USA TSTST 2023 Solutions

United States of America — TST Selection Test

Andrew Gu, Evan Chen, Gopal Goel

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§0 Problems

1. Let ABC be a triangle with centroid G. Points R and S are chosen on rays GB and GC, respectively, such that

$$\angle ABS = \angle ACR = 180° - \angle BGC.$$

   Prove that $\angle RAS + \angle BAC = \angle BGC$.

2. Let $n \geq m \geq 1$ be integers. Prove that

$$\sum_{k=m}^{n} \left(\frac{1}{k^2} + \frac{1}{k^3} \right) \geq m \cdot \left(\sum_{k=m}^{n} \frac{1}{k^2} \right)^2.$$

3. Find all positive integers n for which it is possible to color some cells of an infinite grid of unit squares red, such that each rectangle consisting of exactly n cells (and whose edges lie along the lines of the grid) contains an odd number of red cells.

4. Let $n \geq 3$ be an integer and let K_n be the complete graph on n vertices. Each edge of K_n is colored either red, green, or blue. Let A denote the number of triangles in K_n with all edges of the same color, and let B denote the number of triangles in K_n with all edges of different colors. Prove that

$$B \leq 2A + \frac{n(n-1)}{3}.$$

5. Suppose a, b, and c are three complex numbers with product 1. Assume that none of a, b, and c are real or have absolute value 1. Define

$$p = (a + b + c) + \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) \qquad \text{and} \qquad q = \frac{a}{b} + \frac{b}{c} + \frac{c}{a}.$$

   Given that both p and q are real numbers, find all possible values of the ordered pair (p, q).

6. Let ABC be a scalene triangle and let P and Q be two distinct points in its interior. Suppose that the angle bisectors of $\angle PAQ$, $\angle PBQ$, and $\angle PCQ$ are the altitudes of triangle ABC. Prove that the midpoint of $\overline{PQ}$ lies on the Euler line of ABC.

7. The Bank of Pittsburgh issues coins that have a heads side and a tails side. Vera has a row of 2023 such coins alternately tails-up and heads-up, with the leftmost coin tails-up.

   In a *move*, Vera may flip over one of the coins in the row, subject to the following rules:

   - On the first move, Vera may flip over any of the 2023 coins.
   - On all subsequent moves, Vera may only flip over a coin adjacent to the coin she flipped on the previous move. (We do not consider a coin to be adjacent to itself.)

   Determine the smallest possible number of moves Vera can make to reach a state in which every coin is heads-up.

8. Let ABC be an equilateral triangle with side length 1. Points A_1 and A_2 are chosen on side BC, points B_1 and B_2 are chosen on side CA, and points C_1 and C_2 are chosen on side AB such that $BA_1 < BA_2$, $CB_1 < CB_2$, and $AC_1 < AC_2$.

   Suppose that the three line segments B_1C_2, C_1A_2, and A_1B_2 are concurrent, and the perimeters of triangles AB_2C_1, BC_2A_1, and CA_2B_1 are all equal. Find all possible values of this common perimeter.

9. Let p be a fixed prime and let $a \geq 2$ and $e \geq 1$ be fixed integers. Given a function $f : \mathbb{Z}/a\mathbb{Z} \to \mathbb{Z}/p^e\mathbb{Z}$ and an integer $k \geq 0$, the *kth finite difference*, denoted $\Delta^k f$, is the function from $\mathbb{Z}/a\mathbb{Z}$ to $\mathbb{Z}/p^e\mathbb{Z}$ defined recursively by

   $$\Delta^0 f(n) = f(n)$$
   $$\Delta^k f(n) = \Delta^{k-1} f(n+1) - \Delta^{k-1} f(n) \qquad \text{for } k = 1, 2, \dots.$$

   Determine the number of functions f such that there exists some $k \geq 1$ for which $\Delta^k f = f$.

§1 Solutions to Day 1

§1.1 TSTST 2023/1, proposed by Merlijn Staps
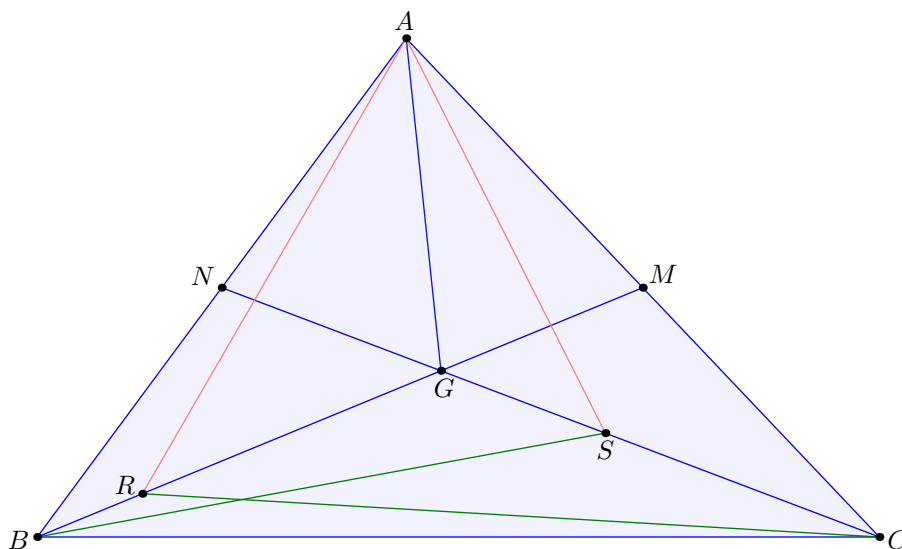
Available online at <https://aops.com/community/p28015679>.

> **Problem statement**
>
> Let ABC be a triangle with centroid G. Points R and S are chosen on rays GB and GC, respectively, such that
>
> $$\angle ABS = \angle ACR = 180° - \angle BGC.$$
>
> Prove that $\angle RAS + \angle BAC = \angle BGC$.

In all the following solutions, let M and N denote the midpoints of $\overline{AC}$ and $\overline{AB}$, respectively.

¶ Solution 1 using power of a point.   From the given condition that $\angle ACR = \angle CGM$, we get that
$$MA^2 = MC^2 = MG \cdot MR \implies \angle RAC = \angle MGA.$$
Analogously,
$$\angle BAS = \angle AGN.$$
Hence,
$$\angle RAS + \angle BAC = \angle RAC + \angle BAS = \angle MGA + \angle AGN = \angle MGN = \angle BGC.$$

¶ Solution 2 using similar triangles.   As before, $\triangle MGC \sim \triangle MCR$ and $\triangle NGB \sim \triangle NBS$. We obtain
$$\frac{|AC|}{|CR|} = \frac{2|MC|}{|CR|} = \frac{2|MG|}{|GC|} = \frac{|GB|}{2|NG|} = \frac{|BS|}{2|BN|} = \frac{|BS|}{|AB|}$$
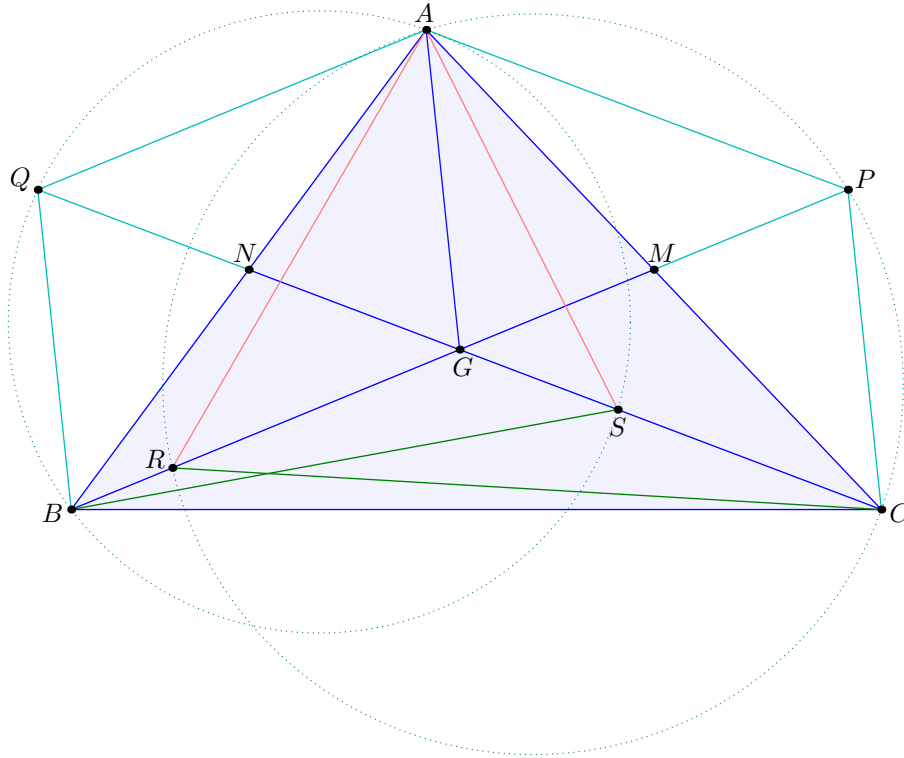which together with $\angle ACR = \angle ABS$ yields
$$\triangle ACR \sim \triangle SBA \implies \angle BAS = \angle CRA.$$

Hence

$$\angle RAS + \angle BAC = \angle RAC + \angle BAS = \angle RAC + \angle CRA = -\angle ACR = \angle BGC,$$

which proves the statement.

¶ Solution 3 using parallelograms. Let M and N be defined as above. Let P be the reflection of G in M and let Q the reflection of G in N. Then $AGCP$ and $AGBQ$ are parallelograms.

> **Claim —** Quadrilaterals $APCR$ and $AQBS$ are concyclic.

