IMO 2010 Solution Notes

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This is a compilation of solutions for the 2010 IMO. Some of the solutions are my own work, but many are from the official solutions provided by the organizers (for which they hold any copyrights), and others were found by users on the Art of Problem Solving forums.

These notes will tend to be a bit more advanced and terse than the "official" solutions from the organizers. In particular, if a theorem or technique is not known to beginners but is still considered "standard", then I often prefer to use this theory anyways, rather than try to work around or conceal it. For example, in geometry problems I typically use directed angles without further comment, rather than awkwardly work around configuration issues. Similarly, sentences like "let $\mathbb{R}$ denote the set of real numbers" are typically omitted entirely.

Corrections and comments are welcome!

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§0 Problems

1. Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that for all $x, y \in \mathbb{R}$,
$$f(\lfloor x \rfloor y) = f(x) \lfloor f(y) \rfloor .$$

2. Let I be the incenter of a triangle ABC and let Γ be its circumcircle. Let line AI intersect Γ again at D. Let E be a point on arc $\widehat{BDC}$ and F a point on side BC such that
$$\angle BAF = \angle CAE < \tfrac{1}{2}\angle BAC.$$
Finally, let G be the midpoint of $\overline{IF}$. Prove that $\overline{DG}$ and $\overline{EI}$ intersect on Γ.

3. Find all functions $g\colon \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ such that
$$(g(m)+n)\,(g(n)+m)$$
is always a perfect square.

4. Let P be a point interior to triangle ABC (with $CA \neq CB$). The lines AP, BP and CP meet again its circumcircle Γ at K, L, M, respectively. The tangent line at C to Γ meets the line AB at S. Show that from $SC = SP$ follows $MK = ML$.

5. Each of the six boxes B_1, B_2, B_3, B_4, B_5, B_6 initially contains one coin. The following two types of operations are allowed:

a) Choose a non-empty box B_j, $1 \le j \le 5$, remove one coin from B_j and add two coins to B_{j+1};

b) Choose a non-empty box B_k, $1 \le k \le 4$, remove one coin from B_k and swap the contents (possibly empty) of the boxes B_{k+1} and B_{k+2}.

Determine if there exists a finite sequence of operations of the allowed types, such that the five boxes B_1, B_2, B_3, B_4, B_5 become empty, while box B_6 contains exactly $2010^{2010^{2010}}$ coins.

6. Let $a_1, a_2, a_3, \dots$ be a sequence of positive real numbers, and s be a positive integer, such that
$$a_n = \max\{a_k + a_{n-k} \mid 1 \le k \le n-1\} \text{ for all } n > s.$$
Prove there exist positive integers $\ell \le s$ and N, such that
$$a_n = a_\ell + a_{n-\ell} \text{ for all } n \ge N.$$

§1 Solutions to Day 1

§1.1 IMO 2010/1

Available online at https://aops.com/community/p1935849.

Problem statement

Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that for all $x, y \in \mathbb{R}$,

$$f(\lfloor x \rfloor y) = f(x) \lfloor f(y) \rfloor .$$

The only solutions are $f(x) \equiv c$, where $c = 0$ or $1 \le c < 2$. It's easy to see these work.

Plug in $x = 0$ to get $f(0) = f(0) \lfloor f(y) \rfloor$, so either

$$1 \le f(y) < 2 \quad \forall y \qquad \text{or} \qquad f(0) = 0$$

In the first situation, plug in $y = 0$ to get $f(x) \lfloor f(0) \rfloor = f(0)$, thus f is constant. Thus assume henceforth $f(0) = 0$.

Now set $x = y = 1$ to get

$$f(1) = f(1) \lfloor f(1) \rfloor$$

so either $f(1) = 0$ or $1 \le f(1) < 2$. We split into cases:

- If $f(1) = 0$, pick $x = 1$ to get $f(y) \equiv 0$.
- If $1 \le f(1) < 2$, then $y = 1$ gives

  $$f(\lfloor x \rfloor) = f(x)$$

  from $y = 1$, in particular $f(x) = 0$ for $0 \le x < 1$. Choose $(x, y) = \left(2, \frac{1}{2}\right)$ to get $f(1) = f(2) \left\lfloor f\left(\frac{1}{2}\right) \right\rfloor = 0$.

§1.2 IMO 2010/2

Available online at https://aops.com/community/p1935927.

Problem statement

Let I be the incenter of a triangle ABC and let Γ be its circumcircle. Let line AI intersect Γ again at D. Let E be a point on arc $\widehat{BDC}$ and F a point on side BC such that

$$\angle BAF = \angle CAE < \tfrac{1}{2}\angle BAC.$$

Finally, let G be the midpoint of $\overline{IF}$. Prove that $\overline{DG}$ and $\overline{EI}$ intersect on Γ.

Let $\overline{EI}$ meet Γ again at K. Then it suffices to show that $\overline{KD}$ bisects $\overline{IF}$. Let $\overline{AF}$ meet Γ again at H, so $\overline{HE} \parallel \overline{BC}$. By Pascal theorem on

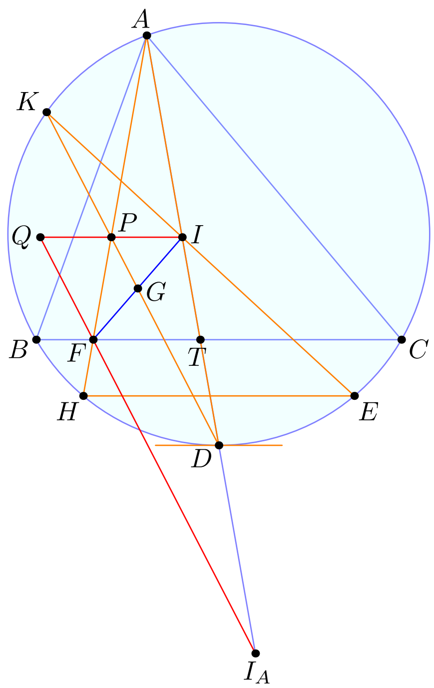
$$AHEKDD$$

we then obtain that $P = \overline{AH} \cap \overline{KD}$ lies on a line through I parallel to $\overline{BC}$.

Let I_A be the A-excenter, and set $Q = \overline{I_A F} \cap \overline{IP}$, and $T = \overline{AIDI_A} \cap \overline{BFC}$. Then

$$-1 = (AI;TI_A) \overset{F}{=} (IQ;\infty P)$$

where ∞ is the point at infinity along $\overline{IPQ}$. Thus P is the midpoint of $\overline{IQ}$. Since D is the midpoint of $\overline{II_A}$ by "Fact 5", it follows that $\overline{DP}$ bisects $\overline{IF}$.



§1.3 IMO 2010/3, proposed by Gabriel Carroll (USA)

Available online at https://aops.com/community/p1935854.

Problem statement

Find all functions $g\colon \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ such that

$$(g(m)+n)\,(g(n)+m)$$

is always a perfect square.

For $c \geq 0$, the function $g(n) = n + c$ works; we prove this is the only possibility.

First, the main point of the problem is that

Claim — We have $g(n) \equiv g(n') \pmod p \implies n \equiv n' \pmod p$.

Proof. Pick a large integer M such that

$$\nu_p(M + g(n)), \quad \nu_p(M + g(n')) \quad \text{are both odd.}$$

(It's not hard to see this is always possible.) Now, since each of

$$\begin{gathered} (M + g(n))\,(n + g(M)) \\ \left(M + g(n')\right)\left(n' + g(M)\right) \end{gathered}$$

is a square, we get $g(n) \equiv g(n') \equiv -M \pmod p$. $\square$

This claim implies that

- The numbers $g(n)$ and $g(n+1)$ differ by ± 1 for any n, and
- The function g is injective.

It follows g is a linear function with slope ± 1, hence done.

§2 Solutions to Day 2

§2.1 IMO 2010/4

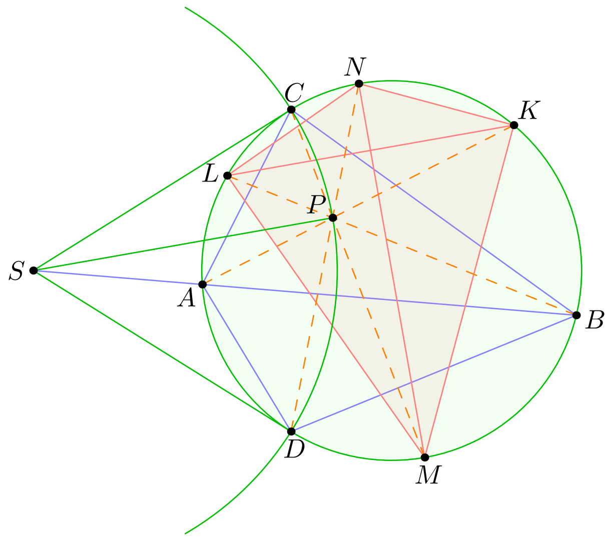
Available online at https://aops.com/community/p1936916.

Problem statement

Let P be a point interior to triangle ABC (with $CA \neq CB$). The lines AP, BP and CP meet again its circumcircle Γ at K, L, M, respectively. The tangent line at C to Γ meets the line AB at S. Show that from $SC = SP$ follows $MK = ML$.

We present two solutions using harmonic bundles.

¶ First solution (Evan Chen). Let N be the antipode of M, and let NP meet Γ again at D. Focus only on $CDMN$ for now (ignoring the condition). Then C and D are feet of altitudes in $\triangle MNP$; it is well-known that the circumcircle of $\triangle CDP$ is orthogonal to Γ (passing through the orthocenter of $\triangle MPN$).



Now, we are given that point S is such that $\overline{SC}$ is tangent to Γ, and $SC = SP$. It follows that S is the circumcenter of $\triangle CDP$, and hence $\overline{SC}$ and $\overline{SD}$ are tangents to Γ.

Then $-1 = (AB; CD) \overset{P}{=} (KL; MN)$. Since $\overline{MN}$ is a diameter, this implies $MK = ML$.

> **Remark.** I think it's more natural to come up with this solution in reverse. Namely, suppose we define the points the other way: let $\overline{SD}$ be the other tangent, so $(AB; CD) = -1$. Then project through P to get $(KL; MN) = -1$, where N is the second intersection of $\overline{DP}$. However, if $ML = MK$ then $KMLN$ must be a kite. Thus one can recover the solution in reverse.

¶ Second solution (Sebastian Jeon). We have

$$SP^2 = SC^2 = SA \cdot SB \implies \measuredangle SPA = \measuredangle PBA = \measuredangle LBA = \measuredangle LKA = \measuredangle LKP$$

(the latter half is Reim's theorem). Therefore $\overline{SP}$ and $\overline{LK}$ are *parallel*.

Now, let $\overline{SP}$ meet Γ again at X and Y, and let Q be the antipode of P on (S). Then

$$SP^2 = SQ^2 = SX \cdot SY \implies (PQ;XY) = -1 \implies \angle QCP = 90^\circ$$

that $\overline{CP}$ bisects $\angle XCY$. Since $\overline{XY} \parallel \overline{KL}$, it follows $\overline{CP}$ bisects to $\angle LCK$ too.

§2.2 IMO 2010/5, proposed by Netherlands

Available online at https://aops.com/community/p1936917.

Problem statement

Each of the six boxes B_1, B_2, B_3, B_4, B_5, B_6 initially contains one coin. The following two types of operations are allowed:

1. Choose a non-empty box B_j, $1 \le j \le 5$, remove one coin from B_j and add two coins to B_{j+1};
2. Choose a non-empty box B_k, $1 \le k \le 4$, remove one coin from B_k and swap the contents (possibly empty) of the boxes B_{k+1} and B_{k+2}.

Determine if there exists a finite sequence of operations of the allowed types, such that the five boxes B_1, B_2, B_3, B_4, B_5 become empty, while box B_6 contains exactly $2010^{2010^{2010}}$ coins.

First,

$$\begin{aligned}(1,1,1,1,1,1) &\to (0,3,1,0,3,1) \to (0,0,7,0,0,7)\\ &\to (0,0,6,2,0,7) \to (0,0,6,1,2,7) \to (0,0,6,1,0,11)\\ &\to (0,0,6,0,11,0) \to (0,0,5,11,0,0).\end{aligned}$$

and henceforth we ignore boxes B_1 and B_2, looking at just the last four boxes; so we write the current position as $(5,11,0,0)$.

We prove a lemma:

Claim — Let $k \ge 0$ and $n > 0$. From $(k,n,0,0)$ we may reach $(k-1,2^n,0,0)$.

Proof. Working with only the last three boxes for now,

$$\begin{aligned}(n,0,0) &\to (n-1,2,0) \to (n-1,0,4)\\ &\to (n-2,4,0) \to (n-2,0,8)\\ &\to (n-3,8,0) \to (n-3,0,16)\\ &\to \dots \to (1,2^{n-1},0) \to (1,0,2^n) \to (0,2^n,0).\end{aligned}$$

Finally we have $(k,n,0,0) \to (k,0,2^n,0) \to (k-1,2^n,0,0)$. $\square$

Now from $(5,11,0,0)$ we go as follows:

$$\begin{aligned}(5,11,0,0) \to (4,2^{11},0,0) \to \left(3,2^{2^{11}},0,0\right) \to \left(2,2^{2^{2^{11}}},0,0\right)\\ \to \left(1,2^{2^{2^{2^{11}}}},0,0\right) \to \left(0,2^{2^{2^{2^{2^{11}}}}},0,0\right).\end{aligned}$$

Let $A = 2^{2^{2^{2^{2^{11}}}}} > 2010^{2010^{2010}} = B$. Then by using move 2 repeatedly on the fourth box (i.e., throwing away several coins by swapping the empty B_5 and B_6), we go from $(0,A,0,0)$ to $(0,B/4,0,0)$. From there we reach $(0,0,0,B)$.

§2.3 IMO 2010/6

Available online at https://aops.com/community/p1936918.

Problem statement

Let $a_1, a_2, a_3, \dots$ be a sequence of positive real numbers, and s be a positive integer, such that

$$a_n = \max\{a_k + a_{n-k} \mid 1 \le k \le n-1\} \text{ for all } n > s.$$

Prove there exist positive integers $\ell \le s$ and N, such that

$$a_n = a_\ell + a_{n-\ell} \text{ for all } n \ge N.$$

Let

$$w_1 = \frac{a_1}{1}, \quad w_2 = \frac{a_2}{2}, \quad \dots, \quad w_s = \frac{a_s}{s}.$$

(The choice of the letter w is for "weight".) We claim the right choice of ℓ is the one maximizing w_ℓ.

Our plan is to view each a_n as a linear combination of the weights $w_1, \dots, w_s$ and track their coefficients.

To this end, let's define an *n-type* to be a vector $T = \langle t_1, \dots, t_s \rangle$ of nonnegative integers such that

- $n = t_1 + \dots + t_s$; and
- t_i is divisible by i for every i.

We then define its *valuation* as $v(T) = \sum_{i=1}^{s} w_i t_i$.

Now we define a n-type to be *valid* according to the following recursive rule. For $1 \le n \le s$ the only valid n-types are

$$\begin{aligned} T_1 &= \langle 1, 0, 0, \dots, 0 \rangle \\ T_2 &= \langle 0, 2, 0, \dots, 0 \rangle \\ T_3 &= \langle 0, 0, 3, \dots, 0 \rangle \\ &\vdots \\ T_s &= \langle 0, 0, 0, \dots, s \rangle \end{aligned}$$

for $n = 1, \dots, s$, respectively. Then for any $n > s$, an n-type is valid if it can be written as the sum of a valid k-type and a valid $(n-k)$-type, componentwise. These represent the linear combinations possible in the recursion; in other words the recursion in the problem is phrased as

$$a_n = \max_{T \text{ is a valid } n\text{-type}} v(T).$$

In fact, we have the following description of valid n-types:

Claim — Assume $n > s$. Then an n-type $\langle t_1, \dots, t_s \rangle$ is valid if and only if either

- there exist indices $i < j$ with $i + j > s$, $t_i \ge i$ and $t_j \ge j$; or
- there exists an index $i > s/2$ with $t_i \ge 2i$.

Proof. Immediate by forwards induction on $n > s$ that all n-types have this property.

The reverse direction is by downwards induction on n. Indeed if $\sum_i \frac{t_i}{i} > 2$, then we may subtract off on of $\{T_1, \dots, T_s\}$ while preserving the condition; and the case $\sum_i \frac{t_i}{i} = 2$ is essentially by definition. □

Remark. The claim is a bit confusingly stated in its two cases; really the latter case should be thought of as the situation $i = j$ but requiring that t_i/i is counted with multiplicity.

Now, for each $n > s$ we pick a valid n-type T_n with $a_n = v(T_n)$; if there are ties, we pick one for which the ℓth entry is as large as possible.

Claim — For any $n > s$ and index $i \neq \ell$, the ith entry of T_n is at most $2s + \ell i$.

Proof. If not, we can go back $i\ell$ steps to get a valid $(n - i\ell)$-type T achieved by decreasing the ith entry of T_n by $i\ell$. But then we can add ℓ to the ℓth entry i times to get another n-type T' which obviously has valuation at least as large, but with larger ℓth entry. □

Now since all other entries in T_n are bounded, eventually the sequence $(T_n)_{n>s}$ just consists of repeatedly adding 1 to the ℓth entry, as required.

Remark. One big step is to consider $w_k = a_k/k$. You can get this using wishful thinking or by examining small cases. (In addition this normalization makes it easier to see why the largest w plays an important role, since then in the definition of type, the n-types all have a sum of n. Unfortunately, it makes the characterization of valid n-types somewhat clumsier too.)