IMO 2014 Solution Notes

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This is a compilation of solutions for the 2014 IMO. Some of the solutions are my own work, but many are from the official solutions provided by the organizers (for which they hold any copyrights), and others were found by users on the Art of Problem Solving forums.

These notes will tend to be a bit more advanced and terse than the "official" solutions from the organizers. In particular, if a theorem or technique is not known to beginners but is still considered "standard", then I often prefer to use this theory anyways, rather than try to work around or conceal it. For example, in geometry problems I typically use directed angles without further comment, rather than awkwardly work around configuration issues. Similarly, sentences like "let $\mathbb{R}$ denote the set of real numbers" are typically omitted entirely.

Corrections and comments are welcome!

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§0 Problems

1. Let $a_0 < a_1 < a_2 < \cdots$ be an infinite sequence of positive integers. Prove that there exists a unique integer $n \ge 1$ such that

$$a_n < \frac{a_0 + a_1 + a_2 + \cdots + a_n}{n} \le a_{n+1}.$$

2. Let $n \ge 2$ be an integer. Consider an $n \times n$ chessboard consisting of n^2 unit squares. A configuration of n rooks on this board is *peaceful* if every row and every column contains exactly one rook. Find the greatest positive integer k such that, for each peaceful configuration of n rooks, there is a $k \times k$ square which does not contain a rook on any of its k^2 unit squares.

3. Convex quadrilateral $ABCD$ has $\angle ABC = \angle CDA = 90^\circ$. Point H is the foot of the perpendicular from A to $\overline{BD}$. Points S and T lie on sides AB and AD, respectively, such that H lies inside triangle SCT and

$$\angle CHS - \angle CSB = 90^\circ, \quad \angle THC - \angle DTC = 90^\circ.$$

Prove that line BD is tangent to the circumcircle of triangle TSH.

4. Let P and Q be on segment BC of an acute triangle ABC such that $\angle PAB = \angle BCA$ and $\angle CAQ = \angle ABC$. Let M and N be points on $\overline{AP}$ and $\overline{AQ}$, respectively, such that P is the midpoint of $\overline{AM}$ and Q is the midpoint of $\overline{AN}$. Prove that $\overline{BM}$ and $\overline{CN}$ meet on the circumcircle of $\triangle ABC$.

5. For every positive integer n, the Bank of Cape Town issues coins of denomination $\frac{1}{n}$. Given a finite collection of such coins (of not necessarily different denominations) with total value at most $99 + \frac{1}{2}$, prove that it is possible to split this collection into 100 or fewer groups, such that each group has total value at most 1.

6. A set of lines in the plane is in *general position* if no two are parallel and no three pass through the same point. A set of lines in general position cuts the plane into regions, some of which have finite area; we call these its *finite regions*. Prove that for all sufficiently large n, in any set of n lines in general position it is possible to colour at least $\sqrt{n}$ lines blue in such a way that none of its finite regions has a completely blue boundary.

§1 Solutions to Day 1

§1.1 IMO 2014/1, proposed by Gerhard Woeginger (AUT)

Available online at https://aops.com/community/p3542095.

Problem statement

Let $a_0 < a_1 < a_2 < \cdots$ be an infinite sequence of positive integers. Prove that there exists a unique integer $n \geq 1$ such that

$$a_n < \frac{a_0 + a_1 + a_2 + \cdots + a_n}{n} \leq a_{n+1}.$$

Fedor Petrov presents the following nice solution. Let us define the sequence

$$b_n = (a_n - a_{n-1}) + \cdots + (a_n - a_1).$$

Since $(a_i)_i$ is increasing, this sequence is unbounded, and moreover $b_1 = 0$. The problem requires an n such that

$$b_n < a_0 \leq b_{n+1}$$

which obviously exists and is unique.

§1.2 IMO 2014/2, proposed by Croatia

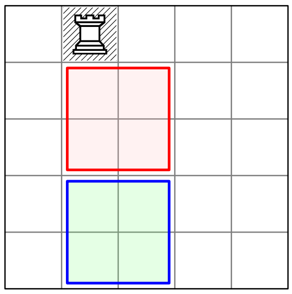
Available online at https://aops.com/community/p3542094.

