8.8: The Trace of a Matrix

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Definition: (Trace) The **trace** of a square matrix is the sum of all of the diagonal elements:

$$\mathrm{Tr}A = A_{11} + A_{22} + A_{33} + ... + A_{nn}$$

Proposition:

$$\mathrm{Tr}(A + B) = \mathrm{Tr}A + \mathrm{Tr}B$$

Proof:

$$\mathrm{Tr}(A + B) = A_{11} + B11 + A_{22} + B_{22} + ... + A_{nn} + Bnn = A_{11} + A_{22} + ... + A_{nn} + B_{11} + B_{22} + ... + B_{nn} = \mathrm{Tr}A + \mathrm{Tr}B$$

Proposition:

$$\mathrm{Tr}AB = \mathrm{Tr}BA$$

Proof:

$$\mathrm{Tr}(AB) = \sum_{i=1}^{n} (AB)_{ii} = \sum_{I=i}^{n} \sum_{j=1}^{n} A_{ij} B_{ji} = \sum_{j=1}^{n} B_{ji} A_{ij} = \sum_{i=1}^{n} (BA)_{ii} = \mathrm{Tr}(BA)$$

Proposition:

$$\mathrm{Tr}A^T = \mathrm{Tr}A$$

Proof: The main diagonal of a matrix is unchanged under a transpose.

Proposition:

$$\mathrm{Tr}A^\dagger = (\mathrm{Tr}A)^*$$

Proof:

$$\mathrm{Tr}A^\dagger = \mathrm{Tr}((A^T)^*) = \sum_{i=1}^{n} (A_{ii})^* = (\mathrm{Tr}A)^*$$