2.1: Diffusion Type Problems

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1 Motivation

We want to see how parabolic type PDEs model diffusion and heat flow, and also develop intuitions on what terms like u_{xx}. In addition, we also introduce IVPs for PDEs

2 Content

Let's work through the steps to develop a PDE model for a physical phenomenon:

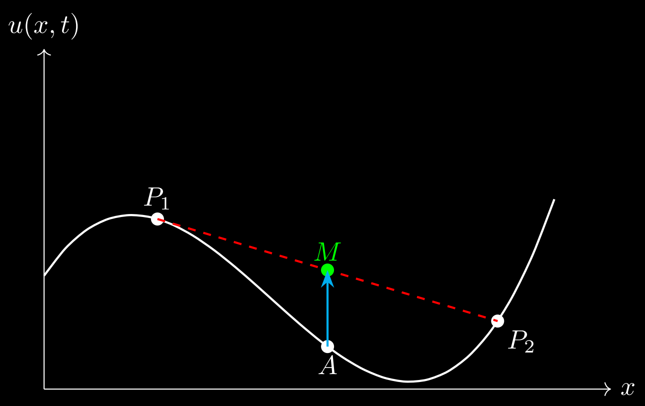
> **Example:** Suppose we had a rod with insulation around it, and at the ends, two different heating/cooling elements, one at T_1, and the other at T_2. Can we make a model to explain how the temperature of points along the rod evolves?

A mathematical model has three components:

1. the PDE equation which describes our model
2. the *boundary conditions* describing the physical limitations of the model (like how we're only measuring temperature along the length of the rod, and not off the ends)
3. the *initial conditions* describing the start of the experiment

Our first component is the heat equation. Lets try an intuitive derivation of the heat equation: if we have two equally spaced points around a point A, the temperature at A will move towards the average of those points over time.

If we graph temperature on the y-axis and position on the x-axis, we can think of the average of two points as the midpoint of a secant line between those points.



As we move P_1 and P_2 arbitrarily close to A, the secant line now becomes a measure of the curvature around A, which is the second derivative. Therefore,

$$u_t = \alpha^2 u_{xx}$$

We also need to define some boundary conditions. One would be that the temperature of the two ends of the rod are fixed, so

$$\begin{cases} u(0,t) = T_1 \\ u(L,t) = T_2 \end{cases} \quad 0 < t < \infty$$

We also need initial conditions, so we will impose $u(x,0) = T_0$, the starting ambient temperature of the rod for $0 \le x \le L$.

We now have an initial-boundary value problem (IBVP), and there is only one solution to these constraints.

2.1 More Diffusion-Type Equations

The equation

$$u_t = \alpha^2 u_{xx} - \beta(u - u_0)$$

where $\beta > 0$ models heat lost to the environment out of the surface of the rod, where u_0 is the ambient temperature.

The nonhomogeneous equation

$$u_t = \alpha^2 u_{xx} + f(x, t)$$

models a scenario where a rod is being supplied with an internal heat source (like a resistive wire).

What if there is some convection, like the concentration of a substance flowing downstream. If we let x be the distance downstream, the flow of the river pushes the stuff downstream, so our equation becomes

$$u_t = \alpha^2 u_{xx} - v u_x$$

If we have a nonhomogeneous material (like a pan and oven mitt) we could have a function as a coefficient, giving

$$u_t = a(x) u_{xx}$$

3 Exercises

Exercise: If the initial temperature of the rod was $u(x, 0) = sin(\pi x)$ for $0 \leq x \leq L$ and if the boundary conditions are $u(0, t) = 0$ and $u(1, t) = 0$, what would the boundary conditions look like for later values?

Solution: Since both ends are always 0 degrees at any point in time, eventually, the entire rod will become zero degrees.

Exercise: Suppose our rod has an internal heat source, so the equation becomes

$$u_t = \alpha^2 u_{xx} + 1$$

for $0 < x < 1$. Suppose we have boundary conditions $u(0, t) = 0$ and $u(1, t) = 1$. Is there a steady state temperature for this rod? What does it look like?

Solution: A steady state temperature is one where the temperature doesn't change over time, or in other words, when $u_t = 0$. If we set $u_t = 0$, we can see what the steady state temperature curve will look like. We get

$$0 = \alpha^2 u_{xx} + 1$$

and rearranging, we get

$$-\frac{1}{\alpha^2} = u_{xx}$$

Integrating with respect to x twice on both sides yields

$$u(x) = -\frac{x^2}{\alpha^2} + Cx + D$$

We know that $u(0, t) = 0$, so plugging in, the D will become 0. We also know that $u(1, t) = 1$, so plugging that in, we get that $C = \frac{1}{\alpha^2} + 1$ and our steady-state solution becomes

$$u(x) = (1 + \frac{1}{\alpha^2})x - \frac{1}{\alpha^2}x^2$$

Exercise: Suppose a metal rod loses heat across its lateral surface (not the ends) via the equation

$$u_t = \alpha^2 u_{xx} - \beta u$$

and $u(0, t) = 1$ and $u(1, t) = 1$. What is the steady state temperature of the rod. Where is the heat flowing?

Solution: We set $u_t = 0$ to find the steady state temperature. We get

$$0 = \alpha^2 u_{xx} - \beta u$$

This is a second order linear equation with constant coefficients. We try $u = e^{rx}$, and our characteristic equation is

$$0 = \alpha^2 r^2 - \beta$$

and our roots are

$$r = \pm \frac{\sqrt{\beta}}{\alpha}$$

Plugging back in, our solutions become

$$u(x) = C_1 e^{\frac{\sqrt{\beta}}{\alpha} x} + C_2 e^{-\frac{\sqrt{\beta}}{\alpha} x}$$

Plugging in the boundary conditions, we get that $C_1 + C_2 = 1$ and $C_1 e^{\frac{\sqrt{\beta}}{\alpha}} + C_2 e^{-\frac{\sqrt{\beta}}{\alpha}} = 1$. We get that $C_1 = \frac{1}{1+e^{\frac{\sqrt{\beta}}{\alpha}}}$ and $C_2 = \frac{e^{\frac{\sqrt{\beta}}{\alpha}}}{1+e^{\frac{\sqrt{\beta}}{\alpha}}}$. In total, we get

$$u(x) = \frac{1}{1 + e^{\frac{\sqrt{\beta}}{\alpha}}} e^{\frac{\sqrt{\beta}}{\alpha} x} + \frac{e^{\frac{\sqrt{\beta}}{\alpha}}}{1 + e^{\frac{\sqrt{\beta}}{\alpha}}} e^{-\frac{\sqrt{\beta}}{\alpha} x}$$

Exercise: Suppose a laterally insulated rod of length $L = 1$ has temperatures fixed at the left and right ends at 0 and 10 degrees Celsius respectively. It also has an initial temperature of $\sin(3\pi x)$. What are the IBVP values for this problem?

Solution: $u(0, t) = 0$ and $u(1, t)$ for all $0 \leq t < \infty$ and $u(x, 0) = \sin(3\pi x)$ for all $0 < x < 1$