USAMO 2008 Solution Notes

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This is a compilation of solutions for the 2008 USAMO. Some of the solutions are my own work, but many are from the official solutions provided by the organizers (for which they hold any copyrights), and others were found by users on the Art of Problem Solving forums.

These notes will tend to be a bit more advanced and terse than the "official" solutions from the organizers. In particular, if a theorem or technique is not known to beginners but is still considered "standard", then I often prefer to use this theory anyways, rather than try to work around or conceal it. For example, in geometry problems I typically use directed angles without further comment, rather than awkwardly work around configuration issues. Similarly, sentences like "let $\mathbb{R}$ denote the set of real numbers" are typically omitted entirely.

Corrections and comments are welcome!

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§0 Problems

1. Prove that for each positive integer n, there are pairwise relatively prime integers $k_0, \dots, k_n$, all strictly greater than 1, such that $k_0 k_1 \dots k_n - 1$ is the product of two consecutive integers.

2. Let ABC be an acute, scalene triangle, and let M, N, and P be the midpoints of $\overline{BC}$, $\overline{CA}$, and $\overline{AB}$, respectively. Let the perpendicular bisectors of $\overline{AB}$ and $\overline{AC}$ intersect ray AM in points D and E respectively, and let lines BD and CE intersect in point F, inside triangle ABC. Prove that points A, N, F, and P all lie on one circle.

3. Let n be a positive integer. Denote by S_n the set of points (x, y) with integer coordinates such that

$$|x| + \left|y + \frac{1}{2}\right| < n.$$

A path is a sequence of distinct points $(x_1, y_1), (x_2, y_2), \dots, (x_\ell, y_\ell)$ in S_n such that, for $i = 2, \dots, \ell$, the distance between (x_i, y_i) and (x_{i-1}, y_{i-1}) is 1.

Prove that the points in S_n cannot be partitioned into fewer than n paths.

4. For which integers $n \geq 3$ can one find a triangulation of regular n-gon consisting only of isosceles triangles?

5. Three nonnegative real numbers r_1, r_2, r_3 are written on a blackboard. These numbers have the property that there exist integers a_1, a_2, a_3, not all zero, satisfying $a_1 r_1 + a_2 r_2 + a_3 r_3 = 0$. We are permitted to perform the following operation: find two numbers x, y on the blackboard with $x \leq y$, then erase y and write $y - x$ in its place. Prove that after a finite number of such operations, we can end up with at least one 0 on the blackboard.

6. At a certain mathematical conference, every pair of mathematicians are either friends or strangers. At mealtime, every participant eats in one of two large dining rooms. Each mathematician insists upon eating in a room which contains an even number of his or her friends. Prove that the number of ways that the mathematicians may be split between the two rooms is a power of two (i.e. is of the form 2^k for some positive integer k).

§1 Solutions to Day 1

§1.1 USAMO 2008/1, proposed by Titu Andreescu

Available online at https://aops.com/community/p1116186.

Problem statement

Prove that for each positive integer n, there are pairwise relatively prime integers $k_0, \dots, k_n$, all strictly greater than 1, such that $k_0 k_1 \dots k_n - 1$ is the product of two consecutive integers.

In other words, if we let

$$P(x) = x(x+1) + 1$$

then we would like there to be infinitely many primes dividing some $P(t)$ for some integer t.

In fact, this result is true in much greater generality. We first state:

Theorem 1.1 (Schur's theorem)

If $P(x) \in \mathbb{Z}[x]$ is nonconstant and $P(0) = 1$, then there are infinitely many primes which divide $P(t)$ for some integer t.

Proof. If $P(0) = 0$, this is clear. So assume $P(0) = c \neq 0$.

Let S be any finite set of prime numbers. Consider then the value

$$P\left(k \prod_{p \in S} p\right)$$

for some integer k. It is $1 \pmod p$ for each prime p, and if k is large enough it should not be equal to 1 (because P is not constant). Therefore, it has a prime divisor not in S. $\square$

Remark. In fact the result holds without the assumption $P(0) \neq 1$. The proof requires only small modifications, and a good exercise would be to write down a similar proof that works first for $P(0) = 20$, and then for any $P(0) \neq 0$. (The $P(0) = 0$ case is vacuous, since then $P(x)$ is divisible by x.)

