USAMO 2019 Solution Notes

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This is a compilation of solutions for the 2019 USAMO. Some of the solutions are my own work, but many are from the official solutions provided by the organizers (for which they hold any copyrights), and others were found by users on the Art of Problem Solving forums.

These notes will tend to be a bit more advanced and terse than the "official" solutions from the organizers. In particular, if a theorem or technique is not known to beginners but is still considered "standard", then I often prefer to use this theory anyways, rather than try to work around or conceal it. For example, in geometry problems I typically use directed angles without further comment, rather than awkwardly work around configuration issues. Similarly, sentences like "let $\mathbb{R}$ denote the set of real numbers" are typically omitted entirely.

Corrections and comments are welcome!

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§0 Problems

1. A function $f\colon \mathbb{N} \to \mathbb{N}$ satisfies

$$\underbrace{f(f(\dots f}_{f(n)\text{ times}}(n)\dots)) = \frac{n^2}{f(f(n))}$$

for all positive integers n. What are all possible values of $f(1000)$?

2. Let $ABCD$ be a cyclic quadrilateral satisfying $AD^2 + BC^2 = AB^2$. The diagonals of $ABCD$ intersect at E. Let P be a point on side $\overline{AB}$ satisfying $\angle APD = \angle BPC$. Show that line PE bisects $\overline{CD}$.

3. Let K be the set of positive integers not containing the decimal digit 7. Determine all polynomials $f(x)$ with nonnegative coefficients such that $f(x) \in K$ for all $x \in K$.

4. Let n be a nonnegative integer. Determine the number of ways to choose sets $S_{ij} \subseteq \{1, 2, \dots, 2n\}$, for all $0 \le i \le n$ and $0 \le j \le n$ (not necessarily distinct), such that

- $|S_{ij}| = i + j$, and
- $S_{ij} \subseteq S_{kl}$ if $0 \le i \le k \le n$ and $0 \le j \le l \le n$.

5. Let m and n be relatively prime positive integers. The numbers $\frac{m}{n}$ and $\frac{n}{m}$ are written on a blackboard. At any point, Evan may pick two of the numbers x and y written on the board and write either their arithmetic mean $\frac{1}{2}(x+y)$ or their harmonic mean $\frac{2xy}{x+y}$. For which (m, n) can Evan write 1 on the board in finitely many steps?

6. Find all polynomials P with real coefficients such that

$$\frac{P(x)}{yz} + \frac{P(y)}{zx} + \frac{P(z)}{xy} = P(x-y) + P(y-z) + P(z-x)$$

for all nonzero real numbers x, y, z obeying $2xyz = x + y + z$.

§1 Solutions to Day 1

§1.1 USAMO 2019/1, proposed by Evan Chen

Available online at https://aops.com/community/p12189527.

Problem statement

A function $f\colon \mathbb{N} \to \mathbb{N}$ satisfies

$$\underbrace{f(f(\dots f}_{f(n) \text{ times}}(n)\dots)) = \frac{n^2}{f(f(n))}$$

for all positive integers n. What are all possible values of $f(1000)$?

Actually, we classify all such functions: f can be any function which fixes odd integers and acts as an involution on the even integers. In particular, $f(1000)$ may be any even integer.

It's easy to check that these all work, so now we check they are the only solutions.

Claim — f is injective.

Proof. If $f(a) = f(b)$, then $a^2 = f^{f(a)}(a) f(f(a)) = f^{f(b)}(b) f(f(b)) = b^2$, so $a = b$. □

Claim — f fixes the odd integers.

Proof. We prove this by induction on odd $n \geq 1$.

Assume f fixes $S = \{1, 3, \dots, n-2\}$ now (allowing $S = \varnothing$ for $n = 1$). Now we have that

$$f^{f(n)}(n) \cdot f^2(n) = n^2.$$

However, neither of the two factors on the left-hand side can be in S since f was injective. Therefore they must both be n, and we have $f^2(n) = n$.

Now let $y = f(n)$, so $f(y) = n$. Substituting y into the given yields

$$y^2 = f^n(y) \cdot y = f^{n+1}(n) \cdot y = ny$$

since $n + 1$ is even. We conclude $n = y$, as desired. □

Thus, f maps even integers to even integers. In light of this, we may let $g = f(f(n))$ (which is also injective), so we conclude that

$$g^{f(n)/2}(n) g(n) = n^2 \qquad \text{for } n = 2, 4, \dots.$$

Claim — The function g is the identity function.

