7.3: Ring Homomorphisms

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Definition: (Ring Homomorphism) Let R and S be rings. Then, a mapping $\varphi : R \to S$ is a **ring homomorphism** if:

1. $\varphi(a + b) = \varphi(a) + \varphi(b)$
2. $\varphi(ab) = \varphi(a)\varphi(b)$

The kernel ker φ is the set of all elements in R which map to 0 in S.

A bijective ring homomorphism is called a ring isomorphism.

Proposition: Let R and S be rings and let $\varphi : R \to S$ be a ring homomorphism. Then:

1. The image of R is a subring of S
2. The kernel of φ is a subring of R

Proof:

1. Elements of the image of R take the form $\varphi(a)$, where a is in R. The image of R is commutative because $\varphi(a) + \varphi(b) = \varphi(a + b) = \varphi(b + a) = \varphi(b) + \varphi(a)$.