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NULL CONTROLLABILITY OF LINEAR SYSTEMS WITH CONSTRAINED CONTROLS*

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Abstract. The paper considers the problem of steering the state of a linear time-varying system to the origin when the control is subject to magnitude constraints. Necessary and sufficient conditions are given for global constrained controllability as well as a necessary and sufficient condition for the existence of a control (satisfying the constraints) which steers the system to the origin from a specified initial epoch (x_0, t_0). The global result does not require zero to be an interior point of the control set Ω, and the theorem for constrained controllability at (x_0, t_0) only requires that Ω be compact, not that it contain zero. The results are compared to those available in the literature. Furthermore, numerical aspects of the problem are discussed as is a technique for determining a steering control.

1. Introduction and formulation. Consider the problem of steering the state of a linear system

$$\text{(S)} \qquad \dot{x}(t) = A(t)x(t) + B(t)u(t), \qquad t \in [t_0, \infty)$$

to the origin from a specified initial condition

$$x(t_0) = x_0$$

by choice of control function $u(\cdot)$. Here $x(t) \in R^n$, $u(t) \in R^m$, and $A(\cdot)$ and $B(\cdot)$ are continuous matrices[1] of appropriate dimension. Unlike the usual controllability problem, where the control values at each instant of time are unconstrained, we insist here that the control values at each instant of time belong to a prespecified set Ω in R^m.

Let $\mathcal{M}(\Omega)$ denote the set of functions from R into Ω that are measurable on $[t_0, \infty)$. Then any control $u(\cdot) \in \mathcal{M}(\Omega)$ is termed admissible. We now define three notions of constrained controllability or, more precisely, Ω-null controllability.

DEFINITION 1.1. The linear system (S) is *Ω-null controllable at* (x_0, t_0) if, given the initial condition $x(t_0) = x_0$, there exists a control $u(\cdot) \in \mathcal{M}(\Omega)$ such that the solution $x(\cdot)$ of (S) satisfies $x(t) = 0$ for some $t \in [t_0, \infty)$.

DEFINITION 1.2. The linear system (S) is *globally Ω-null controllable at* t_0 if (S) is Ω-null controllable at (x_0, t_0) for all $x_0 \in R^n$.

Our major result will pertain to the global type of controllability. To compare our results to those of previous researchers, we also need a local controllability concept.

DEFINITION 1.3. The linear system (S) is *locally Ω-null controllable at* t_0 if there exists an open set $V \subset R^n$, containing the origin, such that (S) is null controllable at (x_0, t_0) for all $x_0 \in V$.

The majority of constrained controllability results are for autonomous systems, i.e., systems where A and B are constant. When $\Omega = R^m$, Kalman [1] showed that a necessary and sufficient condition for global R^m-null controllability is $\operatorname{rank}(Q) = n$, where $Q \triangleq [B, AB, \cdots, A^{n-1}B]$. Lee and Markus [2] considered constraint sets $\Omega \subset R^m$ which contain $u = 0$, and showed that $\operatorname{rank}(Q) = n$ is a necessary and sufficient condition for (S) to be locally Ω-null controllable. Furthermore, if each eigenvalue λ of A satisfies $\operatorname{Re}(\lambda) < 0$, then (S) is globally Ω-null controllable. This result is typical of the results available when Ω contains the origin.

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[1] This requirement can be weakened to local integrability.

Saperstone and Yorke [3] were the first to eliminate the assumption that zero is an interior point of Ω when they considered problems with $m=1$ and $\Omega=[0, 1]$. Their result states that, for these problems, (S) is locally Ω-null controllable if and only if $\operatorname{rank}(Q)=n$ and A has no real eigenvalues. They also extend this result to $m>1$ and consider the m-fold product set $\Omega=\prod_1^m[0, 1]$. Problems with more general constraint sets were studied by Brammer [4] who showed that if there exists a $u\in\Omega$ satisfying $Bu=0$, and the convex hull of Ω has a nonempty interior, then necessary and sufficient conditions for local Ω-null controllability are $\operatorname{rank}(Q)=n$ and the nonexistence of a real eigenvector v of A' satisfying $v'Bu\leqq 0$ for all $u\in\Omega$. In addition, if no eigenvalue of A has a positive real part, then the theorem becomes one for global Ω-null controllability. A similar result for global controllability when $\Omega=[0, 1]$ was obtained by Saperstone [5]. Friedman [6] considers a linear pursuit evasion problem, where the target is a closed convex set, and gives a sufficient condition for the existence of a pursuer control, based on the evader's control, which drives the system from a specified initial condition to the target.

For nonautonomous systems, the most familiar controllability result is that of Kalman [1] when $\Omega=R^m$. He showed that (S) is R^m-null controllable if and only if $W(t_0, t_1)$ is positive definite for some $t_1\in[t_0, \infty)$, where

$$W(t_0, t_1)\triangleq\int_{t_0}^{t_1}\phi(t_1, \tau)B(\tau)B'(\tau)\phi'(t_1, \tau)\,d\tau,$$

and $\phi(t, \tau)$ is the state transition matrix for (S). When the control is constrained, the major global results are those by Conti [7] and Pandolfi [8]. In [7], Conti describes a "divergent integral condition" which is necessary and sufficient for global Ω-null controllability when Ω is the closed unit ball. In order to make Conti's result more compatible with existing theory for time-invariant systems, Pandolfi in [8] defines the notion of pth characteristic exponent for time-varying systems. For the special case when the system is time-invariant, the characteristic exponent turns out to be the real part of some eigenvalue of A. Subsequently, controllability criteria are provided in terms of this exponent.

The Ω-null controllability problem is also studied in papers by Dauer [9], [10], Chukwu and Gronski [11] and Chukwu and Silliman [12]. In order to answer the question of Ω-controllability, one must test a certain *growth condition* which involves searching a function space. In contrast, the results given here are finite-dimensional in nature.

In [13], Grantham and Vincent consider the problem of steering a nonlinear system to a target. They present a technique for determining the boundary between the set of states which can be steered to the target and those which cannot. More recently, Murthy and Evans [14] obtained results comparable to [3]–[5] for discrete linear systems and Pachter and Jacobson [15] developed sufficient conditions for controllability for case where $A(\cdot)$ and $B(\cdot)$ are time-invariant and Ω is a closed convex cone containing the origin. A readable account of the state of the art is contained in the book by Jacobson [16, Chap. 5].

In contrast to much of the work of previous authors, this paper concentrates on the case where $A(\cdot)$ and $B(\cdot)$ are time-varying. Our results for global Ω-null controllability are for constraint sets Ω that are compact and contain zero (but not necessarily as an interior point). One of our main results on global Ω-null controllability is an extension of a theorem of Conti [7] and it degenerates to Conti's theorem when Ω is a unit ball.

Our results for Ω-null controllability at (x_0, t_0) have even wider applicability since they do not require $0 \in \Omega$. Neither do they require the existence of a $u \in \Omega$ such that $Bu = 0$ as in [3]–[5], [7]–[12]. Thus we can analyze controllability of a system with, for example, $m = 1$ and $\Omega = [1, 2]$, whereas, many of the presently available theorems do not apply. Furthermore, as will be illustrated by examples, there are autonomous systems (S) which are neither globally Ω-null controllable nor locally Ω-null controllable but nevertheless are Ω-null controllable at some (x_0, t_0). Our theorem can be used to decompose the state space into two sets. Initial states in one set can be steered to the origin while those in the other cannot be driven to the origin by an admissible control.

2. Main results. In order to describe our necessary and sufficient conditions for global Ω-null controllability, we make use of the *support function* $H_\Omega: R^m \to R \cup \{+\infty\}$ on Ω which for any $\alpha \in R^m$ is given by

$$H_\Omega(\alpha) \triangleq \sup \{\omega'\alpha : \omega \in \Omega\}.$$

Using this notation, we have the following theorem, which is proven in Appendix A.