Proof. The proof is similar to the earlier proof of the claim. Note that g fixes the odd integers already. We proceed by induction to show g fixes the even integers; so assume g fixes the set $S = \{1, 2, \dots, n-1\}$, for some even integer $n \ge 2$. In the equation

$$g^{f(n)/2}(n) \cdot g(n) = n^2$$

neither of the two factors may be less than n. So they must both be n. $\square$

These three claims imply that the solutions we claimed earlier are the only ones.

Remark. The last claim is not necessary to solve the problem; after realizing f and injective fixes the odd integers, this answers the question about the values of $f(1000)$. However, we chose to present the "full" solution anyways.

Remark. After noting f is injective, another approach is outlined below. Starting from any n, consider the sequence

$$n, \ f(n), \ f(f(n)),$$

and so on. We may let m be the smallest term of the sequence; then $m^2 = f(f(m)) \cdot f^{f(m)}(m)$ which forces $f(f(m)) = f^{f(m)}(m) = m$ by minimality. Thus the sequence is 2-periodic. Therefore, $f(f(n)) = n$ always holds, which is enough to finish.

¶ Authorship comments. I will tell you a great story about this problem. Two days before the start of grading of USAMO 2017, I had a dream that I was grading a functional equation. When I woke up, I wrote it down, and it was

$$f^{f(n)}(n) = \frac{n^2}{f(f(n))}.$$

You can guess the rest of the story (and imagine how surprised I was the solution set was interesting). I guess some dreams do come true, huh?

§1.2 USAMO 2019/2, proposed by Ankan Bhattacharya

Available online at https://aops.com/community/p12189455.

Problem statement

Let $ABCD$ be a cyclic quadrilateral satisfying $AD^2 + BC^2 = AB^2$. The diagonals of $ABCD$ intersect at E. Let P be a point on side $\overline{AB}$ satisfying $\angle APD = \angle BPC$. Show that line PE bisects $\overline{CD}$.

Here are three solutions. The first two are similar although the first one makes use of symmedians. The last solution by inversion is more advanced.

¶ First solution using symmedians. We define point P to obey

$$\frac{AP}{BP} = \frac{AD^2}{BC^2} = \frac{AE^2}{BE^2}$$

so that $\overline{PE}$ is the E-symmedian of $\triangle EAB$, therefore the E-median of $\triangle ECD$.

Now, note that

$$AD^2 = AP \cdot AB \quad \text{and} \quad BC^2 = BP \cdot BA.$$

This implies $\triangle APD \sim \triangle ADB$ and $\triangle BPC \sim \triangle BCA$. Thus

$$\measuredangle DPA = \measuredangle ADB = \measuredangle ACB = \measuredangle BCP$$

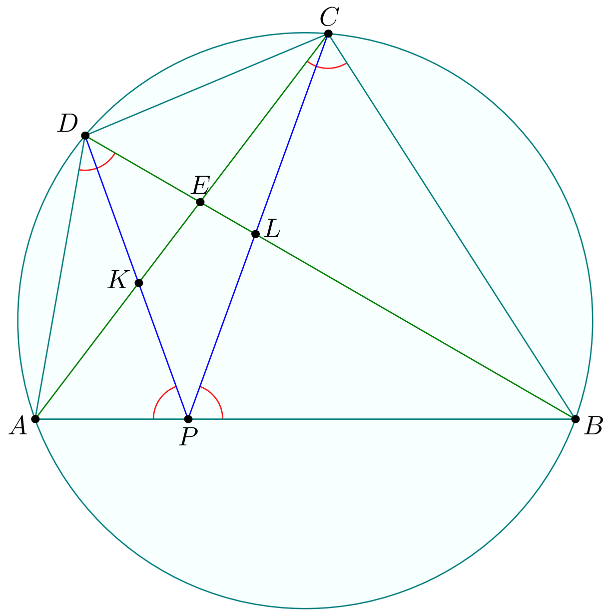
and so P satisfies the condition as in the statement (and is the unique point to do so), as needed.

