2.1: Second Order Linear ODEs

Alex L.

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0.1 Linear Homogeneous Equations

Linear homogeneous equations come in the form $\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = 0$.

Theorem: Superposition Theorem: If y_1 and y_2 are solutions of a homogeneous equation, then $y(x) = C_1y_1(x)+C_2y_2(x)$ are also solutions, where C_1, C_2 are arbitrary constants.

Proof: Let $y = C_1y_1(x)+C_2y_2(x)$. Then, $\frac{d^2y}{dx^2}+p(x)\frac{dy}{dx}+q(x)y = 0 \to \frac{d^2}{dx^2}(C_1y_1(x)+C_2y_2(x))+p(x)\frac{d}{dx}(C_1y_1(x)+C_2y_2(x))+q(x)(C_1y_1(x)+C_2y_2(x)) = \frac{d^2}{dx^2}C_1y_1 + \frac{d^2}{dx^2}C_2y_2 + \frac{d}{dx}C_1p(x)y_1 + \frac{d}{dx}C_2p(x)y_2 + C_1q(x)y_1 + C_2q(x)y_2 = C_1(\frac{d^2y}{dx^2}y_1 + p(x)y_1\frac{dy}{dx} + q(x)y_1) + C_1(\frac{d^2y}{dx^2}y_2 + p(x)y_2\frac{dy}{dx} + q(x)y_2) = 0$. As we can see, substituting in $C_1y_1(x) + C_2y_2(x)$ fulfills the equality, so it is a solution.

Theorem: Existence and Uniqueness: Suppose p, q, f are continuous functions on an interval I, and a is a number in I, and b_0, b_1 are constants. Then, the equation $\frac{d^2y}{dx^2}+p(x)\frac{dy}{dx}+q(x)y = f(x)$ has exactly one solution on the interval I statisfying the initial conditions $y(a) = b_0$ and $y'(a) = b_1$

Def: We say that functions y_1 and y_2 are **linearly independent** if they are not a constant multiple of each other.

Theorem: Let p, q be continuous functions. Let y_1, y_2 be two linearly independent solutions to the homogeneous equation $\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = 0$. Then, every other solution is of the form $y = C_1y_1 + C_2y_2$

0.2 Exercises:

2.1.101: Are $\sin x$ and e^x linearly independent? Justify.

Solution: Yes, as they cannot be written as a linear combination of each other.

2.1.102: Are e^x and e^{x+2} linearly independent? Justify.

Solution: No, $e^{x+2} = e^xe^2$.

2.1.103: Guess the solution to $\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = 5$

Solution: $y = 5$

2.1.104: Guess the solution to $x\frac{d^2y}{dx^2} + \frac{dy}{dx} = 0$

Solution: $y = C_1 \ln x + C_2$

2.1.105: Write down an equation for which we have the solutions e^x and e^{2x}.

Solution: We want an equation of the form $\frac{d^2y}{dx^2} + A\frac{dy}{dx} + By = 0$. By plugging in e^x and e^{2x} we get $e^x + Ae^x + Be^x = 0$ and $4e^{2x} + 2Ae^{2x} + Be^{2x} = 0$. Dividing these equations by e^x and e^{2x} respectively gives us $A + B = -1$ and $2A + B = -2$. Solving the system, we get $A = -3$ and $B = 2$.