IMO 2023 Solution Notes

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This is a compilation of solutions for the 2023 IMO. Some of the solutions are my own work, but many are from the official solutions provided by the organizers (for which they hold any copyrights), and others were found by users on the Art of Problem Solving forums.

These notes will tend to be a bit more advanced and terse than the "official" solutions from the organizers. In particular, if a theorem or technique is not known to beginners but is still considered "standard", then I often prefer to use this theory anyways, rather than try to work around or conceal it. For example, in geometry problems I typically use directed angles without further comment, rather than awkwardly work around configuration issues. Similarly, sentences like "let $\mathbb{R}$ denote the set of real numbers" are typically omitted entirely.

Corrections and comments are welcome!

Contents

§0 Problems

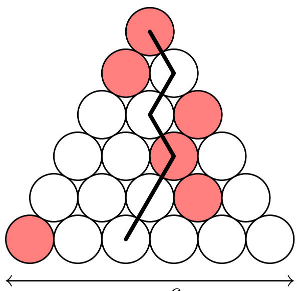
1. Determine all composite integers $n > 1$ that satisfy the following property: if $d_1 < d_2 < \cdots < d_k$ are all the positive divisors of n with then d_i divides $d_{i+1} + d_{i+2}$ for every $1 \le i \le k-2$.

2. Let ABC be an acute-angled triangle with $AB < AC$. Let Ω be the circumcircle of ABC. Let S be the midpoint of the arc CB of Ω containing A. The perpendicular from A to BC meets BS at D and meets Ω again at $E \neq A$. The line through D parallel to BC meets line BE at L. Denote the circumcircle of triangle BDL by ω. Let ω meet Ω again at $P \neq B$. Prove that the line tangent to ω at P meets line BS on the internal angle bisector of $\angle BAC$.

3. For each integer $k \ge 2$, determine all infinite sequences of positive integers a_1, a_2, ... for which there exists a polynomial P of the form
$$P(x) = x^k + c_{k-1}x^{k-1} + \cdots + c_1 x + c_0,$$
where c_0, c_1, ..., c_{k-1} are non-negative integers, such that
$$P(a_n) = a_{n+1}a_{n+2}\cdots a_{n+k}$$
for every integer $n \ge 1$.

4. Let x_1, x_2, ..., x_{2023} be pairwise different positive real numbers such that
$$a_n = \sqrt{(x_1 + x_2 + \cdots + x_n)\left(\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}\right)}$$
is an integer for every $n = 1, 2, \dots, 2023$. Prove that $a_{2023} \ge 3034$.

5. Let n be a positive integer. A *Japanese triangle* consists of $1 + 2 + \cdots + n$ circles arranged in an equilateral triangular shape such that for each $1 \le i \le n$, the i^{th} row contains exactly i circles, exactly one of which is colored red. A *ninja path* in a Japanese triangle is a sequence of n circles obtained by starting in the top row, then repeatedly going from a circle to one of the two circles immediately below it and finishing in the bottom row. Here is an example of a Japanese triangle with $n = 6$, along with a ninja path in that triangle containing two red circles.



In terms of n, find the greatest k such that in each Japanese triangle there is a ninja path containing at least k red circles.

6. Let ABC be an equilateral triangle. Let A_1, B_1, C_1 be interior points of ABC such that $BA_1 = A_1C$, $CB_1 = B_1A$, $AC_1 = C_1B$, and
$$\angle BA_1C + \angle CB_1A + \angle AC_1B = 480°.$$

Let $A_2 = \overline{BC_1} \cap \overline{CB_1}$, $B_2 = \overline{CA_1} \cap \overline{AC_1}$, $C_2 = \overline{AB_1} \cap \overline{BA_1}$. Prove that if triangle $A_1B_1C_1$ is scalene, then the circumcircles of triangles AA_1A_2, BB_1B_2, and CC_1C_2 all pass through two common points.

§1 Solutions to Day 1

§1.1 IMO 2023/1, proposed by Santiago Rodriguez (COL)

Available online at https://aops.com/community/p28097575.

Problem statement

Determine all composite integers $n > 1$ that satisfy the following property: if $d_1 < d_2 < \cdots < d_k$ are all the positive divisors of n with then d_i divides $d_{i+1} + d_{i+2}$ for every $1 \le i \le k-2$.

The answer is prime powers.

¶ Verification that these work. When $n = p^e$, we get $d_i = p^{i-1}$. The i^{th} relationship reads

$$p^{i-1} \mid p^i + p^{i+1}$$

which is obviously true.

¶ Proof that these are the only answers. Conversely, suppose n has at least two distinct prime divisors. Let $p < q$ denote the two smallest ones, and let p^e be the largest power of p which both divides n and is less than q, hence $e \ge 1$. Then the smallest factors of n are 1, p, $\dots$, p^e, q. So we are supposed to have

$$\frac{n}{q} \mid \frac{n}{p^e} + \frac{n}{p^{e-1}} = \frac{(p+1)n}{p^e}$$

which means that the ratio

$$\frac{q(p+1)}{p^e}$$

needs to be an integer, which is obviously not possible.

