JMO 2021 Solution Notes

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This is a compilation of solutions for the 2021 JMO. Some of the solutions are my own work, but many are from the official solutions provided by the organizers (for which they hold any copyrights), and others were found by users on the Art of Problem Solving forums.

These notes will tend to be a bit more advanced and terse than the "official" solutions from the organizers. In particular, if a theorem or technique is not known to beginners but is still considered "standard", then I often prefer to use this theory anyways, rather than try to work around or conceal it. For example, in geometry problems I typically use directed angles without further comment, rather than awkwardly work around configuration issues. Similarly, sentences like "let $\mathbb{R}$ denote the set of real numbers" are typically omitted entirely.

Corrections and comments are welcome!

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§0 Problems

1. Find all functions $f\colon \mathbb{N} \to \mathbb{N}$ which satisfy $f(a^2+b^2) = f(a)f(b)$ and $f(a^2) = f(a)^2$ for all positive integers a and b.

2. Rectangles BCC_1B_2, CAA_1C_2, and ABB_1A_2 are erected outside an acute triangle ABC. Suppose that

$$\angle BC_1C + \angle CA_1A + \angle AB_1B = 180^\circ.$$

Prove that lines B_1C_2, C_1A_2, and A_1B_2 are concurrent.

3. An equilateral triangle Δ of side length $L > 0$ is given. Suppose that n equilateral triangles with side length 1 and with non-overlapping interiors are drawn inside Δ, such that each unit equilateral triangle has sides parallel to Δ, but with opposite orientation. Prove that

$$n \le \frac{2}{3}L^2.$$

4. Carina has three pins, labeled A, B, and C, respectively, located at the origin of the coordinate plane. In a *move*, Carina may move a pin to an adjacent lattice point at distance 1 away. What is the least number of moves that Carina can make in order for triangle ABC to have area 2021?

5. A finite set S of positive integers has the property that, for each $s \in S$, and each positive integer divisor d of s, there exists a unique element $t \in S$ satisfying $\gcd(s,t) = d$. (The elements s and t could be equal.)

Given this information, find all possible values for the number of elements of S.

6. Let $n \ge 4$ be an integer. Find all positive real solutions to the following system of $2n$ equations:

$$\begin{aligned}
a_1 &= \frac{1}{a_{2n}} + \frac{1}{a_2}, & a_2 &= a_1 + a_3, \\
a_3 &= \frac{1}{a_2} + \frac{1}{a_4}, & a_4 &= a_3 + a_5, \\
a_5 &= \frac{1}{a_4} + \frac{1}{a_6}, & a_6 &= a_5 + a_7, \\
&\vdots & &\vdots \\
a_{2n-1} &= \frac{1}{a_{2n-2}} + \frac{1}{a_{2n}}, & a_{2n} &= a_{2n-1} + a_1.
\end{aligned}$$

§1 Solutions to Day 1

§1.1 JMO 2021/1, proposed by Vincent Huang

Available online at https://aops.com/community/p21498724.

Problem statement

Find all functions $f\colon \mathbb{N} \to \mathbb{N}$ which satisfy $f(a^2+b^2) = f(a)f(b)$ and $f(a^2) = f(a)^2$ for all positive integers a and b.

The answer is $f \equiv 1$ only, which works. We prove it's the only one.

The bulk of the problem is:

Claim — If $f(a) = f(b) = 1$ and $a > b$, then $f(a^2 - b^2) = f(2ab) = 1$.

Proof. Write

$$\begin{aligned} 1 = f(a)f(b) = f(a^2+b^2) &= \sqrt{f\left((a^2+b^2)^2\right)} \\ &= \sqrt{f\left((a^2-b^2)^2 + (2ab)^2\right)} \\ &= \sqrt{f(a^2-b^2)f(2ab)}. \end{aligned}$$

□

By setting $a = b = 1$ in the given statement we get $f(1) = f(2) = 1$. Now a simple induction on n shows $f(n) = 1$:

- If $n = 2k$ take $(u,v) = (k,1)$ hence $2uv = n$.
- If $n = 2k+1$ take $(u,v) = (k+1,k)$ hence $u^2 - v^2 = n$.

