JMO 2018 Solution Notes

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This is a compilation of solutions for the 2018 JMO. Some of the solutions are my own work, but many are from the official solutions provided by the organizers (for which they hold any copyrights), and others were found by users on the Art of Problem Solving forums.

These notes will tend to be a bit more advanced and terse than the "official" solutions from the organizers. In particular, if a theorem or technique is not known to beginners but is still considered "standard", then I often prefer to use this theory anyways, rather than try to work around or conceal it. For example, in geometry problems I typically use directed angles without further comment, rather than awkwardly work around configuration issues. Similarly, sentences like "let $\mathbb{R}$ denote the set of real numbers" are typically omitted entirely.

Corrections and comments are welcome!

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§0 Problems

1. For each positive integer n, find the number of n-digit positive integers for which no two consecutive digits are equal, and the last digit is a prime.

2. Let a, b, c be positive real numbers such that $a+b+c = 4\sqrt[3]{abc}$. Prove that

$$2(ab+bc+ca)+4\min(a^2,b^2,c^2) \ge a^2+b^2+c^2.$$

3. Let $ABCD$ be a quadrilateral inscribed in circle ω with $\overline{AC} \perp \overline{BD}$. Let E and F be the reflections of D over $\overline{BA}$ and $\overline{BC}$, respectively, and let P be the intersection of $\overline{BD}$ and $\overline{EF}$. Suppose that the circumcircles of EPD and FPD meet ω at Q and R different from D. Show that $EQ = FR$.

4. Find all real numbers x for which there exists a triangle ABC with circumradius 2, such that $\angle ABC \ge 90^\circ$, and

$$x^4 + ax^3 + bx^2 + cx + 1 = 0$$

where $a = BC$, $b = CA$, $c = AB$.

5. Let p be a prime, and let $a_1, \dots, a_p$ be integers. Show that there exists an integer k such that the numbers

$$a_1 + k,\ a_2 + 2k,\ \dots,\ a_p + pk$$

produce at least $\frac{1}{2}p$ distinct remainders upon division by p.

6. Karl starts with n cards labeled 1, 2, ..., n lined up in random order on his desk. He calls a pair (a,b) of cards *swapped* if $a > b$ and the card labeled a is to the left of the card labeled b.

Karl picks up the card labeled 1 and inserts it back into the sequence in the opposite position: if the card labeled 1 had i cards to its left, then it now has i cards to its right. He then picks up the card labeled 2 and reinserts it in the same manner, and so on, until he has picked up and put back each of the cards 1, ..., n exactly once in that order.

For example, if $n = 4$, then one example of a process is

$$3142 \longrightarrow 3412 \longrightarrow 2341 \longrightarrow 2431 \longrightarrow 2341$$

which has three swapped pairs both before and after.

Show that, no matter what lineup of cards Karl started with, his final lineup has the same number of swapped pairs as the starting lineup.

§1 Solutions to Day 1

§1.1 JMO 2018/1, proposed by Zachary Franco, Zuming Feng

Available online at https://aops.com/community/p10226138.

Problem statement

For each positive integer n, find the number of n-digit positive integers for which no two consecutive digits are equal, and the last digit is a prime.

Almost trivial. Let a_n be the desired answer. We have

$$a_n + a_{n-1} = 4 \cdot 9^{n-1}$$

for all n, by padding the $(n-1)$ digit numbers with a leading zero.

Since $a_0 = 0$, $a_1 = 4$, solving the recursion gives

$$a_n = \frac{2}{5}\left(9^n - (-1)^n\right).$$

The end.

Remark. For concreteness, the first few terms are 0, 4, 32, 292, ...

§1.2 JMO 2018/2, proposed by Titu Andreescu

Available online at https://aops.com/community/p10226140.

Problem statement

Let a, b, c be positive real numbers such that $a+b+c=4\sqrt[3]{abc}$. Prove that

$$2(ab+bc+ca)+4\min(a^2,b^2,c^2)\ge a^2+b^2+c^2.$$

WLOG let $c=\min(a,b,c)=1$ by scaling. The given inequality becomes equivalent to

$$4ab+2a+2b+3\ge(a+b)^2 \qquad \forall a+b=4(ab)^{1/3}-1.$$

Now, let $t=(ab)^{1/3}$ and eliminate $a+b$ using the condition, to get

$$4t^3+2(4t-1)+3\ge(4t-1)^2 \iff 0\le 4t^3-16t^2+16t=4t(t-2)^2$$

which solves the problem.

Equality occurs only if $t=2$, meaning $ab=8$ and $a+b=7$, which gives

$$\{a,b\}=\left\{\frac{7\pm\sqrt{17}}{2}\right\}$$

with the assumption $c=1$. Scaling gives the curve of equality cases.

§1.3 JMO 2018/3, proposed by Ray Li

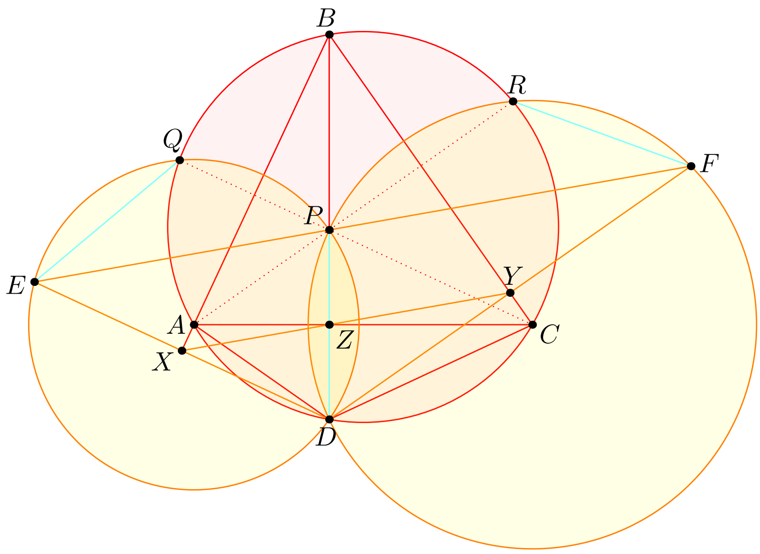
Available online at https://aops.com/community/p10226149.

Problem statement

Let $ABCD$ be a quadrilateral inscribed in circle ω with $\overline{AC} \perp \overline{BD}$. Let E and F be the reflections of D over $\overline{BA}$ and $\overline{BC}$, respectively, and let P be the intersection of $\overline{BD}$ and $\overline{EF}$. Suppose that the circumcircles of EPD and FPD meet ω at Q and R different from D. Show that $EQ = FR$.

Most of this problem is about realizing where the points P, Q, R are.

¶ First solution (Evan Chen). Let X, Y, be the feet from D to $\overline{BA}$, $\overline{BC}$, and let $Z = \overline{BD} \cap \overline{AC}$. By Simson theorem, the points X, Y, Z are collinear. Consequently, the point P is the reflection of D over Z, and so we conclude P is the orthocenter of $\triangle ABC$.



Suppose now we extend ray CP to meet ω again at Q'. Then $\overline{BA}$ is the perpendicular bisector of both $\overline{PQ'}$ and $\overline{DE}$; consequently, $PQ'ED$ is an isosceles trapezoid. In particular, it is cyclic, and so $Q' = Q$. In the same way R is the second intersection of ray $\overline{AP}$ with ω.

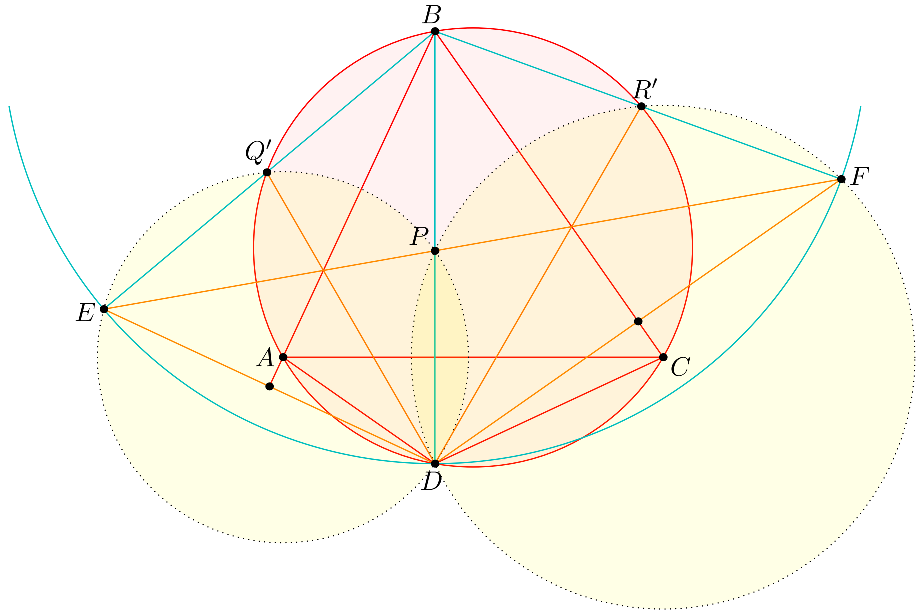
Now, because of the two isosceles trapezoids we have found, we conclude

$$EQ = PD = FR$$

as desired.

> **Remark.** Alternatively, after identifying P, one can note $\overline{BQE}$ and $\overline{BRF}$ are collinear. Since $BE = BD = BF$, upon noticing $BQ = BP = BR$ we are also done.

¶ Second solution (Danielle Wang). Here is a solution which does not identify the point P at all. We know that $BE = BD = BF$, by construction.



Claim — The points B, Q, E are collinear. Similarly the points B, R, F are collinear.

Proof. Work directed modulo 180°. Let Q' be the intersection of $\overline{BE}$ with $(ABCD)$. Let $\alpha = \measuredangle DEB = \measuredangle BDE$ and $\beta = \measuredangle BFD = \measuredangle FDB$.

Observe that $BE = BD = BF$, so B is the circumcenter of $\triangle DEF$. Thus, $\measuredangle DEP = \measuredangle DEF = 90^\circ - \beta$. Then

$$\begin{aligned} \measuredangle DPE &= \measuredangle DEP + \measuredangle PDE = (90^\circ - \beta) + \alpha \\ &= \alpha - \beta + 90^\circ \\ \measuredangle DQ'B &= \measuredangle DCB = \measuredangle DCA + \measuredangle ACB \\ &= \measuredangle DBA - (90^\circ - \measuredangle DBC) = -(90^\circ - \alpha) - (90^\circ - (90^\circ - \beta)) \\ &= \alpha - \beta + 90^\circ. \end{aligned}$$

Thus Q' lies on the desired circle, so $Q' = Q$. $\square$

Now, by power of a point we have $BQ \cdot BE = BP \cdot BD = BR \cdot BF$, so $BQ = BP = BR$. Hence $EQ = PD = FR$.

§2 Solutions to Day 2

§2.1 JMO 2018/4, proposed by Titu Andreescu

Available online at https://aops.com/community/p10232384.

Problem statement

Find all real numbers x for which there exists a triangle ABC with circumradius 2, such that $\angle ABC \ge 90^\circ$, and

$$x^4 + ax^3 + bx^2 + cx + 1 = 0$$

where $a = BC$, $b = CA$, $c = AB$.

The answer is $x = -\frac{1}{2}(\sqrt{6} \pm \sqrt{2})$.