§1.2 IMO 2023/2, proposed by Tiago Mourão and Nuno Arala (POR)

Available online at https://aops.com/community/p28097552.

Problem statement

Let ABC be an acute-angled triangle with $AB < AC$. Let Ω be the circumcircle of ABC. Let S be the midpoint of the arc CB of Ω containing A. The perpendicular from A to BC meets BS at D and meets Ω again at $E \neq A$. The line through D parallel to BC meets line BE at L. Denote the circumcircle of triangle BDL by ω. Let ω meet Ω again at $P \neq B$. Prove that the line tangent to ω at P meets line BS on the internal angle bisector of $\angle BAC$.

Claim — We have LPS collinear.

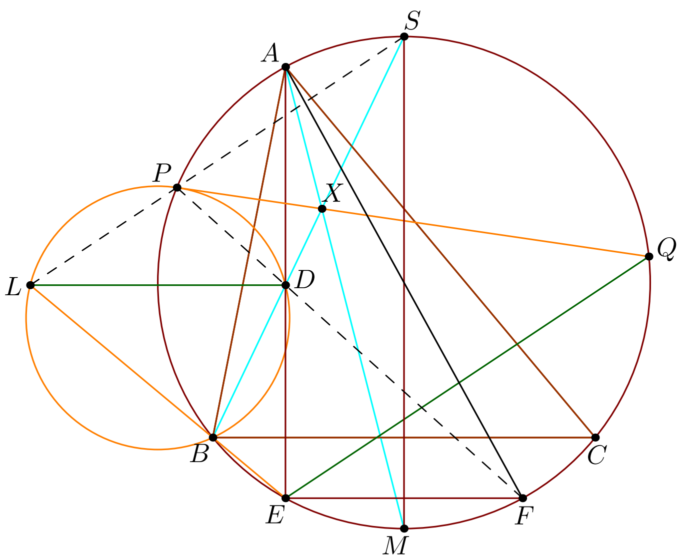
Proof. Because $\measuredangle LPB = \measuredangle LDB = \measuredangle CBD = \measuredangle CBS = \measuredangle SCB = \measuredangle SPB$. $\square$

Let F be the antipode of A, so $AMFS$ is a rectangle.

Claim — We have PDF collinear. (This lets us erase L.)

Proof. Because $\measuredangle SPD = \measuredangle LPD = \measuredangle LBD = \measuredangle SBE = \measuredangle FCS = \measuredangle FPS$. $\square$

Let us define $X = \overline{AM} \cap \overline{BS}$ and complete chord $\overline{PXQ}$. We aim to show that $\overline{PXQ}$ is tangent to $(PDLB)$.

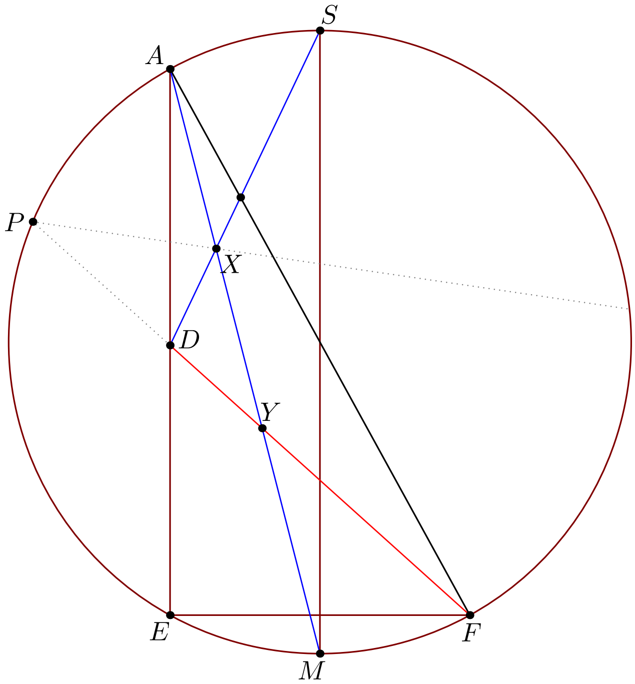


Claim (Main projective claim) — We have $XP = XA$.

Proof. Introduce $Y = \overline{PDF} \cap \overline{AM}$. Note that

$$-1 = (SM; EF) \overset{A}{=} (S, X; D, \overline{AF} \cap \overline{ES}) \overset{F}{=} (\infty X; YA)$$

where $\infty = \overline{AM} \cap \overline{SF}$ is at infinity (because $AMSF$ is a rectangle). Thus, $XY = XA$.