§1.2 JMO 2021/2, proposed by Ankan Bhattacharya

Available online at https://aops.com/community/p21498558.

Problem statement

Rectangles BCC_1B_2, CAA_1C_2, and ABB_1A_2 are erected outside an acute triangle ABC. Suppose that

$$\angle BC_1C + \angle CA_1A + \angle AB_1B = 180^\circ.$$

Prove that lines B_1C_2, C_1A_2, and A_1B_2 are concurrent.

The angle condition implies the circumcircles of the three rectangles concur at a single point P. Then $\measuredangle CPB_2 = \measuredangle CPA_1 = 90^\circ$, hence P lies on A_1B_2 etc., so we're done.

Remark. As one might guess from the two-sentence solution, the entire difficulty of the problem is getting the characterization of the concurrence point.

§1.3 JMO 2021/3, proposed by Alex Zhai

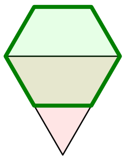
Available online at https://aops.com/community/p21499596.

Problem statement

An equilateral triangle Δ of side length $L > 0$ is given. Suppose that n equilateral triangles with side length 1 and with non-overlapping interiors are drawn inside Δ, such that each unit equilateral triangle has sides parallel to Δ, but with opposite orientation. Prove that

$$n \leq \frac{2}{3}L^2.$$

We present the approach of Andrew Gu. For each triangle, we draw a green regular hexagon of side length $1/2$ as shown below.



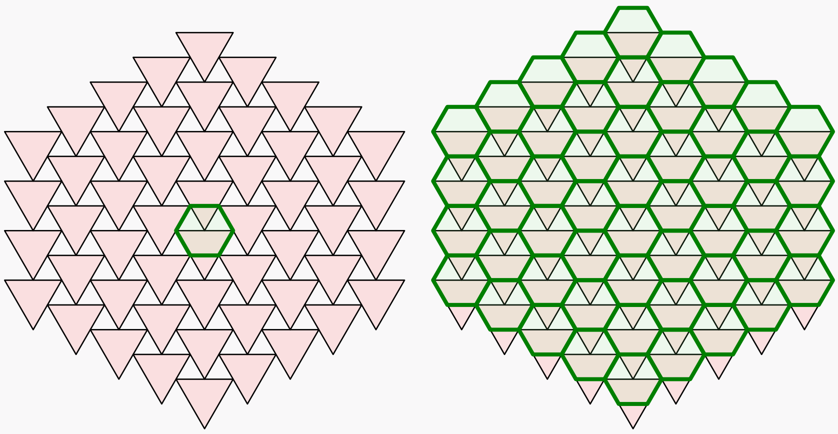
Claim — All the hexagons are disjoint and lie inside Δ.

Proof. Annoying casework. $\square$

Since each hexagon has area $\frac{3\sqrt{3}}{8}$ and lies inside Δ, we conclude

$$\frac{3\sqrt{3}}{8} \cdot n \leq \frac{\sqrt{3}}{4}L^2 \implies n \leq \frac{2}{3}L^2.$$

Remark. The constant $\frac{2}{3}$ is sharp and cannot be improved. The following tessellation shows how to achieve the $\frac{2}{3}$ density. In the figure on the left, one of the green hexagons is drawn in for illustration. The version on the right has all the hexagons.



§2 Solutions to Day 2

§2.1 JMO 2021/4, proposed by Brandon Wang

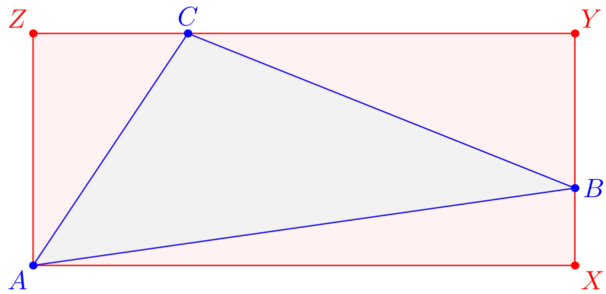
Available online at https://aops.com/community/p21498566.