To finish the proof, let $p_1, \dots, p_n$ be primes and x_i be integers such that

$$\begin{aligned} P(x_1) &\equiv 0 \pmod{p_1} \\ P(x_2) &\equiv 0 \pmod{p_2} \\ &\vdots \\ P(x_n) &\equiv 0 \pmod{p_n} \end{aligned}$$

as promised by Schur's theorem. Then, by Chinese remainder theorem, we can find x such that $x \equiv x_i \pmod{p_i}$ for each i, whence $P(x)$ has at least n prime factor.

§1.2 USAMO 2008/2, proposed by Zuming Feng

Available online at https://aops.com/community/p1116181.

Problem statement

Let ABC be an acute, scalene triangle, and let M, N, and P be the midpoints of $\overline{BC}$, $\overline{CA}$, and $\overline{AB}$, respectively. Let the perpendicular bisectors of $\overline{AB}$ and $\overline{AC}$ intersect ray AM in points D and E respectively, and let lines BD and CE intersect in point F, inside triangle ABC. Prove that points A, N, F, and P all lie on one circle.

We present four solutions.

¶ Barycentric solution. First, we find the coordinates of D. As D lies on $\overline{AM}$, we know $D = (t : 1 : 1)$ for some t. Now by perpendicular bisector formula, we find

$$0 = b^2(t-1) + (a^2 - c^2) \implies t = \frac{c^2 + b^2 - a^2}{b^2}.$$

Thus we obtain

$$D = \left(2S_A : c^2 : c^2\right).$$

Analogously $E = (2S_A : b^2 : b^2)$, and it follows that

$$F = \left(2S_A : b^2 : c^2\right).$$

The sum of the coordinates of F is

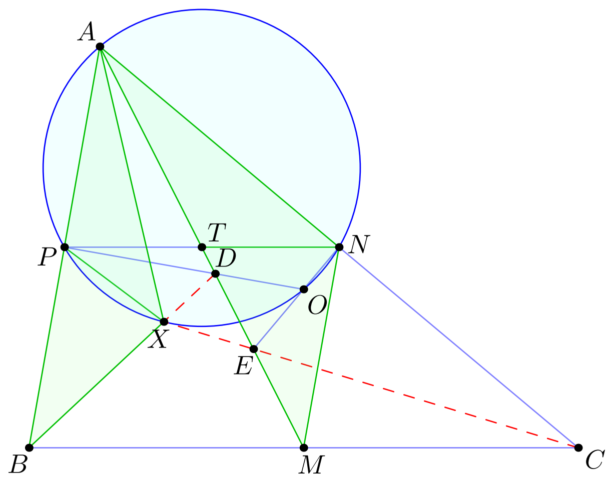
$$(b^2 + c^2 - a^2) + b^2 + c^2 = 2b^2 + 2c^2 - a^2.$$

Hence the reflection of A over F is simply

$$2F - A = \left(2(b^2 + c^2 - a^2) - (2b^2 + 2c^2 - a^2) : 2b^2 : 2c^2\right) = \left(-a^2 : 2b^2 : 2c^2\right).$$

It is evident that F' lies on $(ABC) : -a^2yz - b^2zx - c^2xy = 0$, and we are done.

¶ Synthetic solution (harmonic). Here is a synthetic solution. Let X be the point so that $APXN$ is a cyclic harmonic quadrilateral. We contend that $X = F$. To see this it suffices to prove B, X, D collinear (and hence C, X, E collinear by symmetry).



Let T be the midpoint of $\overline{PN}$, so $\triangle APX \sim \triangle ATN$. So $\triangle ABX \sim \triangle AMN$, ergo

$$\measuredangle XBA = \measuredangle NMA = \measuredangle BAM = \measuredangle BAD = \measuredangle DBA$$

as desired.

¶ **Angle chasing solution (Mason Fang).** Obviously $ANOP$ is concyclic.

Claim — Quadrilateral $BFOC$ is cyclic.