Proof. Because $\angle APR = \angle APG = \angle CGP = -\angle BGC = \angle ACR$. $\qquad \square$

Thus from $\overline{PC} \parallel \overline{GA}$ we get

$$\angle RAC = \angle RPC = \angle GPC = \angle PGA$$

and similarly

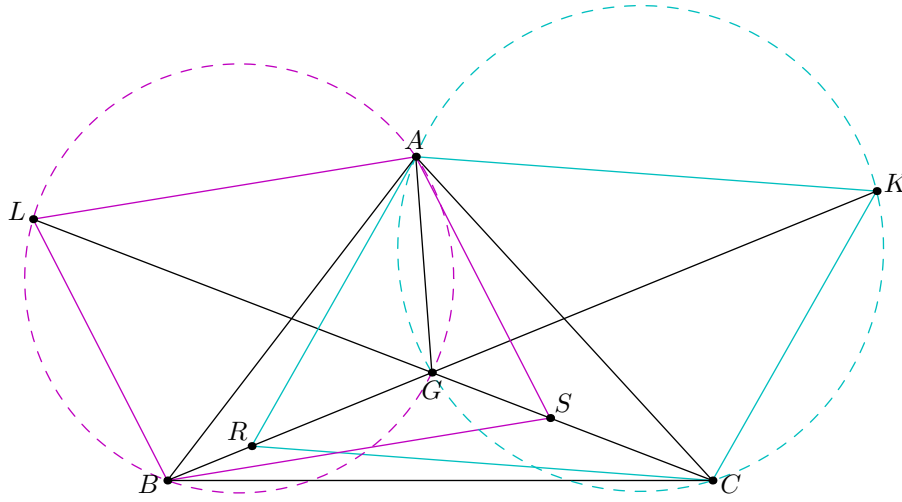$$\angle BAS = \angle BQS = \angle BQG = \angle AGQ.$$

We conclude that

$$\angle RAS + \angle BAC = \angle RAC + \angle BAS = \angle PGA + \angle AGQ = \angle PGQ = \angle BGC.$$

¶ Solution 4 also using parallelograms, by Ankan Bhattacharya. Construct parallelograms $ARCK$ and $ASBL$. Since

$$\angle CAK = \angle ACR = \angle CGB = \angle CGK,$$

it follows that $AGCK$ is cyclic. Similarly, $AGBL$ is also cyclic.

Finally, observe that

$$
\begin{aligned}
\angle RAS + \angle BAC &= \angle BAS + \angle RAC \\
&= \measuredangle ABL + \measuredangle KCA \\
&= \measuredangle AGL + \measuredangle KGA \\
&= \measuredangle KGL \\
&= \angle BGC
\end{aligned}
$$

as requested.

¶ Solution 5 using complex numbers, by Milan Haiman. Note that $\angle RAS + \angle BAC = \angle BAS + \angle RAC$. We compute $\angle BAS$ in complex numbers; then $\angle RAC$ will then be known by symmetry.

Let a, b, c be points on the unit circle representing A, B, C respectively. Let $g = \frac{1}{3}(a + b + c)$ represent the centroid G, and let s represent S.

> **Claim —** We have
> $$
> \frac{s - a}{b - a} = \frac{ab - 2bc + ca}{2ab - bc - ca}.
> $$

Proof. Since S is on line CG, which passes through the midpoint of segment AB, we have that
$$
s = \frac{a + b}{2} + t(c - g)
$$
for some $t \in \mathbb{R}$.

By the given angle condition, we have that
$$
\frac{(s - b)/(b - a)}{(c - g)/(g - b)} \in \mathbb{R}.
$$

Note that
$$
\frac{s - b}{b - a} = t\frac{c - g}{b - a} - \frac{1}{2}.
$$

So,
$$
t\frac{g - b}{b - a} - \frac{g - b}{2(c - g)} \in \mathbb{R}.
$$

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Thus

$$t = \frac{\mathrm{Im}\left(\frac{g-b}{2(c-g)}\right)}{\mathrm{Im}\left(\frac{g-b}{b-a}\right)} = \frac{1}{2} \cdot \frac{\left(\frac{g-b}{c-g}\right) - \overline{\left(\frac{g-b}{c-g}\right)}}{\left(\frac{g-b}{b-a}\right) - \overline{\left(\frac{g-b}{b-a}\right)}}.$$

Let N and D be the numerator and denominator of the second factor above.

We want to compute

$$\frac{s-a}{b-a} = \frac{1}{2} + t\frac{c-g}{b-a} = \frac{(b-a) + 2t(c-g)}{2(b-a)} = \frac{(b-a)D + (c-g)N}{2(b-a)D}.$$

We have

$$(c-g)N = g - b - (c-g)\overline{\left(\frac{g-b}{c-g}\right)}$$
$$= \frac{a+b+c}{3} - b - \left(c - \frac{a+b+c}{3}\right)\frac{\frac{1}{a} + \frac{1}{b} + \frac{1}{c} - \frac{3}{b}}{\frac{3}{c} - \frac{1}{a} - \frac{1}{b} - \frac{1}{c}}$$
$$= \frac{(a+c-2b)(2ab - bc - ca) - (2c - a - b)(ab + bc - 2ca)}{3(2ab - bc - ca)}$$
$$= \frac{3(a^2b + b^2c + c^2a - ab^2 - bc^2 - ca^2)}{3(2ab - bc - ca)}$$
$$= \frac{(a-b)(b-c)(a-c)}{2ab - bc - ca}$$

We also compute

$$(b-a)D = g - b - (b-a)\overline{\left(\frac{g-b}{b-a}\right)}$$
$$= \frac{a+b+c}{3} - b - (b-a)\frac{\frac{1}{a} + \frac{1}{b} + \frac{1}{c} - \frac{3}{b}}{\frac{3}{b} - \frac{3}{a}}$$
$$= \frac{(a+c-2b)c + (ab + bc - 2ca)}{3c}$$
$$= \frac{ab - bc - ca + c^2}{3c}$$
$$= \frac{(a-c)(b-c)}{3c}$$

So, we obtain

$$\frac{s-a}{b-a} = \frac{\frac{1}{3c} + \frac{a-b}{2ab - bc - ca}}{\frac{2}{3c}} = \frac{2ab - bc - ca + 3c(a-b)}{2(2ab - bc - ca)} = \frac{ab - 2bc + ca}{2ab - bc - ca}.$$

$\square$

By symmetry,
$$\frac{r-a}{c-a} = \frac{ab - 2bc + ca}{2ca - ab - bc}.$$

Hence their ratio
$$\frac{s-a}{b-a} \div \frac{r-a}{c-a} = \frac{2ab - bc - ca}{2ca - ab - bc}$$

has argument $\angle RAC + \angle BAS$.

We also have that $\angle BGC$ is the argument of

$$\frac{b-g}{c-g} = \frac{2b-a-c}{2c-a-b}.$$

Note that these two complex numbers are inverse-conjugates, and thus have the same argument. So we're done.

§1.2 TSTST 2023/2, proposed by Raymond Feng, Luke Robitaille

Available online at <https://aops.com/community/p28015692>*.*

> **Problem statement**
>
> Let $n \geq m \geq 1$ be integers. Prove that
> $$\sum_{k=m}^{n} \left(\frac{1}{k^2} + \frac{1}{k^3} \right) \geq m \cdot \left(\sum_{k=m}^{n} \frac{1}{k^2} \right)^2.$$

We show several approaches.

¶ **First solution (authors).** By Cauchy-Schwarz, we have

$$\sum_{k=m}^{n} \frac{k+1}{k^3} = \sum_{k=m}^{n} \frac{\left(\frac{1}{k^2} \right)^2}{\frac{1}{k(k+1)}}$$

$$\geq \frac{\left(\frac{1}{m^2} + \frac{1}{(m+1)^2} + \cdots + \frac{1}{n^2} \right)^2}{\frac{1}{m(m+1)} + \frac{1}{(m+1)(m+2)} + \cdots + \frac{1}{n(n+1)}}$$

$$= \frac{\left(\frac{1}{m^2} + \frac{1}{(m+1)^2} + \cdots + \frac{1}{n^2} \right)^2}{\frac{1}{m} - \frac{1}{n+1}}$$

$$> \frac{\left(\sum_{k=m}^{n} \frac{1}{k^2} \right)^2}{\frac{1}{m}}$$

as desired.

> **Remark** (Bound on error)**.** Let $A = \sum_{k=m}^{n} k^{-2}$ and $B = \sum_{k=m}^{n} k^{-3}$. The inequality above becomes tighter for large m and $n \gg m$. If we use Lagrange's identity in place of Cauchy-Schwarz, we get
>
> $$A + B - mA^2 = m \cdot \sum_{m \leq a < b} \frac{(a-b)^2}{a^3 b^3 (a+1)(b+1)}.$$
>
> We can upper bound this error by
>
> $$\leq m \cdot \sum_{m \leq a < b} \frac{1}{a^3(a+1)b(b+1)} = m \cdot \sum_{m \leq a} \frac{1}{a^3(a+1)^2} \approx m \cdot \frac{1}{m^4} = \frac{1}{m^3},$$
>
> which is still generous as $(a-b)^2 \ll b^2$ for b not much larger than a, so the real error is probably around $\frac{1}{10m^3}$. This exhibits the tightness of the inequality since it implies
>
> $$mA^2 + O(B/m) > A + B.$$

> **Remark** (Construction commentary, from author)**.** My motivation was to write an inequality where Titu could be applied creatively to yield a telescoping sum. This can be difficult because most of the time, such a reverse-engineered inequality will be so loose it's trivial

anyways. My first attempt was the not-so-amazing inequality

$$\frac{n^2+3n}{2} = \sum_1^n i+1 = \sum_1^n \frac{\frac{1}{i}}{\frac{1}{i(i+1)}} > \left(\sum_1^n \frac{1}{\sqrt{i}} \right)^2,$$

which is really not surprising given that $\sum \frac{1}{\sqrt{i}} \ll \frac{n}{\sqrt{2}}$. The key here is that we need "near-equality" as dictated by the Cauchy-Schwarz equality case, i.e. the square root of the numerators should be approximately proportional to the denominators.