THEOREM 2.1. *Suppose Ω is a compact set which contains zero.*[2] *Then,* (S) *is globally Ω-null controllable at t_0 if and only if*

$$\int_{t_0}^{\infty} H_\Omega(B'(\tau)z(\tau))\, d\tau = +\infty \tag{2.1}$$

for all nonzero solutions $z(\cdot)$ of the adjoint system

$$\dot{z}(t) = -A'(t)z(t), \qquad t \in [t_0, \infty), \tag{S$'$}$$

or equivalently, if and only if

$$\int_{t_0}^{\infty} \sup \{\omega' B'(\tau)\phi'(t_0, \tau)\lambda : \omega \in \Omega\}\, d\tau = +\infty$$

for all $\lambda \in R^n$, $\lambda \neq 0$.

We note that $H_\Omega(B'(\tau)z(\tau))$ can be viewed as the composition of a nonnegative Baire function with a measurable function. Hence, the integral in (2.1) is well-defined along all trajectories $z(\cdot)$ of (S).

In the following corollary, we examine the special case of Theorem 2.1 which arises under the strengthened hypothesis "zero is an interior point of Ω." As we might anticipate, for this special case, the structure of the set Ω will not matter other than the requirement that it contains zero in its interior.

COROLLARY 2.2 *Suppose there exists a compact set Ω such that*

(i) *zero is an interior point of Ω;*

(ii) (S) *is globally Ω-null controllable.*

Then (S) *is also globally Ω'-null controllable for any other set Ω'* (*not necessarily compact*) *which contains zero in its interior.*

See Appendix A for proof.

Our proof of Theorem 2.1 will make use of a more fundamental result (also proven in Appendix A) giving conditions for Ω-null controllability at a fixed initial epoch (x_0, t_0). To meet this end, we define the scalar function $J: R^n \times R \times R^n \to R$ by

$$J(x_0, T, \lambda) \triangleq x_0'\phi'(T, t_0)\lambda + \int_{t_0}^{T} H_\Omega(B'(\tau)\phi'(T, \tau)\lambda)\, d\tau. \tag{2.2}$$

[2] The theorem is also valid if the requirement "$0 \in \Omega$" is replaced with "there exists a $u \in \Omega$ such that $Bu = 0$". This type of assumption is used by Brammer [4].

We note that $J(x_0, T, \lambda)$ can be viewed as the support function on the so-called attainable set. This fact is used implicitly in the proof of the next theorem.

THEOREM 2.3. *Let Ω be a compact set. Pick any subset Λ of R^n which contains* 0 *as an interior point. Then* (S) *is Ω-null controllable at* (x_0, t_0) *if and only if*

$$\min \{J(x_0, T, \lambda) : \lambda \in \Lambda\} = 0 \tag{2.3}$$

for some $T \in [t_0, \infty)$, or equivalently, if and only if

$$J(x_0, T, \lambda) \geqq 0 \quad \textit{for all } \lambda \in \Lambda \tag{2.4}$$

for some $T \in [t_0, \infty)$.

Comment. If Ω is also convex and A and B are constant, the sufficiency portion of this theorem is just a special case of Theorem 7.2.1 of [6]. Naturally, the smallest time T for which (2.3) holds will be the minimum *arrival time* at the origin.

Theorem 2.3 can also be stated in terms of the adjoint system (S′), i.e., if we take $\Lambda = R^n$ and notice that $z(t) = \phi'(t_0, t)z(t_0)$ is the response of the adjoint system (S′), then the following theorem is easily proven. (The proof is established by making the change of variables $z(t) \triangleq \phi'(T, t)\lambda$).

THEOREM 2.3′. *Let Ω satisfy the hypothesis of Theorem* 2.3. *Then* (S) *is Ω-null controllable at* (x_0, t_0) *if and only if there exists some $T \in [t_0, \infty)$ such that*

$$x_0' z(t_0) + \int_{t_0}^{T} H_\Omega(B'(\tau)z(\tau))\, d\tau \geqq 0 \tag{2.5}$$

for all solutions $z(\cdot)$ of (S′).

This theorem demonstrates that the question of Ω-null controllability at (x_0, t_0) can be answered by solving a finite dimensional optimization problem. Moreover, the question of global Ω-null controllability can also be answered via a finite dimensional optimization problem.

COROLLARY 2.4. *Let Ω and Λ be as in Theorem* 2.3. *Then* (S) *is globally Ω-null controllable at t_0 if and only if for every $x_0 \in R^n$ there is a time $T_{x_0} \in [t_0, \infty)$ such that*

$$\min \{J(x_0, T_{x_0}, \lambda) : \lambda \in \Lambda\} = 0.$$

The proof of this corollary follows from Theorem 2.3 in conjunction with the definition of global Ω-null controllability.

There is one point worth noting. In using Theorem 2.1 to check for Ω-null controllability at t_0, Ω must be compact and contain 0. If Corollary 2.4 is used, only the compactness assumption must be satisfied.

Next, we present some examples to illustrate how our theorems can be applied and to compare our results to those of [3]–[5].

Example 1. Let $x(t)$ and $u(t)$ be scalars and suppose (S) is described by

$$\dot{x}(t) = x(t) + u(t), \qquad t \in [0, \infty).$$

This system is R^1-null controllable if $\Omega = R^1$. But suppose $\Omega = [0, 1]$. Then the system is *not* globally Ω-null controllable at $t_0 = 0$. This follows from Theorem 2.1 since, for $z_0 < 0$, $H_\Omega(B'(\tau)z(\tau)) = 0$, and thus $\int_0^\infty H_\Omega(B'(\tau)z(\tau))\, d\tau < +\infty$. Also, using [3] or [4], it can be shown that the system is *not* locally Ω-null controllable. Nevertheless, there do exist initial states x_0 from which it is possible to steer the system to the origin. Such states can be determined via Theorem 2.3.

For the above,

$$J(x_0, T, \lambda) = x_0 e^T \lambda + \int_0^T \sup \{\omega e^{T-\tau} \lambda : \omega \in [0, 1]\}\, d\tau.$$

When $\Lambda = [-1, 1]$, this becomes

$$J(x_0, T, \lambda) = \begin{cases} x_0 e^T \lambda, & \lambda \leqq 0, \\ x_0 e^T \lambda + \lambda(e^T - 1), & \lambda > 0, \end{cases}$$

and thus

$$\min\{J(x_0, T, \lambda): \lambda \in [-1, 1]\} = 0$$

if and only if $x_0 \leqq 0$ and $x_0 \geqq e^{-T} - 1$ for some $T \in [0, \infty)$, or equivalently, if and only if $-1 < x_0 \leqq 0$. We conclude that even though (S) is not locally Ω-null controllable, it is Ω-null controllable at $(x_0, 0)$ whenever $-1 < x_0 \leqq 0$.

If $\Omega = [1, 2]$, neither [3]–[5] nor Theorem 2.1 apply. However, we can use Theorem 2.3. Since

$$H_\Omega(B'(\tau)\phi'(T, \tau)\lambda) = \begin{cases} 2\lambda\, e^{(T-\tau)}, & \lambda > 0, \\ \lambda\, e^{(T-\tau)}, & \lambda \leqq 0, \end{cases}$$

$J(x_0, T, \lambda)$ becomes

$$J(x_0, T, \lambda) = \begin{cases} x_0 e^T \lambda + 2\lambda(e^T - 1), & \lambda > 0, \\ x_0 e^T \lambda + \lambda\, (e^T - 1), & \lambda \leqq 0, \end{cases}$$

and

$$\min\{J(x_0, T, \lambda): \lambda \in [-1, 1]\} = 0$$

if and only if $2(e^{-T} - 1) \leqq x_0 \leqq e^{-T} - 1$. Thus (S), with $\Omega = [1, 2]$, is Ω-null controllable at $(x_0, 0)$ whenever $-2 < x_0 \leqq 0$.

As a final variation of this problem, suppose $\Omega = [-a, a]$. Then [4] or Theorem 2.1 shows that (S) is not globally Ω-null controllable. Using [4], it can be demonstrated that (S) is locally Ω-null controllable, while Theorem 2.3 not only tells us that (S) is locally Ω-null controllable but also that the states x_0 which can be steered to the origin are those satisfying $-a < x_0 < a$.

