Rudin Chapter 2: Finite, Countable, and Uncountable Sets

Alex L.

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Definition: (Function) Consider two sets, A and B. Then suppose that each element x of A is associated in some manner with an element of B, which we denote $f(x)$. Then, f is called a **function** from A to B, and A is called the **domain** of f, B is called the **codomain** of f, and all elements $f(x) \subseteq B$ make up the **range** of f, and is denoted $f(A)$.

Definition: (Image) If we have a function $f : A \to B$, and a subset $E \subseteq A$, then the set of all elements $f(e)$, where e is an element in E, is called the **image** of E under f. Likewise, if we have some $E \subseteq f(A)$, that is, some subset of the range, then the **preimage** of E is all of the elements x in A such that $f(x)$ is in E.

Definition: (Injectivity and Surjectivity) Suppose we have $f : A \to B$.

If the range of f is equal to the codomain, that is $f(A) = B$, then the function is **surjective**.

If every element in the range is only mapped to by one element in the domain, that is, $f(x_1) = f(x_2)$ implies $x_1 = x_2$ for all x_1, x_2 in A, then the function is called **injective**.

If a function is both, it is called **bijective**.

Definition: (Set Cardinality) If there exists a bijective mapping $f : A \to B$ between sets A and B, then we say that the sets A and B have the same **cardinality**, denoted $A \sim B$. Alternatively, we can say that these sets have a **one to one correspondence** or they have the same **cardinal number**, or they are **equivalent**.

Proposition: Set equivalence is an equivalence relation, in other words, it obeys the following properties:

1. $A \sim A$
2. $A \sim B$ means that $B \sim A$
3. $A \sim B$ and $B \sim C$ means that $A \sim C$

Proof:

1. We need to show that for any set A, there is some bijective mapping between A and itself. There is always such a mapping, just map the elements of A to themselves, so $f(x) = x$ for all x in A. It seems pretty clear that this mapping is bijective.
2. Suppose there was a bijective mapping $f : A \to B$. Then, does there exist a bijective mapping $g : B \to A$? Yes, if we let g be the inverse of f. Since f was bijective, so is its inverse.
3. Suppose we have $f : A \to B$ and $g : B \to C$ with f, g bijective. Then, to show that set equivalency is transitive, we need to show that $g \circ f$ is bijective. Since the range of f is equal to the domain of g, and the range of g is the entire set C, we know that $g \circ f$ is surjective from A to C. In addition, if we have $g(f(x_1)) = g(f(x_2))$ for some x_1, x_2 in A, since g is injective, we know that $f(x_1) = f(x_2)$ and since f is injective, we know that $x_1 = x_2$, so we know that $g \circ f$ is also injective, so it is bijective, meaning it is a valid set equivalence, meaning that set equivalency is transitive.

Definition: (Finite Sets and Countability) Let J_n be the set of all natural numbers up to n, so $1, 2, 3, \ldots, n$. If $A \sim J_n$ for any natural number n, then A is called **finite** and has cardinality n.

If A is not finite, it is **infinite**.

Let J be the set of all the positive natural numbers. If $A \sim J$, then A is **countably infinite**.

If A is not finite or countably infinite, it is **uncountably infinite**.