Problem statement

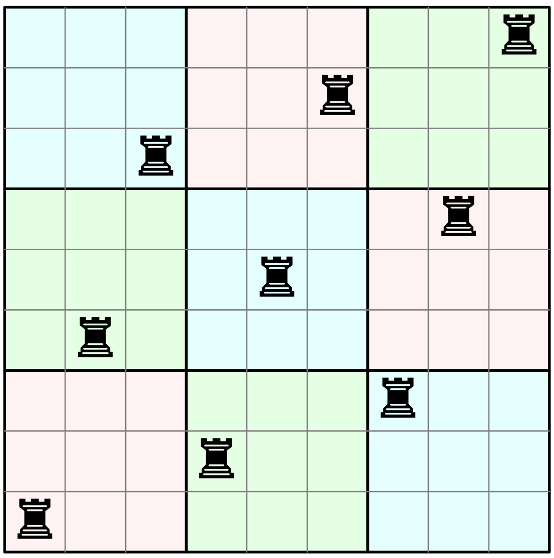
Let $n \geq 2$ be an integer. Consider an $n \times n$ chessboard consisting of n^2 unit squares. A configuration of n rooks on this board is *peaceful* if every row and every column contains exactly one rook. Find the greatest positive integer k such that, for each peaceful configuration of n rooks, there is a $k \times k$ square which does not contain a rook on any of its k^2 unit squares.

The answer is $k = \lfloor \sqrt{n-1} \rfloor$, sir.

First, assume $n > k^2$ for some k. We will prove we can find an empty $k \times k$ square. Indeed, let R be a rook in the uppermost column, and draw k squares of size $k \times k$ directly below it, aligned. There are at most $k-1$ rooks among these squares, as desired.



Now for the construction for $n = k^2$. We draw the example for $k = 3$ (with the generalization being obvious);



To show that this works, consider for each rook drawing an $k \times k$ square of X's whose bottom-right hand corner is the rook (these may go off the board). These indicate positions where one cannot place the upper-left hand corner of any square. It's easy to see that these cover the entire board, except parts of the last $k-1$ columns, which don't matter anyways.

It remains to check that $n \le k^2$ also all work (omitting this step is a common mistake). For this, we can delete rows and column to get an $n \times n$ board, and then fill in any gaps where we accidentally deleted a rook.

§1.3 IMO 2014/3, proposed by Iran

Available online at https://aops.com/community/p3542092.

Problem statement

Convex quadrilateral $ABCD$ has $\angle ABC = \angle CDA = 90^\circ$. Point H is the foot of the perpendicular from A to $\overline{BD}$. Points S and T lie on sides AB and AD, respectively, such that H lies inside triangle SCT and

$$\angle CHS - \angle CSB = 90^\circ, \quad \angle THC - \angle DTC = 90^\circ.$$

Prove that line BD is tangent to the circumcircle of triangle TSH.

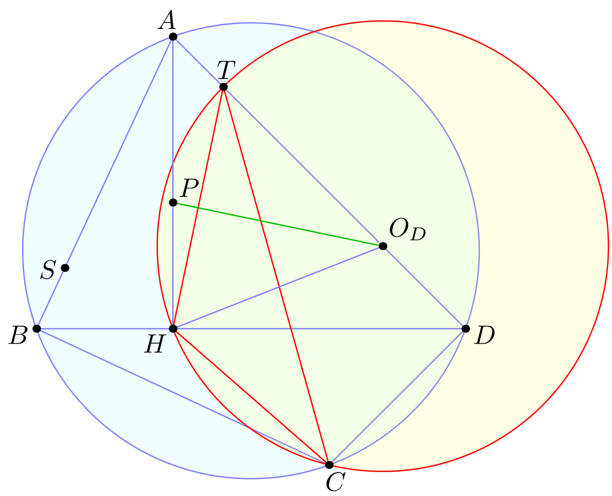
¶ **First solution (mine).** First we rewrite the angle condition in a suitable way.