Since $\triangle APY$ is also right, we get $XP = XA$. $\square$

Alternative proof of claim without harmonic bundles, from Solution 9 of the marking scheme. With $Y = \overline{PDF} \cap \overline{AM}$ defined as before, note that $\overline{AE} \parallel \overline{SM}$ and $\overline{AM} \parallel \overline{SF}$ (as $AMFS$ is a rectangle) gives respectively the similar triangles

$$\triangle AXD \sim \triangle MXS, \qquad \triangle XDY \sim \triangle SDF.$$

From this we conclude

$$\frac{AX}{XD} = \frac{AX + XM}{XD + SX} = \frac{AM}{SD} = \frac{SF}{SD} = \frac{XY}{XD}.$$

So $AX = XY$ and as before we conclude $XP = XA$. $\square$

From $XP = XA$, we conclude that $\widehat{PM}$ and $\widehat{AQ}$ have the same measure. Since $\widehat{AS}$ and $\widehat{EM}$ have the same measure, it follows $\widehat{PE}$ and $\widehat{SQ}$ have the same measure. The desired tangency then follows from

$$\measuredangle QPL = \measuredangle QPS = \measuredangle PQE = \measuredangle PFE = \measuredangle PDL.$$

Remark (Logical ordering)**.** This solution is split into two phases: the "synthetic phase" where we do a bunch of angle chasing, and the "projective phase" where we use cross-ratios because I like projective. For logical readability (so we write in only one logical direction), the projective phase is squeezed in two halves of the synthetic phase, but during an actual solve it's expected to complete the whole synthetic phase first (i.e. to reduce the problem to show $XP = XA$).

Remark. There are quite a multitude of approaches for this problem; the marking scheme for this problem at the actual IMO had 13 different solutions.

§1.3 IMO 2023/3, proposed by Ivan Chan (MAS)

Available online at https://aops.com/community/p28097600.

Problem statement

For each integer $k \ge 2$, determine all infinite sequences of positive integers a_1, a_2, ... for which there exists a polynomial P of the form

$$P(x) = x^k + c_{k-1}x^{k-1} + \cdots + c_1 x + c_0,$$

where c_0, c_1, ..., c_{k-1} are non-negative integers, such that

$$P(a_n) = a_{n+1}a_{n+2}\cdots a_{n+k}$$

for every integer $n \ge 1$.

The answer is a_n being an arithmetic progression. Indeed, if $a_n = d(n-1) + a_1$ for $d \ge 0$ and $n \ge 1$, then

$$a_{n+1}a_{n+2}\dots a_{n+k} = (a_n + d)(a_n + 2d)\dots(a_n + kd)$$

so we can just take $P(x) = (x + d)(x + 2d)\dots(x + kd)$.

The converse direction takes a few parts.

Claim — Either $a_1 < a_2 < \cdots$ or the sequence is constant.

Proof. Note that

$$\begin{aligned} P(a_{n-1}) &= a_n a_{n+1} \cdots a_{n+k-1} \\ P(a_n) &= a_{n+1}a_{n+2}\cdots a_{n+k} \\ \implies a_{n+k} &= \frac{P(a_n)}{P(a_{n-1})} \cdot a_n. \end{aligned}$$

Now the polynomial P is strictly increasing over $\mathbb{N}$.

So assume for contradiction there's an index n such that $a_n < a_{n-1}$. Then in fact the above equation shows $a_{n+k} < a_n < a_{n-1}$. Then there's an index $\ell \in [n+1, n+k]$ such that $a_\ell < a_{\ell-1}$, and also $a_\ell < a_n$. Continuing in this way, we can an infinite descending subsequence of (a_n), but that's impossible because we assumed integers.

Hence we have $a_1 \le a_2 \le \cdots$. Now similarly, if $a_n = a_{n-1}$ for any index n, then $a_{n+k} = a_n$, ergo $a_{n-1} = a_n = a_{n+1} = \cdots = a_{n+k}$. So the sequence is eventually constant, and then by downwards induction, it is fully constant. □

Claim — There exists a constant C (depending only P, k) such that we have $a_{n+1} \le a_n + C$.

Proof. Let C be a constant such that $P(x) < x^k + Cx^{k-1}$ for all $x \in \mathbb{N}$ (for example $C = c_0 + c_1 + \cdots + c_{k-1} + 1$ works). We have

$$a_{n+k} = \frac{P(a_n)}{a_{n+1}a_{n+2}\dots a_{n+k-1}}$$

$$\begin{aligned} &< \frac{P(a_n)}{(a_n+1)(a_n+2)\dots(a_n+k-1)} \\ &< \frac{a_n^k + C \cdot a_n^{k-1}}{(a_n+1)(a_n+2)\dots(a_n+k-1)} \\ &< a_n + C + 1. \end{aligned}$$ □

