Rudin Chapter 3: Upper and Lower Limits

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Definition: (Tending Towards Infinity)

Let $\{s_n\}$ be a sequence of real numbers with the following property:

for every real number M, there is an N with $s_n > M$ for all $n > N$. This sequence is said to **tend towards positive infinity** and is written

$$s_n \to +\infty$$

Likewise, if $\{s_n\}$ had the property for every real number M, there is an N with $s_n < M$ for all $n > N$, then the sequence would **tend towards negative infinity**, and could be written

$$s_n \to -\infty$$

Definition: (Upper and Lower Limits)

Let $\{s_n\}$ be a sequence of real numbers. Let E be the set of all subsequential limits (the set of all numbers to which subsequences can tend to). Then, E is a subset of the extended real numbers (the reals with $\pm\infty$). Then, $s^* = \sup E$ and $s_* = \inf E$, and are called the **upper and lower bounds** of $\{s_n\}$ respectively.

We also can use the notation:

$$s^* = \lim_{n\to\infty} \sup s_n$$

and

$$s_* = \lim_{n\to\infty} \inf s_n$$

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Theorem:

Let $\{s_n\}$ be a sequence of real numbers, and let E be the set of all subsequential limits of $\{s_n\}$. Then,

1. s^* is in E
2. If $x > s^*$, there is an integer N such that $n \geq N$ implies $s_n < x$

The two proofs above can be extrapolated to lower limits as well.

Proof:

1. If $s^* = +\infty$, then there is at least one subsequence which tends towards positive infinity, so its limit is $+\infty$, so s^* is in E.

   If s^* is real, then E is bounded above, with at least one subsequential limit existing, then $s^* \in \bar{E}$, and since the set of all subsequential limits must be closed, $E = \bar{E}$, so s^* is in E.

   If $s^* = -\infty$, then every subsequence must tend towards negative infinity, so $-\infty \in E$.
2. For this, we do a proof by contradiction. Suppose there was some $x > s^*$ where $s_n \geq x$ for infinitely many values of n. Then, we could make a subsequence out of these numbers whose limit will be greater than $s*$, which is a contradiction.

Theorem:

If we have two sequences, $\{s_n\}$ and $\{t_n\}$ and $s_n \leq t_n$ for all $n > N$, where N is a finite number, then

$$s^* \leq t*$$

and

$$s_* \leq t_*$$

Proof:

s^* must be less than or equal to t^* because if we pick a subsequence of $\{t_n\}$ with all elements $n > N$, it will tend towards a limit greater than or equal to any limit of any subsequence of $\{s_n\}$, so t^* must be greater than or equal to s^*. Likewise, any subsequence of $\{t_n\}$ must have infinitely many elements $n > N$, so even the lowest subsequential limit t_* must be greater than or equal to s_*.