Claim — We have $\angle ATH = \angle TCH + 90^\circ$. Thus the circumcenter of $\triangle CTH$ lies on $\overline{AD}$. Similarly the circumcenter of $\triangle CSH$ lies on $\overline{AB}$.

Proof.

$$\begin{aligned} \measuredangle ATH &= \measuredangle DTH \\ &= \measuredangle DTC + \measuredangle CTH \\ &= \measuredangle DTC - \measuredangle THC - \measuredangle HCT \\ &= 90^\circ - \measuredangle HCT = 90^\circ + \measuredangle TCH. \end{aligned}$$

which implies conclusion. □



Let the perpendicular bisector of TH meet AH at P now. It suffices to show that $\frac{AP}{PH}$ is symmetric in $b = AD$ and $d = AB$, because then P will be the circumcenter of $\triangle TSH$. To do this, set $AH = \frac{bd}{2R}$ and $AC = 2R$.

Let O denote the circumcenter of $\triangle CHT$. Use the Law of Cosines on $\triangle ACO$ and $\triangle AHO$, using variables $x = AO$ and $r = HO$. We get that

$$r^2 = x^2 + AH^2 - 2x \cdot AH \cdot \frac{d}{2R} = x^2 + (2R)^2 - 2bx.$$

By the angle bisector theorem, $\frac{AP}{PH} = \frac{AO}{HO}$.

The rest is computation: notice that

$$r^2 - x^2 = h^2 - 2xh \cdot \frac{d}{2R} = (2R)^2 - 2bx$$

where $h = AH = \frac{bd}{2R}$, whence

$$x = \frac{(2R)^2 - h^2}{2b - 2h \cdot \frac{d}{2R}}.$$

Moreover,

$$\frac{1}{2}\left(\frac{r^2}{x^2} - 1\right) = \frac{1}{x}\left(\frac{2}{x}R^2 - b\right).$$

Now, if we plug in the x in the right-hand side of the above, we obtain

$$\frac{2b - 2h \cdot \frac{d}{2R}}{4R^2 - h^2}\left(\frac{2b - 2h \cdot \frac{d}{2R}}{4R^2 - h^2} \cdot 2R^2 - b\right) = \frac{2h}{(4R^2 - h^2)^2}\left(b - h \cdot \frac{d}{2R}\right)\left(-2hdR + bh^2\right).$$

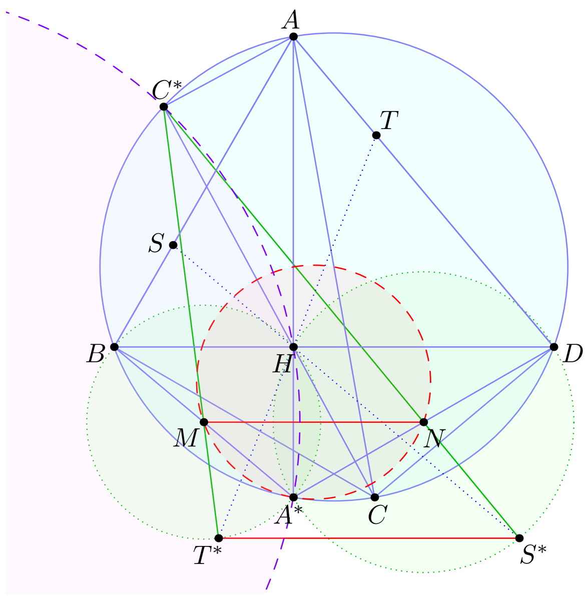
Pulling out a factor of $-2Rh$ from the rightmost term, we get something that is symmetric in b and d, as required.

¶ Second solution (Victor Reis). Here is the fabled solution using inversion at H. First, we rephrase the angle conditions in the following ways:

- $\overline{AD} \perp (THC)$, which is equivalent to the claim from the first solution.
- $\overline{AB} \perp (SHC)$, by symmetry.
- $\overline{AC} \perp (ABCD)$, by definition.

