1.5: Substitution

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0.1 Method

The equation $\frac{dy}{dx} = (x - y + 1)^2$ is not separable or linear, but we can turn it into a solvable form by implementing a change in variables. Let $v = x - y + 1$. We want to know $\frac{dy}{dx}$ in terms of $\frac{dv}{dx}$, v, and x. By differentiating, we get $\frac{dv}{dx} = 1 - \frac{dy}{dx}$. We then plug in and get $\frac{dv}{dx} - 1 = v^2$. This is a separable equation. We get $\frac{1}{1-v^2}dv = dv$ and by integrating, we get $\frac{1}{2}ln|\frac{v+1}{v-1}| = x + C \rightarrow \frac{v+1}{v-1} = C_1e^{2x}$. Now, substitute for $v = x - y + 1$ to get $\frac{x-y+2}{x-y} = De^{2x}$.

When you see	Substitute
$y\frac{dy}{dx}$	$v = y^2$
$y^2\frac{dy}{dx}$	$v = y^3$
$\cos(y)\frac{dy}{dx}$	$v = \sin y$
$\sin(y)\frac{dy}{dx}$	$v = \cos y$
$e^y\frac{dy}{dx}$	$v = e^y$

0.2 Bernoulli Equations

One of the special equations with a predefined substitution is the Bernoulli equations. They come in the form

$$\frac{dy}{dx} + p(x)y = q(x)y^n$$

The substitution $v = y^{1-n}$ turns the equation linear. Keep in mind that n does not need to equal an integer, it can be any number.

Ex: Solve $x\frac{dy}{dx} + y(x+1) + xy^5 = 0$ for $y(1) = 1$
Solution: This is a Bernoulli equation so we substitute $v = y^{1-5} = y^{-4} \rightarrow \frac{dv}{dx} = -4y^{-5}\frac{dy}{dx} \rightarrow \frac{dy}{dx} = \frac{-1}{4}y^5\frac{dv}{dx}$. Then, we substitute and get $(x - \frac{1}{4}y^5)\frac{dv}{dx} + y(x+1) + xy^5 = 0 \rightarrow -\frac{1}{4}\frac{dv}{dx} + y^{-4}(x+1) + x = 0 \rightarrow \frac{dv}{dx} - \frac{4(x+1)}{x}v = 4$. The last part is a linear equation, and so our integrating factor is $e^{\int \frac{-4x-4}{x}dx} = e^{-4x-ln(x)+4} = e^{-4x+4}x^4$, and $e^{-\int \frac{-4x-4}{x}} = e^{4x-4}x^4$. This means our entire linear equation evaluates to $e^{4x-4}x^4(\int 4\frac{e^{-4x+4}}{x^4}dx + 1)$, which is not possible to evaluate in closed form. We then unsubstitute to get $y = \frac{e^{-x+1}}{x(4\int \frac{e^{-4x+4}}{x^4}dx+1)^{\frac{1}{4}}}$

0.3 Homogeneous Equations

Another type of special equation is the homogeneous equation. Suppose we can write a differential in the form $\frac{dy}{dx} = F(\frac{y}{x})$. Then, a substitution might be $v = \frac{y}{x}$ and therefore, $\frac{dy}{dx} = x\frac{dv}{dx} + v$. Then, by substituting, we get $v + x\frac{dv}{dx} = F(v) \rightarrow x\frac{dv}{dx} = F(v) - v \rightarrow \frac{\frac{dv}{dx}}{F(v)-v} = \frac{1}{x}$, and as such, an implicit solution is $\int \frac{1}{F(v)-v}dv = \ln|x| + C$