# 1.4: Linear Equations and the Integrating Factor

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## 0.1 Method

**Def:** A **first order linear equation** is an equation that can be put in the form $\frac{dy}{dx} + p(x)y = f(x)$. Linear refers to the equation being linear in terms of $\frac{dy}{dx}$ and $y$

To solve, we want to find a function $r(x)$ such that $\frac{d}{dx}(r(x)y) = r(x)\frac{dy}{dx} + r(x)p(x)y$. Then, if we multiply the linear equation by $r(x)$, we get $r(x)\frac{dy}{dx} + r(x)p(x)y = r(x)f(x)$. Then, substituting the equality above, we get $\frac{d}{dx}r(x)y = r(x)f(x)$. We then integrate both sides and divide out $r(x)$.

**Def:** The function $r(x)$ is called the **integrating factor**.

We want $\frac{d}{dx}r(x) = r(x)p(x)$, and $e^{\int p(x)dx}$ is a function with this property.

Now we have: $\frac{dy}{dx} + p(x)y = f(x) \to e^{\int p(x)dx}\frac{dy}{dx} + e^{\int p(x)dx}p(x)y = e^{\int p(x)dx}f(x) \to \frac{d}{dx}(e^{\int p(x)dx}y) = e^{\int p(x)dx}f(x) \to e^{\int p(x)dx}y = \int(e^{\int p(x)dx}f(x))dx + C \to y = e^{-\int p(x)dx}(\int e^{\int p(x)dx}f(x)dx + C)$.

## 0.2 Exercises:

1.4.4 Solve $\frac{dy}{dx} + xy = x$

**Solution:** $p(x) = x$, $r(x) = e^{\frac{1}{2}x^2}$, so $\frac{d}{dx}(e^{\frac{1}{2}x^2}y) = e^{\frac{1}{2}x^2}x \to e^{\frac{1}{2}x^2}y = e^{\frac{1}{2}x^2} + C \to y = 1 + \frac{C}{e^{\frac{1}{2}x^2}}$