0: Preliminaries on Sets, Mappings, and Relations

Alex L.

December 13, 2024

Definition: (Families of Sets) We will call a set of sets a **family** to avoid confusion. We will denote it $\mathcal{F}$.

The union of a family $\mathcal{F}$, $\bigcup_{F \in \mathcal{F}} F$ is the set of points that are in at least one of the sets in $\mathcal{F}$.

The intersection of a family, $\bigcap_{F \in \mathcal{F}} F$ is the set of all points that are in all of the sets in $\mathcal{F}$.

Definition: (Choice Function) A **choice function** is a function f that maps a family $\mathcal{F}$ to $\bigcup_{F \in \mathcal{F}} F$, and with the criteria that for every F in $\mathcal{F}$, $f(F)$ maps to an element that is in F.

Definition: (Zermelo's Axiom of Choice) Let $\mathcal{F}$ be a nonempty collection of nonempty sets. Then there is a chocie function on $\mathcal{F}$.

Definition: (Relation) A **relation** between members of a set X is a subset R of $X \times X$. If (a, b) is in R, then we write aRb.

The relation is **reflexive** if aRa for all a in X.

The relation is **transitive** if aRb and bRc implies aRc.

The relation is **symmetric** if aRb implies bRa.

Definition: (Equivalence Relation) A relation which is symmetric, transitive, and reflexive is an **equivalence relation**.

Definition: (Partial Ordering) A relation R on a set X is called a **partial ordering** if it is reflexive, transitive, and for a, b in X, if aRb and bRa then $a = b$.

Definition: (Ordering of a Set) A subset E of X is said to be **totally ordered** if for any a, b in E, either aRb or bRa.

If this is the case, then the **upper bound** of E is an element x such that aRx for all a in E, and it is **maximal** if it is the only element with this property.

Definition: (Ordering of Families) Let $\mathcal{F}$ be a family of sets and let A, B be in F. Then, ARB is true if $A \subseteq B$. This is a partial ordering of $\mathcal{F}$. F is an upper bound if it contains every other set in F and it is maximal if it isn't a proper subset of any set in F.

Lemma: (Zorn's Lemma) Let X be a partially ordered set for which every totally ordered subset has a maximal member. Then, X has a maximal member.