USAMO 2012 Solution Notes

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This is a compilation of solutions for the 2012 USAMO. Some of the solutions are my own work, but many are from the official solutions provided by the organizers (for which they hold any copyrights), and others were found by users on the Art of Problem Solving forums.

These notes will tend to be a bit more advanced and terse than the "official" solutions from the organizers. In particular, if a theorem or technique is not known to beginners but is still considered "standard", then I often prefer to use this theory anyways, rather than try to work around or conceal it. For example, in geometry problems I typically use directed angles without further comment, rather than awkwardly work around configuration issues. Similarly, sentences like "let $\mathbb{R}$ denote the set of real numbers" are typically omitted entirely.

Corrections and comments are welcome!

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§0 Problems

1. Find all integers $n \geq 3$ such that among any n positive real numbers a_1, a_2, $\dots$, a_n with
$$\max(a_1, a_2, \dots, a_n) \leq n \cdot \min(a_1, a_2, \dots, a_n),$$
there exist three that are the side lengths of an acute triangle.

2. A circle is divided into congruent arcs by 432 points. The points are colored in four colors such that some 108 points are colored red, some 108 points are colored green, some 108 points are colored blue, and the remaining 108 points are colored yellow. Prove that one can choose three points of each color in such a way that the four triangles formed by the chosen points of the same color are congruent.

3. Determine which integers $n > 1$ have the property that there exists an infinite sequence a_1, a_2, a_3, $\dots$of nonzero integers such that the equality
$$a_k + 2a_{2k} + \dots + na_{nk} = 0$$
holds for every positive integer k.

4. Find all functions $f\colon \mathbb{N} \to \mathbb{N}$ such that $f(n!) = f(n)!$ for all positive integers n and such that $m - n$ divides $f(m) - f(n)$ for all distinct positive integers m, n.

5. Let P be a point in the plane of $\triangle ABC$, and γ a line through P. Let A', B', C' be the points where the reflections of lines PA, PB, PC with respect to γ intersect lines BC, CA, AB respectively. Prove that A', B', C' are collinear.

6. For integer $n \geq 2$, let x_1, x_2, $\dots$, x_n be real numbers satisfying
$$x_1 + x_2 + \dots + x_n = 0 \quad \text{and} \quad x_1^2 + x_2^2 + \dots + x_n^2 = 1.$$
For each subset $A \subseteq \{1, 2, \dots, n\}$, define $S_A = \sum_{i \in A} x_i$. (If A is the empty set, then $S_A = 0$.) Prove that for any positive number λ, the number of sets A satisfying $S_A \geq \lambda$ is at most $2^{n-3}/\lambda^2$. For which choices of x_1, x_2, $\dots$, x_n, λ does equality hold?

§1 Solutions to Day 1

§1.1 USAMO 2012/1, proposed by Titu Andreescu

Available online at https://aops.com/community/p2669112.

Problem statement

Find all integers $n \geq 3$ such that among any n positive real numbers $a_1, a_2, \dots, a_n$ with

$$\max(a_1, a_2, \dots, a_n) \leq n \cdot \min(a_1, a_2, \dots, a_n),$$

there exist three that are the side lengths of an acute triangle.

The answer is all $n \geq 13$.

Define (F_n) as the sequence of Fibonacci numbers, by $F_1 = F_2 = 1$ and $F_{n+1} = F_n + F_{n-1}$. We will find that Fibonacci numbers show up naturally when we work through the main proof, so we will isolate the following calculation now to make the subsequent solution easier to read.

Claim — For positive integers m, we have $F_m \leq m^2$ if and only if $m \leq 12$.

Proof. A table of the first 14 Fibonacci numbers is given below.

F_1	F_2	F_3	F_4	F_5	F_6	F_7	F_8	F_9	F_{10}	F_{11}	F_{12}	F_{13}	F_{14}
1	1	2	3	5	8	13	21	34	55	89	144	233	377

By examining the table, we see that $F_m \leq m^2$ is true for $m = 1, 2, \dots 12$, and in fact $F_{12} = 12^2 = 144$. However, $F_m > m^2$ for $m = 13$ and $m = 14$.

Now it remains to prove that $F_m > m^2$ for $m \geq 15$. The proof is by induction with base cases $m = 13$ and $m = 14$ being checked already. For the inductive step, if $m \geq 15$ then we have

$$\begin{aligned} F_m &= F_{m-1} + F_{m-2} > (m-1)^2 + (m-2)^2 \\ &= 2m^2 - 6m + 5 = m^2 + (m-1)(m-5) > m^2 \end{aligned}$$

as desired. □

We now proceed to the main problem. The hypothesis $\max(a_1, a_2, \dots, a_n) \leq n \cdot \min(a_1, a_2, \dots, a_n)$ will be denoted by (†).

Proof that all $n \geq 13$ have the property. We first show now that every $n \geq 13$ has the desired property. Suppose for contradiction that no three numbers are the sides of an acute triangle. Assume without loss of generality (by sorting the numbers) that $a_1 \leq a_2 \leq \dots \leq a_n$. Then since a_{i-1}, a_i, a_{i+1} are not the sides of an acute triangle for each $i \geq 2$, we have that $a_{i+1}^2 \geq a_i^2 + a_{i-1}^2$; writing this out gives

$$\begin{aligned} a_3^2 &\geq a_2^2 + a_1^2 \geq 2a_1^2 \\ a_4^2 &\geq a_3^2 + a_2^2 \geq 2a_1^2 + a_1^2 = 3a_1^2 \\ a_5^2 &\geq a_4^2 + a_3^2 \geq 3a_1^2 + 2a_1^2 = 5a_1^2 \\ a_6^2 &\geq a_5^2 + a_4^2 \geq 5a_1^2 + 3a_1^2 = 8a_1^2 \end{aligned}$$

and so on. The Fibonacci numbers appear naturally and by induction, we conclude that $a_i^2 \ge F_i a_1^2$. In particular, $a_n^2 \ge F_n a_1^2$.

However, we know $\max(a_1, \dots, a_n) = a_n$ and $\min(a_1, \dots, a_n) = a_1$, so (†) reads $a_n \le n \cdot a_1$. Therefore we have $F_n \le n^2$, and so $n \le 12$, contradiction!

Proof that no $n \le 12$ have the property. Assume that $n \le 12$. The above calculation also suggests a way to pick the counterexample: we choose $a_i = \sqrt{F_i}$ for every i. Then $\min(a_1, \dots, a_n) = a_1 = 1$ and $\max(a_1, \dots, a_n) = \sqrt{F_n}$, so (†) is true as long as $n \le 12$. And indeed no three numbers form the sides of an acute triangle: if $i < j < k$, then $a_k^2 = F_k = F_{k-1} + F_{k-2} \ge F_j + F_i = a_j^2 + a_i^2$.

§1.2 USAMO 2012/2, proposed by Gregory Galperin

Available online at https://aops.com/community/p2669115.

Problem statement

A circle is divided into congruent arcs by 432 points. The points are colored in four colors such that some 108 points are colored red, some 108 points are colored green, some 108 points are colored blue, and the remaining 108 points are colored yellow. Prove that one can choose three points of each color in such a way that the four triangles formed by the chosen points of the same color are congruent.

First, consider the 431 possible non-identity rotations of the red points, and count overlaps with green points. If we select a rotation randomly, then each red point lies over a green point with probability $\frac{108}{431}$; hence the expected number of red-green incidences is

$$\frac{108}{431} \cdot 108 > 27$$

and so by pigeonhole, we can find a red 28-gon and a green 28-gon which are rotations of each other.

Now, look at the 430 rotations of this 28-gon (that do not give the all-red or all-green configuration) and compare it with the blue points. The same approach gives

$$\frac{108}{430} \cdot 28 > 7$$

incidences, so we can find red, green, blue 8-gons which are similar under rotation.

Finally, the 429 nontrivial rotations of this 8-gon expect

$$\frac{108}{429} \cdot 8 > 2$$

incidences with yellow. So finally we have four monochromatic 3-gons, one of each color, which are rotations of each other.

§1.3 USAMO 2012/3, proposed by Gabriel Carroll

Available online at https://aops.com/community/p2669119.

Problem statement

Determine which integers $n > 1$ have the property that there exists an infinite sequence a_1, a_2, a_3, ...of nonzero integers such that the equality

$$a_k + 2a_{2k} + \cdots + na_{nk} = 0$$

holds for every positive integer k.

Answer: all $n > 2$.

For $n = 2$, we have $a_k + 2a_{2k} = 0$, which is clearly not possible, since it implies $a_{2^k} = \frac{a_1}{2^{k-1}}$ for all $k \geq 1$.

For $n \geq 3$ we will construct a *completely multiplicative* sequence (meaning $a_{ij} = a_i a_j$ for all i and j). Thus (a_i) is determined by its value on primes, and satisfies the condition as long as $a_1 + 2a_2 + \cdots + na_n = 0$. The idea is to take two large primes and use Bezout's theorem, but the details require significant care.

We start by solving the case where $n \geq 9$. In that case, by Bertrand postulate there exists primes p and q such that

$$\lceil n/2 \rceil < q < 2\lceil n/2 \rceil \quad \text{and} \quad \frac{1}{2}(q-1) < p < q-1.$$

Clearly $p \neq q$, and $q \geq 7$, so $p > 3$. Also, $p < q < n$ but $2q > n$, and $4p \geq 4\left(\frac{1}{2}(q+1)\right) > n$. We now stipulate that $a_r = 1$ for any prime $r \neq p, q$ (in particular including $r = 2$ and $r = 3$). There are now three cases, identical in substance.

- If $p, 2p, 3p \in [1, n]$ then we would like to choose nonzero a_p and a_q such that

$$6p \cdot a_p + q \cdot a_q = 6p + q - \frac{1}{2}n(n+1)$$

  which is possible by Bézout lemma, since $\gcd(6p, q) = 1$.

- Else if $p, 2p \in [1, n]$ then we would like to choose nonzero a_p and a_q such that

$$3p \cdot a_p + q \cdot a_q = 3p + q - \frac{1}{2}n(n+1)$$

  which is possible by Bézout lemma, since $\gcd(3p, q) = 1$.

- Else if $p \in [1, n]$ then we would like to choose nonzero a_p and a_q such that

$$p \cdot a_p + q \cdot a_q = p + q - \frac{1}{2}n(n+1)$$

  which is possible by Bézout lemma, since $\gcd(p, q) = 1$. (This case is actually possible in a few edge cases, for example when $n = 9$, $q = 7$, $p = 5$.)

It remains to resolve the cases where $3 \leq n \leq 8$. We enumerate these cases manually:

- For $n = 3$, let $a_n = (-1)^{\nu_3(n)}$.

- For $n = 4$, let $a_n = (-1)^{\nu_2(n)+\nu_3(n)}$.
- For $n = 5$, let $a_n = (-2)^{\nu_5(n)}$.
- For $n = 6$, let $a_n = 5^{\nu_2(n)} \cdot 3^{\nu_3(n)} \cdot (-42)^{\nu_5(n)}$.
- For $n = 7$, let $a_n = (-3)^{\nu_7(n)}$.
- For $n = 8$, we can choose $(p, q) = (5, 7)$ in the prior construction.

This completes the constructions for all $n > 2$.

§2 Solutions to Day 2

§2.1 USAMO 2012/4, proposed by Gabriel Dospinescu

Available online at https://aops.com/community/p2669997.

Problem statement

Find all functions $f\colon \mathbb{N} \to \mathbb{N}$ such that $f(n!) = f(n)!$ for all positive integers n and such that $m - n$ divides $f(m) - f(n)$ for all distinct positive integers m, n.

Answer: $f \equiv 1$, $f \equiv 2$, and f the identity. As these obviously work, we prove these are the only ones.

By putting $n = 1$ and $n = 2$ we give $f(1), f(2) \in \{1, 2\}$. Also, we will use the condition

$$m! - n! \text{ divides } f(m)! - f(n)!.$$

We consider four cases on $f(1)$ and $f(2)$, and dispense with three of them.

- If $f(2) = 1$ then for all $m \geq 3$ we have $m! - 2$ divides $f(m)! - 1$, so $f(m) = 1$ for modulo 2 reasons. Then clearly $f(1) = 1$.
- If $f(1) = f(2) = 2$ we first obtain $3! - 1 \mid f(3)! - 2$, which implies $f(3) = 2$. Then $m! - 3 \mid f(m)! - 2$ for $m \geq 4$ implies $f(m) = 2$ for modulo 3 reasons.

Hence we are left with the case where $f(1) = 1$ and $f(2) = 2$. Continuing, we have

$$3! - 1 \mid f(3)! - 1 \quad \text{and} \quad 3! - 2 \mid f(3)! - 2 \implies f(3) = 3.$$

Continuing by induction, suppose $f(1) = 1, \dots, f(k) = k$.

$$k! \cdot k = (k+1)! - k! \mid f(k+1)! - k!$$

and thus we deduce that $f(k+1) \geq k$, and hence

$$k \mid \frac{f(k+1)!}{k!} - 1.$$

Then plainly $f(k+1) \leq 2k$ for mod k reasons, but also $f(k+1) \equiv 1 \pmod{k}$ so we conclude $f(k+1) = k + 1$.

Remark. Shankar Padmanabhan gives the following way to finish after verifying that $f(3) = 3$. Note that if

$$M = ((((3!)!)!)! \dots)!$$

for any number of iterated factorials then $f(M) = M$. Thus for any n, we have

$$M - n \mid f(M) - f(n) = M - f(n) \implies M - n \mid n - f(n)$$

and so taking M large enough implies $f(n) = n$.

§2.2 USAMO 2012/5, proposed by Titu Andreescu, Cosmin Pohoata

Available online at https://aops.com/community/p2669960.

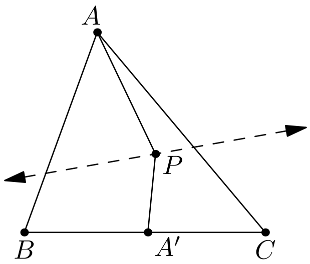
Problem statement

Let P be a point in the plane of $\triangle ABC$, and γ a line through P. Let A', B', C' be the points where the reflections of lines PA, PB, PC with respect to γ intersect lines BC, CA, AB respectively. Prove that A', B', C' are collinear.

We present three solutions.

¶ First solution (complex numbers). Let $p = 0$ and set γ as the real line. Then A' is the intersection of bc and $p\overline{a}$. So, we get

$$a' = \frac{\overline{a}(\overline{b}c - b\overline{c})}{(\overline{b} - \overline{c})\overline{a} - (b - c)a}.$$



Note that

$$\overline{a}' = \frac{a(b\overline{c} - \overline{b}c)}{(b - c)a - (\overline{b} - \overline{c})\overline{a}}.$$

Thus it suffices to prove

$$0 = \det \begin{bmatrix} \frac{\overline{a}(\overline{b}c - b\overline{c})}{(\overline{b}-\overline{c})\overline{a}-(b-c)a} & \frac{a(b\overline{c}-\overline{b}c)}{(b-c)a-(\overline{b}-\overline{c})\overline{a}} & 1 \\ \frac{\overline{b}(\overline{c}a-c\overline{a})}{(\overline{c}-\overline{a})\overline{b}-(c-a)b} & \frac{b(c\overline{a}-\overline{c}a)}{(c-a)b-(\overline{c}-\overline{a})\overline{b}} & 1 \\ \frac{\overline{c}(\overline{a}b-a\overline{b})}{(\overline{a}-\overline{b})\overline{c}-(a-b)c} & \frac{c(a\overline{b}-\overline{a}b)}{(a-b)c-(\overline{a}-\overline{b})\overline{c}} & 1 \end{bmatrix}.$$

This is equivalent to

$$0 = \det \begin{bmatrix} \overline{a}(\overline{b}c - b\overline{c}) & a(\overline{b}c - b\overline{c}) & (\overline{b} - \overline{c})\overline{a} - (b - c)a \\ \overline{b}(\overline{c}a - c\overline{a}) & b(\overline{c}a - c\overline{a}) & (\overline{c} - \overline{a})\overline{b} - (c - a)b \\ \overline{c}(\overline{a}b - a\overline{b}) & c(\overline{a}b - a\overline{b}) & (\overline{a} - \overline{b})\overline{c} - (a - b)c \end{bmatrix}.$$

This determinant has the property that the rows sum to zero, and we're done.