¶ Second solution using only angle chasing (by proposer). We again re-define P to obey $AD^2 = AP \cdot AB$ and $BC^2 = BP \cdot BA$. As before, this gives $\triangle APD \sim \triangle ABD$ and $\triangle BPC \sim \triangle BDP$ and so we let

$$\theta := \measuredangle DPA = \measuredangle ADB = \measuredangle ACB = \measuredangle BCP.$$

Our goal is to now show $\overline{PE}$ bisects $\overline{CD}$.

Let $K = \overline{AC} \cap \overline{PD}$ and $L = \overline{AD} \cap \overline{PC}$. Since $\measuredangle KPA = \theta = \measuredangle ACB$, quadrilateral $BPKC$ is cyclic. Similarly, so is $APLD$.



Finally $AKLB$ is cyclic since

$$\measuredangle BKA = \measuredangle BKC = \measuredangle BPC = \theta = \measuredangle DPA = \measuredangle DLA = \measuredangle BLA.$$

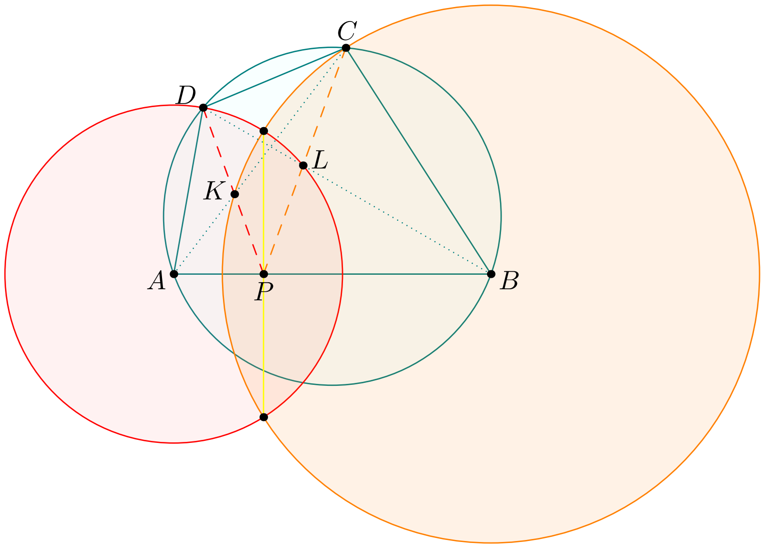
This implies $\measuredangle CKL = \measuredangle LBA = \measuredangle DCK$, so $\overline{KL} \parallel \overline{BC}$. Then PE bisects $\overline{BC}$ by Ceva's theorem on $\triangle PCD$.

¶ Third solution (using inversion). By hypothesis, the circle ω_a centered at A with radius AD is orthogonal to the circle ω_b centered at B with radius BC. For brevity, we let $\mathbf{I}_a$ and $\mathbf{I}_b$ denote inversion with respect to ω_a and ω_b.

We let P denote the intersection of $\overline{AB}$ with the radical axis of ω_a and ω_b; hence $P = \mathbf{I}_a(B) = \mathbf{I}_b(A)$. This already implies that

$$\measuredangle DPA \overset{\mathbf{I}_a}{=} \measuredangle ADB = \measuredangle ACB \overset{\mathbf{I}_b}{=} \measuredangle BPC$$

so P satisfies the angle condition.



Claim — The point $K = \mathbf{I}_a(C)$ lies on ω_b and $\overline{DP}$. Similarly $L = \mathbf{I}_b(D)$ lies on ω_a and $\overline{CP}$.

Proof. The first assertion follows from the fact that ω_b is orthogonal to ω_a. For the other, since (BCD) passes through A, it follows $P = \mathbf{I}_a(B)$, $K = \mathbf{I}_a(C)$, and $D = \mathbf{I}_a(D)$ are collinear. □

Finally, since C, L, P are collinear, we get A is concyclic with $K = \mathbf{I}_a(C)$, $L = \mathbf{I}_a(L)$, $B = \mathbf{I}_a(B)$, i.e. that $AKLB$ is cyclic. So $\overline{KL} \parallel \overline{CD}$ by Reim's theorem, and hence $\overline{PE}$ bisects $\overline{CD}$ by Ceva's theorem.

§1.3 USAMO 2019/3, proposed by Titu Andreescu, Vlad Matei, Cosmin Pohoata

Available online at https://aops.com/community/p12189457.

Problem statement

Let K be the set of positive integers not containing the decimal digit 7. Determine all polynomials $f(x)$ with nonnegative coefficients such that $f(x) \in K$ for all $x \in K$.