This motivates using $\frac{1}{i^4}$ as the numerator, which works like a charm. After working out the resulting statement, the LHS and RHS even share a sum, which adds to the simplicity of the problem.

The final touch was to unrestrict the starting value of the sum, since this allows the strength of the estimate $\frac{1}{i^2} \approx \frac{1}{i(i+1)}$ to be fully exploited.

¶ **Second approach by inducting down, Luke Robitaille and Carl Schildkraut.** Fix n; we'll induct downwards on m. For the base case of $n=m$ the result is easy, since the left side is $\frac{m+1}{m^3}$ and the right side is $\frac{m}{m^4} = \frac{1}{m^3}$.

For the inductive step, suppose we have shown the result for $m+1$. Let

$$A = \sum_{k=m+1}^n \frac{1}{k^2} \qquad \text{and} \qquad B = \sum_{k=m+1}^n \frac{1}{k^3}.$$

We know $A+B \geq (m+1)A^2$, and we want to show

$$\left(A + \frac{1}{m^2} \right) + \left(B + \frac{1}{m^3} \right) \geq m \left(A + \frac{1}{m^2} \right)^2.$$

Indeed,

$$\left(A + \frac{1}{m^2} \right) + \left(B + \frac{1}{m^3} \right) - m \left(A + \frac{1}{m^2} \right)^2 = A + B + \frac{m+1}{m^3} - mA^2 - \frac{2A}{m} - \frac{1}{m^3}$$

$$= (A + B - (m+1)A^2) + \left(A - \frac{1}{m} \right)^2 \geq 0,$$

and we are done.

¶ **Third approach by reducing $n \to \infty$, Michael Ren and Carl Schildkraut.** First, we give:

> **Claim** (Reduction to $n \to \infty$) — If the problem is true when $n \to \infty$, it is true for all n.

Proof. Let $A = \sum_{k=m}^n k^{-2}$ and $B = \sum_{k=m}^n k^{-3}$. Consider the region of the xy-plane defined by $y > mx^2 - x$. We are interested in whether (A, B) lies in this region.

However, the region is bounded by a convex curve, and the sequence of points $(0,0)$, $\left(\frac{1}{m^2}, \frac{1}{m^3} \right)$, $\left(\frac{1}{m^2} + \frac{1}{(m+1)^2}, \frac{1}{m^3} + \frac{1}{(m+1)^3} \right)$, ... has successively decreasing slopes between consecutive points. Thus it suffices to check that the inequality is true when $n \to \infty$. $\square$

Set $n = \infty$ henceforth. Let

$$A = \sum_{k=m}^\infty \frac{1}{k^2} \text{ and } B = \sum_{k=m}^\infty \frac{1}{k^3};$$

we want to show $B \geq mA^2 - A$, which rearranges to

$$1 + 4mB \geq (2mA - 1)^2.$$

Write

$$C = \sum_{k=m}^{\infty} \frac{1}{k^2(2k-1)(2k+1)} \text{ and } D = \sum_{k=m}^{\infty} \frac{8k^2 - 1}{k^3(2k-1)^2(2k+1)^2}.$$

Then

$$\frac{2}{2k-1} - \frac{2}{2k+1} = \frac{1}{k^2} + \frac{1}{k^2(2k-1)(2k+1)},$$

and

$$\frac{2}{(2k-1)^2} - \frac{2}{(2k+1)^2} = \frac{1}{k^3} + \frac{8k^2 - 1}{k^3(2k-1)^2(2k+1)^2},$$

so that

$$A = \frac{2}{2m-1} - C \text{ and } B = \frac{2}{(2m-1)^2} - D.$$

Our inequality we wish to show becomes

$$\frac{2m+1}{2m-1}C \geq D + mC^2.$$

We in fact show two claims:

> **Claim —** We have
> $$\frac{2m + 1/2}{2m - 1}C \geq D.$$

Proof. We compare termwise; we need

$$\frac{2m + 1/2}{2m - 1} \cdot \frac{1}{k^2(2k-1)(2k+1)} \geq \frac{8k^2 - 1}{k^3(2k-1)^2(2k+1)^2}$$

for $k \geq m$. It suffices to show

$$\frac{2k + 1/2}{2k - 1} \cdot \frac{1}{k^2(2k-1)(2k+1)} \geq \frac{8k^2 - 1}{k^3(2k-1)^2(2k+1)^2},$$

which is equivalent to $k(2k + 1/2)(2k+1) \geq 8k^2 - 1$. This holds for all $k \geq 1$. $\square$

> **Claim —** We have
> $$\frac{1/2}{2m - 1}C \geq mC^2.$$

Proof. We need $C \leq 1/(2m(2m-1))$; indeed,

$$\frac{1}{2m(2m-1)} = \sum_{k=m}^{\infty} \left(\frac{1}{2k(2k-1)} - \frac{1}{2(k+1)(2k+1)} \right) = \sum_{k=m}^{\infty} \frac{4k+1}{2k(2k-1)(k+1)(2k+1)};$$

comparing term-wise with the definition of C and using the inequality $k(4k+1) \geq 2(k+1)$ for $k \geq 1$ gives the desired result. $\square$

Combining the two claims finishes the solution.

¶ Fourth approach by bashing, Carl Schildkraut. With a bit more work, the third approach can be adapted to avoid the $n \to \infty$ reduction. Similarly to before, define

$$A = \sum_{k=m}^{n} \frac{1}{k^2} \text{ and } B = \sum_{k=m}^{n} \frac{1}{k^3};$$

we want to show $1 + 4mB \geq (2mA - 1)^2$. Writing

$$C = \sum_{k=m}^{n} \frac{1}{k^2(2k-1)(2k+1)} \text{ and } D = \sum_{k=m}^{n} \frac{8k^2 - 1}{k^3(2k-1)^2(2k+1)^2}.$$

We compute

$$A = \frac{2}{2m-1} - \frac{2}{2n+1} - C \text{ and } B = \frac{2}{(2m-1)^2} - \frac{2}{(2n+1)^2} - D.$$

Then, the inequality we wish to show reduces (as in the previous solution) to

$$\frac{2m+1}{2m-1}C + \frac{2(2m+1)}{(2m-1)(2n+1)} \geq D + mC^2 + \frac{2(2m+1)}{(2n+1)^2} + \frac{4m}{2n+1}C.$$

We deal first with the terms not containing the variable n, i.e. we show that

$$\frac{2m+1}{2m-1}C \geq D + mC^2.$$

For this part, the two claims from the previous solution go through exactly as written above, and we have $C \leq 1/(2m(2m-1))$. We now need to show

$$\frac{2(2m+1)}{(2m-1)(2n+1)} \geq \frac{2(2m+1)}{(2n+1)^2} + \frac{4m}{2n+1}C$$

(this is just the inequality between the remaining terms); our bound on C reduces this to proving

$$\frac{4(2m+1)(n-m+1)}{(2m-1)(2n+1)^2} \geq \frac{2}{(2m-1)(2n+1)}.$$

Expanding and writing in terms of n, this is equivalent to

$$n \geq \frac{1 + 2(m-1)(2m+1)}{4m} = m - \frac{2m+1}{4m},$$

which holds for all $n \geq m$.

§1.3 TSTST 2023/3, proposed by Merlijn Staps

Available online at https://aops.com/community/p28015682.

Problem statement

Find all positive integers n for which it is possible to color some cells of an infinite grid of unit squares red, such that each rectangle consisting of exactly n cells (and whose edges lie along the lines of the grid) contains an odd number of red cells.

We claim that this is possible for all positive integers n. Call a positive integer for which such a coloring is possible *good*. To show that all positive integers n are good we prove the following:

(i) If n is good and p is an odd prime, then pn is good;

(ii) For every $k \geq 0$, the number $n = 2^k$ is good.

Together, (i) and (ii) imply that all positive integers are good.

¶ Proof of (i). We simply observe that if every rectangle consisting of n cells contains an odd number of red cells, then so must every rectangle consisting of pn cells. Indeed, because p is prime, a rectangle consisting of pn cells must have a dimension (length or width) divisible by p and can thus be subdivided into p rectangles consisting of n cells.

Thus every coloring that works for n automatically also works for pn.

¶ Proof of (ii). Observe that rectangles with $n = 2^k$ cells have $k + 1$ possible shapes: $2^m \times 2^{k-m}$ for $0 \leq m \leq k$.

Claim — For each of these $k + 1$ shapes, there exists a coloring with two properties:

- Every rectangle with n cells and shape $2^m \times 2^{k-m}$ contains an odd number of red cells.

- Every rectangle with n cells and a different shape contains an even number of red cells.

*Proof.* This can be achieved as follows: assuming the cells are labeled with $(x, y) \in \mathbb{Z}^2$, color a cell red if $x \equiv 0 \pmod{2^m}$ and $y \equiv 0 \pmod{2^{k-m}}$. For example, a 4×2 rectangle gets the following coloring:
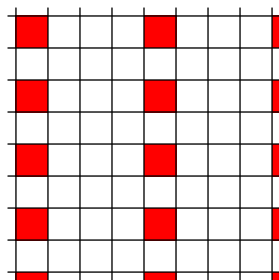
A $2^m \times 2^{k-m}$ rectangle contains every possible pair $(x \bmod 2^m, y \bmod 2^{k-m})$ exactly once, so such a rectangle will contain one red cell (an odd number).

