2.2: Constant Coefficients

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A linear homogeneous equation with constant coefficients comes in the form $\frac{d^2y}{dx^2} + C_1\frac{dy}{dx} + C_2y = 0$

A solution of this equation will need to stay pretty much the same when we differentiate it, so we can add up a combination of it and its derivativess to get 0.

Lets try a solution of the form $y(x) = e^{rx}$. Then, $\frac{dy}{dx} = re^{rx}$ and $\frac{d^2y}{dx^2} = r^2e^{rx}$. We can then divide through by e^{rx} to get a polynomial we can solve for to find solutions for r.

Ex: $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 8y = 0$. Lets replace y and its derivatives with e^{rx}. We get $r^2e^{rx} - 6re^{rx} + 8e^{rx} = 0$. Then, divide by e^{rx} to get $r^2 - 6r + 8 = 0$. This is an easy polynomial to solve for. We get $(r-2)(r-4) = 0$, and our solutions become $y_1 = e^{2x}$ and $y_2 = e^{4x}$.

Since this is a second order linear homogeneous differential equation, our general solution is $y = C_1e^{4x} + C_2e^{2x}$.

0.1 Complex Roots

Sometimes, the roots to the r polynomial (characteristic polynomial) are complex. We then get solutions that look like $e^{(a\pm bi)x}$. We can separate the exponential to get $e^{ax}e^{\pm bix}$, and then using De Moivre's theorem, put the solution into the form $e^{ax}(\cos bx \pm i \sin bx)$. And we can linearly combine both solutions to isolate cos and sin as well.

0.2 Exercises

2.2.101: Find solutions to the equation $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 2y = 0$

Solution: If we substitute $y = e^{rx}$ we get $r^2e^{rx} + 4re^{rx} + 2e^{rx} = 0$, then we can divide by e^{rx} to get $r^2 + 4r + 2 = 0$. Solving for the roots, we get $r = -2 + \sqrt{2}, -2 - \sqrt{2}$. This means our solution is $y = C_1e^{(-2+\sqrt{2})x} + C_2e^{(-2-\sqrt{2})x}$

2.2.102: Find solutions to the equation $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = 0$

Solution: Using the technique above, we get $r^2 - 6r + 9 = 0$, and the factor for this is $r = 3$. If we have a duplicated root, we need to make our combination linearly independent, so we add an x to one of them. Our solution becomes $y = C_1e^{3x} + C_2xe^{3x}$