The answer is only the obvious ones: $f(x) = 10^e x$, $f(x) = k$, and $f(x) = 10^e x + k$, for any choice of $k \in K$ and $e > \log_{10} k$ (with $e \ge 0$).

Now assume f satisfies $f(K) \subseteq K$; such polynomials will be called *stable*. We first prove the following claim which reduces the problem to the study of monomials.

Lemma (Reduction to monomials)
If $f(x) = a_0 + a_1x + a_2x^2 + \dots$ is stable, then each monomial a_0, a_1x, a_2x^2, $\dots$is stable.

Proof. For any $x \in K$, plug in $f(10^e x)$ for large enough e: the decimal representation of f will contain a_0, a_1x, a_2x^2 with some zeros padded in between. $\square$

Let's tackle the linear case next. Here is an ugly but economical proof.

Claim (Linear classification) — If $f(x) = cx$ is stable, then $c = 10^e$ for some nonnegative integer e.

Proof. We will show when $c \neq 10^e$ then we can find $x \in K$ such that cx starts with the digit 7. This can actually be done with the following explicit cases in terms of how c starts in decimal notation:

- For $9 \cdot 10^e \le c < 10 \cdot 10^e$, pick $x = 8$.
- For $8 \cdot 10^e \le c < 9 \cdot 10^e$, pick $x = 88$.
- For $7 \cdot 10^e \le c < 8 \cdot 10^e$, pick $x = 1$.
- For $4.4 \cdot 10^e \le c < 7 \cdot 10^e$, pick $11 \le x \le 16$.
- For $2.7 \cdot 10^e \le c < 4.4 \cdot 10^e$, pick $18 \le x \le 26$.
- For $2 \cdot 10^e \le c < 2.7 \cdot 10^e$, pick $28 \le x \le 36$.
- For $1.6 \cdot 10^e \le c < 2 \cdot 10^e$, pick $38 \le x \le 46$.
- For $1.3 \cdot 10^e \le c < 1.6 \cdot 10^e$, pick $48 \le x \le 56$.
- For $1.1 \cdot 10^e \le c < 1.3 \cdot 10^e$, pick $58 \le x \le 66$.
- For $1 \cdot 10^e \le c < 1.1 \cdot 10^e$, pick $x = 699\dots9$ for suitably many 9's. $\square$

The hardest part of the problem is the case where $\deg f > 1$. We claim that no solutions exist then:

Claim (Higher-degree classification) — No monomial of the form $f(x) = cx^d$ is stable for any $d > 1$.

Proof. Note that $f(10x+3)$ is stable too. Thus

$$f(10x+3) = 3^d + 10d \cdot 3^{d-1}x + 100\binom{d}{2} \cdot 3^{d-1}x^2 + \dots$$

is stable. By applying the lemma the linear monomial $10d \cdot 3^{d-1}x$ is stable, so $10d \cdot 3^{d-1}$ is a power of 10, which can only happen if $d = 1$. □

Thus the only nonconstant stable polynomials with nonnegative coefficients must be of the form $f(x) = 10^e x + k$ for $e \geq 0$. It is straightforward to show we then need $k < 10^e$ and this finishes the proof.

Remark. The official solution replaces the proof for $f(x) = cx$ with Kronecker density. From $f(1) = c \in K$, we get $f(c) = c^2 \in K$, et cetera and hence $c^n \in K$. But it is known that when c is not a power of 10, some power of c starts with any specified prefix.

§2 Solutions to Day 2

§2.1 USAMO 2019/4, proposed by Ricky Liu

Available online at https://aops.com/community/p12195861.

Problem statement

Let n be a nonnegative integer. Determine the number of ways to choose sets $S_{ij} \subseteq \{1, 2, \dots, 2n\}$, for all $0 \le i \le n$ and $0 \le j \le n$ (not necessarily distinct), such that

- $|S_{ij}| = i + j$, and
- $S_{ij} \subseteq S_{kl}$ if $0 \le i \le k \le n$ and $0 \le j \le l \le n$.

