7.7: Equations of Lines, Spheres, and Planes

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0.1 Line

Consider the fact that a line has a fixed position vector a, and from that point, a vector b that decides where the line points from a. As such, we can write the equation of a line as

$$\vec{r}(\lambda) = \vec{a} + \lambda\vec{b}$$

Remember that all vectors are pointing from the origin to a point. Different values of λ give different points on the line.

Taking the components of the vector equation, we get

$$\vec{r}(\lambda) =< a_x + \lambda b_x, a_y + \lambda b_y, a_z + \lambda b_z >$$

giving three parametric equations for the line:

$$x(\lambda) = a_x \lambda b_x, \quad y(\lambda) = a_y \lambda b_y, \quad z(\lambda) = a_z \lambda b_z$$

Solving for λ, and setting all of them equal to each other, and turn $x(\lambda)$ into x and so on, and we get three simultaneous equations:

$$\frac{x - a_x}{b_x} = \frac{y - a_y}{b_y} = \frac{z - a_z}{b_z} = c$$

where c is some constant.

Alternatively, if we subtract $\vec{a}$ to the other side of the original equation, and take the cross product with respect to $\vec{b}$ on both sides, since $\vec{b} \times \vec{b} = 0$, we get

$$(\vec{r} - \vec{a}) \times \vec{b} = 0$$

and since the equation doesn't depend on λ anymore, we remove it.

We can find the equation of a line passing through two fixed points by setting one of our fixed points to be a, and set b to be the vector pointing from the first point to the second point. Given two points $\vec{a}$ and $\vec{c}$, the equation becomes

$$\vec{r}(\lambda) = \vec{a} + \lambda(\vec{c} - \vec{a})$$

0.2 Planes

A plane can be defined by a point $\vec{a}$ and a unit normal vector perpendicular to the plane $\hat{n}$. The equation for a plane is given by

$$(\vec{r} - \vec{a}) \cdot \hat{n} = 0$$

This make sense because given a point $\vec{r}$, the vector pointing from $\vec{a}$ to $\vec{r}$ is $\vec{r} - \vec{a}$, and if that is perpendicular to our normal vector, it is in the plane, so the dot product between the two is zero.

If the components of $\hat{n}$ are $< n_x, n_y, n_z >$, then the plane can be expressed as

$$n_x x + n_y y + n_z z = d$$

where d is the length a vector perpendicular to the plane pointing to the origin is.

If we have three points $\vec{a}, \vec{b}, \vec{c}$, the equation of a plane can be written

$$r(\lambda, \mu) = \vec{a} + \lambda(\vec{b} - \vec{a}) + \mu(\vec{c} - \vec{a})$$

Again, we start with a point $\vec{a}$, and find vectors pointing from $\vec{a}$ to the other points. These three vectors form a plane.

Another equation is given by

$$\vec{r} = \alpha\vec{a} + \beta\vec{b} + \gamma\vec{c}$$

so long as $\alpha + \beta + \gamma = 1$.

0.3 Spheres

The equation of a sphere can be given by

$$|\vec{r} - \vec{c}|^2 = a^2$$

where $\vec{c}$ is the center and a is the radius.