

# IMO 2011 Solution Notes

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This is a compilation of solutions for the 2011 IMO. Some of the solutions are my own work, but many are from the official solutions provided by the organizers (for which they hold any copyrights), and others were found by users on the Art of Problem Solving forums.

These notes will tend to be a bit more advanced and terse than the “official” solutions from the organizers. In particular, if a theorem or technique is not known to beginners but is still considered “standard”, then I often prefer to use this theory anyways, rather than try to work around or conceal it. For example, in geometry problems I typically use directed angles without further comment, rather than awkwardly work around configuration issues. Similarly, sentences like “let  $\mathbb{R}$  denote the set of real numbers” are typically omitted entirely.

Corrections and comments are welcome!

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## §0 Problems

1. Given any set  $A = \{a_1, a_2, a_3, a_4\}$  of four distinct positive integers, we denote the sum  $a_1 + a_2 + a_3 + a_4$  by  $s_A$ . Let  $n_A$  denote the number of pairs  $(i, j)$  with  $1 \leq i < j \leq 4$  for which  $a_i + a_j$  divides  $s_A$ . Find all sets  $A$  of four distinct positive integers which achieve the largest possible value of  $n_A$ .
2. Let  $\mathcal{S}$  be a finite set of at least two points in the plane. Assume that no three points of  $\mathcal{S}$  are collinear. A *windmill* is a process that starts with a line  $\ell$  going through a single point  $P \in \mathcal{S}$ . The line rotates clockwise about the *pivot*  $P$  until the first time that the line meets some other point belonging to  $\mathcal{S}$ . This point,  $Q$ , takes over as the new pivot, and the line now rotates clockwise about  $Q$ , until it next meets a point of  $\mathcal{S}$ . This process continues indefinitely.

Show that we can choose a point  $P$  in  $\mathcal{S}$  and a line  $\ell$  going through  $P$  such that the resulting windmill uses each point of  $\mathcal{S}$  as a pivot infinitely many times.

3. Let  $f: \mathbb{R} \rightarrow \mathbb{R}$  be a real-valued function defined on the set of real numbers that satisfies

$$f(x+y) \leq yf(x) + f(f(x))$$

for all real numbers  $x$  and  $y$ . Prove that  $f(x) = 0$  for all  $x \leq 0$ .

4. Let  $n > 0$  be an integer. We are given a balance and  $n$  weights of weight  $2^0, 2^1, \dots, 2^{n-1}$ . We are to place each of the  $n$  weights on the balance, one after another, in such a way that the right pan is never heavier than the left pan. At each step we choose one of the weights that has not yet been placed on the balance, and place it on either the left pan or the right pan, until all of the weights have been placed. Determine the number of ways in which this can be done.
5. Let  $f: \mathbb{Z} \rightarrow \mathbb{Z}_{\geq 0}$  be a function such that  $f(m-n) \mid f(m) - f(n)$  for  $m, n \in \mathbb{Z}$ . Prove that if  $m, n \in \mathbb{Z}$  satisfy  $f(m) \leq f(n)$  then  $f(m) \mid f(n)$ .
6. Let  $ABC$  be an acute triangle with circumcircle  $\Gamma$ . Let  $\ell$  be a tangent line to  $\Gamma$ , and let  $\ell_a, \ell_b, \ell_c$  be the lines obtained by reflecting  $\ell$  in the lines  $BC, CA$ , and  $AB$ , respectively. Show that the circumcircle of the triangle determined by the lines  $\ell_a, \ell_b$ , and  $\ell_c$  is tangent to the circle  $\Gamma$ .

## §1 Solutions to Day 1

### §1.1 IMO 2011/1, proposed by Fernando Campos (MEX)

Available online at <https://aops.com/community/p2363530>.

#### Problem statement

Given any set  $A = \{a_1, a_2, a_3, a_4\}$  of four distinct positive integers, we denote the sum  $a_1 + a_2 + a_3 + a_4$  by  $s_A$ . Let  $n_A$  denote the number of pairs  $(i, j)$  with  $1 \leq i < j \leq 4$  for which  $a_i + a_j$  divides  $s_A$ . Find all sets  $A$  of four distinct positive integers which achieve the largest possible value of  $n_A$ .

There are two curves of solutions, namely  $\{x, 5x, 7x, 11x\}$  and  $\{x, 11x, 19x, 29x\}$ , for any positive integer  $x$ , achieving  $n_A = 4$  (easy to check). We'll show that  $n_A \leq 4$  and equality holds only in one of the curves.

