IMO 2009 Solution Notes

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11 December 2023

This is a compilation of solutions for the 2009 IMO. Some of the solutions are my own work, but many are from the official solutions provided by the organizers (for which they hold any copyrights), and others were found by users on the Art of Problem Solving forums.

These notes will tend to be a bit more advanced and terse than the "official" solutions from the organizers. In particular, if a theorem or technique is not known to beginners but is still considered "standard", then I often prefer to use this theory anyways, rather than try to work around or conceal it. For example, in geometry problems I typically use directed angles without further comment, rather than awkwardly work around configuration issues. Similarly, sentences like "let $\mathbb{R}$ denote the set of real numbers" are typically omitted entirely.

Corrections and comments are welcome!

Contents

§0 Problems

1. Let $n, k \geq 2$ be positive integers and let a_1, a_2, a_3, ..., a_k be distinct integers in the set $\{1, 2, \ldots, n\}$ such that n divides $a_i(a_{i+1} - 1)$ for $i = 1, 2, \ldots, k - 1$. Prove that n does not divide $a_k(a_1 - 1)$.

2. Let ABC be a triangle with circumcenter O. The points P and Q are interior points of the sides CA and AB respectively. Let K, L, M be the midpoints of $\overline{BP}$, $\overline{CQ}$, $\overline{PQ}$, respectively, and let Γ be the circumcircle of $\triangle KLM$. Suppose that $\overline{PQ}$ is tangent to Γ. Prove that $OP = OQ$.

3. Suppose that $s_1, s_2, s_3, \ldots$ is a strictly increasing sequence of positive integers such that the sub-sequences $s_{s_1}, s_{s_2}, s_{s_3}, \ldots$ and $s_{s_1+1}, s_{s_2+1}, s_{s_3+1}, \ldots$ are both arithmetic progressions. Prove that the sequence $s_1, s_2, s_3, \ldots$ is itself an arithmetic progression.

4. Let ABC be a triangle with $AB = AC$. The angle bisectors of $\angle CAB$ and $\angle ABC$ meet the sides BC and CA at D and E, respectively. Let K be the incenter of triangle ADC. Suppose that $\angle BEK = 45°$. Find all possible values of $\angle CAB$.

5. Find all functions $f \colon \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ such that for positive integers a and b, the numbers

$$a, \qquad f(b), \qquad f(b + f(a) - 1)$$

are the sides of a non-degenerate triangle.

6. Let a_1, a_2, ..., a_n be distinct positive integers and let M be a set of $n - 1$ positive integers not containing $s = a_1 + \cdots + a_n$. A grasshopper is to jump along the real axis, starting at the point 0 and making n jumps to the right with lengths a_1, a_2, ..., a_n in some order. Prove that the order can be chosen in such a way that the grasshopper never lands on any point in M.

§1 Solutions to Day 1

§1.1 IMO 2009/1

Available online at https://aops.com/community/p1561571.

> **Problem statement**
>
> Let $n, k \geq 2$ be positive integers and let a_1, a_2, a_3, ..., a_k be distinct integers in the set $\{1, 2, \ldots, n\}$ such that n divides $a_i(a_{i+1} - 1)$ for $i = 1, 2, \ldots, k - 1$. Prove that n does not divide $a_k(a_1 - 1)$.

We proceed indirectly and assume that

$$a_i(a_{i+1} - 1) \equiv 0 \pmod{n}$$

for $i = 1, \ldots, k$ (indices taken modulo k). We claim that this implies all the a_i are equal modulo n.

Let $q = p^e$ be any prime power dividing n. Then, $a_1(a_2 - 1) \equiv 0 \pmod{q}$, so p divides either a_1 or a_2.

- If $p \mid a_1$, then $p \nmid a_1 - 1$. Then

$$a_k(a_1 - 1) \equiv 0 \pmod{q} \implies a_k \equiv 0 \pmod{q}.$$

  In particular, $p \mid a_k$. So repeating this argument, we get $a_{k-1} \equiv 0 \pmod{q}$, $a_{k-2} \equiv 0 \pmod{q}$, and so on.

- Similarly, if $p \mid a_2 - 1$ then $p \nmid a_2$, and from

$$a_2(a_3 - 1) \equiv 0 \pmod{q} \implies a_3 \equiv 1 \pmod{q}.$$

  In particular, $p \mid a_3 - 1$. So repeating this argument, we get $a_4 \equiv 0 \pmod{q}$, $a_5 \equiv 0 \pmod{q}$, and so on.

Either way, we find $a_i \pmod{q}$ is constant (and either 0 or 1).

Since q was an arbitrary prime power dividing n, by Chinese remainder theorem we conclude that $a_i \pmod{n}$ is constant as well. But this contradicts the assumption of distinctness.

§1.2 IMO 2009/2

Available online at https://aops.com/community/p1561572.

Problem statement

Let ABC be a triangle with circumcenter O. The points P and Q are interior points of the sides CA and AB respectively. Let K, L, M be the midpoints of $\overline{BP}$, $\overline{CQ}$, $\overline{PQ}$, respectively, and let Γ be the circumcircle of $\triangle KLM$. Suppose that $\overline{PQ}$ is tangent to Γ. Prove that $OP = OQ$.

By power of a point, we have $-AQ \cdot QB = OQ^2 - R^2$ and $-AP \cdot PC = OP^2 - R^2$. Therefore, it suffices to show $AQ \cdot QB = AP \cdot PC$.
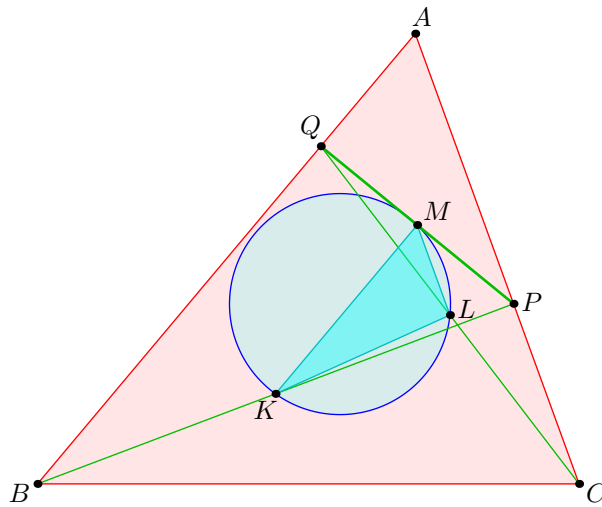
As $\overline{ML} \parallel \overline{AC}$ and $\overline{MK} \parallel \overline{AB}$ we have that

$$\angle APQ = \angle LMP = \angle LKM$$
$$\angle PQA = \angle KMQ = \angle MLK$$

and consequently we have the (opposite orientation) similarity

$$\triangle APQ \stackrel{-}{\sim} \triangle MKL.$$

Therefore

$$\frac{AQ}{AP} = \frac{ML}{MK} = \frac{2ML}{2MK} = \frac{PC}{QB}$$

id est $AQ \cdot QB = AP \cdot PC$, which is what we wanted to prove.

§1.3 IMO 2009/3, proposed by Gabriel Carroll

Available online at https://aops.com/community/p1561573.

Problem statement

Suppose that $s_1, s_2, s_3, \ldots$ is a strictly increasing sequence of positive integers such that the sub-sequences $s_{s_1}, s_{s_2}, s_{s_3}, \ldots$ and $s_{s_1+1}, s_{s_2+1}, s_{s_3+1}, \ldots$ are both arithmetic progressions. Prove that the sequence $s_1, s_2, s_3, \ldots$ is itself an arithmetic progression.

We present two solutions.

¶ **First solution (Alex Zhai).** Let $s(n) \coloneqq s_n$ and write

$$s(s(n)) = Dn + A$$
$$s(s(n) + 1) = D'n + B.$$

In light of the bounds $s(s(n)) \le s(s(n) + 1) \le s(s(n+1))$ we right away recover $D = D'$ and $A \le B$.

Let $d_n = s(n+1) - s(n)$. Note that $\sup d_n < \infty$ since d_n is bounded above by A. Then we let

$$m \coloneqq \min d_n, \qquad M \coloneqq \max d_n.$$

Now suppose a achieves the maximum, meaning $s(a+1) - s(a) = M$. Then

$$\underbrace{d_{s(s(a))} + \cdots + d_{s(s(a+1))-1}}_{D \text{ terms}} = \boxed{s(s(s(a+1))) - s(s(s(a)))}$$

$$= (D \cdot s(a+1) + A) - (D \cdot s(a) + A) = DM.$$

Now M was maximal hence $M = d_{s(s(a))} = \cdots = d_{s(s(a+1))-1}$. But $d_{s(s(a))} = B - A$ is a constant. Hence $M = B - A$. In the same way $m = B - A$ as desired.

¶ **Second solution.** We retain the notation D, A, B above, as well as $m = \min_n s(n+1) - s(n) \ge B - A$. We do the involution trick first as:

$$D = \boxed{s(s(s(n)+1)) - s(s(s(n)))} = s(Dn + B) - s(Dn + A)$$

and hence we recover $D \ge m(B - A)$.

The edge case $D = B - A$ is easy since then $m = 1$ and $D = s(Dn + B) - s(Dn + A)$ forces s to be a constant shift. So henceforth assume $D > B - A$.