Assume henceforth a_n is nonconstant, and hence unbounded. For each index n and term a_n in the sequence, consider the associated differences $d_1 = a_{n+1} - a_n$, $d_2 = a_{n+2} - a_{n+1}$, $\dots$, $d_k = a_{n+k} - a_{n+k-1}$, which we denote by

$$\Delta(n) \coloneqq (d_1, \dots, d_k).$$

This Δ can only take up to C^k different values. So in particular, some tuple $(d_1, \dots, d_n)$ must appear infinitely often as $\Delta(n)$; for that tuple, we obtain

$$P(a_N) = (a_N + d_1)(a_N + d_1 + d_2) \dots (a_N + d_1 + \dots + d_k)$$

for infinitely many N. But because of that, we actually must have

$$P(X) = (X + d_1)(X + d_1 + d_2) \dots (X + d_1 + \dots + d_k).$$

However, this *also* means that *exactly* one output to Δ occurs infinitely often (because that output is determined by P). Consequently, it follows that Δ is eventually constant. For this to happen, a_n must eventually coincide with an arithmetic progression of some common difference d, and $P(X) = (X+d)(X+2d)\dots(X+kd)$. Finally, this implies by downwards induction that a_n is an arithmetic progression on all inputs.

§2 Solutions to Day 2

§2.1 IMO 2023/4, proposed by Merlijn Staps (NLD)

Available online at https://aops.com/community/p28104298.

Problem statement

Let $x_1, x_2, \dots, x_{2023}$ be pairwise different positive real numbers such that

$$a_n = \sqrt{(x_1 + x_2 + \dots + x_n)\left(\frac{1}{x_1} + \frac{1}{x_2} + \dots + \frac{1}{x_n}\right)}$$

is an integer for every $n = 1, 2, \dots, 2023$. Prove that $a_{2023} \ge 3034$.

Note that $a_{n+1} > \sqrt{\sum_1^n x_i \sum_1^n \frac{1}{x_i}} = a_n$ for all n, so that $a_{n+1} \ge a_n + 1$. Observe $a_1 = 1$. We are going to prove that

$$a_{2m+1} \ge 3m + 1 \qquad \text{for all } m \ge 0$$

by induction on m, with the base case being clear.

We now present two variations of the induction. The first shorter solution compares a_{n+2} directly to a_n, showing it increases by at least 3. Then we give a longer approach that compares a_{n+1} to a_n, and shows it cannot increase by 1 twice in a row.

¶ Induct-by-two solution. Let $u = \sqrt{\frac{x_{n+1}}{x_{n+2}}} \neq 1$. Note that by using Cauchy-Schwarz with three terms:

$$\begin{aligned} a_{n+2}^2 &= \left[(x_1 + \dots + x_n) + x_{n+1} + x_{n+2}\right]\left[\left(\frac{1}{x_1} + \dots + \frac{1}{x_n}\right) + \frac{1}{x_{n+2}} + \frac{1}{x_{n+1}}\right] \\ &\ge \left(\sqrt{(x_1 + \dots + x_n)\left(\frac{1}{x_1} + \dots + \frac{1}{x_n}\right)} + \sqrt{\frac{x_{n+1}}{x_{n+2}}} + \sqrt{\frac{x_{n+2}}{x_{n+1}}}\right)^2 \\ &= \left(a_n + u + \frac{1}{u}\right)^2. \\ \implies a_{n+2} &\ge a_n + u + \frac{1}{u} > a_n + 2 \end{aligned}$$

where the last equality $u + \frac{1}{u} > 2$ is by AM-GM, strict as $u \neq 1$. It follows that $a_{n+2} \ge a_n + 3$, completing the proof.

¶ Induct-by-one solution. The main claim is:

Claim — It's impossible to have $a_n = c$, $a_{n+1} = c + 1$, $a_{n+2} = c + 2$ for any c and n.

Proof. Let $p = x_{n+1}$ and $q = x_{n+2}$ for brevity. Let $s = \sum_1^n x_i$ and $t = \sum_1^n \frac{1}{x_n}$, so $c^2 = a_n^2 = st$.

From $a_n = c$ and $a_{n+1} = c$ we have

$$(c+1)^2 = a_{n+1}^2 = (p + s)\left(\frac{1}{p} + t\right)$$

$$\begin{aligned} &= st + pt + \frac{1}{p}s + 1 = c^2 + pt + \frac{1}{p}s + 1 \\ &\overset{\text{AM-GM}}{\geq} c^2 + 2\sqrt{st} + 1 = c^2 + 2\sqrt{c^2} + 1 = (c+1)^2. \end{aligned}$$

Hence, equality must hold in the AM-GM we must have exactly

$$pt = \frac{1}{p}s = c.$$

If we repeat the argument again on $a_{n+1} = c + 1$ and $a_{n+2} = c_{n+2}$, then

$$p\left(\frac{1}{q} + t\right) = \frac{1}{p}(q + s) = c + 1.$$

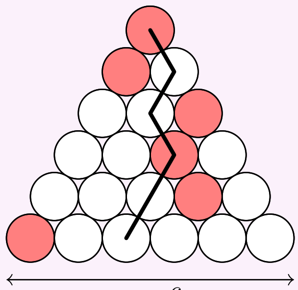
However this forces $\frac{p}{q} = \frac{q}{p} = 1$ which is impossible. $\square$

§2.2 IMO 2023/5, proposed by Merlijn Staps and Daniël Kroes (NLD)

Available online at https://aops.com/community/p28104367.