Now for concreteness we will use a negative inversion at H which swaps B and D and overlay it on the original diagram. As usual we denote inverses with stars.

Let us describe the inverted problem. We let M and N denote the midpoints of $\overline{A^*B^*}$ and $\overline{A^*D^*}$, which are the centers of (HA^*B^*) and (HA^*D^*). From $\overline{T^*C^*} \perp (HA^*D^*)$, we know have C^*, M, T^* collinear. Similarly, C^*, N, S^* are collinear. We have that (A^*HC^*) is orthogonal to $(ABCD)$ which remains fixed. We wish to show $\overline{T^*S^*}$ and $\overline{MN}$ are parallel.



Lot ω denote the circumcircle of $\triangle A^*HC^*$, which is orthogonal to the original circle $(ABCD)$. It would suffices to show (A^*HC^*) is an H-Apollonius circle with respect to $\overline{MN}$, from which we would get $C^*M/HM = C^*N/HN$.

However, ω through H and A, hence it center lies on line MN. Moreover ω is orthogonal to (A^*MN) (since (A^*MN) and (A^*BD) are homothetic). This is enough (for example, if we let O denote the center of ω, we now have $\mathrm{r}(\omega)^2 = OH^2 = OM \cdot ON$). (Note in this proof that the fact that C^* lies on $(ABCD)$ is not relevant.)

§2 Solutions to Day 2

§2.1 IMO 2014/4, proposed by Giorgi Arabidze (GEO)

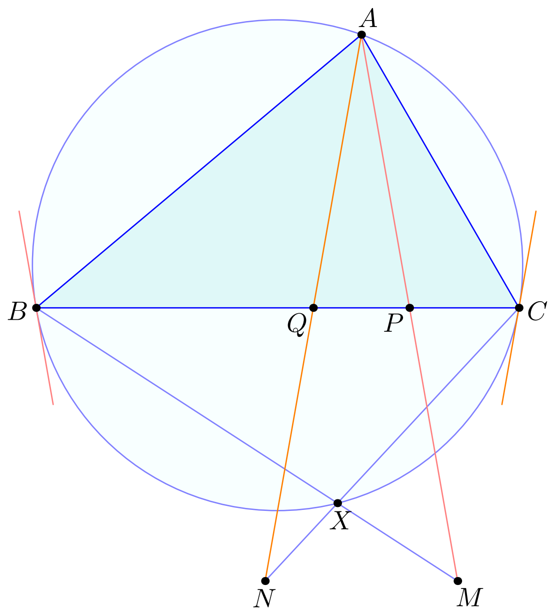
Available online at https://aops.com/community/p3543136.

Problem statement

Let P and Q be on segment BC of an acute triangle ABC such that $\angle PAB = \angle BCA$ and $\angle CAQ = \angle ABC$. Let M and N be points on $\overline{AP}$ and $\overline{AQ}$, respectively, such that P is the midpoint of $\overline{AM}$ and Q is the midpoint of $\overline{AN}$. Prove that $\overline{BM}$ and $\overline{CN}$ meet on the circumcircle of $\triangle ABC$.

We give three solutions.

¶ First solution by harmonic bundles. Let $\overline{BM}$ intersect the circumcircle again at X.



The angle conditions imply that the tangent to (ABC) at B is parallel to $\overline{AP}$. Let ∞ be the point at infinity along line AP. Then

$$-1 = (AM; P\infty) \overset{B}{=} (AX; BC).$$

Similarly, if $\overline{CN}$ meets the circumcircle at Y then $(AY; BC) = -1$ as well. Hence $X = Y$, which implies the problem.

¶ Second solution by similar triangles. Once one observes $\triangle CAQ \sim \triangle CBA$, one can construct D the reflection of B across A, so that $\triangle CAN \sim \triangle CBD$. Similarly, letting E be the reflection of C across A, we get $\triangle BAP \sim \triangle BCA \implies \triangle BAM \sim \triangle BCE$. Now to show $\angle ABM + \angle ACN = 180^\circ$ it suffices to show $\angle EBC + \angle BCD = 180^\circ$, which follows since $BCDE$ is a parallelogram.

