\dot{x}_i = f_i(w_i) + g_i(w_i)x_{i+1} \tag{2}$$

$$\dot{x}_n = f_n(x) + g_n(x)u \tag{3}$$

where $i = 1, \ldots, n-1$, $x = [x_1, \cdots, x_n]^\top \in \Re^n$ is the state vector with initial condition $x(0) = x_0$, $w_i = [x_1, \cdots, x_i]^\top$, the first state x_1 is considered as the scalar output, and u is the scalar control signal. The functions f_i and g_i are assumed to be known. The backstepping and CFBS approaches require slightly different assumptions on the continuity of f_i and g_i, which will be stated later in Assumptions 2 and 4. To ensure controllability, we will invoke the following assumption, which is standard in backstepping.

Assumption 1: There exists a constant $g_o > 0$ such that for $i = 1, \ldots, n$ each function $|g_i(w_i)| \geq g_o$.

Our objective is trajectory tracking. Therefore, we assume there is a desired trajectory $x_d(t) : \Re^+ \mapsto \Re$ with derivative $\dot{x}_d(t)$, both of which are available and bounded for all $t > 0$. The objective of the control design is to specify a control signal u to steer $x_1(t)$ from any initial conditions to track the reference input $x_d(t)$ and to achieve boundedness for the states x_i for $i = 2, \ldots, n$. Note that standard backstepping requires knowledge of the first n derivatives of $x_d(t)$.

III. Standard Backstepping

This section summarizes the standard backstepping design. The design is a point-of-reference in the subsequent discussion and in the stability proof of Section VI-B.

The tracking error vector is defined as $\bar{x} = [\bar{x}_1, \ldots, \bar{x}_n]$ with $\bar{x}_i = x_i - \bar{\alpha}_{i-1}$ for $i = 1, \ldots, n$ where $\bar{\alpha}_0 = x_d$. The vector of functions $\bar{\alpha} = [\bar{\alpha}_1, \ldots, \bar{\alpha}_n]^\top$, referred to as virtual control signals, is defined as

$$\bar{\alpha}_1(w_1, x_d) = \frac{1}{g_1}(-k_1\bar{x}_1 + \dot{x}_d - f_1) \tag{4}$$

$$\bar{\alpha}_i(w_i, x_d) = \frac{1}{g_j}(-k_j\bar{x}_j + \dot{\bar{\alpha}}_{j-1} - f_j - g_{j-1}\bar{x}_{j-1}) \tag{5}$$

where $j = 2, \ldots, n$, w_i are defined after (3) and $k_i > 0$ for $i = 1, \ldots, n$. The control variable is assigned the value $u(t) = \bar{\alpha}_n(x(t), x_d(t))$. In the interest of presenting a specific formulation, in the above definition, we have canceled the natural dynamics of the system. However, if certain nonlinearities are considered 'beneficial', then they need not be canceled by the control signal. For (4), (5) to be well-defined the following technical assumption concerning $f_i(w_i), g_i(w_i)$, for $i = 1, \cdots, n$ must be satisfied.

Assumption 2: For each $i \in [1, n]$, $f_i, g_i \in \mathcal{C}^{n-i}$.

In addition, we require the following assumption.

Assumption 3: For $t \geq 0$, for $i \in [0, n-1]$, the signals $x_d^{(i)}(t)$ are continuous, bounded, and available; and, the signal $x_d^{(n)}(t)$ is bounded and available.

The closed-loop tracking error differential equations are

$$\dot{\bar{x}}_1 = -k_1\bar{x}_1 + g_1\bar{x}_2 \tag{6}$$

$$\dot{\bar{x}}_i = -k_i\bar{x}_i + g_i\bar{x}_{i+1} - g_{i-1}\bar{x}_{i-1} \tag{7}$$

$$\dot{\bar{x}}_n = -k_n\bar{x}_n - g_{n-1}\bar{x}_{n-1} \tag{8}$$

for $i = 2, \ldots, n-1$ with initial conditions defined by $\bar{x}_i(0) = x_i(0) - \bar{\alpha}_{i-1}(w_i(0), x_d(0))$.