Problem statement

Let n be a positive integer. A *Japanese triangle* consists of $1+2+\cdots+n$ circles arranged in an equilateral triangular shape such that for each $1 \le i \le n$, the i^{th} row contains exactly i circles, exactly one of which is colored red. A *ninja path* in a Japanese triangle is a sequence of n circles obtained by starting in the top row, then repeatedly going from a circle to one of the two circles immediately below it and finishing in the bottom row. Here is an example of a Japanese triangle with $n = 6$, along with a ninja path in that triangle containing two red circles.

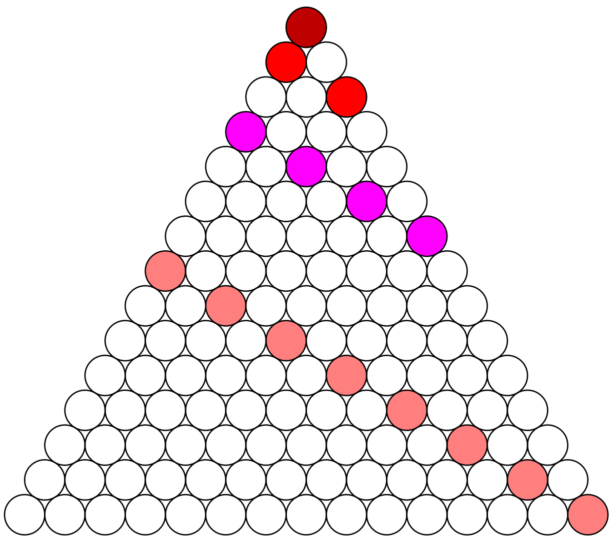


In terms of n, find the greatest k such that in each Japanese triangle there is a ninja path containing at least k red circles.

The answer is

$$k = \lfloor \log_2(n) \rfloor + 1.$$

¶ Construction. It suffices to find a Japanese triangle for $n = 2^e - 1$ with the property that at most e red circles in any ninja path. The construction shown below for $e = 4$ obviously generalizes, and works because in each of the sets $\{1\}$, $\{2,3\}$, $\{4,5,6,7\}$, $\ldots$, $\{2^{e-1}, \ldots, 2^e - 1\}$, at most one red circle can be taken. (These sets are colored in different shades of red for visual clarity).



¶ Bound. Conversely, we show that in any Japanese triangle, one can find a ninja path containing at least

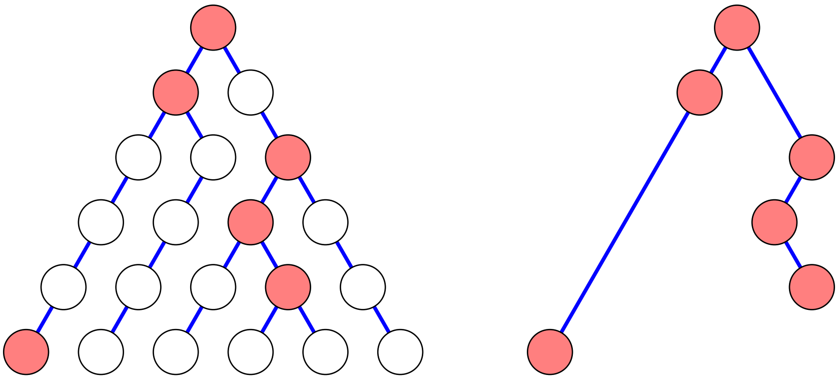
$$k = \lfloor \log_2(n) \rfloor + 1.$$

The following short solution was posted at https://aops.com/community/p28134004, apparently first found by the team leader for Iran.

We construct a rooted binary tree T_1 on the set of all circles as follows. For each row, other than the bottom row:

- Connect the red circle to both circles under it;
- White circles to the left of the red circle in its row are connected to the left;
- White circles to the right of the red circle in its row are connected to the right.

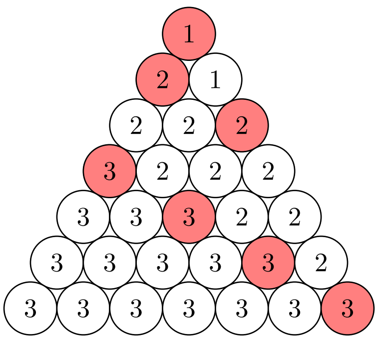
The circles in the bottom row are all leaves of this tree. For example, the $n = 6$ construction in the beginning gives the tree shown on the left half of the figure below:



T_1 T_2

Now focus on only the red circles, as shown in the right half of the figure. We build a new rooted tree T_2 where each red circle is joined to the red circle below it if there was a path of (zero or more) white circles in T_1 between them. Then each red circle has at most 2 direct descendants in T_2. Hence the depth of the new tree T_2 exceeds $\log_2(n)$, which produces the desired path.