Problem statement

Carina has three pins, labeled A, B, and C, respectively, located at the origin of the coordinate plane. In a *move*, Carina may move a pin to an adjacent lattice point at distance 1 away. What is the least number of moves that Carina can make in order for triangle ABC to have area 2021?

The answer is 128.

Define the **bounding box** of triangle ABC to be the smallest axis-parallel rectangle which contains all three of the vertices A, B, C.



Lemma

The area of a triangle ABC is at most half the area of the bounding box.

Proof. This can be proven by explicit calculation in coordinates. Nonetheless, we outline a geometric approach. By considering the smallest/largest x coordinate and the smallest/largest y coordinate, one can check that some vertex of the triangle must coincide with a corner of the bounding box (there are four "extreme" coordinates across the $3 \cdot 2 = 6$ coordinates of our three points).

So, suppose the bounding box is $AXYZ$. Imagine fixing C and varying B along the perimeter entire rectangle. The area is a linear function of B, so the maximal area should be achieved when B coincides with one of the vertices $\{A, X, Y, Z\}$. But obviously the area of $\triangle ABC$ is

- exactly 0 if $B = A$,
- at most half the bounding box if $B \in \{X, Z\}$ by one-half-base-height,
- at most half the bounding box if $B = Y$, since $\triangle ABC$ is contained inside either $\triangle AYZ$ or $\triangle AXZ$. □

We now proceed to the main part of the proof.

Claim — If n moves are made, the bounding box has area at most $(n/2)^2$. (In other words, a bounding box of area A requires at least $\left\lceil 2\sqrt{A} \right\rceil$ moves.)

Proof. The sum of the width and height of the bounding box increases by at most 1 each move, hence the width and height have sum at most n. So, by AM-GM, their product is at most $(n/2)^2$. □

This immediately implies $n \geq 128$, since the bounding box needs to have area at least $4042 > 63.5^2$.

On the other hand, if we start all the pins at the point $(3, 18)$ then we can reach the following three points in 128 moves:

$$\begin{aligned} A &= (0,0) \\ B &= (64,18) \\ C &= (3,64) \end{aligned}$$

and indeed triangle ABC has area exactly 2021.

§2.2 JMO 2021/5, proposed by Carl Schildkraut

Available online at https://aops.com/community/p21498580.

Problem statement

A finite set S of positive integers has the property that, for each $s \in S$, and each positive integer divisor d of s, there exists a unique element $t \in S$ satisfying $\gcd(s,t) = d$. (The elements s and t could be equal.)

Given this information, find all possible values for the number of elements of S.

The answer is that $|S|$ must be a power of 2 (including 1), or $|S| = 0$ (a trivial case we do not discuss further).

Construction: For any nonnegative integer k, a construction for $|S| = 2^k$ is given by

$$S = \{(p_1 \text{ or } q_1) \times (p_2 \text{ or } q_2) \times \cdots \times (p_k \text{ or } q_k)\}$$

for $2k$ distinct primes $p_1, \ldots, p_k, q_1, \ldots, q_k$.

Converse: the main claim is as follows.

Claim — In any valid set S, for any prime p and $x \in S$, $\nu_p(x) \le 1$.

Proof. Assume for contradiction $e = \nu_p(x) \ge 2$.

- On the one hand, by taking x in the statement, we see $\frac{e}{e+1}$ of the elements of S are divisible by p.
- On the other hand, consider a $y \in S$ such that $\nu_p(y) = 1$ which must exist (say if $\gcd(x,y) = p$). Taking y in the statement, we see $\frac{1}{2}$ of the elements of S are divisible by p.