On the other hand, consider a $2^\ell \times 2^{k-\ell}$ rectangle with $\ell > m$. The set of cells this covers is (x, y) where x covers a range of size 2^ℓ and y covers a range of size $2^{k-\ell}$. The number of red cells is the count of x with $x \equiv 0 \mod 2^m$ multiplied by the count of y with $y \equiv 0 \mod 2^{k-m}$. The former number is exactly $2^{\ell-k}$ because 2^k divides 2^ℓ (while the latter is 0 or 1) so the number of red cells is even. The $\ell < m$ case is similar. $\qquad\square$

Finally, given these $k + 1$ colorings, we can add them up modulo 2, i.e. a cell will be colored red if it is red in an odd number of these $k + 1$ colorings. We illustrate $n = 4$ as an example; the coloring is 4-periodic in both axes so we only show one 4×4 cell.
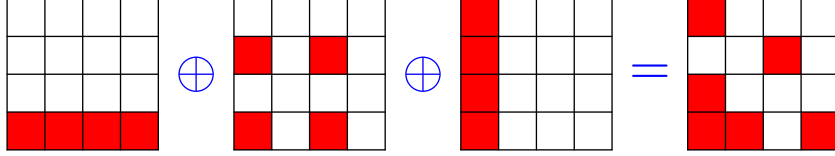
This solves the problem.

> **Remark.** The final coloring can be described as follows: color (x, y) red if
> $$\max(0, \min(\nu_2(x), k) + \min(\nu_2(y), k) - k + 1)$$
> is odd.

> **Remark** (Luke Robitaille). Alternatively for (i), if $n = 2^e k$ for odd k then one may dissect an $a \times b$ rectangle with area n into k rectangles of area 2^e, each $2^{\nu_2(a)} \times 2^{\nu_2(b)}$. This gives a way to deduce the problem from (ii) without having to consider odd prime numbers.

¶ **Alternate proof of (ii) using generating functions.** We will commit to constructing a coloring which is n-periodic in both directions. (This is actually forced, so it's natural to do so.) With that in mind, let

$$f(x, y) = \sum_{i=0}^{2^k-1} \sum_{j=0}^{2^k-1} \lambda_{i,j} x^i y^j$$

denote its generating function, where $f \in \mathbb{F}_2[x, y]$.

For this to be valid, we need that for any $2^p \times 2^q$ rectangle with area n, the sum of the coefficients of f over it should be one, modulo $x^{2^k} = y^{2^k} = 1$. In other words, whenever $p + q = k$, we must have

$$f(x, y)(1 + \cdots + x^{2^p-1})(1 + \cdots + y^{2^q-1}) = (1 + \cdots + x^{2^k-1})(1 + \cdots + y^{2^k-1}),$$

taken modulo $x^{2^k} = y^{2^k} = 1$. The idea is to rewrite these expressions: because we're in characteristic 2, the given assertion is $(x + 1)^{2^k} = (y + 1)^{2^k} = 0$, and the requested property is

$$f(x, y)(x + 1)^{2^p-1}(y + 1)^{2^q-1} = (x + 1)^{2^k-1}(y + 1)^{2^k-1}.$$

This suggests the substitution $g(x, y) = f(x+1, y+1)$: then we can replace $(x+1, y+1) \mapsto (x, y)$ to simplify the requested property significantly:

> Whenever $p + q = k$, we must have
> $$g(x, y)x^{2^p-1}y^{2^q-1} = x^{2^k-1}y^{2^k-1},$$
> modulo x^{2^k} and y^{2^k}.

However, now the construction of g is very simple: for example, the choice

$$g(x, y) = \sum_{p+q=k} x^{2^k - 2^p} y^{2^k - 2^q}$$

works. The end.

> **Remark.** Unraveling the substitutions seen here, it's possible to show that this is actually the same construction provided in the first solution.

§2 Solutions to Day 2

§2.1 TSTST 2023/4, proposed by Ankan Bhattacharya

Available online at https://aops.com/community/p28015691.

> **Problem statement**
>
> Let $n \geq 3$ be an integer and let K_n be the complete graph on n vertices. Each edge of K_n is colored either red, green, or blue. Let A denote the number of triangles in K_n with all edges of the same color, and let B denote the number of triangles in K_n with all edges of different colors. Prove that
> $$B \leq 2A + \frac{n(n-1)}{3}.$$

Consider all unordered pairs of different edges which share exactly one vertex (call these *vees* for convenience). Assign each vee a *charge* of $+2$ if its edge colors are the same, and a charge of -1 otherwise.

We compute the total charge in two ways.

¶ **Total charge by summing over triangles.**   Note that

- each monochromatic triangle has a charge of $+6$,

- each bichromatic triangle has a charge of 0, and

- each trichromatic triangle has a charge of -3.

Since each vee contributes to exactly one triangle, we obtain that the total charge is $6A - 3B$.

¶ **Total charge by summing over vertices.**   We can also calculate the total charge by examining the centers of the vees. If a vertex has a red edges, b green edges, and c blue edges, the vees centered at that vertex contribute a total charge of

$$2\left[\binom{a}{2} + \binom{b}{2} + \binom{c}{2}\right] - (ab + ac + bc)$$
$$= (a^2 - a + b^2 - b + c^2 - c) - (ab + ac + bc)$$
$$= (a^2 + b^2 + c^2 - ab - ac - bc) - (a + b + c)$$
$$= (a^2 + b^2 + c^2 - ab - ac - bc) - (n - 1)$$
$$\geq -(n - 1).$$

In particular, the total charge is at least $-n(n-1)$.

¶ **Conclusion.**   Thus, we obtain

$$6A - 3B \geq -n(n-1) \iff B \leq 2A + \frac{n(n-1)}{3}$$

as desired.

§2.2 TSTST 2023/5, proposed by David Altizio

Available online at <https://aops.com/community/p28015713>.

> **Problem statement**
>
> Suppose a, b, and c are three complex numbers with product 1. Assume that none of a, b, and c are real or have absolute value 1. Define
>
> $$p = (a + b + c) + \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) \qquad \text{and} \qquad q = \frac{a}{b} + \frac{b}{c} + \frac{c}{a}.$$
>
> Given that both p and q are real numbers, find all possible values of the ordered pair (p, q).

We show $(p, q) = (-3, 3)$ is the only possible ordered pair.

¶ First solution.

Setup for proof   Let us denote $a = y/x$, $b = z/y$, $c = x/z$, where x, y, z are nonzero complex numbers. Then

$$p + 3 = 3 + \sum_{\text{cyc}} \left(\frac{x}{y} + \frac{y}{x}\right) = 3 + \frac{x^2(y + z) + y^2(z + x) + z^2(x + y)}{xyz}$$

$$= \frac{(x + y + z)(xy + yz + zx)}{xyz}.$$

$$q - 3 = -3 + \sum_{\text{cyc}} \frac{y^2}{zx} = \frac{x^3 + y^3 + z^3 - 3xyz}{xyz}$$

$$= \frac{(x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx)}{xyz}.$$

It follows that

$$\mathbb{R} \ni 3(p + 3) + (q - 3)$$

$$= \frac{(x + y + z)(x^2 + y^2 + z^2 + 2(xy + yz + zx))}{xyz}$$

$$= \frac{(x + y + z)^3}{xyz}.$$

Now, note that if $x + y + z = 0$, then $p = -3$, $q = 3$ so we are done.

Main proof   We will prove that if $x + y + z \neq 0$ then we contradict either the hypothesis that $a, b, c \notin \mathbb{R}$ or that a, b, c do not have absolute value 1.

Scale x, y, z in such a way that $x + y + z$ is nonzero and real; hence so is xyz. Thus, as $p + 3 \in \mathbb{R}$, we conclude $xy + yz + zx \in \mathbb{R}$ as well. Hence, x, y, z are the roots of a cubic with real coefficients. Thus,

- either all three of $\{x, y, z\}$ are real (which implies $a, b, c \in \mathbb{R}$),

- or two of $\{x, y, z\}$ are a complex conjugate pair (which implies one of a, b, c has absolute value 1).

Both of these were forbidden by hypothesis.

Construction As we saw in the setup, $(p, q) = (-3, 3)$ will occur as long as $x+y+z = 0$, and no two of x, y, z to share the same magnitude or are collinear with the origin. This is easy to do; for example, we could choose $(x, y, z) = (3, 4i, -(3 + 4i))$. Hence $a = \frac{3}{4i}$, $b = -\frac{4i}{3+4i}$, $c = -\frac{3+4i}{3}$ satisfies the hypotheses of the problem statement.

¶ **Second solution, found by contestants.** The main idea is to make the substitution

$$x = a + \frac{1}{c}, \qquad y = b + \frac{1}{a}, \qquad z = c + \frac{1}{b}.$$

Then we can check that

$$x + y + z = p$$
$$xy + yz + zx = p + q + 3$$
$$xyz = p + 2.$$

Therefore x, y, z are the roots of a cubic with real coefficients. As in the previous solution, we note that this cubic must either have all real roots, or a complex conjugate pair of roots. We also have the relation $a(y + 1) = ab + a + 1 = x + 1$, and likewise $b(z + 1) = y + 1$, $c(x + 1) = z + 1$. This means that if any of x, y, z are equal to -1, then all are equal to -1.

Assume for the sake of contradiction that none are equal to -1. In the case where the cubic has three real roots, $a = \frac{x+1}{y+1}$ would be real. On the other hand, if there is a complex conjugate pair (without loss of generality, x and y) then a has magnitude 1. Therefore this cannot occur.

We conclude that $x = y = z = -1$, so $p = -3$ and $q = 3$. The solutions (a, b, c) can then be parameterized as $(a, -1 - \frac{1}{a}, -\frac{1}{1+a})$. To construct a solution, we need to choose a specific value of a such that none of the wrong conditions hold; when $a = 2i$, say, we obtain the solution $(2i, -1 + \frac{i}{2}, \frac{-1+2i}{5})$.

¶ **Third solution by Luke Robitaille and Daniel Zhu.** The answer is $p = -3$ and $q = 3$. Let's first prove that no other (p, q) work.

Let $e_1 = a + b + c$ and $e_2 = a^{-1} + b^{-1} + c^{-1} = ab + ac + bc$. Also, let $f = e_1 e_2$. Note that $p = e_1 + e_2$.