The derivative of the Lyapunov function $V_o = (1/2)\sum_{i=1}^n \bar{x}_i^2$ along the solutions of (6)–(8) satisfies $\dot{V}_o \leq -\underline{k}V_o$ where $\underline{k} = min(k_i)$. Therefore, by Theorem 4.10 in [7] the origin of the tracking error system of (6)–(8) is exponentially stable. In addition, for $i = 1, \ldots, n$: $\bar{x}_i \in \mathcal{L}_2 \cap \mathcal{L}_\infty$, $\bar{\alpha}_i, x_i \in \mathcal{L}_\infty$.

Equation (5) has a deceptively simple form. As n increases, analytic computation of $\dot{\bar{\alpha}}_i$ for $i = 1, \ldots, n-1$ becomes increasingly complicated. Practitioners have used various ad-hoc methods to address the issue. The ad-hoc methods can be problematic in adaptive backstepping approaches, because they can result in the breakdown of the proofs of certain stability properties of the closed-loop adaptive approach.

IV. Command Filtered Approach

This section presents a modification of the backstepping approach that eliminates the analytic computation of $\dot{\bar{\alpha}}_i$ for $i = 1, \ldots, n-1$, while allowing rigorous stability analysis and allowing extension to the adaptive case. Section IV-A discusses the definition of the signals involved in the approach. Fig. 1 depicts the signal flow in block diagram form. Section IV-B presents two theorems that summarize the properties of the approach.

A. Design Approach

Define the pseudocontrol signals α_i for the CFBS procedure, for $\mathrm{i} = 2, \ldots, \mathrm{n}$, as

$$\alpha_1 = \frac{1}{g_1}(-k_1\tilde{x}_1 + \dot{x}_{1,c} - f_1) \tag{9}$$

$$\alpha_i = \frac{1}{g_i}(-k_i\tilde{x}_i + \dot{x}_{i,c} - f_i - g_{i-1}v_{i-1}) \tag{10}$$

$$u = \alpha_n\left(x(t), x_{n,c}(t), v_{n-1}(t)\right). \tag{11}$$

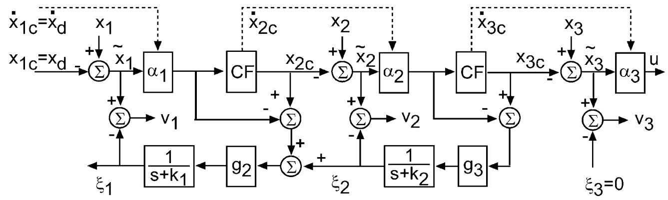

Fig. 1. Block diagram of the CFBS approach for $n = 3$. Each command filter (CF) is defined in (15), (16). The α_i are computed according to (9), (10). The dotted lines from the CF to the α_i represent the communication of the command derivative $\dot{x}_{i,c}$ from the CF to the computation of α_i.

The control gains, $k_i > 0$, $i = 1, \cdots, n$ are designer specified constants as in the standard approach. The tracking error vector is defined as $\tilde{x} = [\tilde{x}_1, \cdots, \tilde{x}_n]^\top$ where

$$\tilde{x}_i = x_i - x_{i,c} \quad \text{for } i = 1, \ldots, n. \tag{12}$$

Define the compensated tracking error signals v_i as

$$v_i = \tilde{x}_i - \xi_i, \text{ for } i = 1, \cdots, n. \tag{13}$$

The ξ_i signals for $i = 1, \ldots, n-1$ are defined as

$$\dot{\xi}_i = -k_i \xi_i + g_i(x_{i+1,c} - \alpha_i) + g_i \xi_{i+1} \tag{14}$$

with $\xi_i(0) = 0$. For $i = n$, define $\xi_n = 0$.