Example 2. Our second example illustrates the application of Theorem 2.1 for a nonautonomous system. We consider the time-varying two-dimensional system (S) described by

$$\dot{x}_1(t) = u(t) \sin t,$$

$$\dot{x}_2(t) = -\frac{1}{(t+1)^2} x_1(t) + u(t) t \sin t, \qquad t \in [0, \infty).$$

The control constraint set is taken to be $\Omega = [0, 1]$. By a straightforward computation, the state transition matrix for the adjoint system (S′) is found to be

$$\phi_*(t, t_0) = \begin{bmatrix} 1 & \dfrac{t - t_0}{(t+1)(t_0+1)} \\ 0 & 1 \end{bmatrix}.$$

Hence, in accordance with Theorem 2.1, (S) is globally Ω-null controllable at $t_0 = 0$ if and only if

$$\int_0^\infty \sup_{\omega \in [0, 1]} \omega[\sin\tau \quad \tau \sin\tau] \begin{bmatrix} 1 & \dfrac{\tau}{\tau+1} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} z_{01} \\ z_{02} \end{bmatrix} d\tau = +\infty$$

for *all* nonzero initial conditions $z_0 \triangleq [z_{01}\ z_{02}]'$. Evaluating above, this reduces to the requirement that

$$\int_0^\infty I(\tau)\,d\tau \triangleq \int_0^\infty \max\left\{0, z_{01}\sin\tau + z_{02}\tau\sin\tau\left(1+\frac{1}{\tau+1}\right)\right\}d\tau = +\infty \tag{2.6}$$

for all $z_0 \neq 0$. We shall show that this condition is indeed satisfied.

Case 1. $z_{01} \neq 0$, $z_{02} = 0$. For this case, we have

$$\int_0^\infty I(\tau)\,d\tau = \int_0^\infty \max\{0, z_{01}\sin\tau\}\,d\tau$$
$$= \int_{\mathcal{T}_1} z_{01}\sin\tau\,d\tau,$$

where $\mathcal{T}_1 \triangleq \{\tau \geqq 0 : z_{01}\sin\tau > 0\}$. Because the range set $\mathcal{T}_1$ of integration is the union of infinitely many intervals of length π, it follows that

$$\int_0^\infty I(\tau)\,d\tau = +\infty.$$

Case 2. z_{01} = anything, $z_{02} \neq 0$. Let $T^* \triangleq (|z_{01}|+1)/|z_{02}|$. Then to verify (2.6), it suffices to show that

$$\int_{\mathcal{T}_2} I(\tau)\,d\tau = +\infty,$$

where $\mathcal{T}_2 = \{\tau \geqq T^* : z_{02}\sin\tau > 0\}$. (Recall that the integrand is nonnegative.) Now, for $\tau \in \mathcal{T}_2$, we notice that the integrand $I(\tau)$ can be bounded from below as follows:

$$\begin{aligned} z_{01}\sin\tau + z_{02}\tau\sin\tau\left(1+\frac{1}{\tau+1}\right) &\geqq |z_{02}|\,|\sin\tau|\,\tau\left(1+\frac{1}{\tau+1}\right) - |z_{01}|\,|\sin\tau| \\ &\geqq (|z_{02}|\tau - |z_{01}|)|\sin\tau| \\ &\geqq (|z_{02}|T^* - |z_{01}|)|\sin\tau| \\ &= |\sin\tau|. \end{aligned}$$

Hence,

$$\int_{\mathcal{T}_2} I(\tau)\,d\tau \geqq \int_{\mathcal{T}_2} |\sin\tau|\,d\tau = +\infty$$

because the range of integration is once again the union of infinitely many intervals of length π.

We conclude that (S) is globally Ω-null controllable.

3. Relationship with other controllability results. In this section, we compare our controllability results with those of Conti [7] and Brammer [4]. We also consider, as a limiting case of our theory, the usual controllability problem obtained when magnitude constraints are not present.

Result of Conti. An important special case of Theorem 2.1 occurs when Ω is a closed unit ball in R^m, i.e.,

$$\Omega = \{\omega \in R^m : \|\omega\| \leqq 1\},$$

where $\|\cdot\|$ is a prespecified norm on R^m. For this situation we have

$$H_\Omega(B'(\tau)z(\tau)) = \sup\{\omega' B'(\tau)z(\tau) : \|\omega\| \leqq 1\} = \|B'(\tau)z(\tau)\|_*,$$

where $\|\cdot\|_*$ is the norm on R^m which is dual to $\|\cdot\|$. (For example $\|\cdot\|_*$ is the l^1 norm when $\|\cdot\|$ is the l^∞ norm; $\|\cdot\|$ and $\|\cdot\|_*$ coincide when $\|\cdot\|$ is the usual l^2 (Euclidean) norm.)

By Theorem 2.1, we conclude that (S) is globally Ω-null controllable at t_0 if and only if

$$\int_{t_0}^{\infty} \|B'(\tau)z(\tau)\|_* \, d\tau = +\infty \tag{3.1}$$

for all nonzero solutions $z(\cdot)$ of (S'). This result is established independently in Conti [7] and also discussed in Pandolfi [8]. This result, in conjunction with Corollary 2.2 leads immediately to the following proposition.

PROPOSITION 3.1. *Let Ω be any set containing zero in its interior. Then* (3.1) *is a necessary and sufficient condition for global Ω-null controllability.*

Thus, Conti's condition is a necessary and sufficient condition for global Ω-null controllability for any set Ω containing zero in its interior, not just when Ω is the closed unit ball.

Result of Brammer. Consider the case when $A(t)\equiv A$ and $B(t)\equiv B$ are time-invariant. For these autonomous problems, the following necessary conditions can be obtained directly from Theorem 2.1. Recall that $Q=[B, AB, \cdots, A^{n-1}B]$.

THEOREM 3.2. *Assume $A(t)\equiv A$ and $B(t)\equiv B$ are time-invariant and that Ω is a compact set which contains the origin. If* (S) *is globally Ω-null controllable then*

(i) rank $(Q)=n$;

(ii) *there is no real eigenvector v of A' satisfying $v'B\omega \leqq 0$ for all $\omega \in \Omega$;*

(iii) *no eigenvalue of A has a positive real part.*

The proof of this result is in Appendix B.

In [4], Brammer has obtained the same result using a different method of proof. There, he also shows that the above three conditions are also sufficient for global Ω-null controllability in the time invariant case if it is also assumed that the convex hull of Ω has a nonempty interior. Alternative proofs of the sufficiency results have been given by Heymann and Stern [25] and Hajek. The latter proof is in [5].

We note that the system of Example 1 of § 2 does not satisfy these three conditions. Nevertheless, it is Ω-null controllable at $(x_0, 0)$ for some initial states x_0.

The Case $\Omega=R^m$. When $\Omega=R^m$, it is well-known [17, p. 171] that the time-varying system (S) is completely controllable (globally R^m-null controllable at t_0 in our notation) if and only if the rows of $\phi(t_0,\cdot)B(\cdot)$ are linearly independent on some bounded interval $[t_0, T]$. Here we show that when $\Omega=R^m$, (2.1) is a necessary and sufficient condition for global R^m-null controllability. This is accomplished by showing that (2.1) is equivalent to the rows of $\phi(t_0,\cdot)B(\cdot)$ being linearly independent on some bounded interval $[t_0, T]$.

PROPOSITION 3.3. (S) *is globally R^m-null controllable if and only if*

$$\int_{t_0}^{\infty} H_{R^m}(B'(\tau)z(\tau)) \, d\tau = +\infty$$

for all nonzero solutions $z(\cdot)$ of (S').

The proof of this result is in Appendix B.

4. Some computational aspects. In a large number of problems, one may have to resort to the computer to check whether or not a system is Ω-null controllable. When using (2.3), a solution of the minimization problem $\min\{J(x_0, T, \lambda): \lambda \in \Lambda\}$ is needed. Direct application of so-called gradient or descent algorithms to compute $\min\{J(x_0, T, \lambda): \lambda \in \Lambda\}$ is precluded by the fact that $J(x_0, T, \lambda)$ is, in general, not differentiable in λ. This fact is a consequence of the sup-operation involved in the definition of $H_\Omega(B'(\tau)\phi'(T,\tau)\lambda)$. Fortunately, however, numerical computation of $\min\{J(x_0, T, \lambda): \lambda \in \Lambda\}$ is feasible if "generalized steepest descent" schemes are used. These schemes rely on subdifferential[3] rather than gradient information. The next two lemmas develop a description of the subdifferential of $J(x_0, T, \lambda)$. The proofs are given in Appendix C.

LEMMA 4.1. *For fixed $(x_0, T) \in R^n \times R$, $J(x_0, T, \lambda)$ is a lower semicontinuous convex function of λ.*

LEMMA 4.2. *For fixed $(x_0, T) \in R^n \times R$, the subdifferential of $J(x_0, T, \cdot)$ at $\lambda \in R^n$ consists of all vectors $\lambda_* \in R^n$ of the form*

$$\lambda_* = \phi(T, t_0)x_0 + \int_{t_0}^{T} \phi(T,\tau)B(\tau)\omega_*(\tau)\, d\tau, \tag{4.1}$$

where

$$\begin{aligned}\omega_*(\tau) &\in \arg\max\{\omega' B'(\tau)\phi'(T,\tau)\lambda : \omega \in \Omega\} \\ &= \{\omega \in \Omega : \omega' B'(\tau)\phi'(T,\tau)\lambda \geqq \eta' B'(\tau)\phi'(T,\tau)\lambda \ \forall \eta \in \Omega\}\end{aligned} \tag{4.2}$$

for almost all $\tau \in [0, T]$.

Remark. Since $J(x_0, T, \lambda)$ is the support function on the attainable set (see discussion preceding Theorem 2.3), a geometric interpretation of the subdifferential at λ is available: This set consists of all vectors in the normal cone to the attainable set at λ. (See Goodman [24, p. 285].)

Formulae (4.1) and (4.2) hold for arbitrary compact-convex Ω. Often, however, more structural information is known about Ω. In such cases, (4.1) and (4.2) may simplify. To illustrate, suppose

$$\Omega = [-M_1, M_1] \times [-M_2, M_2] \times \cdots \times [-M_m, M_m], \qquad M_i > 0.$$

Then, the maximum in (4.2) is achieved in the ith component by

$$[\omega_*(\tau)]_i \in M_i \operatorname{sgn}[B'(\tau)\phi'(T,\tau)\lambda]_i, \qquad i = 1, 2, \cdots, m,$$

where $\operatorname{sgn} x \triangleq 1$ if $x > 0$; $\operatorname{sgn} x \triangleq -1$ if $x < 0$; $\operatorname{sgn} 0 \triangleq [-1, 1]$. Consequently, for this case, we can substitute into (4.1), and show that the subdifferential $\partial J(x_0, T, \lambda)$ consists of all vectors $\lambda_* \in R^n$ of the form

$$\lambda_* = \phi(T, 0)x_0 + \int_0^T \sum_{i=1}^{m} M_i h_i(T,\tau) \operatorname{sgn} \lambda' h_i(T,\tau)\, d\tau, \tag{4.3}$$

where $h_i(T,\tau)$ is the ith column of $H(T,\tau) \triangleq \phi(T,\tau)B(\tau)$. This description of the subdifferentials of $J(x_0, T, \cdot)$ can be used in conjunction with the generalized steepest descent algorithms to compute $\min\{J(x_0, T, \lambda): \lambda \in \Lambda\}$.

We also note that λ_* is uniquely specified by (4.3) if

$$\text{measure}\{\tau : \lambda' h_i(T,\tau) = 0\} = 0 \quad \textit{for } i = 1, 2, \cdots, m.$$

For such λ, $\partial J(x_0, T, \lambda)$ is precisely $\nabla_\lambda J(x_0, T, \lambda)$, the gradient of $J(x_0, T, \cdot)$ at λ.

5. The steering control. Using the results of § 2, we can determine if (S) is Ω-null controllable. However, those results do not give a method for determining a *steering control* $u_*(\cdot) \in \mathcal{M}(\Omega)$ which accomplishes this objective.

[3] $\lambda_* \in \partial J(x_0, T, \lambda)$, the subdifferential of $J(x_0, T, \cdot)$ at λ, if and only if

$$J(x_0, T, z) \geqq J(x_0, T, \lambda) + (z - \lambda)'\lambda_* \quad \text{for all } z \in R^n.$$

One method of determining an appropriate $u_*(\cdot)$ is to solve the time optimal control problem, i.e., find $u_*(\cdot)\in\mathcal{M}(\Omega)$ which steers (S) from given (x_0, t_0) to the origin and does so in minimum time. If there is a control which steers the system to the origin, then there is a time optimal one [2]. Hence, in principle, a steering control can be numerically computed using any of a wide variety of algorithms which are available for solution of the time optimal control problem.

Since the solution of the time optimal problem is determined by solving a two point boundary value problem, it can be quite difficult to obtain the steering control this way. In this section, a "simpler" alternative method for generating a steering control is presented. This technique does not involve a two point boundary value problem and leads to a control which steers the system arbitrarily close to the origin. Our result is obtained from the following minimum norm problem:[4] Given initial point (x_0, t_0) and a final time T, find $u(\cdot)\in\mathcal{M}(\Omega)$ which leads to the smallest value of $\|x(T)\|$. The solution of this minimum norm problem is characterized in the next theorem.

THEOREM 5.1. *Let (x_0, t_0) and T be given. Suppose that $\lambda_*\in R^n$ achieves the minimum of $J(x_0, T, \lambda)$ over the closed unit ball. Then any solution of the minimum norm problem satisfies*

$$u_*(\tau)\in\arg\max\{\omega' B'(\tau)\phi'(T,\tau)\lambda_*: \omega\in\Omega\} \tag{5.1}$$

for almost all $\tau\in[t_0, T]$.

See Appendix D for proof.

We note that condition (5.1) will uniquely determine $u_*(\cdot)$ whenever the minimum of $\omega' B'(\tau)\phi'(T,\tau)\lambda_*$ is uniquely achieved. For example, suppose

$$\Omega=[-M_1, M_1]\times[-M_2, M_2]\times\cdots\times[-M_m, M_m],\qquad M_i>0.$$

Then (5.1) requires

$$[u_*(\tau)]_i\in M_i \operatorname{sgn}[B'(\tau)\phi'(T,\tau)\lambda_*]_i,\qquad i=1,2,\cdots,m. \tag{5.2}$$

For the case when the minimum of $\|x(T)\|=0$, $\lambda_*=0$ and (5.1) will not determine a control which steers (S) to the origin. The following heuristic procedure can be used to determine a control which steers (S) arbitrarily close to the origin: Choose a T such that the minimum of $\|x(T)\|$ is nonzero. As T is increased, the minimum of $\|x(T)\|$ approaches zero and the corresponding solution $u_*(\cdot)$, generated via (5.2), of the minimum norm problem results in a control which steers the system progressively closer to the origin.

In our next theorem, we provide another useful characterization of steering controls. For fixed $T\in[0,\infty)$, $x_0\in R^n$, we define the functional $V_T: R^n\times\mathcal{M}(\Omega)\to R$ by

$$V_T(\lambda, u(\cdot))=\lambda'\phi(T,0)x_0+\int_0^T \lambda'\phi(T,\tau)B(\tau)u(\tau)\,d\tau.$$

THEOREM 5.2. *Pick any compact convex set Λ containing zero as an interior point. Then $V_T(\lambda, u(\cdot))$ possesses at least one saddle point $(\lambda_*, u_*(\cdot))\in\Lambda\times\mathcal{M}(\Omega)$. Moreover, $u_*(\cdot)$ steers x_0 to zero at time T if and only if $V_T(\lambda_*, u_*(\cdot))=0$.*

See Appendix D for proof.

6. Additional applications. In this section, we use our results to obtain an existence theorem for the time optimal control problem and also apply our results to a pursuit game.

[4] Here the linear system (S) is required to be R^m-null controllable.

6.1. Existence of time optimal controls. Consider the following time optimal control problem: Find $u(\cdot)\in\mathcal{M}(\Omega)$ which drives the state $x(\cdot)$ of (S) from an initial position $x(t_0)=x_0$ to the origin and minimizes

$$C(u(\cdot))=\int_{t_0}^{t_f} dt; \qquad t_f=\text{arrival time at the origin.}$$

The classical theorem for existence of a time optimal control (e.g., Lee and Markus [2]) requires that there is at least one control which transfers the state $x(\cdot)$ of (S) to the origin. Combining the result of [2] with our Theorem 2.3, we obtain the following existence lemma.

LEMMA 6.1. *There exists a solution to the time optimal control problem if and only if there is some finite $t_f\in[t_0,\infty)$ such that*

$$\min\{J(x_0,t_f,\lambda):\lambda\in\Lambda\}=0.$$

Furthermore, the time optimal cost is given by

$$C^*(u_*(\cdot))=\min\{t_f:\min[J(x_0,t_f,\lambda):\lambda\in\Lambda]=0\}.$$

6.2. Pursuit Games. Next, we consider the pursuit game studied by Hájek [18]. The system is described by

$$\dot{x}(t)=Ax(t)-p(t)+q(t),\qquad p(t)\in P,\quad q(t)\in Q,\quad x(t_0)=x_0, \tag{6.1}$$

where P and Q are compact convex subsets of R^n. The pursuer $p(\cdot)$ seeks a strategy $\sigma:Q\times[t_0,\infty)\to P$ which steers $x(\cdot)$ to the origin for all possible quarry controls $q(\cdot):[t_0,\infty)\to Q$. A quarry control is admissible if it is measurable and a strategy is admissible if $\sigma(\cdot)$ preserves measurability.

In [18], a solution to this problem is obtained in terms of the *associated control system*

$$\dot{y}(t)=Ay(t)-u(t),\qquad u(t)\in P\overset{*}{-}Q,\qquad y(t_0)=x_0, \tag{6.2}$$

where $P\overset{*}{-}Q$ is the Pontryagin difference; i.e.,

$$P\overset{*}{-}Q\triangleq\{x\in R^n: x+Q\subseteq P\}.$$

Admissible controls $u(\cdot)$ above must be measurable.

Simply put, Hájek's result says that the state $x(\cdot)$ of (6.1) can be forced to the origin, for all admissible $q(\cdot)$, if and only if the state $y(\cdot)$ of (6.2) can be steered to the origin. More precisely, the following theorem is available.

FIRST RECIPROCITY THEOREM [18]. *Initial position x_0 in (6.1) can be (stroboscopically) forced to the origin at time $T\geqq t_0$ by a strategy $\sigma(\cdot)$ if and only if x_0 in (6.2) can be steered to the origin at time T by an admissible control $u(\cdot)$. Furthermore, $\sigma(\cdot)$ and $u(\cdot)$ are related by*

$$\sigma(q,t)=u(t)+q. \tag{6.3}$$

By applying Theorem 2.3 to (6.2), we obtain another condition for determining if (6.1) can be forced to the origin.

LEMMA 6.2. *Assume $P\overset{*}{-}Q$ compact. Pick any subset Λ of R^n containing zero as an interior point. Then x_0 in (6.1) can be forced to the origin at time $T\geqq t_0$ by a strategy $\sigma(\cdot)$ if and only if*

$$\min\{K(x_0,T,\lambda):\lambda\in\Lambda\}=0,$$

where

$$K(x_0, T, \lambda) \triangleq x_0' e^{A'(T-t_0)}\lambda + \int_{t_0}^{T} H_{P \pm Q}(e^{A'(T-\tau)}\lambda)\, d\tau.$$

It should be pointed out that in addition to pursuit game interpretation of (6.1), (6.1) can also be viewed as a problem of steering a system with disturbances to the origin if $q(\cdot)$ is thought of as a disturbance. Also, the results apply to systems described by

$$\dot{x}(t) = Ax(t) + Bp(t) + Cq(t), \qquad p(t) \in P, \quad q(t) \in Q$$

if one replaces $Bp(t)$ by $p'(t)$, $Cq(t)$ by $-q'(t)$, P by BP, and Q by CQ.

Appendix A. Proof of Theorems 2.1, 2.3 and Corollary 2.2. Since Theorem 2.3 is used in the proof of Theorem 2.1, we first present the proof of Theorem 2.3. There are many ways to prove Theorem 2.3; our proof exploits the convexity of the attainable set in conjunction with a measurable selection theorem. We note that a proof of the sufficiency part of the theorem is given in [6, Thm. 7.2.1]. To simplify our notation, we henceforth take $t_0 = 0$ without loss of generality. This will apply to subsequent appendices as well.

Proof of Theorem 2.3. Let $A_T(x_0)$ be the set of states which can be attained from x_0 at time T, i.e.,

$$A_T(x_0) = \left\{ \phi(T, 0)x_0 + \int_0^T \phi(T, \tau)B(\tau)u(\tau)\, d\tau \colon u(\cdot) \in \mathcal{M}(\Omega) \right\}. \tag{A.1}$$

The set $A_T(x_0)$ is convex and compact [2]. From Definition 1.1, it follows that x_0 can be steered to 0 at time T if and only if $0 \in A_T(x_0)$ or, equivalently, by the separating hyperplane theorem [21],

$$0 \leqq \sup \{\lambda' a \colon a \in A_T(x_0)\} \tag{A.2}$$

for all vectors $\lambda \in R^n$. Using (A.1), requirement (A.2) becomes

$$\lambda'\phi(T, 0)x_0 + \sup \left\{ \int_0^T \lambda'\phi(T, \tau)B(\tau)u(\tau)\, d\tau \colon u(\cdot) \in \mathcal{M}(\Omega) \right\} \geqq 0 \tag{A.3}$$

for all $\lambda \in R^n$. As a consequence of the measurable selection theory of [19], we can commute the supremum and integral operations in (A.3)[5]. Thus, $0 \in A_T(x_0)$ if and only if

$$0 \leqq \lambda'\phi(T, 0)x_0 + \int_0^T H_\Omega(B'(\tau)\phi'(T, \tau)\lambda)\, d\tau = J(x_0, T, \lambda) \tag{A.4}$$

for all $\lambda \in R^n$. Since $J(x_0, T, \lambda)$ is positively homogeneous in λ, we can restrict λ to Λ in (A.4), Theorem 2.3 now follows. □

Next, we present the proof of Theorem 2.1. In the proof, Theorem 2.3 is used.

Proof of Theorem 2.1 (*Necessity*). We suppose that (S) is globally Ω-null controllable at $t_0 = 0$. Let $z(\cdot)$ be any nonzero solution of (S'); we must prove that

$$\int_0^\infty H_\Omega(B'(\tau)z(\tau))\, d\tau = +\infty. \tag{A.5}$$

[5] $\phi(T, \tau)\, B(\tau)$ being a Carthéodory function enables us to apply the results of [19].

Proceeding by contradiction, suppose there is a nonzero solution $\hat{z}(\cdot)$ such that

$$\int_0^\infty H_\Omega(B'(\tau)\hat{z}(\tau))\, d\tau = \alpha, \qquad \alpha < \infty.$$

Then there is a positive constant $\beta < \infty$ such that

$$\int_0^\infty H_\Omega(B'(\tau)\hat{z}(\tau))\, d\tau < \beta.$$

Define

$$x_0^* \triangleq \frac{-2\beta \hat{z}(0)}{\hat{z}'(0)\hat{z}(0)}, \qquad x_0^* \neq 0.$$

We now claim that x_0^* *cannot* be steered to zero by an admissible control $u(\cdot) \in \mathcal{M}(\Omega)$. To prove our claim, for each $t \in [0, \infty)$, define

$$\lambda_t \triangleq \phi'(0, t)\hat{z}(0), \qquad \lambda_t \neq 0.$$

Now, given any $t \in [0, \infty)$,

$$\begin{aligned} J(x_0^*, t, \lambda_t) &= x_0^{*\prime}\phi'(t, 0)\lambda_t + \int_0^t H_\Omega(B'(\tau)\phi'(t, \tau)\lambda_t)\, d\tau \\ &= x_0^{*\prime}\hat{z}(0) + \int_0^t H_\Omega(B'(\tau)\hat{z}(\tau))\, d\tau \\ &\leqq -2\beta + \beta \\ &< 0. \end{aligned}$$

Taking $\Lambda = R^n$ in Theorem 2.3, it follows that

$$\min\{J(x_0^*, t, \lambda) : \lambda \in \Lambda\} \leqq J(x_0^*, t, \lambda_t) < 0$$

for all $t \in [0, \infty)$. By Theorem 2.3, (S) is *not* Ω-null controllable at $(x_0^*, 0)$. □

(*Sufficiency*). Now, we assume that (A.5) holds. Again, we proceed by contradiction, i.e., suppose (S) is *not* globally Ω-null controllable at $t_0 = 0$. Hence, there exists an initial condition $x_0^* \neq 0$ which cannot be steered to zero. By Theorem 2.3 (with $\Lambda = R^n$), we can find a sequence of times $\langle t_k \rangle_{k=1}^\infty$ and a sequence of vectors $\langle \lambda_k \rangle_{k=1}^\infty$ having the following properties:

(P1) $$\lim_{k \to \infty} t_k = +\infty,$$

(P2) $$J(x_0^*, t_k, \lambda_k) < 0 \quad \text{for } k = 1, 2, 3 \cdots.$$

We are going to construct an initial condition $\tilde{z}_0 \neq 0$ for (S′) which makes the integral in (A.5) finite. To meet this end, let

$$z_k = \frac{\phi'(t_k, 0)\lambda_k}{\|\phi'(t_k, 0)\lambda_k\|}, \qquad k = 1, 2, \cdots.$$

We note that each z_k above is nonzero because $\lambda_k \neq 0$, and $\phi(t_k, 0)$ is invertible. Then $\langle z_k \rangle_{k=1}^\infty$ is a sequence in R^n belonging to the set

$$S \triangleq \{z \in R^n : \|z\| = 1\}.$$

Since S is compact, we can extract a subsequence $\langle z_{k_j} \rangle_{j=1}^\infty$ which converges to some vector $\tilde{z}_0 \in S$. We will now show that $\tilde{z}_0$ is the initial condition which we seek. Let $\tilde{z}(\cdot)$ be the trajectory of (S′) generated by $z(0) \triangleq \tilde{z}_0$; let $\langle t_{k_j}\rangle_{j=1}^{\infty}$ denote the subsequence of times corresponding to $\langle z_{k_j}\rangle_{j=1}^{\infty}$. By (P1), we have

$$\lim_{j\to\infty} t_{k_j} = +\infty,$$

and by (P2), it follows that

$$x_0^{*\prime}\phi'(t_{k_j}, 0)\lambda_{k_j} + \int_0^{t_{k_j}} H_\Omega(B'(\tau)\phi'(t_{k_j}, \tau)\lambda_{k_j})\, d\tau < 0 \quad \textit{for } j = 1, 2, 3, \cdots.$$

Dividing by $\|\phi'(t_{k_j}, 0)\lambda_{k_j}\|$ and noting that H_Ω is positively homogeneous, we obtain

$$\begin{aligned}\int_0^{t_{k_j}} H_\Omega(B'(\tau)\phi'(0, \tau)z_{k_j})\, d\tau &\leqq \|x_0^*\|\,\|z_{k_j}\| \quad \textit{for } j = 1, 2, 3, \cdots,\\ &\leqq \|x_0^*\| \qquad\qquad \textit{for } j = 1, 2, 3, \cdots.\end{aligned}$$

We would like to obtain an inequality involving $\tilde{z}_0$ with an infinite upper limit on this integral. To accomplish this, we define

$$f_{k_j}(\tau) \triangleq \begin{cases} H_\Omega(B'(\tau)\phi'(0, \tau)z_{k_j}) & \text{if } \tau \in [0, t_{k_j}],\\ 0 & \text{otherwise,}\end{cases} \qquad j = 1, 2, 3, \cdots,$$

$$f(\tau) \triangleq H_\Omega(B'(\tau)\phi'(0, \tau)\tilde{z}_0), \qquad \tau \in [0, \infty),$$

and make the following observations.

(i) $\int_0^\infty f_{k_j}(\tau)\, d\tau$ is bounded (by $\|x_0^*\|$) for $j = 1, 2, 3, \cdots$.

(ii) $f_{k_j}(\tau)$ converges pointwise to $f(\tau)$ on $[0, \infty)$. This observation is proven using the facts that $z_{k_j} \to \tilde{z}_0$, $t_{k_j} \to +\infty$, and H_Ω depends continuously on its argument.

Applying Fatou's lemma [20, p. 83], we have

$$\begin{aligned}\int_0^\infty f(\tau)\, d\tau &\leqq \liminf_{j\to\infty} \int_0^\infty f_{k_j}(\tau)\, d\tau\\ &\leqq \limsup_{j\to\infty} \int_0^\infty f_{k_j}(\tau)\, d\tau\\ &\leqq \|x_0^*\|.\end{aligned}$$

Substitution for $f(\tau)$ above gives

$$\int_0^\infty H_\Omega(B'(\tau)\phi'(0, \tau)\tilde{z}_0)\, d\tau \leqq \|x_0^*\|,$$

i.e.,

$$\begin{aligned}\int_0^\infty H_\Omega(B'(\tau)\tilde{z}(\tau))\, d\tau &\leqq \|x_0^*\|\\ &< \infty,\end{aligned}$$

which is the contradiction that we seek. This completes the proof of the theorem. □

Proof of Corollary 2.2. Suppose Ω and Ω' satisfy the hypotheses of the corollary. We are going to show that (S) is globally Ω'-null controllable. To prove this, it is sufficient to find a subset $\Omega'_\delta \subseteq \Omega'$ such that (S) is globally Ω'_δ-null controllable: Pick $\delta > 0$ such that

$$\Omega'_\delta \triangleq \{\omega : \|\omega\| \leqq \delta\} \subseteq \Omega'.$$

(This can be accomplished because zero is interior to Ω'.) Now, to prove that Ω'_δ has the desired property, we pick $R>0$ such that

$$\Omega_R \triangleq \{\omega: \|\omega\| \leqq R\} \supseteq \Omega.$$

(This can also be done since Ω is compact, hence bounded.) Let $z(\cdot)$ be any nonzero solution of (S'). Then we have

$$\begin{aligned}
\int_0^\infty H_{\Omega'_\delta}(B'(\tau)z(\tau))\,d\tau &= \int_0^\infty \sup\{\omega'B'(\tau)z(\tau): \|\omega\| \leqq \delta\}\,d\tau \\
&= \delta \int_0^\infty \|B'(\tau)z(\tau)\|\,d\tau \\
&= \frac{\delta}{R}\int_0^\infty R\|B'(\tau)z(\tau)\|\,d\tau \\
&= \frac{\delta}{R}\int_0^\infty \sup\{\omega'B'(\tau)z(\tau): \|\omega\| \leqq R\}\,d\tau \\
&= \frac{\delta}{R}\int_0^\infty H_{\Omega_R}(B'(\tau)z(\tau))\,d\tau \\
&= +\infty
\end{aligned}$$

since (S) is globally Ω_R-null controllable. (Ω_R-null controllability follows from Ω-null controllability in conjunction with the fact that $\Omega_R \supseteq \Omega$.) By Theorem 2.1, we conclude that (S) must be globally Ω'_δ-null controllable and hence Ω'-null controllable. □

Appendix B.

Proof of Theorem 3.2. (*i*) This condition follows immediately from the fact that global R^m-null controllability is necessary for global Ω-null controllability.

It is also possible to prove (i) directly from Theorem 2.1. Suppose (S) is globally Ω-null controllable but rank $(Q)<n$. Then there exists a $v \in R^n$, $v \neq 0$, such that $B' e^{-A't}v = 0$ for all $t \geqq 0$. Let $z(0)=v$. Then $z(\tau)=e^{-A't}v$ and

$$\int_0^\infty \sup_{\omega\in\Omega} (\omega'B'z(\tau))\,d\tau = \int_0^\infty \sup_{\omega\in\Omega} (\omega'B'\,e^{-A't}v)\,d\tau = 0$$

which contradicts Theorem 2.1.

(ii) Suppose (S) is globally Ω-null controllable but there exists a real eigenvector v of A' satisfying $\omega'B'v \leqq 0$ for all $\omega \in \Omega$. Denoting by λ the real eigenvalue associated with v, we have $e^{-A't}v = e^{-\lambda t}v$. With $z(0)=v$, $z(\tau)=e^{-A'\tau}v = e^{-\lambda\tau}v$, and

$$\begin{aligned}
\int_0^\infty \sup_{\omega\in\Omega} (\omega'B'z(\tau)\,d\tau &= \int_0^\infty \sup_{\omega\in\Omega} (\omega'B'e^{-\lambda\tau}v)\,d\tau \\
&= \int_0^\infty e^{-\lambda\tau} \sup_{\omega\in\Omega} (\omega'B'v)\,d\tau.
\end{aligned}$$

Now this integral is less than or equal to zero since $\sup\{\omega'B'v: \omega\in\Omega\} \leqq 0$ and $e^{-\lambda t} \geqq 0$. This contradicts Theorem 2.1.

(iii) Again the proof is by contradiction. Assume (S) is globally Ω-null controllable but A has an eigenvalue λ with a positive real part. Then λ is also an eigenvalue of A' so that $A'v=\lambda v$, where v is an eigenvector corresponding to A'. Let $\bar\lambda$ and $\bar v$ denote the complex conjugate of λ and v. They satisfy $A\bar{v}=\bar{\lambda}\bar{v}$. Hence,

$$e^{-A't}v=e^{\lambda t}v \quad \text{and} \quad e^{-A't}\bar{v}=e^{\bar{\lambda}t}\bar{v}.$$

Consider the solution of the adjoint equation corresponding to the initial condition $z(0)=v+\bar{v}$. (Note that $z(0)$ is real.) For this $z(0)$,

$$\begin{aligned}\sup_{\omega\in\Omega}(\omega'B'z(\tau)) &= \sup_{\omega\in\Omega}(\omega'B'\,e^{-A'\tau}(v+\bar{v}))\\ &= \sup_{\omega\in\Omega}[\omega'B'(e^{-\lambda t}v+e^{-\bar{\lambda}t}\bar{v})]\\ &= \sup_{\omega\in\Omega}\{\omega'B'\,e^{-at}[2m\cos bt+2n\sin bt]\},\end{aligned}$$

where a and b are the real part and imaginary part of λ and n and m are the real part and imaginary part of v. Let $M\triangleq\sup\{\sup[\omega'B'(2n\cos bt+2n\sin bt):\omega\in\Omega]:t\geqq 0\}$. M is finite since Ω is compact, i.e., $M\leqq 2\max\{|n|,|m|\}\,\|B\|\sup\{\|\omega\|:\omega\in\Omega\}$. Thus

$$\sup_{\omega\in\Omega}(\omega'B'z(\tau))\leqq Me^{-at},$$

and

$$\int_0^\infty \sup_{\omega\in\Omega}(\omega'B'z(\tau))\,d\tau\leqq M\int_0^\infty e^{-at}\,dt.$$

The integral on the right is finite since $a>0$ and we have a contradiction to Theorem 2.1. □

Proof of Proposition 3.3. (*Necessity*). Suppose (S) is globally R^m-null controllable. Then there is a finite interval $[0,T]$ on which the rows of $\phi(0,\cdot)B(\cdot)$ are linearly independent. Thus, for every nonzero vector $z_0\in R^n$, it follows that $B'(t)\phi'(0,t)z_0\neq 0$ for some $t\in[0,T]$. Since $B'(\cdot)\phi'(0,\cdot)z_0$ is continuous, there must be an interval $I=[t-\delta,t+\delta]$ on which $B'(\tau)\phi'(0,\tau)z_0\neq 0$ for all $\tau\in I$. On this interval, we have

$$\sup\{\omega B'(\tau)\phi'(0,\tau)z_0:\ \omega\in R^m\}=+\infty.$$

Hence, using the nonnegativity of $H_\Omega(\cdot)$, we conclude that

$$\begin{aligned}\int_0^\infty H_{R^m}(B'(\tau)z(\tau))\,d\tau &\geqq \int_I H_{R^m}(B'(\tau)\phi'(0,\tau)z_0)\,d\tau\\ &= \int_I \sup\{\omega'B'(\tau)\phi'(0,\tau)z_0:\ \omega\in R^m\}\,d\tau\\ &= +\infty.\end{aligned}$$

(*Sufficiency*). Proceeding by contradiction, we suppose that for all nonzero solutions $z(\cdot)$ of (S′), we have

$$\int_0^\infty H_{R^m}(B'(\tau)z(\tau))\,d\tau=+\infty,$$

but the columns of $B'(\cdot)\phi'(0,\cdot)$ are linearly dependent on every bounded interval $[0,T]$. Let $\langle T_n\rangle_{n=1}^\infty$ be a monotone increasing sequence of times such that $T_n\to\infty$. Then, for each n, we can find a nonzero vector $\tilde{z}_n$ such that $B'(\tau)\phi'(0, \tau)\tilde{z}_n \equiv 0$ on $[0, T_n]$. Let

$$z_n \triangleq \frac{\tilde{z}_n}{\|\tilde{z}_n\|} \quad \text{for } n = 1, 2, \cdots,$$

Then, $\langle z_n \rangle_{n=1}^{\infty}$ is a sequence in the (compact) unit ball. Hence, we can extract a subsequence z_{n_j} converging to some $\hat{z}_0$, $\|\hat{z}_0\| = 1$. We notice that the corresponding subsequence of times T_{n_j} still converges to $+\infty$. Furthermore, for each fixed $\tau \in [0, \infty)$, we have

$$\begin{aligned} B'(\tau)\phi'(0, \tau)\hat{z}_0 &= \lim_{j \to \infty} B'(\tau)\phi'(0, \tau)z_{n_j} \\ &= 0. \end{aligned}$$

Consequently, if $\hat{z}(\tau)$ is the trajectory mate of $\hat{z}_0$,

$$\int_0^{\infty} H_{R^m}(B'(\tau)\hat{z}(\tau))\, d\tau = \int_0^{\infty} \sup \{\omega' B'(\tau)\phi'(0, \tau)\hat{z}_0 : \omega \in R^m\}\, d\tau = 0$$

which contradicts the assumed hypothesis. □

Appendix C.

Proof of Lemma 4.1. For (x_0, T) fixed, $J(x_0, T, \lambda)$ can be expressed as

$$J(x_0, T, \lambda) = \sup \{H_\omega(\lambda) : \omega(\cdot) \in \mathcal{M}(\Omega)\},$$

where

$$H_\omega(\lambda) \triangleq \lambda'\phi(T, 0)x_0 + \int_0^T \lambda'\phi(T, \tau)B(\tau)\omega(\tau)\, d\tau.$$

Consequently, $J(x_0, T, \cdot)$ is the pointwise supremum over an indexed collection of continuous linear (hence convex) functions. Hence $J(x_0, T, \cdot)$ itself must be convex and at least lower semicontinuous (in fact, continuous). □

Proof of Lemma 4.2. We prove this lemma using some of the standard properties of subdifferentials given in Rockafellar [21], [22]. Since both functions in the definition of $J(x_0, T, \lambda)$ are finite and convex, $\lambda_* \in \partial J(x_0, T, \lambda)$ if and only if

$$\begin{aligned} \lambda_* &\in \partial(x_0'\phi'(T, 0)\lambda) + \partial \int_0^T H_\Omega(B'(\tau)\phi'(T, \tau)\lambda)\, d\tau && \text{(by Theorem 23.8 of [22])}. \\ &= \phi(T, 0)x_0 + \int_0^T \partial H_\Omega(B'(\tau)\phi'(T, \tau)\lambda)\, d\tau && \text{(by Theorem 23 of [22])} \\ &= \phi(T, 0)x_0 + \int_0^T \phi(T, \tau)B(\tau) \cdot \partial H_\Omega(\hat{\omega}(\tau))\big|_{\hat{\omega}(\tau) = B'(\tau)\phi'(T, \tau)\lambda}\, d\tau \\ & && \text{(by Theorem 23.9 of [21])}. \end{aligned}$$

Now, by Corollary 23.5.3 of [21], $\omega_*(\tau) \in \partial H_\Omega(\hat{\omega}(\tau))$ if and only if $\omega_*(\tau) \in \arg\max \{\omega'\hat{\omega}(\tau) : \omega \in \Omega\}$. Substituting the required form for $\hat{\omega}$ above, we obtain our desired representation for λ_*. □

Appendix D.

Sketch of a proof of Theorem 5.1. Let $f: L^1(0, T; R^m) \to R$, $g: R^n \to R$, $\Lambda_T: L^1(0, T; R^m) \to R^n$ be given by

$$f(u) \triangleq \begin{cases} 0 & \text{if } u(\cdot) \in \mathcal{M}(\Omega) \\ +\infty & \text{otherwise,} \end{cases}$$

$$g(z) \triangleq -\|\phi(T, 0)x_0 + z\|, \qquad z \in R^n,$$

$$\Lambda_T u \triangleq \int_0^T \phi(T, \tau)B(\tau)u(\tau)\, d\tau.$$

Then, using the notation above

$$\begin{aligned} \inf (MN) &\triangleq \inf \{\|x(T)\|: u(\cdot) \in \mathcal{M}(\Omega)\} \\ &= \inf \{f(u) - g(\Lambda_T u): u \in L^1(0, T; R^m)\}. \end{aligned}$$

Written in this way, inf (MN) is in the standard form for application of Rockafellar's extension of Fenchel's duality theorem (cf. [23, Thm.1]). The functionals f and g are, respectively, proper convex and concave functions; it can be easily shown that inf (MN) is "stably set"—a technical precondition for Rockafellar's theorem.

By carrying out the computations involved in Theorem 1 of [23], it can be shown that the problem

$$\min (MN)^* \triangleq \min \{J(x_0, T, \lambda): \lambda \in \Lambda\}$$

is dual to inf (MN) in the following sense:

$$\inf (MN) + \min (MN)^* = 0.$$

The "extremality condition" in Rockafellar's theorem provides a necessary condition which must be satisfied by all solution pairs λ_* solving $(MN)^*$ and $u_*(\cdot)$ solving (MN). This extremality condition requires

$$\Lambda_T^* \lambda_* \in \partial f(u_*),$$

where Λ_T^* is the adjoint of Λ_T, and $\partial f(u_*)$ is the subdifferential of f at u_*. For our choice of f, this necessary condition particularizes to

$$\lambda'_* \phi(T, \tau)B(\tau) \in (\text{normal cone of } \mathcal{M}(\Omega) \text{ at } u_*(\cdot)).$$

We denote this normal cone at u_* by $N_c(u_*)$. By definition of the normal cone, we have $v(\cdot) \in N_c(u^*)$ if and only if

$$\int_0^T u'_*(\tau)B'(\tau)\phi'(T, \tau)\lambda_* d\tau = \int_0^T \sup \{\omega' B'(\tau)\phi'(T, \tau)\lambda_*: \omega \in \Omega\}\, d\tau.$$

This is possible only if $\omega = u_*(\tau)$ achieves the supremum of $\omega' B'(\tau)\phi'(T, \tau)\lambda_*$ for almost all $\tau \in [0, T]$. Equivalently, we must have

$$u_*(\tau) \in \arg\max \{\omega' B'(\tau)\phi'(T, \tau)\lambda_*: \omega \in \Omega\}$$

for almost all $\tau \in [0, T]$.

Proof of Theorem 5.2. As in the proof of Theorem 2.3, let $A_T(x_0)$ be the set of states which can be attained from x_0 at time T. We recall that this set is compact and convex. Define $W_T: \Lambda \times A_T(x_0) \to R$ by

$$W_T(\lambda, \xi) \triangleq \lambda' \xi. \tag{D.1}$$

In accordance with Proposition 2.3 of [19, p. 171], $W_T(\lambda, \xi)$ will possess a saddle point because the following conditions are satisfied:

(D.2.1) For all $\lambda \in \Lambda$, $W(\lambda, \cdot)$ is concave and upper semicontinuous.

(D.2.2) For all $\xi \in A_T(x_0)$, $W(\cdot, \xi)$ is convex and lower semicontinuous.

We note that

$$\min_{\lambda \in \Lambda} \max_{u(\cdot) \in \mathcal{M}(\Omega)} V_T(\lambda, u(\cdot)) = \min_{\lambda \in \Lambda} \max_{\xi \in A_T(x_0)} W_T(\lambda, \xi).$$

Furthermore,

$$\max_{u(\cdot) \in \mathcal{M}(\Omega)} \min_{\lambda \in \Lambda} V_T(\lambda, u(\cdot)) = \max_{\xi \in A_T(x_0)} \min_{\lambda \in \Lambda} W_T(\lambda, \xi).$$

These equalities, in conjunction with the fact that W_T possesses a saddle point, imply that V_T also has a saddle point.

To prove the last part of the theorem, we take $(\lambda_*, u_*(\cdot))$ to be a given saddle point of $V_T(\lambda, u(\cdot))$. Hence we have

$$V_T(\lambda_*, u_*(\cdot)) = \min_{\lambda \in \Lambda} \max_{u(\cdot) \in \mathcal{M}(\Omega)} V_T(\lambda, u(\cdot)). \tag{D.3}$$

Using a measurable selection argument, as in the proof of Theorem 2.3, it is also apparent that

$$\min_{\lambda \in \Lambda} \max_{u(\cdot) \in \mathcal{M}(\Omega)} V_T(\lambda, u(\cdot)) = \min_{\lambda \in \Lambda} J(x_0, T, \lambda). \tag{D.4}$$

From (D.3) and (D.4) we conclude that

$$V_T(\lambda_*, u_*(\cdot)) = \min_{\lambda \in \Lambda} J(x_0, T, \lambda). \tag{D.5}$$

From Theorem 2.3 and the comments following the theorem, we know that x_0 can be steered to zero at time T if and only if

$$\begin{aligned} 0 &= \min_{\lambda \in \Lambda} J(x_0, T, \lambda) \\ &= V_T(\lambda_*, u_*(\cdot)) \qquad \text{(by (D.5))}. \end{aligned}$$

To complete the proof, we must show that if $V_T(\lambda_*, u_*(\cdot)) = 0$, then $u^*(\cdot)$ steers x_0 to 0. Now

$$0 = V_T(\lambda_*, u_*(\cdot)) \leqq V_T(\lambda, u_*(\cdot)) \quad \text{for all } \lambda \in \Lambda$$

or

$$0 \leqq \lambda' \left[\phi(T, 0)x_0 + \int_0^T \phi(T, \tau)B(\tau)u_*(\tau)\, d\tau \right] \quad \text{for all } \lambda \in \Lambda.$$

Thus

$$0 \leqq \lambda' x(T, x_0, u_*(\cdot)) \quad \text{for all } \lambda \in \Lambda. \tag{D.6}$$

Since 0 is an interior point of the convex, compact set Λ, (D.6) implies $x(T, x_0, u_*(\cdot)) = 0$ and $u_*(\cdot)$ is a steering control. □

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