4.5: Variance

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1 Motivation:

We want to know about the spread of a random variable, or how much the outputs can differ from one another.

2 Content:

Definition: (Variance) If X is a random variable with expected value (mean) $E[X]$, then the **variance** of X, $\mathrm{Var}(X)$, is defined as:

$$\mathrm{Var}(X) = E[(X - E[X])^2]$$

Proposition: Another formula for $\mathrm{Var}(X)$ is

$$\mathrm{Var}(X) = E[X^2] - E[X]^2$$

Proof:

$$\mathrm{Var}(X) = E[(X - E[X])^2]$$

By definintion of the expected value of a random variable, we get

$$\sum_x (x - E[X])^2 p(x)$$

or the average distance of all values of X from the mean of X, weighted by the probability of getting those variables. Expanding, we get

$$\sum_x (x^2 - 2E[X]x + E[X]^2)p(x)$$

and by spliting the summation, we get

$$\sum_x x^2 p(x) - 2E[X] \sum_x xp(x) + E[X]^2 \sum_x p(x)$$

For the first term, this is the definition of the expected value of X^2. For the second term, $E[X] = \sum_x xp(x)$, so this entire term evaluates to $-2E[X]^2$, and for the third term, the sum of the mass density function $p(x)$ is equal to 1 by definition, so the entire term evaluates to $E[X]^2$.

This means the entire expression becomes

$$\mathrm{Var}(X) = E[X^2] - E[X]^2$$

Example: Find $\mathrm{Var}(X)$ if X represents the outcome of a fair dice.

Solution: We will use the formula $\mathrm{Var}(X) = E[X^2] - E[X]^2$. $E[X] = 1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + 3 \cdot \frac{1}{6} + 4 \cdot \frac{1}{6} + 5 \cdot \frac{1}{6} + 6 \cdot \frac{1}{6} = \frac{21}{6} = \frac{7}{2}$
$E[X^2] = 1^2 \cdot \frac{1}{6} + 2^2 \cdot \frac{1}{6} + 3^2 \cdot \frac{1}{6} + 4^2 \cdot \frac{1}{6} + 5^2 \cdot \frac{1}{6} + 6^2 \cdot \frac{1}{6} = \frac{91}{6}$

$$\mathrm{Var}(X) = \frac{91}{6} - \frac{49}{4} = \frac{35}{12}$$

Proposition:

$$\text{Var}(aX+b) = a^2\text{Var}(X)$$

Proof: Let $\mu = E[X]$ and remember that $E[aX+b] = a\mu + b$. Then,

$$\text{Var}(aX+b) = E[(aX+b-a\mu-b)^2]$$

Then, cancelling b on the inside, we get

$$E[(aX-a\mu)^2]$$

and we can factor out an a to get

$$E[a^2(X-\mu)^2]$$

and this is equal to

$$a^2E[(X-\mu)^2]$$

which is equal to

$$a^2\text{Var}(x)$$

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Definition: (Standard Deviation) The **standard devation** of a random variable X is given by

$$\text{SD}(X) = \sqrt{\text{Var}(X)}$$