Equations (9) and (10) use the signals $x_{i,c}$ and $\dot{x}_{i,c}$ for $i = 1, \ldots, n$ that are defined in this paragraph. For $i = 1$, $x_{1,c} = x_d = \bar{\alpha}_0$ and $\dot{x}_{1,c} = \dot{x}_d = \dot{\bar{\alpha}}_0$. For $i = 1, \ldots, n-1$, define the state space implementation of the command filters as

$$\dot{z}_{i,1} = \omega_n z_{i,2} \tag{15}$$

$$\dot{z}_{i,2} = -2\zeta\omega_n z_{i,2} - \omega_n(z_{i,1} - \alpha_i) \tag{16}$$

with $x_{i+1,c}(t) = z_{i,1}$ and $\dot{x}_{i+1,c}(t) = \omega_n z_{i,2}$ as the outputs of each filter. The filter initial conditions are $z_{i,1}(0) = \alpha_i(w_i(0), x_{i,c}(0), v_{i-1}(0))$ and $z_{i,2}(0) = 0$. The filter design parameters are $\omega_n > 0$ and $\zeta \in (0, 1]$. Each command filter is designed to compute $x_{i+1,c}$ and $\dot{x}_{i+1,c}$ without differentiation. The transfer functions corresponding to (15), (16) are

$$\frac{\begin{bmatrix} \omega_n^2 \\ s\omega_n^2 \end{bmatrix}}{s^2 + 2\zeta\omega_n s + \omega_n^2}.$$

The natural frequency of the command filter is equal to the parameter ω_n; the filter has unit dc gain to the first output; and the first output is the integral of the second output. The designer would typically select $\omega_n > k_{i+1}$ for all i so that $x_{i+1,c}$ and $\dot{x}_{i+1,c}$ will accurately track α_i and $\dot{\alpha}_i$, respectively. The effect of the errors $(x_{i+1,c} - \alpha_i)$ and $(\dot{x}_{i+1,c} - \dot{\alpha}_i)$ is a crucial issue to be analyzed in the stability of this approach.

The signal $x_{i+1,c}$ is the command filtered version of α_i; therefore, $(x_{i+1,c} - \alpha_i)$ represents the unachieved portion of α_i. Each signal ξ_i is a filtered version of $(x_{i+1,c} - \alpha_i)$. The variables v_i are referred as *compensated tracking errors* because they are obtained by removing the filtered unachieved portion of α_i, as represented by ξ_i, from the tracking error (see (13)). Given these signal definitions, we are able to prove the results summarized in Section IV-B.

B. Summary of Stability Properties

The two theorems of this section summarize the stability properties of the CFBS approach. Prior to stating the theorems we state the following assumptions.

Assumption 4: For each $i = 1, \ldots, n$, f_i and g_i and their first partial derivatives are continuous and bounded on any compact set $D_i \subset \Re^i$.

Assumption 5: For $t \geq 0$, the signals $x_d(t)$ and $\dot{x}_d^{(1)}(t)$ are continuous, bounded, and available.

Whereas the backstepping approach of Section III required Assumptions 2 and 3, the command filtered approach will invoke Assumptions 4 and 5. In comparison: for $i = n$, Assumption 4 is more stringent than Assumption 2; for $i = n-1$, Assumption 4 is somewhat equivalent to Assumption 2; and, for $i = 1, \ldots, n-2$, Assumption 4 is less stringent than Assumption 2. Assumptions 5 is always less stringent than Assumption 3. Note that Assumption 4 implies that each f_i is (locally) Lipshitz in w_i.

Theorem 1: For the system described by (2), (3) that satisfies Assumptions 1, 4 and 5 with the feedback control law defined in (11), we have the following properties for $i = 1, \ldots, n$: (1) $\tilde{x}_n, v_i \in \mathcal{L}_\infty \bigcap \mathcal{L}_2$; and, (2) $\tilde{x}_n(t)$ and $v_i(t)$ converge to zero exponentially. ◇

Theorem 1 states that the compensated tracking errors of the command filtered backstepping approach (i.e., v) have the same properties as the tracking errors of the standard backstepping approach (i.e., $\bar{x}$). The proof of Theorem 1 is in Section VI-A and mainly uses the equations derived in Section V-B. Theorem 1 leaves open the question of the properties of the signals $\tilde{x}_i$ and ξ_i for $i = 1, \ldots, n-1$, which are discussed in Theorem 2. Let $y(t, \epsilon)$ be a time signal, where ϵ is a filter parameter influencing the generation of the signal. Next, we will use the notation $y(t, \epsilon) = \mathcal{O}(\epsilon)$ which is defined as follows [7].