We prove this the only possible answer. Evidently $x < 0$. Now, note that

$$a^2 + c^2 \le b^2 \le 4b$$

since $b \le 4$ (the diameter of its circumcircle). Then,

$$\begin{aligned} 0 &= x^4 + ax^3 + bx^2 + cx + 1 \\ &= x^2 \left[\left(x + \frac{1}{2}a \right)^2 + \left(\frac{1}{x} + \frac{1}{2}c \right)^2 + \left(b - \frac{a^2+c^2}{4} \right) \right] \\ &\ge 0 + 0 + 0 = 0. \end{aligned}$$

In order for equality to hold, we must have $x = -\frac{1}{2}a$, $1/x = -\frac{1}{2}c$, and $a^2 + c^2 = b^2 = 4b$. This gives us $b = 4$, $ac = 4$, $a^2 + c^2 = 16$. Solving for $a, c > 0$ implies

$$\{a, c\} = \left\{ \sqrt{6} \pm \sqrt{2} \right\}.$$

This gives the x values claimed above; by taking a, b, c as deduced here, we find they work too.

> **Remark.** Note that by perturbing $\triangle ABC$ slightly, we see *a priori* that the set of possible x should consist of unions of intervals (possibly trivial). So it makes sense to try inequalities no matter what.

§2.2 JMO 2018/5, proposed by Ankan Bhattacharya

Available online at https://aops.com/community/p10232389.

Problem statement

Let p be a prime, and let $a_1, \dots, a_p$ be integers. Show that there exists an integer k such that the numbers

$$a_1 + k, \ a_2 + 2k, \ \dots, \ a_p + pk$$

produce at least $\frac{1}{2}p$ distinct remainders upon division by p.

For each $k = 0, \dots, p-1$ let G_k be the graph on $\{1, \dots, p\}$ where we join $\{i, j\}$ if and only if

$$a_i + ik \equiv a_j + jk \pmod p \iff k \equiv -\frac{a_i - a_j}{i - j} \pmod p.$$

So we want a graph G_k with at least $\frac{1}{2}p$ connected components.

However, each $\{i, j\}$ appears in exactly one graph G_k, so some graph has at most $\frac{1}{p}\binom{p}{2} = \frac{1}{2}(p-1)$ edges (by "pigeonhole"). This graph has at least $\frac{1}{2}(p+1)$ connected components, as desired.

Remark. Here is an example for $p = 5$ showing equality can occur:

$$\begin{bmatrix} 0 & 0 & 3 & 4 & 3 \\ 0 & 1 & 0 & 2 & 2 \\ 0 & 2 & 2 & 0 & 1 \\ 0 & 3 & 4 & 3 & 0 \\ 0 & 4 & 1 & 1 & 4 \end{bmatrix}.$$

Ankan Bhattacharya points out more generally that $a_i = i^2$ is sharp in general.

§2.3 JMO 2018/6, proposed by Maria Monks Gillespie

Available online at https://aops.com/community/p10232393.

Problem statement

Karl starts with n cards labeled 1, 2, ..., n lined up in random order on his desk. He calls a pair (a, b) of cards *swapped* if $a > b$ and the card labeled a is to the left of the card labeled b.

Karl picks up the card labeled 1 and inserts it back into the sequence in the opposite position: if the card labeled 1 had i cards to its left, then it now has i cards to its right. He then picks up the card labeled 2 and reinserts it in the same manner, and so on, until he has picked up and put back each of the cards 1, ..., n exactly once in that order.

For example, if $n = 4$, then one example of a process is

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The official solution is really tricky. Call the process P.

We define a new process P' where, when re-inserting card i, we additionally change its label from i to $n + i$. An example of P' also starting with 3142 is:

$$3142 \longrightarrow 3452 \longrightarrow 6345 \longrightarrow 6475 \longrightarrow 6785.$$

Note that now, each step of P' preserves the number of inversions. Moreover, the final configuration of P' is the same as the final configuration of P with all cards incremented by n, and of course thus has the same number of inversions. Boom.