Our main insight is to consider the quantity $q' = \frac{b}{a} + \frac{c}{b} + \frac{a}{c}$. Note that $f = q + q' + 3$. Also,

$$\begin{aligned} qq' &= 3 + \frac{a^2}{bc} + \frac{b^2}{ac} + \frac{c^2}{ab} + \frac{bc}{a^2} + \frac{ac}{b^2} + \frac{ab}{c^2} \\ &= 3 + a^3 + b^3 + c^3 + a^{-3} + b^{-3} + c^{-3} \\ &= 9 + a^3 + b^3 + c^3 - 3abc + a^{-3} + b^{-3} + c^{-3} - 3a^{-1}b^{-1}c^{-1} \\ &= 9 + e_1(e_1^2 - 3e_2) + e_2(e_2^2 - 3e_1) \\ &= 9 + e_1^3 + e_2^3 - 6e_1 e_2 \\ &= 9 + p(p^2 - 3f) - 6f \\ &= p^3 - (3p + 6)f + 9. \end{aligned}$$

As a result, the quadratic with roots q and q' is $x^2 - (f - 3)x + (p^3 - (3p + 6)f + 9)$, which implies that

$$q^2 - qf + 3q + p^3 - (3p + 6)f + 9 = 0 \iff (3p + q + 6)f = p^3 + q^2 + 3q + 9.$$

At this point, two miracles occur. The first is the following claim:

> **Claim** — f is not real.

Proof. Suppose f is real. Since $(x - e_1)(x - e_2) = x^2 - px + f$, there are two cases:

- e_1 and e_2 are real. Then, a, b, and c are the roots of $x^3 - e_1 x^2 + e_2 x - 1$, and since every cubic with real coefficients has at least one real root, at least one of a, b, and c is real, contradiction.

- e_1 and e_2 are conjugates. Then, the polynomial $x^3 - \bar{e}_2 x^2 + \bar{e}_1 x - 1$, which has roots $\bar{a}^{-1}$, $\bar{b}^{-1}$, and $\bar{c}^{-1}$, is the same as the polynomial with a, b, c as roots. We conclude that the multiset $\{a, b, c\}$ is invariant under inversion about the unit circle, so one of a, b, and c must lie on the unit circle. This is yet another contradiction. $\square$

As a result, we know that $3p + q + 6 = p^3 + q^2 + 3q + 9 = 0$. The second miracle is that substituting $q = -3p - 6$ into $q^2 + 3q + p^3 + 9 = 0$, we get

$$0 = p^3 + 9p^2 + 27p + 27 = (p + 3)^3,$$

so $p = -3$. Thus $q = 3$.

It remains to construct valid a, b, and c. To do this, let's pick some e_1, let $e_2 = -3 - e_1$, and let a, b, and c be the roots of $x^3 - e_1 x^2 + e_2 x - 1$. It is clear that this guarantees $p = -3$. By our above calculations, q and q' are the roots of the quadratic $x^2 - (f - 3)x + (3f - 18)$, so one of q and q' must be 3; by changing the order of a, b, and c if needed, we can guarantee this to be q. It suffices to show that for some choice of e_1, none of a, b, or c are real or lie on the unit circle.

To do this, note that we can rewrite $x^3 - e_1 x^2 + (-3 - e_1)x - 1 = 0$ as

$$e_1 = \frac{x^3 - 3x - 1}{x^2 + x},$$

so all we need is a value of e_1 that is not $\frac{x^3 - 3x - 1}{x^2 + x}$ for any real x or x on the unit circle. One way to do this is to choose any nonreal e_1 with $|e_1| < 1/2$. This clearly rules out any real x. Also, if $|x| = 1$, by the triangle inequality $|x^3 - 3x - 1| \geq |3x| - |x^3| - |1| = 1$ and $|x^2 + x| \leq 2$, so $\left|\frac{x^3 - 3x - 1}{x^2 + x}\right| \geq 1/2$.

§2.3 TSTST 2023/6, proposed by Holden Mui

Available online at https://aops.com/community/p28015708.

> **Problem statement**
>
> Let ABC be a scalene triangle and let P and Q be two distinct points in its interior. Suppose that the angle bisectors of $\angle PAQ$, $\angle PBQ$, and $\angle PCQ$ are the altitudes of triangle ABC. Prove that the midpoint of $\overline{PQ}$ lies on the Euler line of ABC.

We present three approaches.

¶ Solution 1 (Ankit Bisain). Let H be the orthocenter of ABC, and construct P' using the following claim.

> **Claim —** There is a point P' for which
> $$\measuredangle APH + \measuredangle AP'H = \measuredangle BPH + \measuredangle BP'H = \measuredangle CPH + \measuredangle CP'H = 0.$$

Proof. After inversion at H, this is equivalent to the fact that P's image has an isogonal conjugate in ABC's image. $\square$

Now, let X, Y, and Z be the reflections of P over $\overline{AH}$, $\overline{BH}$, and $\overline{CH}$ respectively. Additionally, let Q' be the image of Q under inversion about $(PXYZ)$.
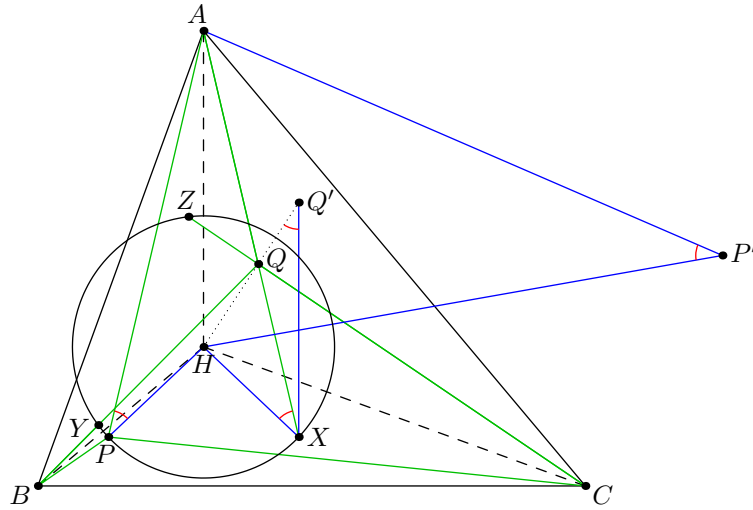
> **Claim —** $ABCP' \stackrel{-}{\sim} XYZQ'$.

Proof. Since
$$\measuredangle YXZ = \measuredangle YPZ = \measuredangle(\overline{BH}, \overline{CH}) = -\measuredangle BAC$$
and cyclic variants, triangles ABC and XYZ are similar. Additionally,
$$\measuredangle HQ'X = -\measuredangle HXQ = -\measuredangle HXA = \measuredangle HPA = -\measuredangle HP'A$$
and cyclic variants, so summing in pairs gives $\measuredangle YQ'Z = -\measuredangle BP'C$ and cyclic variants; this implies the similarity. $\square$

> **Claim** — Q' lies on the Euler line of triangle XYZ.

Proof. Let O be the circumcenter of ABC so that $ABCOP' \stackrel{-}{\sim} XYZHQ'$. Then $\angle HP'A = -\angle HQ'X = \angle OP'A$, so P' lies on $\overline{OH}$. By the similarity, Q' must lie on the Euler line of XYZ. $\qquad\square$

To finish the problem, let G_1 be the centroid of ABC and G_2 be the centroid of XYZ. Then with signed areas,

$$
\begin{aligned}
[G_1HP] + [G_1HQ] &= \frac{[AHP] + [BHP] + [CHP]}{3} + \frac{[AHQ] + [BHQ] + [CHQ]}{3} \\
&= \frac{[AHQ] - [AHX] + [BHQ] - [BHY] + [CHQ] - [CHZ]}{3} \\
&= \frac{[HQX] + [HQY] + [HQZ]}{3} \\
&= [QG_2H] \\
&= 0
\end{aligned}
$$

where the last line follows from the last claim. Therefore $\overline{G_1H}$ bisects $\overline{PQ}$, as desired.

> **Remark.** This solution characterizes the set of all points P for which such a point Q exists. It is the image of the Euler line under the mapping described in the first claim.

¶ **Solution 2 using complex numbers (Carl Schildkraut and Milan Haiman).** Let (ABC) be the unit circle in the complex plane, and let $A = a$, $B = b$, $C = c$ such that $|a| = |b| = |c| = 1$. Let $P = p$ and $Q = q$, and $O = 0$ and $H = h = a + b + c$ be the circumcenter and orthocenter of ABC respectively.