The idea is that right now the B terms are "too big", so we want to use the involution trick in a way that makes as many "A minus B" shape expressions as possible. This motivates considering $s(s(s(n+1))) - s(s(s(n)+1)+1) > 0$, since the first expression will have all A's and the second expression will have all B's. Calculation gives:

$$s(D(n+1) + A) - s(Dn + B + 1) = \boxed{s(s(s(n+1))) - s(s(s(n)+1)+1)}$$

$$= (Ds(n+1) + A) - (D(s(n) + 1) + B)$$

$$= D\left(s(n+1) - s(n)\right) + A - B - D.$$

Then by picking n achieving the minimum m,

$$\underbrace{m(D + A - B - 1)}_{>0} \le s(s(s(n + 1))) - s(s(s(n) + 1) + 1) \le Dm + A - B - D$$

which becomes

$$(D - m(B - A)) + ((B - A) - m) \le 0.$$

Since both of these quantities were supposed to be nonnegative, we conclude $m = B - A$ and $D = m^2$. Now the estimate $D = s(Dn + B) - s(Dn + A) \ge m(B - A)$ is actually sharp, so it follows that $s(n)$ is arithmetic.

§2 Solutions to Day 2

§2.1 IMO 2009/4

*Available online at https://aops.com/community/p1562847.*

Problem statement

Let ABC be a triangle with $AB = AC$. The angle bisectors of $\angle CAB$ and $\angle ABC$ meet the sides BC and CA at D and E, respectively. Let K be the incenter of triangle ADC. Suppose that $\angle BEK = 45°$. Find all possible values of $\angle CAB$.

Here is the solution presented in my book *EGMO.*

Let I be the incenter of ABC, and set $\angle DAC = 2x$ (so that $0° < x < 45°$). From $\angle AIE = \angle DIC$, it is easy to compute

$$\angle KIE = 90° - 2x, \ \angle ECI = 45° - x, \ \angle IEK = 45°, \ \angle KEC = 3x.$$

Having chased all the angles we want, we need a relationship. We can find it by considering the side ratio $\frac{IK}{KC}$. Using the angle bisector theorem, we can express this in terms of triangle IDC; however we can also express it in terms of triangle IEC.
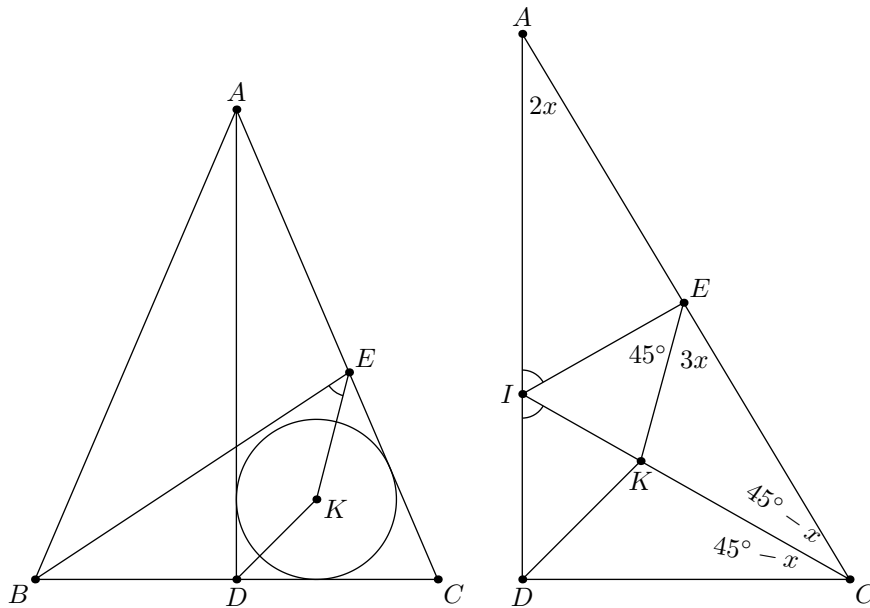
By the law of sines, we obtain

$$\frac{IK}{KC} = \frac{\sin 45° \cdot \frac{EK}{\sin(90° - 2x)}}{\sin(3x) \cdot \frac{EK}{\sin(45° - x)}} = \frac{\sin 45° \sin(45° - x)}{\sin(3x) \sin(90° - 2x)}.$$

Also, by the angle bisector theorem on $\triangle IDC$, we have

$$\frac{IK}{KC} = \frac{ID}{DC} = \frac{\sin(45° - x)}{\sin(45° + x)}.$$

Equating these and cancelling $\sin(45° - x) \neq 0$ gives

$$\sin 45° \sin(45° + x) = \sin 3x \sin(90° - 2x).$$

Applying the product-sum formula (again, we are just trying to break down things as much as possible), this just becomes

$$\cos(x) - \cos(90° + x) = \cos(5x - 90°) - \cos(90° + x)$$

or $\cos x = \cos(5x - 90°)$.