Proof. Write

$$\begin{aligned}\measuredangle BFC = \measuredangle FBC + \measuredangle BCF &= \measuredangle FBA + \measuredangle ABC + \measuredangle BCA + \measuredangle ACF \\ &= \measuredangle DBA + \measuredangle ABC + \measuredangle BCA + \measuredangle ACE \\ &= \measuredangle BAD + \measuredangle ABC + \measuredangle BCA + \measuredangle EAC \\ &= 2\angle BAC = \angle BOC.\end{aligned}$$ □

Define $Q = \overline{AA} \cap \overline{BC}$.

Claim — Point Q lies on $\overline{FO}$.

Proof. Write

$$\begin{aligned}\measuredangle BOQ = \measuredangle BOA + \measuredangle AOQ &= 2\measuredangle BCA + 90^\circ + \measuredangle AQO \\ &= 2\measuredangle BCA + 90^\circ + \measuredangle AMO \\ &= 2\measuredangle BCA + 90^\circ + \measuredangle AMC + 90^\circ \\ &= \measuredangle BCA + \measuredangle MAC = \measuredangle BCA + \measuredangle ACE \\ &= \measuredangle BCE = \measuredangle BOF.\end{aligned}$$ □

As Q is the radical center of $(ANOP)$, (ABC) and $(BFOC)$, this implies the result.

¶ **Inversive solution (Kelin Zhu).** Invert about A with radius $\sqrt{bc}$ followed by a reflection over the angle bisector of $\angle A$, and denote the image of a point X by X'. The inverted problem now states the following:

> In triangle AP^*N^*, let B^*, C^* be the midpoints of AP^*, AN^* and D^*, E^* be the intersection of the A symmedian with (AP^*), (AN^*), respectively. (AB^*D^*), (AC^*E^*) intersect at a point F^*; prove that it lies on P^*N^*.

I claim that, in fact, the midpoint of P^*N^* is the desired intersection. Redefine that point as F^* and I will prove that (AB^*D^*), (AC^*E^*) pass through it.

Note that

$$\angle AD^*B^* = \angle D^*AB^* = \angle F^*AN^* = \angle AF^*B^*,$$

where the first equality is due to B^* being the circumcircle of AD^*P^*, the second equality is due to the definition of the symmedian, and the third equality is due to the parallelogram $AB^*F^*C^*$. A symmetric argument for C finishes.

§1.3 USAMO 2008/3, proposed by Gabriel Carroll

Available online at https://aops.com/community/p1116367.

Problem statement

Let n be a positive integer. Denote by S_n the set of points (x, y) with integer coordinates such that

$$|x| + \left|y + \frac{1}{2}\right| < n.$$

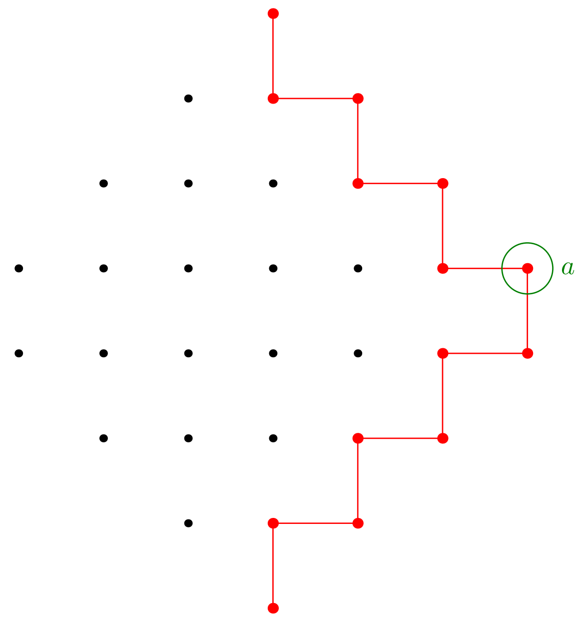
A path is a sequence of distinct points $(x_1, y_1), (x_2, y_2), \dots, (x_\ell, y_\ell)$ in S_n such that, for $i = 2, \dots, \ell$, the distance between (x_i, y_i) and (x_{i-1}, y_{i-1}) is 1.