¶ **Third solution by barycentric coordinates.** Since $PB = c^2/a$ we have

$$P = (0 : a^2 - c^2 : c^2)$$

so the reflection $\vec{M} = 2\vec{P} - \vec{A}$ has coordinates

$$M = (-a^2 : 2(a^2 - c^2) : 2c^2).$$

Similarly $N = (-a^2 : 2b^2 : 2(b^2 - a^2))$. Thus

$$\overline{BM} \cap \overline{CN} = (-a^2 : 2b^2 : 2c^2)$$

which clearly lies on the circumcircle, and is in fact the point identified in the first solution.

§2.2 IMO 2014/5, proposed by Luxembourg

Available online at https://aops.com/community/p3543144.

Problem statement

For every positive integer n, the Bank of Cape Town issues coins of denomination $\frac{1}{n}$. Given a finite collection of such coins (of not necessarily different denominations) with total value at most $99 + \frac{1}{2}$, prove that it is possible to split this collection into 100 or fewer groups, such that each group has total value at most 1.

We'll prove the result for at most $k - \frac{k}{2k+1}$ with k groups. First, perform the following optimizations.

- If any coin of size $\frac{1}{2m}$ appears twice, then replace it with a single coin of size $\frac{1}{m}$.
- If any coin of size $\frac{1}{2m+1}$ appears $2m+1$ times, group it into a single group and induct downwards.

Apply this operation repeatedly until it cannot be done anymore.

Now construct boxes B_0, B_1, …, B_{k-1}. In box B_0 put any coins of size $\frac{1}{2}$ (clearly there is at most one). In the other boxes B_m, put coins of size $\frac{1}{2m+1}$ and $\frac{1}{2m+2}$ (at most $2m$ of the former and at most one of the latter). Note that the total weight in the box is less than 1. Finally, place the remaining "light" coins of size at most $\frac{1}{2k+1}$ in a pile.

Then just toss coins from the pile into the boxes arbitrarily, other than the proviso that no box should have its weight exceed 1. We claim this uses up all coins in the pile. Assume not, and that some coin remains in the pile when all the boxes are saturated. Then all the boxes must have at least $1 - \frac{1}{2k+1}$, meaning the total amount in the boxes is strictly greater than

$$k\left(1 - \frac{1}{2k+1}\right) > k - \tfrac{1}{2}$$

which is a contradiction.

Remark. This gets a stronger bound $k - \frac{k}{2k+1}$ than the requested $k - \frac{1}{2}$.

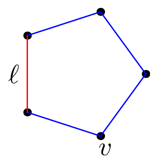
§2.3 IMO 2014/6, proposed by Austria

Available online at https://aops.com/community/p3543151.

Problem statement

A set of lines in the plane is in *general position* if no two are parallel and no three pass through the same point. A set of lines in general position cuts the plane into regions, some of which have finite area; we call these its *finite regions*. Prove that for all sufficiently large n, in any set of n lines in general position it is possible to colour at least $\sqrt{n}$ lines blue in such a way that none of its finite regions has a completely blue boundary.

Suppose we have colored k of the lines blue, and that it is not possible to color any additional lines. That means any of the $n-k$ non-blue lines is the side of some finite region with an otherwise entirely blue perimeter. For each such line ℓ, select one such region, and take the next counterclockwise vertex; this is the intersection of two blue lines v. We'll say ℓ is the *eyelid* of v.



You can prove without too much difficulty that every intersection of two blue lines has at most two eyelids. Since there are $\binom{k}{2}$ such intersections, we see that

$$n - k \le 2\binom{k}{2} = k^2 - k$$

so $n \le k^2$, as required.

> **Remark.** In fact, $k = \sqrt{n}$ is "sharp for greedy algorithms", as illustrated below for $k = 3$: