SIRS Model with Vaccination and Spatial Heterogenaity

First, define and encode the parameters according to the following differential equations:

$$
\begin{aligned}
\frac{dS_p}{dt} &= \mu(S_p + I_p + R + I_s + S_s + V) - \mu S_p - \nu S_p - \beta S_p \sum_{j \in \mathbf{C}} I_{p,j} - \alpha\beta S_p \sum_{j \in \mathbf{C}} I_{s,j} \\
\frac{dI_p}{dt} &= \beta S_p \sum_{j \in \mathbf{C}} I_{p,j} + \alpha\beta S_p \sum_{j \in \mathbf{C}} I_{s,j} - \mu I_p - \gamma I - p \\
\frac{dR}{dt} &= \gamma I_p + \gamma I_s - \mu R - \delta R \\
\frac{dS_s}{dt} &= \delta R + \delta_{vax} V - \mu S_s - \nu S_s - \epsilon\beta S_s \sum_{j \in \mathbf{C}} I_{p,j} - \epsilon\beta\alpha S_s \sum_{j \in \mathbf{C}} I_{s,j} \\
\frac{dI_s}{dt} &= \epsilon\beta S_s \sum_{j \in \mathbf{C}} I_{p,j} + \epsilon\beta\alpha S_s \sum_{j \in \mathbf{C}} I_{s,j} - \mu I_s - \gamma I_s \\
\frac{dV}{dt} &= \nu S_p + \nu S_s - \mu V - \delta_{vax} V
\end{aligned}
$$

Where:

- S_p: primary susceptible [individuals]
- I_p: primary infected [individuals]
- R: recovered [individuals]
- I_s: secondary susceptible [individuals]
- S_s: secondary infected [individuals]
- V: vaccinated [individuals]
- μ: birth and death rate $[T^{-1}]$
- β: transmission potential $[T^{-1}\text{individual}^{-1}]$
- ν: vaccination rate $[T^{-1}]$
- γ: recovery rate (inverse of duration of infection) $[T^{-1}]$
- δ: rate of loss of natural immunity (inverse of duration of immunity) $[T^{-1}]$
- δ_{vax}: rate of loss of vaccine-derived immunity $[T^{-1}]$
- ϵ: susceptibility factor for secondary susceptible $[\emptyset]$
- α: transmission factor for secondary infected $[\emptyset]$
- $\mathbf{C}$: set of commuting-related counties $[\{\emptyset\}]$

We can rewrite the model in vector-matrix form:

$$
\frac{d}{dt}\begin{bmatrix} S_p \\ I_p \\ R \\ S_s \\ I_s \\ V \end{bmatrix} = \begin{bmatrix}
-(\beta \sum_{j\in\mathbf{C}} I_{p,j} + \alpha\beta \sum_{j\in\mathbf{C}} I_{s,j} + \nu) & \mu & \mu & \mu & \mu \\
\beta \sum_{j\in\mathbf{C}} I_{p,j} + \alpha\beta \sum_{j\in\mathbf{C}} I_{s,j} & -(\gamma+\mu) & 0 & 0 & 0 \\
0 & \gamma & -(\mu+\delta) & 0 & \gamma \\
0 & 0 & \delta & -(\epsilon\beta \sum_{j\in\mathbf{C}} I_{p,j} + \epsilon\beta\alpha \sum_{j\in\mathbf{C}} I_{s,j}) & 0 \\
0 & 0 & 0 & \epsilon\beta \sum_{j\in\mathbf{C}} I_{p,j} + \epsilon\beta\alpha \sum_{j\in\mathbf{C}} I_{s,j} & -(\gamma+\mu) \\
\nu & 0 & 0 & \nu & 0
\end{bmatrix}
$$