Definition 1: The signal $y(t, \epsilon)$ is said to be of order ϵ, denoted as $y(t, \epsilon) = \mathcal{O}(\epsilon)$, if there exists positive constants k and c such that $|y(t, \epsilon)| \leq k|\epsilon|, \forall |\epsilon| < c$ and $t \geq 0$.

Theorem 2: For the system described by (2), (3) that satisfies Assumptions 1, 2, 3, and 4 with the feedback control law defined in (11), and the initial conditions specified in Section IV-A, we have the following properties: (1) $\tilde{x}(t, \epsilon) - \bar{x}(t) = \mathcal{O}(\epsilon)$; (2) $z_{i,1}(t, \epsilon) - \bar{\alpha}_i = \mathcal{O}(\epsilon)$; and, (3) $z_{i,2}(t, \epsilon) - \dot{\bar{\alpha}}_i = \mathcal{O}(\epsilon)$; for $i = 1, \ldots, n-1$ where $\bar{\alpha}_i$ is defined in (5), $\bar{x}(t)$ is the backstepping tracking error solution to (6), (8), $[z_{i,1}, z_{i,2}]^\top$ is the solution to (15), (16), and the notation $\tilde{x}(t, \epsilon)$ represents the tracking error of (12) for a specific choice of the command filter parameter ω_n, and $\epsilon = (1/\omega_n)$. ◇

Theorem 2 shows that, by increasing the command filter natural frequency ω_n, the solution to the CFBS closed-loop system can be made arbitrarily close to the backstepping solution that relies on analytic derivatives. The proof of Theorem 2 is in Section VI-B. The proof uses the command filtered tracking error differential equations derived in Section V-A and Tikhonov's theorem (Theorem 11.2 in [7]).

Remark 1: Because Theorem 2 compares the solutions of the command filtered and standard backstepping approaches, Assumptions 1–4 are all required for its proof. However, implementation of the CFBS controller only requires Assumptions 1, 4, and 5. In fact, Assumption 4 is more conservative than is required for implementation.

V. Error Dynamics

The analysis of Section VI requires the dynamics of the tracking error $\tilde{x}_i$ and the dynamics of the compensated tracking error v_i. These equations are derived in this section. The analysis of Section VI-B will use Tikhonov's theorem which requires analysis of the dependence of the initial conditions of the system on the parameter ϵ. Therefore, we explicitly state this dependence in the following subsections.

A. Tracking Error

This subsection uses the control approach defined in Section IV to derive the differential equations for the tracking error. This analysis can be divided into three cases

$$\begin{aligned}\dot{\tilde{x}}_1 &= f_1 + g_1\alpha_1 - \dot{x}_{1,c} + g_1(x_{2,c} - \alpha_1) + g_1(x_2 - x_{2,c}) \\ &= -k_1\tilde{x}_1 + g_1(x_{2,c} - \alpha_1) + g_1\tilde{x}_2;\end{aligned} \tag{17}$$

$$\begin{aligned}\dot{\tilde{x}}_i &= f_i + g_i\alpha_i - \dot{x}_{i,c} + g_i(x_{i+1,c} - \alpha_i) + g_i(x_{i+1} - x_{i+1,c})\\ &= -k_i\tilde{x}_i - g_{i-1}v_{i-1} + g_i(x_{i+1,c} - \alpha_i) + g_i\tilde{x}_{i+1}; \qquad (18)\\ \dot{\tilde{x}}_n &= f_n + g_n u - \dot{x}_{n,c}\\ &= -k_n\tilde{x}_n - g_{n-1}v_{n-1} \qquad (19)\end{aligned}$$

for $2 < i < (n-1)$. The initial conditions for the tracking error differential (17)–(19) are

$$\tilde{x}_i(0) = x_i(0) - x_{i,c}(0) \qquad (20)$$

which are independent of ϵ.