The first step is to translate the given geometric conditions into a single usable equation:

> **Claim** — We have the equation
> $$
> (p + q) \sum_{\text{cyc}} a^3(b^2 - c^2) = (\overline{p} + \overline{q})abc \sum_{\text{cyc}}(bc(b^2 - c^2)). \tag{1}
> $$

Proof. The condition that the altitude $\overline{AH}$ bisects $\angle PAQ$ is equivalent to

$$
\begin{aligned}
&\frac{(p - a)(q - a)}{(h - a)^2} = \frac{(p - a)(q - a)}{(b + c)^2} \in \mathbb{R} \\
\implies &\frac{(p - a)(q - a)}{(b + c)^2} = \overline{\left(\frac{(p - a)(q - a)}{(b + c)^2}\right)} = \frac{(a\overline{p} - 1)(a\overline{q} - 1)b^2c^2}{(b + c)^2 a^2} \\
\implies &a^2(p - a)(q - a) = b^2c^2(a\overline{p} - 1)(a\overline{q} - 1) \\
\implies &a^2pq - a^2b^2c^2\overline{pq} + (a^4 - b^2c^2) = a^3(p + q) - ab^2c^2(\overline{p} + \overline{q}).
\end{aligned}
$$

Writing the symmetric conditions that $\overline{BH}$ and $\overline{CH}$ bisect $\angle PBQ$ and $\angle PCQ$ gives three equations:

$$
\begin{aligned}
a^2pq - a^2b^2c^2\overline{pq} + (a^4 - b^2c^2) &= a^3(p + q) - ab^2c^2(\overline{p} + \overline{q}) \\
b^2pq - a^2b^2c^2\overline{pq} + (b^4 - c^2a^2) &= b^3(p + q) - bc^2a^2(\overline{p} + \overline{q})
\end{aligned}
$$

$$c^2 pq - a^2 b^2 c^2 \overline{pq} + (c^4 - a^2 b^2) = c^3(p+q) - ca^2 b^2 (\overline{p} + \overline{q}).$$

Now, sum $(b^2 - c^2)$ times the first equation, $(c^2 - a^2)$ times the second equation, and $(a^2 - b^2)$ times the third equation. On the left side, the coefficients of pq and $\overline{pq}$ are 0. Additionally, the coefficient of 1 (the parenthesized terms on the left sides of each equation) sum to 0, since

$$\sum_{\text{cyc}} (a^4 - b^2 c^2)(b^2 - c^2) = \sum_{\text{cyc}} (a^4 b^2 - b^4 c^2 - a^4 c^2 + c^4 b^2).$$

This gives (1) as desired. $\qquad\square$

We can then factor (1):

> **Claim** — The left-hand side of (1) factors as
>
> $$-(p+q)(a-b)(b-c)(c-a)(ab+bc+ca)$$
>
> while the right-hand side factors as
>
> $$-(\overline{p}+\overline{q})(a-b)(b-c)(c-a)(a+b+c).$$

Proof. This can of course be verified by direct expansion, but here is a slightly more economic indirect proof.

Consider the cyclic sum on the left as a polynomial in a, b, and c. If $a = b$, then it simplifies as $a^3(a^2 - c^2) + a^3(c^2 - a^2) + c^3(a^2 - a^2) = 0$, so $a - b$ divides this polynomial. Similarly, $a - c$ and $b - c$ divide it, so it can be written as $f(a,b,c)(a-b)(b-c)(c-a)$ for some symmetric quadratic polynomial f, and thus it is some linear combination of $a^2 + b^2 + c^2$ and $ab + bc + ca$. When $a = 0$, the whole expression is $b^2 c^2 (b-c)$, so $f(0,b,c) = -bc$, which implies that $f(a,b,c) = -(ab+bc+ca)$.

Similarly, consider the cyclic sum on the right as a polynomial in a, b, and c. If $a = b$, then it simplifies as $ac(a^2 - c^2) + ca(c^2 - a^2) + a^2(a^2 - a^2) = 0$, so $a - b$ divides this polynomial. Similarly, $a - c$ and $b - c$ divide it, so it can be written as $g(a,b,c)(a-b)(b-c)(c-a)$ where g is a symmetric linear polynomial; hence, it is a scalar multiple of $a+b+c$. When $a = 0$, the whole expression is $bc(b^2 - c^2)$, so $g(0,b,c) = -b-c$, which implies that $g(a,b,c) = -(a+b+c)$. $\qquad\square$

Since A, B, and C are distinct, we may divide by $(a-b)(b-c)(c-a)$ to obtain

$$(p+q)(ab+bc+ca) = (\overline{p}+\overline{q})abc(a+b+c) \implies (p+q)\overline{h} = (\overline{p}+\overline{q})h.$$

This implies that $\frac{\frac{p+q}{2} - 0}{h - 0}$ is real, so the midpoint of $\overline{PQ}$ lies on line $\overline{OH}$.

¶ Solution 3 also using complex numbers (Michael Ren). We use complex numbers as in the previous solution. The angle conditions imply that $\frac{(a-p)(a-q)}{(b-c)^2}$, $\frac{(b-p)(b-q)}{(c-a)^2}$, and $\frac{(c-p)(c-q)}{(a-b)^2}$ are real numbers. Take a linear combination of these with real coefficients X, Y, and Z to be determined; after expansion, we obtain

$$\left[\frac{X}{(b-c)^2} + \frac{Y}{(c-a)^2} + \frac{Z}{(a-b)^2} \right] pq$$

$$- \left[\frac{aX}{(b-c)^2} + \frac{bY}{(c-a)^2} + \frac{cZ}{(a-b)^2} \right] (p+q)$$

$$+ \left[\frac{a^2 X}{(b-c)^2} + \frac{b^2 Y}{(c-a)^2} + \frac{c^2 Z}{(a-b)^2} \right]$$

which is a real number. To get something about the midpoint of PQ, the pq coefficient should be zero, which motivates the following lemma.

> **Lemma**
>
> There exist real X, Y, Z for which
>
> $$\frac{X}{(b-c)^2} + \frac{Y}{(c-a)^2} + \frac{Z}{(a-b)^2} = 0 \text{ and}$$
>
> $$\frac{aX}{(b-c)^2} + \frac{bY}{(c-a)^2} + \frac{cZ}{(a-b)^2} \neq 0.$$

Proof. Since $\mathbb{C}$ is a 2-dimensional vector space over $\mathbb{R}$, there exist real X, Y, Z such that $(X, Y, Z) \neq (0, 0, 0)$ and the first condition holds. Suppose for the sake of contradiction that $\frac{aX}{(b-c)^2} + \frac{bY}{(c-a)^2} + \frac{cZ}{(a-b)^2} = 0$. Then

$$\frac{(b-a)Y}{(c-a)^2} + \frac{(c-a)Z}{(a-b)^2}$$

$$= \frac{aX}{(b-c)^2} + \frac{bY}{(c-a)^2} + \frac{cZ}{(a-b)^2} - a \left(\frac{X}{(b-c)^2} + \frac{Y}{(c-a)^2} + \frac{Z}{(a-b)^2} \right)$$

$$= 0.$$

We can easily check that $(Y, Z) = (0, 0)$ is impossible, therefore $\frac{(b-a)^3}{(c-a)^3} = -\frac{Z}{Y}$ is real. This means $\angle BAC = 60°$ or $120°$. By symmetry, the same is true of $\angle CBA$ and $\angle ACB$. This is impossible because ABC is scalene. $\qquad \square$

With the choice of X, Y, Z as in the lemma, there exist complex numbers α and β, depending only on a, b, and c, such that $\alpha \neq 0$ and $\alpha(p+q) + \beta$ is real. Therefore the midpoint of PQ, which corresponds to $\frac{p+q}{2}$, lies on a fixed line. It remains to show that this line is the Euler line. First, choose $P = Q$ to be the orthocenter to show that the orthocenter lies on the line. Secondly, choose P and Q to be the foci of the Steiner circumellipse to show that the centroid lies on the line. (By some ellipse properties, the external angle bisector of $\angle PAQ$ is the tangent to the circumellipse at A, which is the line through A parallel to BC. Therefore these points are valid.) Therefore the fixed line of the midpoint is the Euler line.

> **Remark.** This solution does not require fixing the origin of the complex plane or setting (ABC) to be the unit circle.

§3 Solutions to Day 3

§3.1 TSTST 2023/7, proposed by Luke Robitaille

Available online at https://aops.com/community/p28015706.

> **Problem statement**
>
> The Bank of Pittsburgh issues coins that have a heads side and a tails side. Vera has a row of 2023 such coins alternately tails-up and heads-up, with the leftmost coin tails-up.
>
> In a *move*, Vera may flip over one of the coins in the row, subject to the following rules:
>
> - On the first move, Vera may flip over any of the 2023 coins.
>
> - On all subsequent moves, Vera may only flip over a coin adjacent to the coin she flipped on the previous move. (We do not consider a coin to be adjacent to itself.)
>
> Determine the smallest possible number of moves Vera can make to reach a state in which every coin is heads-up.

The answer is $\boxed{4044}$. In general, replacing 2023 with $4n + 3$, the answer is $8n + 4$.

¶ **Bound.** Observe that the first and last coins must be flipped, and so every coin is flipped at least once. Then, the $2n + 1$ even-indexed coins must be flipped at least twice, so they are flipped at least $4n + 2$ times.

The $2n + 2$ odd-indexed coins must then be flipped at least $4n + 1$ times. Since there are an even number of these coins, the total flip count must be even, so they are actually flipped a total of at least $4n + 2$ times, for a total of at least $8n + 4$ flips in all.

¶ **Construction.** For $k = 0, 1, \ldots, n - 1$, flip $(4k + 1, 4k + 2, 4k + 3, 4k + 2, 4k + 3, 4k + 4, 4k + 3, 4k + 4)$ in that order; then at the end, flip $4n + 1, 4n + 2, 4n + 3, 4n + 2$. This is illustrated below for $4n + 3 = 15$.
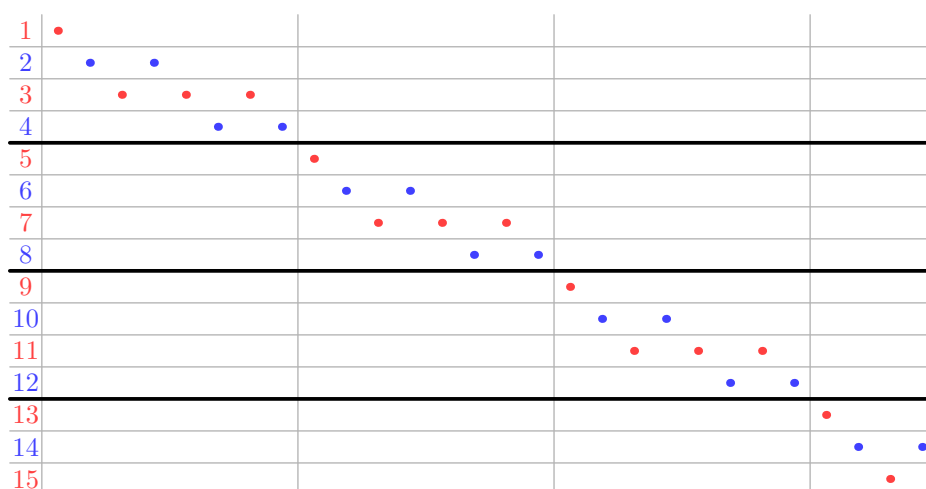
It is easy to check this works, and there are 4044 flips, as desired.