The answer is $(2n)! \cdot 2^{n^2}$. First, we note that $\varnothing = S_{00} \subsetneq S_{01} \subsetneq \dots \subsetneq S_{nn} = \{1, \dots, 2n\}$ and thus multiplying by $(2n)!$ we may as well assume $S_{0i} = \{1, \dots, i\}$ and $S_{in} = \{1, \dots, n+i\}$. We illustrate this situation by placing the sets in a grid, as below for $n = 4$; our goal is to fill in the rest of the grid.

$$\begin{bmatrix} 1234 & 12345 & 123456 & 1234567 & 12345678 \\ 123 & & & & \\ 12 & & & & \\ 1 & & & & \\ \varnothing & & & & \end{bmatrix}$$

We claim the number of ways to do so is 2^{n^2}. In fact, more strongly even the partial fillings are given exactly by powers of 2.

Claim — Fix a choice T of cells we wish to fill in, such that whenever a cell is in T, so are all the cells above and left of it. (In other words, T is a Young tableau.) The number of ways to fill in these cells with sets satisfying the inclusion conditions is $2^{|T|}$.

An example is shown below, with an indeterminate set marked in red (and the rest of T marked in blue).

$$\begin{bmatrix} 1234 & 12345 & 123456 & 1234567 & 12345678 \\ 123 & \color{blue}{1234} & \color{blue}{12346} & \color{blue}{123467} & \\ 12 & \color{blue}{124} & \color{red}{1234 \text{ or } 1246} & & \\ 1 & \color{blue}{12} & & & \\ \varnothing & \color{blue}{2} & & & \end{bmatrix}$$

Proof. The proof is by induction on $|T|$, with $|T| = 0$ being vacuous.

Now suppose we have a corner $\begin{bmatrix} B & C \\ A & \color{red}{S} \end{bmatrix}$ where A, B, C are fixed and S is to be chosen. Then we may write $B = A \cup \{x\}$ and $C = A \cup \{x, y\}$ for $x, y \notin A$. Then the two choices of S are $A \cup \{x\}$ (i.e. B) and $A \cup \{y\}$, and both of them are seen to be valid.

In this way, we gain a factor of 2 any time we add one cell as above to T. Since we can achieve any Young tableau in this way, the induction is complete. $\square$

§2.2 USAMO 2019/5, proposed by Yannick Yao

Available online at https://aops.com/community/p12195834.

Problem statement

Let m and n be relatively prime positive integers. The numbers $\frac{m}{n}$ and $\frac{n}{m}$ are written on a blackboard. At any point, Evan may pick two of the numbers x and y written on the board and write either their arithmetic mean $\frac{1}{2}(x+y)$ or their harmonic mean $\frac{2xy}{x+y}$. For which (m,n) can Evan write 1 on the board in finitely many steps?

We claim this is possible if and only $m+n$ is a power of 2. Let $q = m/n$, so the numbers on the board are q and $1/q$.

Impossibility: The main idea is the following.

Claim — Suppose p is an odd prime. Then if the initial numbers on the board are $-1 \pmod p$, then all numbers on the board are $-1 \pmod p$.

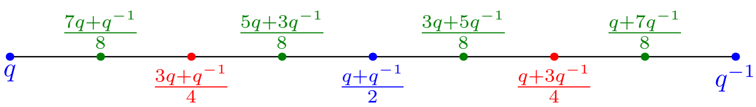
Proof. Let $a \equiv b \equiv -1 \pmod p$. Note that $2 \not\equiv 0 \pmod p$ and $a+b \equiv -2 \not\equiv 0 \pmod p$. Thus $\frac{a+b}{2}$ and $\frac{2ab}{a+b}$ both make sense modulo p and are equal to $-1 \pmod p$. □

Thus if there exists *any* odd prime divisor p of $m+n$ (implying $p \nmid mn$), then

$$q \equiv \frac{1}{q} \equiv -1 \pmod p.$$

and hence all numbers will be $-1 \pmod p$ forever. This implies that it's impossible to write 1, whenever $m+n$ is divisible by some odd prime.

Construction: Conversely, suppose $m+n$ is a power of 2. We will actually construct 1 without even using the harmonic mean.



Note that

$$\frac{n}{m+n} \cdot q + \frac{m}{m+n} \cdot \frac{1}{q} = 1$$

and obviously by taking appropriate midpoints (in a binary fashion) we can achieve this using arithmetic mean alone.

§2.3 USAMO 2019/6, proposed by Titu Andreescu, Gabriel Dospinescu

Available online at https://aops.com/community/p12195858.