At this point we are basically done; the rest is making sure we do not miss any solutions and write up the completion nicely. One nice way to do this is by using product-sum in reverse as

$$0 = \cos(5x - 90°) - \cos x = 2\sin(3x - 45°)\sin(2x - 45°).$$

This way we merely consider the two cases

$$\sin(3x - 45°) = 0 \text{ and } \sin(2x - 45°) = 0.$$

Notice that $\sin\theta = 0$ if and only θ is an integer multiple of $180°$. Using the bound $0° < x < 45°$, it is easy to see that that the permissible values of x are $x = 15°$ and $x = \frac{45}{2}°$. As $\angle A = 4x$, this corresponds to $\angle A = 60°$ and $\angle A = 90°$, which can be seen to work.

§2.2 IMO 2009/5

Available online at https://aops.com/community/p1562848.

Problem statement

Find all functions $f\colon \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ such that for positive integers a and b, the numbers

$$a, \qquad f(b), \qquad f(b+f(a)-1)$$

are the sides of a non-degenerate triangle.

The only function is the identity function (which works). We prove it is the only one.
   Let $P(a,b)$ denote the given statement.

> **Claim —** We have $f(1) = 1$, and $f(f(n)) = n$. (In particular f is a bijection.)

Proof. Note that
$$P(1,b) \implies f(b) = f(b + f(1) - 1).$$
Otherwise, the function f is periodic modulo $N = f(1) - 1 \geq 1$. This is impossible since we can fix b and let a be arbitrarily large in some residue class modulo N.
   Hence $f(1) = 1$, so taking $P(n,1)$ gives $f(f(n)) = n$. $\qquad\square$

> **Claim —** Let $\delta = f(2) - 1 > 0$. Then for every n,
> $$f(n+1) = f(n) + \delta \quad \text{or} \quad f(n-1) = f(n) + \delta$$

Proof. Use
$$P(2, f(n)) \implies n - 2 < f(f(n) + \delta) < n + 2.$$
Let $y = f(f(n) + \delta)$, hence $n - 2 < y < n + 2$ and $f(y) = f(n) + \delta$. But, remark that if $y = n$, we get $\delta = 0$, contradiction. So $y \in \{n+1, n-1\}$ and that is all. $\qquad\square$

We now show f is an arithmetic progression with common difference $+\delta$. Indeed we already know $f(1) = 1$ and $f(2) = 1 + \delta$. Now suppose $f(1) = 1, \ldots, f(n) = 1 + (n-1)\delta$. Then by induction for any $n \geq 2$, the second case can't hold, so we have $f(n+1) = f(n) + \delta$, as desired.
   Combined with $f(f(n)) = n$, we recover that f is the identity.

§2.3 IMO 2009/6

Available online at https://aops.com/community/p1562840.

Problem statement

Let a_1, a_2, ..., a_n be distinct positive integers and let M be a set of $n-1$ positive integers not containing $s = a_1 + \cdots + a_n$. A grasshopper is to jump along the real axis, starting at the point 0 and making n jumps to the right with lengths a_1, a_2, ..., a_n in some order. Prove that the order can be chosen in such a way that the grasshopper never lands on any point in M.

The proof is by induction on n. Assume $a_1 < \cdots < a_n$ and call each element of M a *mine*. Let $x = s - a_n$. We consider four cases, based on whether x has a mine and whether there is a mine past x.

- If x has no mine, and there is a mine past x, then at most $n-2$ mines in $[0, x]$ and so we use induction to reach x, then leap from x to s and win.

- If x has no mine but there is also no mine to the right of x, then let m be the maximal mine. By induction hypothesis on $M \setminus \{m\}$, there is a path to x using $\{a_1, \ldots, a_{n-1}\}$ which avoids mines except possibly m. If the path hits the mine m on the hop of length a_k, we then swap that hop with a_n, and finish.

- If x has a mine, but there are no mines to the right of x, we can repeat the previous case with $m = x$.

- Now suppose x has a mine, *and* there is a mine past x. There should exist an integer $1 \le i \le n-1$ such that $s - a_i$ and $y = s - a_i - a_n$ both have no mine. By induction hypothesis, we can then reach y in $n-2$ steps (as there are two mines to the right of y); then $y \to s - a_i \to s$ finishes.

Remark. It seems much of the difficulty of the problem is realizing induction will actually work. Attempts at induction are, indeed, a total minefield (ha!), and given the position P6 of the problem, it is expected that many contestants will abandon induction after some cursory attempts fail.