Let  $A = \{a < b < c < d\}$ .

**Claim —** We have  $n_A \leq 4$  with equality iff

$$a + b \mid c + d, \quad a + c \mid b + d, \quad a + d = b + c.$$

*Proof.* Note  $a + b \mid s_A \iff a + b \mid c + d$  etc. Now  $c + d \nmid a + b$  and  $b + d \nmid a + c$  for size reasons, so we already have  $n_A \leq 4$ ; moreover  $a + d \mid b + c$  and  $b + c \mid a + d$  if and only if  $a + d = b + c$ .  $\square$

We now show the equality curve is the one above.

$$a + c \mid b + d \iff a + c \mid -a + 2b + c \iff a + c \mid 2(b - a).$$

Since  $a + c > |b - a|$ , so we must have  $a + c = 2(b - a)$ . So we now have

$$\begin{aligned} c &= 2b - 3a \\ d &= b + c - a = 3b + c - 4a. \end{aligned}$$

The last condition is

$$a + b \mid c + d = 5b - 7a \iff a + b \mid 12a.$$

Now, let  $x = \gcd(a, b)$ . The expressions for  $c$  and  $d$  above imply that  $x \mid c, d$  so we may scale down so that  $x = 1$ . Then  $\gcd(a + b, a) = \gcd(a, b) = 1$  and so  $a + b \mid 12$ .

We have  $c > b$ , so  $3a < b$ . The only pairs  $(a, b)$  with  $3a < 2b$ ,  $\gcd(a, b) = 1$  and  $a + b \mid 12$  are  $(a, b) \in \{(1, 5), (1, 11)\}$  which give the solutions earlier.

**§1.2 IMO 2011/2, proposed by Geoff Smith (UNK)**

Available online at <https://aops.com/community/p2363537>.

**Problem statement**

Let  $\mathcal{S}$  be a finite set of at least two points in the plane. Assume that no three points of  $\mathcal{S}$  are collinear. A *windmill* is a process that starts with a line  $\ell$  going through a single point  $P \in \mathcal{S}$ . The line rotates clockwise about the *pivot*  $P$  until the first time that the line meets some other point belonging to  $\mathcal{S}$ . This point,  $Q$ , takes over as the new pivot, and the line now rotates clockwise about  $Q$ , until it next meets a point of  $\mathcal{S}$ . This process continues indefinitely.

Show that we can choose a point  $P$  in  $\mathcal{S}$  and a line  $\ell$  going through  $P$  such that the resulting windmill uses each point of  $\mathcal{S}$  as a pivot infinitely many times.

Orient  $\ell$  in some direction, and color the plane such that its left half is red and right half is blue. The critical observation is that:

**Claim —** The number of points on the red side of  $\ell$  does not change, nor does the number of points on the blue side (except at a moment when  $\ell$  contains two points).

Thus, if  $|\mathcal{S}| = n + 1$ , it suffices to pick the initial configuration so that there are  $\lfloor n/2 \rfloor$  red and  $\lceil n/2 \rceil$  blue points. Then when the line  $\ell$  does a full  $180^\circ$  rotation, the red and blue sides “switch”, so the windmill has passed through every point.

(See official shortlist for verbose write-up; this is deliberately short to make a point.)

### §1.3 IMO 2011/3, proposed by Igor Voronovich, BLR

Available online at <https://aops.com/community/p2363539>.

#### Problem statement

Let  $f: \mathbb{R} \rightarrow \mathbb{R}$  be a real-valued function defined on the set of real numbers that satisfies

$$f(x+y) \leq yf(x) + f(f(x))$$

for all real numbers  $x$  and  $y$ . Prove that  $f(x) = 0$  for all  $x \leq 0$ .

We begin by rewriting the given as

$$f(z) \leq (z-x)f(x) + f(f(x)) \quad \forall x, z \in \mathbb{R} \quad (\heartsuit)$$

(which is better anyways since control over inputs to  $f$  is more valuable). We start by eliminating the double  $f$ : let  $z = f(w)$  to get

$$f(f(w)) \leq (f(w)-x)f(x) + f(f(x))$$

and then use the *symmetry trick* to write

$$f(f(x)) \leq (f(x)-w)f(w) + f(f(w))$$

so that when we sum we get

$$wf(w) + xf(x) \leq 2f(x)f(w).$$

Next we use *cancellation trick*: set  $w = 2f(x)$  in the above to get

$$xf(x) \leq 0 \quad \forall x \in \mathbb{R}. \quad (\spadesuit)$$

**Claim —** For every  $p \in \mathbb{R}$ , we have  $f(p) \leq 0$ .

*Proof.* Assume  $f(p) > 0$  for some  $p \in \mathbb{R}$ . Then for any negative number  $z$ ,

$$0 \stackrel{(\spadesuit)}{\leq} f(z) \stackrel{(\heartsuit)}{\leq} (z-p)f(p) + f(f(p)).$$

which is false if we let  $z \rightarrow -\infty$ . □

Together with  $(\spadesuit)$  we derive  $f(x) = 0$  for  $x < 0$ . Finally, letting  $x$  and  $z$  be any negative numbers in  $(\heartsuit)$ , we get  $f(0) \geq 0$ , so  $f(0) = 0$  too.

**Remark.** As another corollary of the claim,  $f(f(x)) = 0$  for all  $x$ .

**Remark.** A nontrivial example of a working  $f$  is to take

$$f(x) = \begin{cases} -\exp(\exp(\exp(x))) & x > 0 \\ 0 & x \leq 0. \end{cases}$$

or some other negative function growing rapidly in absolute value for  $x > 0$ .

■

## §2 Solutions to Day 2

### §2.1 IMO 2011/4, proposed by Morteza Saghaian (IRN)

Available online at <https://aops.com/community/p2365036>.

#### Problem statement

Let  $n > 0$  be an integer. We are given a balance and  $n$  weights of weight  $2^0, 2^1, \dots, 2^{n-1}$ . We are to place each of the  $n$  weights on the balance, one after another, in such a way that the right pan is never heavier than the left pan. At each step we choose one of the weights that has not yet been placed on the balance, and place it on either the left pan or the right pan, until all of the weights have been placed. Determine the number of ways in which this can be done.

The answer is  $a_n = (2n - 1)!!$ .

We refer to what we're counting as a *valid  $n$ -sequence*: an order of which weights to place, and whether to place them on the left or right pan.

We use induction, with  $n = 1$  being obvious. Now consider the weight  $2^0 = 1$ .

- If we delete it from any valid  $n$ -sequence, we get a valid  $(n - 1)$ -sequence with all weights doubled.
- Given a valid  $(n - 1)$ -sequence with all weights doubled, we may insert  $2^0 = 1$  it into  $2n - 1$  ways. Indeed, we may insert it anywhere, and designate it either left or right, except we may not designate right if we choose to insert  $2^0 = 1$  at the very beginning.

Consequently, we have that

$$a_n = (2n - 1) \cdot a_{n-1}.$$

Since  $a_1 = 1$ , the conclusion follows.

**Remark** (Gabriel Levin). An alternate approach can be done by considering the heaviest weight  $2^{n-1}$  instead of the lightest weight  $2^0 = 1$ . This gives a more complicated recursion:

$$a_n = \sum_{k=0}^{n-1} \binom{n-1}{k} 2^{n-k-1} (n-k-1)! a_k$$

by summing over the index  $k$  corresponding to the number of weights placed before the heaviest weight  $2^{n-1}$  is placed.

This recursion can then be rewritten as

$$\begin{aligned} a_n &= \sum_{k=0}^{n-1} \binom{n-1}{k} 2^{n-k-1} (n-k-1)! a_k \\ &= (n-1)! \sum_{k=0}^{n-1} \frac{2^{n-k-1} a_k}{k!} \\ &= 2(n-1)! \sum_{k=0}^{n-1} \frac{2^{(n-1)-k-1} a_k}{k!} \\ &= 2(n-1)(n-2)! \left( \sum_{k=0}^{(n-1)-1} \frac{2^{(n-1)-k-1} a_k}{k!} \right) + 2(n-1)! \frac{\frac{1}{2} a_{n-1}}{(n-1)!} = 2(n-1)a_{n-1} + a_{n-1} = (2n-1)a_{n-1} \end{aligned}$$

and since  $a_1 = 1$ , we have  $a_n = (2n - 1)!!$  by induction on  $n$ .



## §2.2 IMO 2011/5, proposed by Mahyar Sefidgaran (IRN)

Available online at <https://aops.com/community/p2365041>.

### Problem statement

Let  $f: \mathbb{Z} \rightarrow \mathbb{Z}_{>0}$  be a function such that  $f(m-n) \mid f(m) - f(n)$  for  $m, n \in \mathbb{Z}$ . Prove that if  $m, n \in \mathbb{Z}$  satisfy  $f(m) \leq f(n)$  then  $f(m) \mid f(n)$ .