¶ Another recursive proof of bound, communicated by Helio Ng. We give another proof that $\lfloor \log_2 n \rfloor + 1$ is always achievable. Define $f(i, j)$ to be the maximum number of red circles contained in the portion of a ninja path from $(1, 1)$ to (i, j), including the endpoints $(1, 1)$ and (i, j). (If (i, j) is not a valid circle in the triangle, define $f(i, j) = 0$ for convenience.) An example is shown below with the values of $f(i, j)$ drawn in the circles.



We have that

$$f(i,j) = \max\{f(i-1,j-1), f(i,j-1)\} + \begin{cases} 1 & \text{if } (i,j) \text{ is red} \\ 0 & \text{otherwise} \end{cases}$$

since every ninja path passing through (i,j) also passes through either $(i-1,j-1)$ or $(i,j-1)$. Now consider the quantity $S_j = f(0,j) + \cdots + f(j,j)$. We obtain the following recurrence:

Claim — $S_{j+1} \ge S_j + \left\lceil \frac{S_j}{j} \right\rceil + 1$.

Proof. Consider a maximal element $f(m,j)$ of $\{f(0,j), \dots, f(j,j)\}$. We perform the following manipulations:

$$\begin{aligned} S_{j+1} &= \sum_{i=0}^{j+1} \max\{f(i-1,j), f(i,j)\} + \sum_{i=0}^{j+1} \begin{cases} 1 & \text{if } (i,j+1) \text{ is red} \\ 0 & \text{otherwise} \end{cases} \\ &= \sum_{i=0}^{m} \max\{f(i-1,j), f(i,j)\} + \sum_{i=m+1}^{j} \max\{f(i-1,j), f(i,j)\} + 1 \\ &\ge \sum_{i=0}^{m} f(i,j) + \sum_{i=m+1}^{j} f(i-1,j) + 1 \\ &= S_j + f(m,j) + 1 \\ &\ge S_j + \left\lceil \frac{S_j}{j} \right\rceil + 1 \end{aligned}$$

where the last inequality is due to Pigeonhole. □

This is actually enough to solve the problem. Write $n = 2^c + r$, where $0 \le r \le 2^c - 1$.

Claim — $S_n \ge cn + 2r + 1$. In particular, $\left\lceil \frac{S_n}{n} \right\rceil \ge c + 1$.

Proof. First note that $S_n \ge cn + 2r + 1$ implies $\left\lceil \frac{S_n}{n} \right\rceil \ge c + 1$ because

$$\left\lceil \frac{S_n}{n} \right\rceil \ge \left\lceil \frac{cn + 2r + 1}{n} \right\rceil = c + \left\lceil \frac{2r+1}{n} \right\rceil = c + 1.$$

We proceed by induction on n. The base case $n = 1$ is clearly true as $S_1 = 1$. Assuming that the claim holds for some $n = j$, we have

$$\begin{aligned} S_{j+1} &\ge S_j + \left\lceil \frac{S_j}{j} \right\rceil + 1 \\ &\ge cj + 2r + 1 + c + 1 + 1 \\ &= c(j+1) + 2(r+1) + 1 \end{aligned}$$

so the claim is proved for $n = j+1$ if $j+1$ is not a power of 2. If $j + 1 = 2^{c+1}$, then by writing $c(j+1) + 2(r+1) + 1 = c(j+1) + (j+1) + 1 = (c+2)(j+1) + 1$, the claim is also proved. □

Now $\left\lceil \frac{S_n}{n} \right\rceil \ge c + 1$ implies the existence of some ninja path containing at least $c + 1$ red circles, and we are done.

§2.3 IMO 2023/6, proposed by Ankan Bhattacharya, Luke Robitaille (USA)

Available online at https://aops.com/community/p28104331.

Problem statement

Let ABC be an equilateral triangle. Let A_1, B_1, C_1 be interior points of ABC such that $BA_1 = A_1C$, $CB_1 = B_1A$, $AC_1 = C_1B$, and

$$\angle BA_1C + \angle CB_1A + \angle AC_1B = 480^\circ.$$

Let $A_2 = \overline{BC_1} \cap \overline{CB_1}$, $B_2 = \overline{CA_1} \cap \overline{AC_1}$, $C_2 = \overline{AB_1} \cap \overline{BA_1}$. Prove that if triangle $A_1B_1C_1$ is scalene, then the circumcircles of triangles AA_1A_2, BB_1B_2, and CC_1C_2 all pass through two common points.

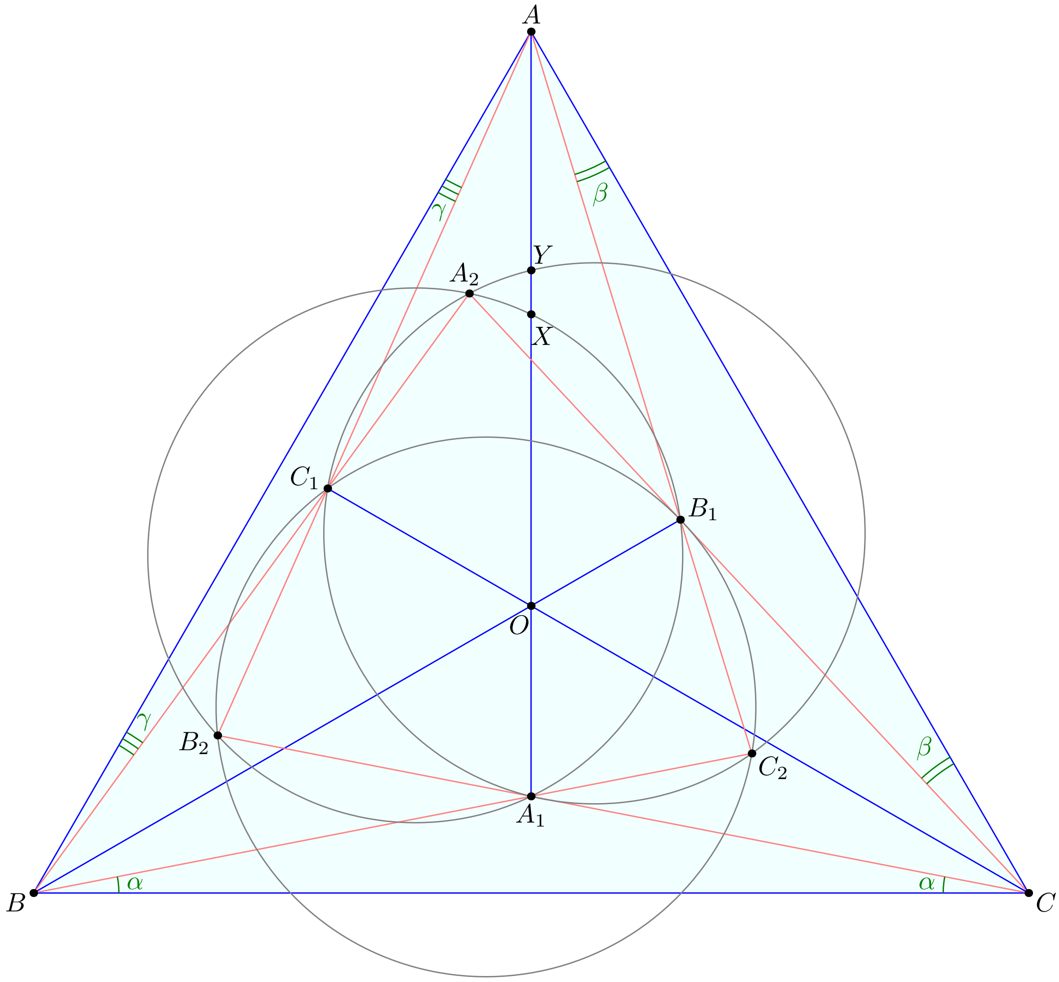
This is the second official solution from the marking scheme, also communicated to me by Michael Ren. Define O as the center of ABC and set the angles

$$\begin{aligned} \alpha &:= \angle A_1CB = \angle CBA_1 \\ \beta &:= \angle ACB_1 = \angle B_1AC \\ \gamma &:= \angle C_1AB = \angle C_1BA \end{aligned}$$

so that

$$\alpha + \beta + \gamma = 30^\circ.$$

In particular, $\max(\alpha, \beta, \gamma) < 30^\circ$, so it follows that A_1 lies inside $\triangle OBC$, and similarly for the others. This means for example that C_1 lies between B and A_2, and so on. Therefore the polygon $A_2C_1B_2A_1C_2B_1$ is convex.



We start by providing the "interpretation" for the 480° angle in the statement:

> **Claim** — Point A_1 is the circumcenter of $\triangle A_2BC$, and similarly for the others.

Proof. We have $\angle BA_1C = 180^\circ - 2\alpha$, and

$$\begin{aligned}\angle BA_2C &= 180^\circ - \angle CBC_1 - \angle B_1CB \\ &= 180^\circ - (60^\circ - \gamma) - (60^\circ - \beta) \\ &= 60^\circ + \beta + \gamma = 90^\circ - \alpha = \frac{1}{2}\angle BA_1C.\end{aligned}$$

Since A_1 lies inside $\triangle BA_2C$, it follows A_1 is exactly the circumcenter. □

> **Claim** — Quadrilateral $B_2C_1B_1C$ can be inscribed in a circle, say γ_a. Circles γ_b and γ_c can be defined similarly. Finally, these three circles are pairwise distinct.