So $e = 1$, contradiction. □

Now since $|S|$ equals the number of divisors of any element of S, we are done.

§2.3 JMO 2021/6, proposed by Mohsen Jamaali

Available online at https://aops.com/community/p21498967.

Problem statement

Let $n \geq 4$ be an integer. Find all positive real solutions to the following system of $2n$ equations:

$$\begin{aligned} a_1 &= \frac{1}{a_{2n}} + \frac{1}{a_2}, & a_2 &= a_1 + a_3, \\ a_3 &= \frac{1}{a_2} + \frac{1}{a_4}, & a_4 &= a_3 + a_5, \\ a_5 &= \frac{1}{a_4} + \frac{1}{a_6}, & a_6 &= a_5 + a_7, \\ &\vdots & &\vdots \\ a_{2n-1} &= \frac{1}{a_{2n-2}} + \frac{1}{a_{2n}}, & a_{2n} &= a_{2n-1} + a_1. \end{aligned}$$

The answer is that the only solution is $(1, 2, 1, 2, \dots, 1, 2)$ which works.

We will prove a_{2k} is a constant sequence, at which point the result is obvious.

¶ First approach (Andrew Gu). Apparently, with indices modulo $2n$, we should have

$$a_{2k} = \frac{1}{a_{2k-2}} + \frac{2}{a_{2k}} + \frac{1}{a_{2k+2}}$$

for every index k (this eliminates all a_{odd}'s). Define

$$m = \min_k a_{2k} \qquad \text{and} \qquad M = \max_k a_{2k}.$$

Look at the indices i and j achieving m and M to respectively get

$$\begin{aligned} m &= \frac{2}{m} + \frac{1}{a_{2i-2}} + \frac{1}{a_{2i+2}} \geq \frac{2}{m} + \frac{1}{M} + \frac{1}{M} = \frac{2}{m} + \frac{2}{M} \\ M &= \frac{2}{M} + \frac{1}{a_{2j-2}} + \frac{1}{a_{2j+2}} \leq \frac{2}{M} + \frac{1}{m} + \frac{1}{m} = \frac{2}{m} + \frac{2}{M}. \end{aligned}$$

Together this gives $m \geq M$, so $m = M$. That means a_{2i} is constant as i varies, solving the problem.

¶ Second approach (author's solution). As before, we have

$$a_{2k} = \frac{1}{a_{2k-2}} + \frac{2}{a_{2k}} + \frac{1}{a_{2k+2}}$$

The proof proceeds in three steps.

- Define
$$S = \sum_k a_{2k}, \quad \text{and} \quad T = \sum_k \frac{1}{a_{2k}}.$$
Summing gives $S = 4T$. On the other hand, Cauchy-Schwarz says $S \cdot T \geq n^2$, so $T \geq \frac{1}{2}n$.

- On the other hand,
$$1 = \frac{1}{a_{2k-2}a_{2k}} + \frac{2}{a_{2k}^2} + \frac{1}{a_{2k}a_{2k+2}}$$
Sum this modified statement to obtain
$$n = \sum_k \left(\frac{1}{a_{2k}} + \frac{1}{a_{2k+2}}\right)^2 \overset{\text{QM-AM}}{\geq} \frac{1}{n}\left(\sum_k \frac{1}{a_{2k}} + \frac{1}{a_{2k+2}}\right)^2 = \frac{1}{n}(2T)^2$$
So $T \leq \frac{1}{2}n$.
- Since $T \leq \frac{1}{2}n$ and $T \geq \frac{1}{2}n$, we must have equality everywhere above. This means a_{2k} is a constant sequence.

Remark. The problem is likely intractable over $\mathbb{C}$, in the sense that one gets a high-degree polynomial which almost certainly has many complex roots. So it seems likely that most solutions must involve some sort of inequality, using the fact we are over $\mathbb{R}_{>0}$ instead.