Let  $P(m, n)$  denote the given assertion. First, we claim  $f$  is even. This is straight calculation:

- $P(x, 0) \implies f(x) \mid f(x) - f(0) \implies f(x) \mid M := f(0)$ .
- $P(0, x) \implies f(-x) \mid M - f(x) \implies f(-x) \mid f(x)$ . Analogously,  $f(x) \mid f(-x)$ . So  $f(x) = f(-x)$  and  $f$  is even.

**Claim** — Let  $x, y, z$  be integers with  $x + y + z = 0$ . Then among  $f(x), f(y), f(z)$ , two of them are equal and divide the third.

*Proof.* Let  $a = f(\pm x), b = f(\pm y), c = f(\pm z)$  be positive integers. Note that

$$\begin{aligned} a &\mid b - c \\ b &\mid c - a \\ c &\mid a - b \end{aligned}$$

from  $P(y, -z)$  and similarly. WLOG  $c = \max(a, b, c)$ ; then  $c > |a - b|$  so  $a = b$ . Thus  $a = b \mid c$  from the first two.  $\square$

This implies the problem, by taking  $x$  and  $y$  in the previous claim to be the integers  $m$  and  $n$ .

**Remark.** At <https://aops.com/community/c6h418981p2381909>, Davi Medeiros gives the following characterization of functions  $f$  satisfying the hypothesis.

Pick  $f(0)$ ,  $k$  positive integers, a chain  $d_1 \mid d_2 \mid \dots \mid d_k$  of divisors of  $f(0)$ , and positive integers  $a_1, a_2, \dots, a_{k-1}$ , greater than 1 (if  $k = 1$ ,  $a_i$  doesn't exist, for every  $i$ ). We'll define  $f$  as follows:

- $f(n) = d_1$ , for every integer  $n$  that is not divisible by  $a_1$ ;
- $f(a_1 n) = d_2$ , for every integer  $n$  that is not divisible by  $a_2$ ;
- $f(a_1 a_2 n) = d_3$ , for every integer  $n$  that is not divisible by  $a_3$ ;
- $f(a_1 a_2 a_3 n) = d_4$ , for every integer  $n$  that is not divisible by  $a_4$ ;
- ...
- $f(a_1 a_2 \dots a_{k-1} n) = d_k$ , for every integer  $n$ ;

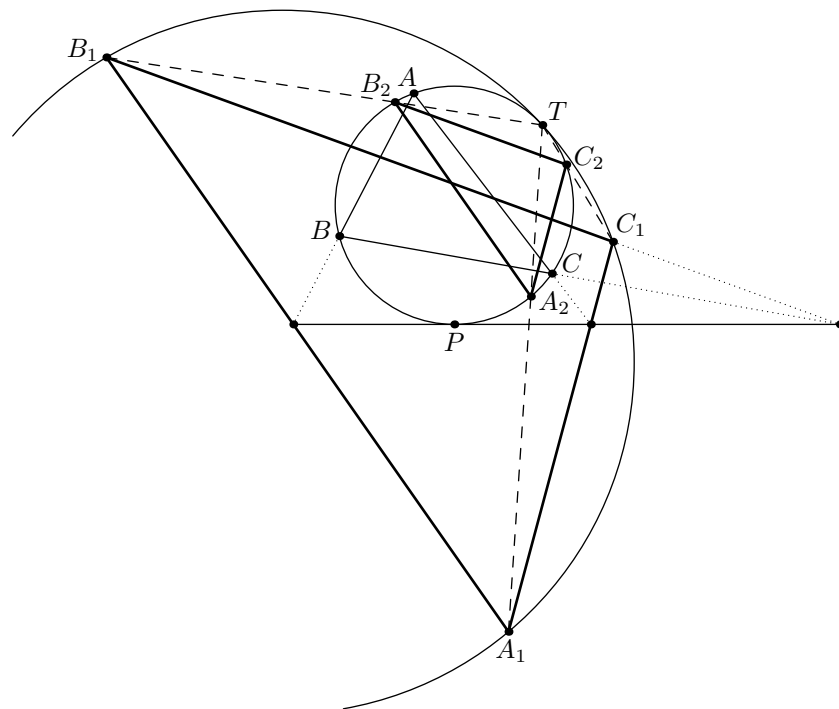
### §2.3 IMO 2011/6, proposed by Japan

Available online at <https://aops.com/community/p2365045>.

#### Problem statement

Let  $ABC$  be an acute triangle with circumcircle  $\Gamma$ . Let  $\ell$  be a tangent line to  $\Gamma$ , and let  $\ell_a, \ell_b, \ell_c$  be the lines obtained by reflecting  $\ell$  in the lines  $BC, CA$ , and  $AB$ , respectively. Show that the circumcircle of the triangle determined by the lines  $\ell_a, \ell_b$ , and  $\ell_c$  is tangent to the circle  $\Gamma$ .

This is a hard problem with many beautiful solutions. The following solution is not very beautiful but not too hard to find during an olympiad, as the only major insight it requires is the construction of  $A_2, B_2$ , and  $C_2$ .



We apply complex numbers with  $\omega$  the unit circle and  $p = 1$ . Let  $A_1 = \ell_B \cap \ell_C$ , and let  $a_2 = a^2$  (in other words,  $A_2$  is the reflection of  $P$  across the diameter of  $\omega$  through  $A$ ). Define the points  $B_1, C_1, B_2, C_2$  similarly.

We claim that  $A_1A_2, B_1B_2, C_1C_2$  concur at a point on  $\Gamma$ .

We begin by finding  $A_1$ . If we reflect the points  $1 + i$  and  $1 - i$  over  $\overline{AB}$ , then we get two points  $Z_1, Z_2$  with

$$z_1 = a + b - ab(1 - i) = a + b - ab + abi$$

$$z_2 = a + b - ab(1 + i) = a + b - ab - abi.$$

Therefore,

$$z_1 - z_2 = 2abi$$

$$\overline{z_1}z_2 - \overline{z_2}z_1 = -2i \left( a + b + \frac{1}{a} + \frac{1}{b} - 2 \right).$$

Now  $\ell_C$  is the line  $\overline{Z_1 Z_2}$ , so with the analogous equation  $\ell_B$  we obtain:

$$\begin{aligned}
 a_1 &= \frac{-2i \left(a + b + \frac{1}{a} + \frac{1}{b} - 2\right) (2aci) + 2i \left(a + c + \frac{1}{a} + \frac{1}{c} - 2\right) (2abi)}{\left(-\frac{2}{ab}i\right) (2aci) - \left(-\frac{2}{ac}i\right) (2abi)} \\
 &= \frac{[c - b] a^2 + \left[\frac{c}{b} - \frac{b}{c} - 2c + 2b\right] a + (c - b)}{\frac{c}{b} - \frac{b}{c}} \\
 &= a + \frac{(c - b) [a^2 - 2a + 1]}{(c - b)(c + b)/bc} \\
 &= a + \frac{bc}{b + c} (a - 1)^2.
 \end{aligned}$$

Then the second intersection of  $\overline{A_1 A_2}$  with  $\omega$  is given by

$$\begin{aligned}
 \frac{a_1 - a_2}{1 - a_2 \overline{a_1}} &= \frac{a + \frac{bc}{b+c} (a - 1)^2 - a^2}{1 - a - a^2 \cdot \frac{(1-1/a)^2}{b+c}} \\
 &= \frac{a + \frac{bc}{b+c} (1 - a)}{1 - \frac{1}{b+c} (1 - a)} \\
 &= \frac{ab + bc + ca - abc}{a + b + c - 1}.
 \end{aligned}$$

Thus, the claim is proved.

Finally, it suffices to show  $\overline{A_1 B_1} \parallel \overline{A_2 B_2}$ . One can also do this with complex numbers; it amounts to showing  $a^2 - b^2$ ,  $a - b$ ,  $i$  (corresponding to  $\overline{A_2 B_2}$ ,  $\overline{A_1 B_1}$ ,  $\overline{PP}$ ) have their arguments an arithmetic progression, equivalently

$$\frac{(a - b)^2}{i(a^2 - b^2)} \in \mathbb{R} \iff \frac{(a - b)^2}{i(a^2 - b^2)} = \frac{\left(\frac{1}{a} - \frac{1}{b}\right)^2}{\frac{1}{i} \left(\frac{1}{a^2} - \frac{1}{b^2}\right)}$$

which is obvious.

**Remark.** One can use directed angle chasing for this last part too. Let  $\overline{BC}$  meet  $\ell$  at  $K$  and  $\overline{B_2 C_2}$  meet  $\ell$  at  $L$ . Evidently

$$\begin{aligned}
 -\angle B_2 L P &= \angle L P B_2 + \angle P B_2 L \\
 &= 2\angle K P B + \angle P B_2 C_2 \\
 &= 2\angle K P B + 2\angle P B C \\
 &= -2\angle P K B \\
 &= \angle P K B_1
 \end{aligned}$$

as required.