Prove that the points in S_n cannot be partitioned into fewer than n paths.

¶ First solution (local). We proceed by induction on n. The base case $n = 1$ is clear, so suppose $n > 1$. Let S denote the set of points

$$S = \left\{(x, y) : x + \left|y + \frac{1}{2}\right| \geq n - 2\right\}.$$

An example when $n = 4$ is displayed below.



For any minimal partition $\mathcal{P}$ of S_n, let P denote the path passing through the point $a = (n - 1, 0)$. Then the intersection of P with S consists of several disconnected paths; let N be the number of nodes in the component containing a, and pick $\mathcal{P}$ such that N is maximal. We claim that in this case $P = S$.

Assume not. First, note $a = (n - 1, 0)$ must be connected to $b = (n - 1, -1)$ (otherwise join them to decrease the number of paths).

Now, starting from $a = (n - 1, 0)$ walk along P away from b until one of the following three conditions is met:

- We reach a point v not in S. Let w be the point before v, and x the point in S adjacent to w. Then delete vw and add wx. This increases N while leaving the number of edges unchanged: so this case can't happen.

- We reach an endpoint v of P (which may be a), lying inside the set S, which is not the topmost point $(0, n-1)$. Let w be the next point of S. Delete any edge touching w and add edge vw. This increases N while leaving the number of edges unchanged: so this case can't happen.

- We reach the topmost point $(0, n-1)$.

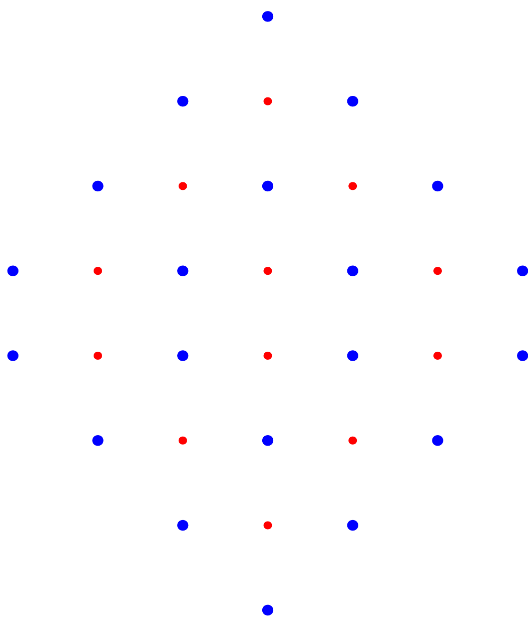
Thus we see that P must follow S until reaching the topmost point $(0, n-1)$. Similarly it must reach the bottom-most point $(0, -n)$. Hence $P = S$.

The remainder of S_n is just S_{n-1}, and hence this requires at least $n-1$ paths to cover by the inductive hypothesis. So S_n requires at least n paths, as desired.

> **Remark** (Motivational comments from Evan)**.** Basically the idea is that I wanted to peel away the right path S highlighted in red in the figure, so that one could induct. But the problem is that the red path might not actually exist, e.g. the set of paths might contain the mirror of S instead.
>
> Nonetheless, in those equality cases I found I could perturb some edges (e.g. change from $(-1, n-2)$–$(0, n-2)$ to $(0, n-2)$–$(1, n-2)$). So the idea then was to do little changes and try to convert the given partition into one where the red path S exists, (and then peel it away for induction) without decreasing the total number of paths.
>
> To make this work, you actually want the incisions to begin ear the points a and b, because that's the point of S that is most constrained (e.g. you get a–b right away for free), and assemble the path from there. (If you try to do it from the top, it's much less clear what's happening.) That's why the algorithm starts the mutations from around a.

¶ Second solution (global). Here is a much shorter official solution, which is much trickier to find, and "global" in nature.

Color the upper half of the diagram with a blue/red checkerboard pattern such that the uppermost point $(n-1, 0)$ is blue. Reflect it over to the bottom, as shown.



Assume there are m paths. Cut in two any paths with two adjacent blue points; this occurs only along the horizontal symmetry axis. Thus:

- After cutting there are at most $m + n$ paths, since at most n cuts occur.