Proof. Using directed angles now, we have

$$\measuredangle B_2B_1C_2 = 180^\circ - \measuredangle AB_1B_2 = 180^\circ - 2\measuredangle ACB = 180^\circ - 2(60^\circ - \alpha) = 60^\circ + 2\alpha.$$

By the same token, $\measuredangle B_2C_1C_2 = 60^\circ + 2\alpha$. This establishes the existence of γ_a.

The proof for γ_b and γ_c is the same. Finally, to show the three circles are distinct, it would be enough to verify that the convex hexagon $A_2C_1B_2A_1C_2B_1$ is not cyclic.

Assume for contradiction it was cyclic. Then

$$360^\circ = \angle C_2A_1B_1 + \angle B_2C_1A_2 + \angle A_2B_1C_2 = \angle BA_1C + \angle CB_1A + \angle AC_1B = 480^\circ$$

which is absurd. This contradiction eliminates the degenerate case, so the three circles are distinct. $\square$

For the remainder of the solution, let $\operatorname{Pow}(P, \omega)$ denote the power of a point P with respect to a circle ω.

Let line AA_1 meet γ_b and γ_c again at X and Y, and set $k_a := \frac{AX}{AY}$. Consider the locus of all points P such that

$$\mathcal{C}_a := \Big\{\text{points } P \text{ in the plane satisfying } \operatorname{Pow}(P, \gamma_b) = k_a \operatorname{Pow}(P, \gamma_c)\Big\}.$$

We recall the *coaxiality lemma*[1], which states that (given γ_b and γ_c are not concentric) the locus $\mathcal{C}_a$ must be either a circle (if $k_a \neq 1$) or a line (if $k_a = 1$).

On the other hand, A_1, A_2, and A all obviously lie on $\mathcal{C}_a$. (For A_1 and A_2, the powers are both zero, and for the point A, we have $\operatorname{Pow}(P, \gamma_b) = AX \cdot AA_1$ and $\operatorname{Pow}(P, \gamma_c) = AY \cdot AA_1$.) So $\mathcal{C}_a$ must be exactly the circumcircle of $\triangle AA_1A_2$ from the problem statement.

We turn to evaluating k_a more carefully. First, note that

$$\angle A_1XB_1 = \angle A_1B_2B_1 = \angle CB_2B_1 = 90^\circ - \angle B_2AC = 90^\circ - (60^\circ - \gamma) = 30^\circ + \gamma.$$

Now using the law of sines, we derive

$$\begin{aligned} \frac{AX}{AB_1} = \frac{\sin \angle AB_1X}{\sin \angle AXB_1} &= \frac{\sin(\angle A_1XB_1 - \angle XAB_1)}{\sin \angle A_1XB_1} \\ &= \frac{\sin\left((30^\circ + \gamma) - (30^\circ - \beta)\right)}{\sin(30^\circ + \gamma)} = \frac{\sin(\beta + \gamma)}{\sin(30^\circ + \gamma)}. \end{aligned}$$

Similarly, $AY = AC_1 \cdot \frac{\sin(\beta+\gamma)}{\sin(30^\circ+\beta)}$, so

$$k_a = \frac{AX}{AY} = \frac{AB_1}{AC_1} \cdot \frac{\sin(30^\circ + \beta)}{\sin(30^\circ + \gamma)}.$$

Now define analogous constants k_b and k_c and circles $\mathcal{C}_b$ and $\mathcal{C}_c$. Owing to the symmetry of the expressions, we have the key relation

$$k_a k_b k_c = 1.$$

In summary, the three circles in the problem statement may be described as

$$\begin{aligned} \mathcal{C}_a &= (AA_1A_2) = \{\text{points } P \text{ such that } \operatorname{Pow}(P, \gamma_b) = k_a \operatorname{Pow}(P, \gamma_c)\} \\ \mathcal{C}_b &= (BB_1B_2) = \{\text{points } P \text{ such that } \operatorname{Pow}(P, \gamma_c) = k_b \operatorname{Pow}(P, \gamma_a)\} \\ \mathcal{C}_c &= (CC_1C_2) = \{\text{points } P \text{ such that } \operatorname{Pow}(P, \gamma_a) = k_c \operatorname{Pow}(P, \gamma_b)\}. \end{aligned}$$

Since k_a, k_b, k_c have product 1, it follows that any point on at least two of the circles must lie on the third circle as well. The convexity of hexagon $A_2C_1B_2A_1C_2B_1$ mentioned earlier ensures these any two of these circles do intersect at two different points, completing the solution.

[1] We quickly outline a proof of this lemma: in the Cartesian coordinate system, the expression $\operatorname{Pow}((x, y), \omega)$ is an expression of the form $x^2 + y^2 + \bullet x + \bullet y + \bullet$ for some constants $\bullet$ whose value does not matter. Substituting this into the equation $\frac{k_a \operatorname{Pow}(P,\gamma_c) - \operatorname{Pow}(P,\gamma_b)}{k_a - 1} = 0$ gives the equation of a circle provided $k_a \neq 1$, and when $k_a = 1$, one instead recovers the radical axis.