2.4: Mechanical Vibrations

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Proposition: (Damped Harmonic Oscillators) Consider a mass m on a spring with spring constant k, and let the distance from equilibrium be x. Also, lets suppose there is some other force opposing the motion of the mass, $c\frac{dx}{dt}$, which we call the **damping force**. There also might be some external force $F_{ex}(t)$ on the spring as well. Lets sum all of the forces and get an equation. The sum of the forces, $F_{ex}(t) - c\frac{dx}{dt} - kx = m\frac{d^2x}{dt^2}$, is equal to mass times acceleration. Rearranging, we get

$$m\frac{d^2x}{dt^2} - c\frac{dx}{dt} - kx = F(T)$$

Definition: Some common terminology:

1. When $F(t) = 0$ for all t, the system is **free**
2. When $F(t) \neq 0$ for all t, the system is **forced**
3. When $c > 0$, the system is **damped**
4. When $c = 0$, the system is **undamped**

0.1 Free Undamped Motion

Definition: (Undamped Harmonic Oscillation) The equation describing undamped harmonic oscillators is:

$$m\frac{d^2x}{dt^2} + kx = 0$$

Lets divide through by m to get

$$\frac{d^2x}{dt^2} - \frac{k}{m}x = 0$$

.

By setting $\omega_0 = \sqrt{\frac{k}{m}}$ we get

$$\frac{d^2x}{dt^2} - \omega_0^2 x = 0$$

The general solution to the above equation is

$$x(t) = A\cos(\omega_0 t) + B\sin(\omega_0 t)$$

which is equal to

$$x(t) = C\cos(\omega_0 t - \gamma)$$

for some constant C and γ.

Through some algebra, we arrive at $C = \sqrt{A^2 + B^2}$ and $\tan\gamma = \frac{B}{A}$

0.2 Free Damped Motion

Definition: (Free Damped Harmonic Oscillation) The equation describing free damped harmonic oscillation is:

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = 0$$

Now, lets divide through by m to get $\frac{d^2x}{dt^2} + \frac{c}{m}\frac{dx}{dt} + \frac{k}{m}x = 0$Let set variables $\gamma = \frac{c}{2m}$ and $\omega_0 = \sqrt{\frac{k}{m}}$

Substituting, we get

$$\frac{d^2x}{dt^2} + 2\gamma\frac{dx}{dt} + \omega_0^2 = 0x = 0$$

and since the mass, spring constant, and damping factor probably aren't changing, this is a linear homogeneous constant coefficients second order ODE.

The characteristic equation is

$$r^2 + 2\gamma r + \omega_0^2 = 0$$

treating ω_0^2 as a single variable.

The roots of this equation are $r = -\gamma \pm \sqrt{\gamma^2 - \omega_0^2}$. The sign of the determinant, $\gamma^2 - \omega_0^2$, is the same as $c^2 - 4km$, so we get real roots only if $c^2 \geq 4km$.

> **Definition:** (Overdamping) If we have two real roots, the system is **overdamped**, and the solution becomes
>
> $$x(t) = C_1e^{r_1t} + C_2e^{r_2t}$$
>
> r_1 and r_2 are both negative because γ is always greater than $\sqrt{\gamma^2 - \omega_0^2}$. Over time, the motion in the system will approach zero.

> **Definition:** (Critical Damping) If $c^2 = 4km$, the system is **critically damped**, and there is one root with multiplicity 2. Solutions look like:
>
> $$x(t) = C_1e^{r_1t} + C_2te^{r_1t}$$

> **Definition:** (Underdamped) If $c^2 < 4km$, the system is **underdamped** , and we have two complex roots. The solution becomes
>
> $$x(t) = Ce^{-\gamma t}(A\cos(\omega_1 t) + B\sin(\omega_1 t))$$
>
> Our system develops limitng constraints specifying the maximum value $x(t)$ can take at any given time, and stays the same even under phase shifts. This constraint is called the **envelope curve**.