- On the other hand, there are $2n$ more blue points than red points. Hence after cutting there must be at least $2n$ paths (since each path alternates colors, except possibly for double-red pairs).

So $m + n \geq 2n$, hence $m \geq n$.

Remark. This problem turned out to be known already. It appears in this reference:

> Nikolai Beluhov, Nyakolko Zadachi po Shahmatna Kombinatorika, *Matematika Plyus*, 2006, issue 4, pages 61–64.

Section 1 of 2 was reprinted with revisions as Nikolai Beluhov, Dolgii Put Korolya, *Kvant*, 2010, issue 4, pages 39–41. The reprint is available at http://kvant.ras.ru/pdf/2010/2010-04.pdf.

Remark (Nikolai Beluhov)**.** As pointed out in the reference above, this problem arises naturally when we try to estimate the greatest possible length of a closed king tour on the chessboard of size $n \times n$ with n even, a question posed by Igor Akulich in Progulki Korolya, *Kvant*, 2000, issue 3, pages 47–48. Each one of the two references above contains a proof that the answer is $n + \sqrt{2}(n^2 - n)$.

§2 Solutions to Day 2

§2.1 USAMO 2008/4, proposed by Gregory Galperin

Available online at https://aops.com/community/p1116177.

Problem statement

For which integers $n \geq 3$ can one find a triangulation of regular n-gon consisting only of isosceles triangles?

The answer is n of the form $2^a(2^b + 1)$ where a and b are nonnegative integers not both zero.

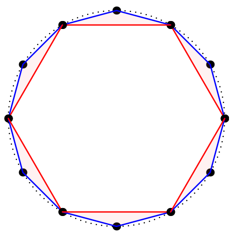
Call the polygon $A_1 \dots A_n$ with indices taken modulo n. We refer to segments A_1A_2, A_2A_3, ..., A_nA_1 as *short sides*. Each of these must be in the triangulation. Note that

- when n is even, the isosceles triangles triangle using a short side A_1A_2 are $\triangle A_nA_1A_2$ and $\triangle A_1A_2A_3$ only, which we call *small*.
- when n is odd, in addition to the small triangles, we have $\triangle A_{\frac{1}{2}(n+3)}A_1A_2$, which we call *big*.

This leads to the following two claims.

Claim — If $n > 4$ is even, then n works iff $n/2$ does.

Proof. All short sides must be part of a small triangle; after drawing these in, we obtain an $n/2$-gon.



Thus the sides of $\mathcal{P}$ must pair off, and when we finish drawing we have an $n/2$-gon. □

Since $n = 4$ works, this implies all powers of 2 work and it remains to study the case when n is odd.

Claim — If $n > 1$ is odd, then n works if and only if $n = 2^b + 1$ for some positive integer b.

Proof. We cannot have all short sides part of small triangles for parity reasons, so some side, must be part of a big triangle. Since big triangles contain the center O, there can be at most one big triangle too.

Then we get $\frac{1}{2}(n-1)$ small triangles, pairing up the remaining sides. Now repeating the argument with the triangles on each half shows that the number $n-1$ must be a power of 2, as needed. □

§2.2 USAMO 2008/5, proposed by Kiran Kedlaya

Available online at https://aops.com/community/p1116189.

Problem statement

Three nonnegative real numbers r_1, r_2, r_3 are written on a blackboard. These numbers have the property that there exist integers a_1, a_2, a_3, not all zero, satisfying $a_1r_1 + a_2r_2 + a_3r_3 = 0$. We are permitted to perform the following operation: find two numbers x, y on the blackboard with $x \le y$, then erase y and write $y - x$ in its place. Prove that after a finite number of such operations, we can end up with at least one 0 on the blackboard.

We first show we can decrease the quantity $|a_1| + |a_2| + |a_3|$ as long as $0 \notin \{a_1, a_2, a_3\}$. Assume $a_1 > 0$ and $r_1 > r_2 > r_3$ without loss of generality and consider two cases.