B. Compensated Tracking Error

Due to the fact that $\dot{v}_i = \dot{\tilde{x}}_i - \dot{\xi}_i$ for $1 < i < n$, it is straightforward to show that for $2 < i < n-1$

$$\begin{aligned}\dot{v}_1 &= -k_1v_1 + g_1v_2; \qquad (21)\\ \dot{v}_i &= -k_iv_i - g_{i-1}v_{i-1} + g_iv_{i+1} \qquad (22)\\ \dot{v}_n &= \dot{\tilde{x}}_n = -k_nv_n - g_{n-1}v_{n-1}. \qquad (23)\end{aligned}$$

The initial conditions for differential (21)–(23) are $v_i(0) = \tilde{x}_i(0)$ which are defined in (20) and are independent of ϵ.

VI. Stability Analysis

Section IV-B presented two theorems that summarized the properties of the CFBS approach. These theorems are proved in the following two subsections.

A. Proof of Theorem 1

The properties of the vector v are analyzed by considering the following Lyapunov function $V = \sum_{i=1}^n V_i(v_i) = (1/2)\|v\|_2^2$, where $V_i = (1/2)v_i^2$. The time derivative of the V is $\dot{V} = \sum_{i=1}^n \dot{V}_i$, which along solutions of (21)–(23) are

$$\begin{aligned}\dot{V}_1 &= -k_1v_1^2 + v_1g_1v_2,; \qquad (24)\\ \dot{V}_i &= -k_iv_i^2 - v_{i-1}g_{i-1}v_i + v_ig_iv_{i+1} \qquad (25)\\ \dot{V}_n &= -k_nv_n^2 - v_{n-1}g_{n-1}v_n \qquad (26)\end{aligned}$$

for $1 < i < n$. Therefore, the derivative of $V(t)$ is $\dot{V} \le -\underline{k}\|v\|_2^2 = -2\underline{k}V$ where $\underline{k} = \min_i(k_i)$. By Theorem 4.10 in [7], the origin of the system described by (21)–(23) is globally exponentially stable. The state $\tilde{x}_n$ converges exponentially to zero, because $\tilde{x}_n = v_n$. Also, by integration of $\dot{V} \le -\underline{k}\|v\|_2^2$, it is straightforward to show that $v \in \mathcal{L}_2$.

Note that the structure of the command filtered system is intentionally designed to follow a standard backstepping proof [8]. However, this proof shows the exponential stability of the compensated tracking error v, not $\tilde{x}$. The properties of $\tilde{x}$, z, and $\tilde{x} - \bar{x}$ are addressed by Theorem 2.

B. Proof of Theorem 2

This proof uses singular perturbation theory. The proof will show that all preconditions of Theorem 11.2 in [7] are satisfied, so that the theorem can be applied. The proof uses the compact set $\mathcal{D}_{\hat{x}} \times \mathcal{D}_{\hat{z}}$ where $\mathcal{D}_{\hat{x}} \subset \Re^{2n-1}$ and $\mathcal{D}_{\hat{z}} \subset \Re^{2n-2}$ are compact sets that contain the origin.

Remark 2: Theorem 11.2 in [7] is too long to allow its inclusion. To allow straightforward interpretation of the results of this section in terms of that theorem, (27), (28) are in the form of [7, (11.6)–(11.7)]. The terminology of this section and the technical statements following each numbered equation correspond to the requirements of Theorem 11.2.