§3.2 TSTST 2023/8, proposed by Ankan Bhattacharya

Available online at https://aops.com/community/p28015680.

> **Problem statement**
>
> Let ABC be an equilateral triangle with side length 1. Points A_1 and A_2 are chosen on side BC, points B_1 and B_2 are chosen on side CA, and points C_1 and C_2 are chosen on side AB such that $BA_1 < BA_2$, $CB_1 < CB_2$, and $AC_1 < AC_2$.
>
> Suppose that the three line segments B_1C_2, C_1A_2, and A_1B_2 are concurrent, and the perimeters of triangles AB_2C_1, BC_2A_1, and CA_2B_1 are all equal. Find all possible values of this common perimeter.

The only possible value of the common perimeter, denoted p, is 1.

**¶ Synthetic approach (from author).**   We prove the converse of the problem first:

> **Claim** ($p = 1$ implies concurrence) — Suppose the six points are chosen so that triangles AB_2C_1, BC_2A_1, CA_2B_1 all have perimeter 1. Then lines $\overline{B_1C_2}$, $\overline{C_1A_2}$, and $\overline{A_1B_2}$ are concurrent.

Proof. The perimeter conditions mean that $\overline{B_2C_1}$, $\overline{C_2A_1}$, and $\overline{A_2B_1}$ are tangent to the incircle of $\triangle ABC$.
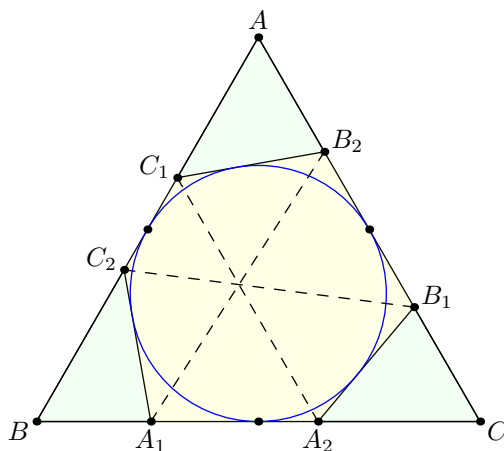
Hence the result follows by *Brianchon's theorem.*   □

Now suppose $p \neq 1$. Let $\overline{B'_2C'_1}$ be the dilation of $\overline{B_2C_1}$ with ratio $\frac{1}{p}$ at center A, and define C'_2, A'_1, A'_2, B'_1 similarly. The following diagram showcases the situation $p < 1$.
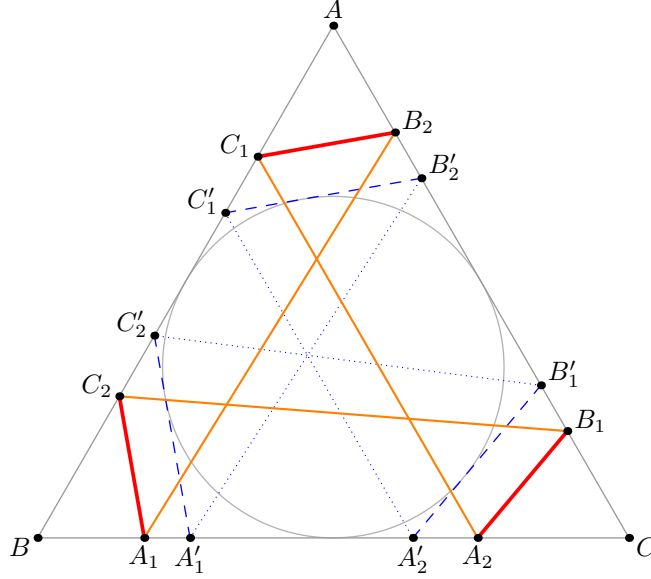
By the reasoning in the $p = 1$ case, note that $\overline{B_1'C_2'}$, $\overline{C_1'A_2'}$, and $\overline{A_1'B_2'}$ are concurrent. However, $\overline{B_1C_2}$, $\overline{C_1A_2}$, $\overline{A_1B_2}$ lie in the interior of quadrilaterals $BCB_1'C_2'$, $CAC_1'A_2'$, and $ABA_1'B_2'$, and these quadrilaterals do not share an interior point, a contradiction.

Thus $p \geq 1$. Similarly, we can show $p \leq 1$, and so $p = 1$ is forced (and achieved if, for example, the three triangles are equilateral with side length $1/3$).

¶ Barycentric solution (by Carl, Krit, Milan). We show that, if the common perimeter is 1, then the lines concur. To do this, we use barycentric coordinates. Let $A = (1 : 0 : 0)$, $B = (0 : 1 : 0)$, and $C = (0 : 0 : 1)$. Let $A_1 = (0 : 1 - a_1 : a_1)$, $A_2 = (0 : a_2 : 1 - a_2)$, $B_1 = (b_1 : 0 : 1 - b_1)$, $B_2 = (1 - b_2 : 0 : b_2)$, $C_1 = (1 - c_1 : c_1 : 0)$, and $C_2 = (c_2 : 1 - c_2 : 0)$. The line B_1C_2 is defined by the equation

$$\det \begin{bmatrix} x & y & z \\ b_1 & 0 & 1 - b_1 \\ c_2 & 1 - c_2 & 0 \end{bmatrix} = 0;$$

i.e.

$$x\big(-(1 - b_1)(1 - c_2)\big) + y\big((1 - b_1)c_2\big) + z\big(b_1(1 - c_2)\big) = 0.$$

Computing the equations for the other lines cyclically, we get that the lines B_1C_2, C_1A_2, and A_1B_2 concur if and only if

$$\det \begin{bmatrix} -(1 - b_1)(1 - c_2) & (1 - b_1)c_2 & b_1(1 - c_2) \\ c_1(1 - a_2) & -(1 - c_1)(1 - a_2) & (1 - c_1)a_2 \\ (1 - a_1)b_2 & a_1(1 - b_2) & -(1 - a_1)(1 - b_2) \end{bmatrix} = 0.$$

Let this matrix be M. We also define the similar matrix

$$N = \begin{bmatrix} -(1 - b_2)(1 - c_1) & (1 - b_2)c_1 & b_2(1 - c_1) \\ c_2(1 - a_1) & -(1 - c_2)(1 - a_1) & (1 - c_2)a_1 \\ (1 - a_2)b_1 & a_2(1 - b_1) & -(1 - a_2)(1 - b_1) \end{bmatrix}.$$

Geometrically, $\det N = 0$ if and only if $B_2'C_1'$, $C_2'A_1'$, and $A_2'B_1'$ concur, where for a point P on a side of triangle ABC, P' denotes its reflection over that side's midpoint.

> **Claim —** We have $\det M = \det N$.

Proof. To show $\det M = \det N$, it suffices to demonstrate that the determinant above is invariant under swapping subscripts of "1" and "2," an operation we call Ψ.

We use the definition of the determinant as a sum over permutations. The even permutations give us the following three terms:

$$-(1-b_1)(1-c_2)(1-c_1)(1-a_2)(1-a_1)(1-b_2) = -\prod_{i=1}^{2}\big((1-a_i)(1-b_i)(1-c_i)\big)$$

$$(1-a_1)b_2(1-b_1)c_2(1-c_1)a_2 = \big((1-a_1)(1-b_1)(1-c_1)\big)\big(a_2b_2c_2\big)$$

$$c_1(1-a_2)a_1(1-b_2)b_1(1-c_2) = \big((1-a_2)(1-b_2)(1-c_2)\big)\big(a_1b_1c_1\big).$$

The first term is invariant under Ψ, while the second and third terms are swapped under Ψ. For the odd permutations, we have a contribution to the determinant of

$$\sum_{\text{cyc}}(1-b_1)(1-c_2)(1-c_1)a_2a_1(1-b_2);$$

each summand is invariant under Ψ. This finishes the proof of our claim. $\square$

Now, it suffices to show that, if AB_2C_1, BC_2A_1, and CA_2B_1 each have perimeter 1, then

$$\det\begin{bmatrix} -(1-b_2)(1-c_1) & (1-b_2)c_1 & b_2(1-c_1) \\ c_2(1-a_1) & -(1-c_2)(1-a_1) & (1-c_2)a_1 \\ (1-a_2)b_1 & a_2(1-b_1) & -(1-a_2)(1-b_1). \end{bmatrix} = 0.$$

Indeed, we have $AB_2 = b_2$ and $AC_1 = c_1$, so by the law of cosines,

$$1 - b_2 - c_1 = 1 - AB_2 - AC_1 = B_2C_1 = \sqrt{b_2^2 + c_1^2 - b_2c_1}.$$

This gives

$$(1-b_2-c_1)^2 = b_2^2 + c_1^2 - b_2c_1 \implies 1 - 2b_2 - 2c_1 + 3b_2c_1 = 0.$$

Similarly, $1 - 2c_2 - 2a_1 + 3c_2a_1 = 0$ and $1 - 2a_2 - 2b_1 + 3a_2b_1 = 0$.

Now,

$$N\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix} -(1-b_2)(1-c_1) + (1-b_2)c_1 + b_2(1-c_1) \\ -(1-c_2)(1-a_1) + (1-c_2)a_1 + c_2(1-a_1) \\ -(1-a_2)(1-b_1) + (1-a_2)b_1 + a_2(1-b_1) \end{bmatrix}$$

$$= \begin{bmatrix} -1 + 2b_2 + 2c_1 - 3b_2c_1 \\ -1 + 2c_2 + 2a_1 - 3c_2a_1 \\ -1 + 2a_2 + 2b_1 - 2a_2b_1 \end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}.$$

So it follows $\det N = 0$, as desired.

§3.3 TSTST 2023/9, proposed by Holden Mui

Available online at .

> **Problem statement**
>
> Let p be a fixed prime and let $a \geq 2$ and $e \geq 1$ be fixed integers. Given a function $f \colon \mathbb{Z}/a\mathbb{Z} \to \mathbb{Z}/p^e\mathbb{Z}$ and an integer $k \geq 0$, the *kth finite difference*, denoted $\Delta^k f$, is the function from $\mathbb{Z}/a\mathbb{Z}$ to $\mathbb{Z}/p^e\mathbb{Z}$ defined recursively by
>
> $$\Delta^0 f(n) = f(n)$$
> $$\Delta^k f(n) = \Delta^{k-1} f(n+1) - \Delta^{k-1} f(n) \qquad \text{for } k = 1, 2, \ldots.$$
>
> Determine the number of functions f such that there exists some $k \geq 1$ for which $\Delta^k f = f$.

The answer is

$$(p^e)^a \cdot p^{-e p^{\nu_p(a)}} = p^{e(a - p^{\nu_p(a)})}.$$

¶ **First solution by author.** For convenience in what follows, set $d = \nu_p(a)$, let $a = p^d \cdot b$, and let a function $f \colon \mathbb{Z}/a\mathbb{Z} \to \mathbb{Z}/p^e\mathbb{Z}$ be *essential* if it equals one of its iterated finite differences.

The key claim is the following.

> **Claim** (Characterization of essential functions) — A function f is essential if and only if
> $$f(x) + f(x + p^d) + \cdots + f(x + (b-1)p^d) = 0 \tag{2}$$
> for all x.

As usual, we split the proof into two halves.

Proof that essential implies the equation First, suppose that f is essential, with $\Delta^N f = f$. Observe that f is in the image of Δ^k for any k, because $\Delta^{mN} f = f$ for any m. The following lemma will be useful.

> **Lemma**
>
> Let $g \colon \mathbb{Z}/a\mathbb{Z} \to \mathbb{Z}/p^e\mathbb{Z}$ be any function, and let $h = \Delta^{p^d} g$. Then
>
> $$h(x) + h(x + p^d) + \cdots + h(x + (b-1)p^d) \equiv 0 \pmod{p}$$
>
> for all x.

Proof. By definition,

$$h(x) = \Delta^{p^d} g(x) = \sum_{k=0}^{p^d} (-1)^k \binom{p^d}{k} g(x + p^d - k).$$

However, it is known that $\binom{p^d}{k}$ is a multiple of p if $1 \leq k \leq p^d - 1$, so

$$h(x) \equiv g(x + p^d) + (-1)^{p^d} g(x) \pmod{p}.$$

Using this, we easily obtain

$$
h(x) + h(x + p^d) + \cdots + h(x + (b-1)p^d)
$$
$$
\equiv \begin{cases} 0 & p > 2 \\ 2(g(x) + g(x + p^d) + \cdots + g(x + (b-1)p^d)) & p = 2 \end{cases}
$$
$$
\equiv 0 \pmod{p},
$$

as desired. $\square$

> **Corollary**
>
> Let $g \colon \mathbb{Z}/a\mathbb{Z} \to \mathbb{Z}/p^e\mathbb{Z}$ be any function, and let $h = \Delta^{ep^d} g$. Then
>
> $$
> h(x) + h(x + p^d) + \cdots + h(x + (b-1)p^d) = 0
> $$
>
> for all x.

Proof. Starting with the lemma, define

$$
h_1(x) = \frac{h(x) + h(x + p^d) + \cdots + h(x + (b-1)p^d)}{p}.
$$

Applying the lemma to h_1 shows the corollary for $e = 2$, since $h_1(x)$ is divisible by p, hence the numerator is divisible by p^2. Continue in this manner to get the result for general $e > 2$. $\square$

This immediately settles this direction, since f is in the image of Δ^{ep^d}.

Proof the equation implies essential   Let $\mathcal{S}$ be the set of all functions satisfying 2; then it's easy to see that Δ is a function on $\mathcal{S}$. To show that all functions in $\mathcal{S}$ are essential, it's equivalent to show that Δ is a permutation on $\mathcal{S}$.

We will show that Δ is injective on $\mathcal{S}$. Suppose otherwise, and consider two functions f, g in $\mathcal{S}$ with $\Delta f = \Delta g$. Then, we obtain that f and g differ by a constant; say $g = f + \lambda$. However, then

$$
g(0) + g(p^e) + \cdots + g((b-1)p^e)
$$
$$
= (f(0) + \lambda) + (f(p^e) + \lambda) + \cdots + (f((b-1)p^e) + \lambda)
$$
$$
= b\lambda.
$$

This should also be zero. Since $p \nmid b$, we obtain $\lambda = 0$, as desired.

Counting   Finally, we can count the essential functions: all but the last p^d entries can be chosen arbitrarily, and then each remaining entry has exactly one possible choice. This leads to a count of
$$
(p^e)^{a-p^d} = p^{e(a-p^{\nu_p(a)})},
$$

as promised.

¶ Second solution by Daniel Zhu.   There are two parts to the proof: solving the $e = 1$ case, and using the $e = 1$ result to solve the general problem by induction on e. These parts are independent of each other.

The case $e = 1$ Represent functions f as elements

$$\alpha_f := \sum_{k \in \mathbb{Z}/a\mathbb{Z}} f(-k)x^k \in \mathbb{F}_p[x]/(x^a - 1)$$

Then, since $\alpha_{\Delta f} = (x - 1)\alpha_f$, we wish to find the number of $\alpha \in \mathbb{F}_p[x]/(x^a - 1)$ such that $(x - 1)^m \alpha = \alpha$ for some m.

Now, make the substitution $y = x - 1$ and let $P(y) = (y + 1)^a - 1$; we want to find $\alpha \in \mathbb{F}_p[y]/(P(y))$ such that $y^m \alpha = \alpha$ for some m.

If we write $P(y) = y^d Q(y)$ with $Q(0) \neq 0$, then by the Chinese Remainder Theorem we have the ring isomorphism

$$\mathbb{F}_p[y]/(P(y)) \cong \mathbb{F}_p[y]/(y^d) \times \mathbb{F}_p[y]/(Q(y)).$$

Note that y is nilpotent in the first factor, while it is a unit in the second factor. So the α that work are exactly those that are zero in the first factor; thus there are p^{a-d} such α. We can calculate $d = p^{v_p(a)}$ (via, say, Lucas's Theorem), so we are done.

The general problem   The general idea is as follows: call a $f \colon \mathbb{Z}/a\mathbb{Z} \to \mathbb{Z}/p^e\mathbb{Z}$ *e-good* if $\Delta^m f = f$ for some m. Our result above allows us to count the 1-good functions. Then, if $e \geq 1$, every $(e + 1)$-good function, when reduced mod p^e, yields an e-good function, so we count $(e + 1)$-good functions by counting how many reduce to any given e-good function.

Formally, we use induction on e, with the $e = 1$ case being treated above. Suppose now we have solved the problem for a given $e \geq 1$, and we now wish to solve it for $e + 1$. For any function $g \colon \mathbb{Z}/a\mathbb{Z} \to \mathbb{Z}/p^{e+1}\mathbb{Z}$, let $\bar{g} \colon \mathbb{Z}/a\mathbb{Z} \to \mathbb{Z}/p^e\mathbb{Z}$ be its reduction mod p^e. For a given e-good f, let $n(f)$ be the number of $(e+1)$-good g with $\bar{g} = f$. The following two claims now finish the problem:

> **Claim** — If f is e-good, then $n(f) > 0$.

Proof. Suppose m is such that $\Delta^m f = f$. Pick any g with $\bar{g} = f$, and consider the sequence of functions

$$g, \Delta^m g, \Delta^{2m} g, \ldots.$$

Since there are finitely many functions $\mathbb{Z}/a\mathbb{Z} \to \mathbb{Z}/p^{e+1}\mathbb{Z}$, there must exist $a < b$ such that $\Delta^{am} g = \Delta^{bm} g$. We claim $\Delta^{am} g$ is the desired $(e + 1)$-good function. To see this, first note that since $\overline{\Delta^k g} = \Delta^k \bar{g}$, we must have $\overline{\Delta^{am} g} = \Delta^{am} f = f$. Moreover,

$$\Delta^{(b-a)m}(\Delta^{am} g) = \Delta^{bm} g = \Delta^{am} g,$$

so $\Delta^{am} g$ is $(e + 1)$-good. $\square$

> **Claim** — If f is e-good, and $n(f) > 0$, then $n(f)$ is exactly the number of 1-good functions, i.e. $p^{a - p^{v_p(a)}}$.

Proof. Let g be any $(e + 1)$-good function with $\bar{g} = f$. We claim that the $(e + 1)$-good g_1 with $\bar{g}_1 = f$ are exactly the functions of the form $g + p^e h$ for any 1-good h. Since these functions are clearly distinct, this characterization will prove the claim.

To show that this condition is sufficient, note that $\overline{g + p^e h} = \bar{g} = f$. Moreover, if $\Delta^m g = g$ and $\Delta^{m'} h = h$, then

$$\Delta^{mm'}(g + p^e h) = \Delta^{mm'} g + p^e \Delta^{mm'} h = g + p^e h.$$

To show that this condition is necessary, let g_1 be any $(e+1)$-good function such that $\bar{g}_1 = f$. Then $g_1 - g$ is also $(e+1)$-good, since if $\Delta^m g = g$, $\Delta^{m'} g_1 = g_1$, we have

$$\Delta^{mm'}(g_1 - g) = \Delta^{mm'} g_1 - \Delta^{mm'} g = g_1 - g.$$

On the other hand, we also know that $g_1 - g$ is divisible by p^e. This means that it must be $p^e h$ for some function $f : \mathbb{Z}/a\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z}$, and it is not hard to show that $g_1 - g$ being $(e+1)$-good means that h is 1-good. $\qquad\square$