- $r_2 > 0$ or $r_3 > 0$; these cases are identical. If $r_2 > 0$ then $r_3 < 0$ and get

  $$0 = a_1r_1 + a_2r_2 + a_3r_3 > a_1r_3 + a_3r_3 \implies a_1 + a_3 < 0$$

  so $|a_1 + a_3| < |a_3|$, and hence we perform $(r_1, r_2, r_3) \mapsto (r_1 - r_3, r_2, r_3)$.

- Both r_2 and r_3 are less than zero. Assume for contradiction that $|a_1 + a_2| \ge -a_2$ and $|a_1 + a_3| \ge -a_3$ both hold (if either fails then we use $(r_1, r_2, r_3) \mapsto (r_1 - r_2, r_2, r_3)$ and $(r_1, r_2, r_3) \mapsto (r_1 - r_3, r_2, r_3)$, respectively). Clearly $a_1 + a_2$ and $a_1 + a_3$ are both positive in this case, so we get $a_1 + 2a_2$ and $a_1 + 2a_3 \ge 0$; adding gives $a_1 + a_2 + a_3 \ge 0$. But

  $$\begin{aligned} 0 &= a_1r_1 + a_2r_2 + a_3r_3 \\ &> a_1r_2 + a_2r_2 + a_3r_2 \\ &= r_2(a_1 + a_2 + a_3) \\ \implies 0 &< a_1 + a_2 + a_3. \end{aligned}$$

Since this covers all cases, we see that we can always decrease $|a_1|+|a_2|+|a_3|$ whenever $0 \notin \{a_1, a_2, a_3\}$. Because the a_i are integers this cannot occur indefinitely, so eventually one of the a_i's is zero. At this point we can just apply the Euclidean Algorithm, so we're done.

§2.3 USAMO 2008/6, proposed by Sam Vandervelde

Available online at https://aops.com/community/p1116182.

Problem statement

At a certain mathematical conference, every pair of mathematicians are either friends or strangers. At mealtime, every participant eats in one of two large dining rooms. Each mathematician insists upon eating in a room which contains an even number of his or her friends. Prove that the number of ways that the mathematicians may be split between the two rooms is a power of two (i.e. is of the form 2^k for some positive integer k).

Take the obvious graph interpretation where we are trying to 2-color a graph. Let A be the adjacency matrix of the graph over $\mathbb{F}_2$, except the diagonal of A has $\deg v \pmod 2$ instead of zero. Then let $\vec{d}$ be the main diagonal. Splittings then correspond to $A\vec{v} = \vec{d}$. It's then immediate that the number of ways is either zero or a power of two, since if it is nonempty it is a coset of $\ker A$.

Thus we only need to show that:

Claim — At least one coloring exists.

Proof. If not, consider a minimal counterexample G. Clearly there is at least one odd vertex v. Consider the graph with vertex set $G - v$, where all pairs of neighbors of v have their edges complemented. By minimality, we have a good coloring here. One can check that this extends to a good coloring on G by simply coloring v with the color matching an even number of its neighbors. This breaks minimality of G, and hence all graphs G have a coloring. □

It's also possible to use linear algebra. We prove the following lemma:

Lemma (grobber)

Let V be a finite dimensional vector space, $T: V \to V$ and $w \in V$. Then w is in the image of T if and only if there are no $\xi \in V^\vee$ for which $\xi(w) \neq 0$ and yet $\xi \circ T = 0$.

Proof. Clearly if $T(v) = w$, then no ξ exists. Conversely, assume w is not in the image of T. Then the image of T is linearly independent from w. Take a basis $e_1, \dots, e_m$ for the image of T, add w, and then extend it to a basis for all of V. Then have ξ kill all e_i but not w. □

Corollary

In a symmetric matrix A mod 2, there exists a vector v such that Av is a copy of the diagonal of A.

Proof. Let ξ be such that $\xi \circ T = 0$. Look at ξ as a column vector $w^\top$, and let d be the diagonal. Then

$$0 = w^\top \cdot T \cdot w = \xi(d)$$

because this extracts the sum of coefficients submatrix of T, and all the symmetric entries cancel off. Thus no ξ as in the previous lemma exists. $\square$

This corollary gives the desired proof.