Define the vectors $\hat{x} = [\tilde{x}_1, \ldots, \tilde{x}_n, \xi_1, \ldots, \xi_{n-1}]^\top \in \Re^{2n-1}$ and $\hat{z} = [z_{1,1}, z_{1,2}, \ldots, z_{n-1,1}, z_{n-1,2}] \in \Re^{2n-2}$. The differential equations for these vectors are

$$\begin{aligned}\dot{\hat{x}} &= \hat{f}(t, \hat{x}, \hat{z}, \epsilon) \qquad (27)\\ \epsilon\dot{\hat{z}} &= \hat{g}(t, \hat{x}, \hat{z}, \epsilon) \qquad (28)\end{aligned}$$

where $\hat{f}$ and $\hat{g}$ are defined below. The initial conditions $\hat{x}(0) = [\tilde{x}_1(0), \ldots, \tilde{x}_n(0), 0, \ldots, 0]^\top$ and $\hat{z}(0) = [z_{i,1}(0), 0, \ldots, z_{n-1,1}(0), 0]$ are independent of ϵ.

The vector field $\hat{f}$, as derived based on (14) and (17)–(19), is given by

$$\begin{aligned}\hat{f}_1(t, \hat{x}, \hat{z}, \epsilon) &= -k_1\tilde{x}_1 + g_1(z_{1,1} - \alpha_1) + g_1\tilde{x}_2\\ \hat{f}_i(t, \hat{x}, \hat{z}, \epsilon) &= -k_i\tilde{x}_i - g_{i-1}(\tilde{x}_{i-1} - \xi_{i-1})\\ &\quad + g_i(z_{i,1} - \alpha_i) + g_i\tilde{x}_{i+1}\\ \hat{f}_n(t, \hat{x}, \hat{z}, \epsilon) &= -k_n\tilde{x}_n - g_{n-1}(\tilde{x}_{n-1} - \xi_{n-1})\\ \hat{f}_{j+n}(t, \hat{x}, \hat{z}, \epsilon) &= -k_j\xi_j + g_j(z_{j,1} - \alpha_j) + g_j\xi_{j+1}\end{aligned}$$

for $i = 2, \ldots, n-1$ and for $j = 1, \ldots, n-1$. Note that $\hat{f}$ is independent of ϵ. Therefore, on any compact set $\mathcal{D}_{\hat{x}} \times \mathcal{D}_{\hat{z}}$, with Assumption 4: the function $\hat{f}$ and its first partial derivatives with respect to $(\hat{x}, \hat{z}, \epsilon)$ are continuous and bounded; and, $(\partial\hat{f}/\partial t)$ is Lipschitz in $\hat{x}$ uniformly in t.

Note that $\hat{z}$ is just the concatenation of the states of each of the command filters defined in (15), (16). Therefore, $\hat{g}$ is the concatenation of these same equations. For $i = 1, \ldots, n-1$, the elements of the vector field g are determined from (15), (16) as

$$\begin{aligned}\hat{g}_{2i}(t, \hat{x}, \hat{z}, \epsilon) &= z_{i,2}\\ \hat{g}_{2i+1}(t, \hat{x}, \hat{z}, \epsilon) &= -2\zeta z_{i,2} - (z_{i,1} - \alpha_i) \qquad (29)\end{aligned}$$

which shows that $\hat{g}$ is independent of ϵ. Therefore, on any compact set $\mathcal{D}_{\hat{x}} \times \mathcal{D}_{\hat{z}}$, with Assumptions 4 and 1: the function $\hat{g}$ and its first partial derivatives with respect to $(\hat{x}, \hat{z}, \epsilon)$ are continuous and bounded; the first partial derivative of $\hat{g}$ with respect to t is continuous and bounded; and, $\partial\hat{g}(t, \hat{x}, \hat{z}, 0)/\partial\hat{z}$ has bounded first partial derivatives with respect to its arguments.

For $\epsilon = 0$ the unique solution to (28) is defined by $z_{i,1} = \alpha_i$ and $z_{i,2} = 0$ which in vector form will be denoted by $\hat{z} = \hat{h}(t, \hat{x})$ where for $i = 1, \ldots, n-1$

$$\hat{h}_{2i}(t, \hat{x}) = \alpha_i \quad \text{and} \quad \hat{h}_{2i+1}(t, \hat{x}) = 0.$$

With Assumptions 4 and 1, on any compact set $\mathcal{D}_{\hat{x}}$, the function $\hat{h}(t, \hat{x})$ has bounded first partial derivatives with respect to its arguments.