Remark. Alternatively, if you don't notice that you could just blindly expand:

$$\begin{aligned} &\sum_{\text{cyc}} ((\overline{b} - \overline{c})\overline{a} - (b - c)a) \cdot -\det \begin{bmatrix} b & \overline{b} \\ c & \overline{c} \end{bmatrix} (\overline{c}a - c\overline{a}) \left(\overline{a}b - a\overline{b}\right) \\ &= (\overline{b}c - c\overline{b})(\overline{c}a - c\overline{a})(\overline{a}b - a\overline{b}) \sum_{\text{cyc}} \left(ab - ac + \overline{ca} - \overline{ba}\right) = 0. \end{aligned}$$

¶ Second solution (Desargues involution). We let $C'' = \overline{A'B'} \cap \overline{AB}$. Consider complete quadrilateral $ABCA'B'C''C$. We see that there is an involutive pairing τ at P swapping (PA, PA'), (PB, PB'), (PC, PC''). From the first two, we see τ coincides with reflection about ℓ, hence conclude $C'' = C$.

¶ Third solution (barycentric), by Catherine Xu. We will perform barycentric coordinates on the triangle PCC', with $P = (1,0,0)$, $C' = (0,1,0)$, and $C = (0,0,1)$. Set $a = CC'$, $b = CP$, $c = C'P$ as usual. Since A, B, C' are collinear, we will define $A = (p : k : q)$ and $B = (p : \ell : q)$.

Claim — Line γ is the angle bisector of $\angle APA'$, $\angle BPB'$, and $\angle CPC'$.

Proof. Since $A'P$ is the reflection of AP across γ, etc. □

Thus B' is the intersection of the isogonal of B with respect to $\angle P$ with the line CA; that is,

$$B' = \left(\frac{p}{k} \frac{b^2}{\ell} : \frac{b^2}{\ell} : \frac{c^2}{q} \right).$$

Analogously, A' is the intersection of the isogonal of A with respect to $\angle P$ with the line CB; that is,

$$A' = \left(\frac{p}{\ell} \frac{b^2}{k} : \frac{b^2}{k} : \frac{c^2}{q} \right).$$

The ratio of the first to third coordinate in these two points is both $b^2 pq : c^2 k\ell$, so it follows A', B', and C' are collinear.

Remark (Problem reference)**.** The converse of this problem appears as problem 1052 attributed S. V. Markelov in the book *Geometriya: 9–11 Klassy: Ot Uchebnoy Zadachi k Tvorcheskoy, 1996*, by I. F. Sharygin.

§2.3 USAMO 2012/6, proposed by Gabriel Carroll

Available online at https://aops.com/community/p2670037.

Problem statement

For integer $n \ge 2$, let $x_1, x_2, \dots, x_n$ be real numbers satisfying

$$x_1 + x_2 + \cdots + x_n = 0 \quad \text{and} \quad x_1^2 + x_2^2 + \cdots + x_n^2 = 1.$$

For each subset $A \subseteq \{1, 2, \dots, n\}$, define $S_A = \sum_{i \in A} x_i$. (If A is the empty set, then $S_A = 0$.) Prove that for any positive number λ, the number of sets A satisfying $S_A \ge \lambda$ is at most $2^{n-3}/\lambda^2$. For which choices of $x_1, x_2, \dots, x_n, \lambda$ does equality hold?

Let ε_i be a coin flip of 0 or 1. Then we have

$$\begin{aligned} \mathbb{E}[S_A^2] &= \mathbb{E}\left[\left(\sum \varepsilon_i x_i\right)^2\right] = \sum_i \mathbb{E}[\varepsilon_i^2] x_i^2 + \sum_{i<j} \mathbb{E}[\varepsilon_i \varepsilon_j] 2 x_i x_j \\ &= \frac{1}{2} \sum x_i^2 + \frac{1}{2} \sum x_i x_j = \frac{1}{2} + \frac{1}{2} \sum_{i<j} x_i x_j = \frac{1}{2} + \frac{1}{2}\left(-\frac{1}{2}\right) = \frac{1}{4}. \end{aligned}$$

In other words, $\sum_A S_A^2 = 2^{n-2}$. Since can always pair A with its complement, we conclude

$$\sum_{S_A > 0} S_A^2 = 2^{n-3}.$$

Equality holds iff $S_A \in \{\pm\lambda, 0\}$ for every A. This occurs when $x_1 = 1/\sqrt{2}$, $x_2 = -1/\sqrt{2}$, $x_3 = \cdots = 0$ (or permutations), and $\lambda = 1/\sqrt{2}$.