Problem statement

Find all polynomials P with real coefficients such that

$$\frac{P(x)}{yz} + \frac{P(y)}{zx} + \frac{P(z)}{xy} = P(x-y) + P(y-z) + P(z-x)$$

for all nonzero real numbers x, y, z obeying $2xyz = x + y + z$.

The given can be rewritten as saying that

$$\begin{aligned} Q(x,y,z) &:= xP(x) + yP(y) + zP(z) \\ &\quad - xyz\left(P(x-y) + P(y-z) + P(z-x)\right) \end{aligned}$$

is a polynomial vanishing whenever $xyz \neq 0$ and $2xyz = x + y + z$, for real numbers x, y, z.

Claim — This means $Q(x,y,z)$ vanishes also for any complex numbers x, y, z obeying $2xyz = x + y + z$.

Proof. Indeed, this means that the rational function

$$R(x,y) := Q\left(x, y, \frac{x+y}{2xy-1}\right)$$

vanishes for any real numbers x and y such that $xy \neq \frac{1}{2}$, $x \neq 0$, $y \neq 0$, $x + y \neq 0$. This can only occur if R is identically zero as a rational function with real coefficients. If we then regard R as having complex coefficients, the conclusion then follows. □

Remark (Algebraic geometry digression on real dimension)**.** Note here we use in an essential way that z can be solved for in terms of x and y. If $s(x,y,z) = 2xyz - (x+y+z)$ is replaced with some general condition, the result may become false; e.g. we would certainly not expect the result to hold when $s(x,y,z) = x^2 + y^2 + z^2 - (xy + yz + zx)$ since for real numbers $s = 0$ only when $x = y = z$!

The general condition we need here is that $s(x,y,z) = 0$ should have "real dimension two". Here is a proof using this language, in our situation.

Let $M \subset \mathbb{R}^3$ be the surface $s = 0$. We first contend M is two-dimensional manifold. Indeed, the gradient $\nabla s = \langle 2yz - 1, 2zx - 1, 2xy - 1 \rangle$ vanishes only at the points $(\pm 1/\sqrt{2}, \pm 1/\sqrt{2}, \pm 1/\sqrt{2})$ where the $\pm$ signs are all taken to be the same. These points do not lie on M, so the result follows by the *regular value theorem*. In particular the topological closure of points on M with $xyz \neq 0$ is all of M itself; so Q vanishes on all of M.

If we now identify M with the semi-algebraic set consisting of maximal ideals $(x - a, y - b, z - c)$ in $\operatorname{Spec} \mathbb{R}[x,y,z]$ satisfying $2abc = a + b + c$, then we have real dimension two, and thus the Zariski closure of M is a two-dimensional closed subset of $\operatorname{Spec} \mathbb{R}[x,y,z]$. Thus it must be $Z = \mathcal{V}(2xyz - (x + y + z))$, since this Z is an irreducible two-dimensional closed subset (say, by *Krull's principal ideal theorem*) containing M. Now Q is a global section vanishing on all of Z, therefore Q is contained in the (radical, principal) ideal $(2xyz - (x + y + z))$ as needed. So it is actually divisible by $2xyz - (x + y + z)$ as desired.

Now we regard P and Q as complex polynomials instead. First, note that substituting $(x, y, z) = (t, -t, 0)$ implies P is even. We then substitute

$$(x, y, z) = \left(x, \frac{i}{\sqrt{2}}, \frac{-i}{\sqrt{2}}\right)$$

to get

$$\begin{aligned} &xP(x) + \frac{i}{\sqrt{2}}\left(P\left(\frac{i}{\sqrt{2}}\right) - P\left(\frac{-i}{\sqrt{2}}\right)\right) \\ &= \frac{1}{2}x\left(P(x - i/\sqrt{2}) + P(x + i/\sqrt{2}) + P(\sqrt{2}i)\right) \end{aligned}$$

which in particular implies that

$$P\left(x + \frac{i}{\sqrt{2}}\right) + P\left(x - \frac{i}{\sqrt{2}}\right) - 2P(x) \equiv P(\sqrt{2}i)$$

identically in x. The left-hand side is a second-order finite difference in x (up to scaling the argument), and the right-hand side is constant, so this implies $\deg P \leq 2$.

Since P is even and $\deg P \leq 2$, we must have $P(x) = cx^2 + d$ for some real numbers c and d. A quick check now gives the answer $P(x) = c(x^2 + 3)$ which all work.