Let $\bar{x}(t)$ denote the solution of the reduced order problem [7, eqn. (11.5) p. 424]

$$\dot{\bar{x}} = \hat{f}\left(t, \bar{x}, \hat{h}(t, \hat{x}), 0\right) \qquad (30)$$

with $\hat{x}(0) = [\tilde{x}_1(0), \ldots, \tilde{x}_n(0), 0, \ldots, 0]^\top$. Because of the initial condition and the fact that $z_{i,1} = \alpha_i$, the solution of the reduced order problem has $\bar{x}_i(t) = 0$ for $i = n+1, \ldots, 2n-1$ for all $t > 0$. Given the facts in the previous sentence, the solution for states $\bar{x}_i$ for $i = 1, \ldots, n$, for the same initial conditions, is the solution to the standard backstepping problem presented in Section III which is exponentially stable. Therefore, for the reduced order problem, the states $\bar{x}_i$ with $i = 1, \ldots, n$ converge exponentially to zero. Because $\bar{x}_i$ converge exponentially to zero for $i = 1, \ldots, n$ and $\bar{x}_i(t) = 0$ for all $t > 0$ for $i = n+1, \ldots, 2n-1$, the origin is an exponentially stable equilibrium of the reduced order system.

By defining $\hat{y} = \hat{z} - \hat{h}(t, \hat{x})$, the boundary layer model $(dy/d\tau) = \hat{g}(t, \hat{x}, \hat{y} + \hat{h}(t, \hat{x}))$ with $(t, \hat{x})$ considered fixed and $\tau = (t/\epsilon)$ (see in

[7, eqn. (11.14) p. 433]) is $(d\hat{y}/d\tau) = A\hat{y}$ where A is a block diagonal matrix with $(n-1)$ blocks each defined by

$$J_i = \begin{bmatrix} 0 & 1 \\ -1 & -2\zeta \end{bmatrix}.$$

The boundary layer model is independent of $\hat{x}$. The matrix A is Hurwitz. Therefore, the origin is a globally exponentially stable equilibrium of the boundary layer model.

All conditions of Theorem 11.2 in [7] hold on any compact set $\mathcal{D}_{\hat{x}} \times \mathcal{D}_{\hat{z}}$. If we denote the solutions to (27), (28), and (30) respectively, as $\hat{x}(t,\epsilon)$, $\hat{z}(t,\epsilon)$, and $\bar{x}(t)$ then for all $t > 0$, $\hat{x}(t,\epsilon) - \bar{x}(t) = \mathcal{O}(\epsilon)$ and $\hat{z}(t,\epsilon) - \hat{h}(t,\bar{x}) = \mathcal{O}(\epsilon)$ which proves Theorem 2.

VII. Conclusion

This article presented a rigorous analysis and proof of stability for a practical extension to backstepping. A main motivation was facilitation of backstepping implementation by offering a means to determine the time derivatives of the virtual control signals, denoted herein by $\bar{\alpha}$. The new approach is feasible even when the number of iterations of the backstepping method is large (i.e., greater than three). The required derivatives are determined by a method referred to as *command filtering*. The method was described in Section IV-A. Its properties were summarized in Theorems 1 and 2 of Section IV-B. In particular, Theorem 2 states that by increasing the bandwidth of the command filters, the performance of the command filtered backstepping approach can be made arbitrarily close to that of the standard backstepping approach that uses analytic calculation of derivatives. Additional benefits of the command filtered approach presented herein are that it is applicable to a wider class of systems than standard backstepping (see Remark 1) and the command filters can also be used to enforce constraints on the state trajectories [3], [5], [19].

Herein, we have presented and analyzed the CFBS approach for the nonadaptive case. The extension of CFBS to the adaptive case is considered in [3]–[5], [13] with a partial stability analysis. These articles rigorously consider the stability of the parameter adaptation process, but not the command filter variables. A complete analysis of the adaptive case is beyond the scope of the present technical note, but will be considered in a subsequent article.

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