4.8: Other Discrete Probability Distributions

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1 Geometric Random Variables

Definition: (Geometric Random Variable) Given independent bernoulli trials with probability p, a **geometric random variable** X is the number of trials until a single success, and is given by

$$P\{X = n\} = (1-p)^{n-1}p$$

as there need to be $n-1$ failures and 1 success to get a first success at the nth trial.

Proposition: If X is a geometric random variable then

$$E[X] = \frac{1}{p}$$

$$\text{Var}[X] = \frac{1-p}{p^2}$$

Proof: The expected value of a geometric random variable is

$$E[X] = \sum_{i=i}^{\infty} i(1-p)^{i-1}p$$

We can split i to get

$$E[X] = \sum_{i=1}^{\infty}(i+1-1)(1-p)^{i-1}p$$

And we can distribute and split the sum to get

$$E[X] = \sum_{i=1}^{\infty}(i-1)(1-p)^{i-1}p + \sum_{i=1}^{\infty}(1-p)^{i-1}p$$

The second term equals one. After infinite trials, we are bound to get a success, no moatter how small p is. Therefore, our equation becomes

$$E[X] = \sum_{i=1}^{\infty}(i-1)(1-p)^{i-1}p + 1$$

We can now use a change of index to get

$$E[X] = \sum_{i=0}^{\infty} i(1-p^i)p + 1$$

Taking out a $(1-p)$, and discarding the first term of our sum, we get

$$E[X] = (1-p)\sum_{i=1}^{\infty} i(1-p)^{i-1}p + 1$$

Notice now, that our sum is now an exact replica of $E[X]$. Substituting, we get that

$$E[X] = (1-p)E[X] + 1$$

Doing some algebra, we get that

$$pE[X] = 1$$

so

$$E[X] = \frac{1}{p}$$

2 Negative Binomial Random Variable

Definition: (Negative Binomial Random Variables) Suppose that bernoulli trials are conducted with probability p. Let X be a random variable that r successes are obtained in X trials and when the last trial is a success. Then, X is a **negative binomial random variable**, and is given by

$$P\{X = n\} = \binom{n-1}{r-1} p^{r-1}(1-p)^{n-r}$$

Proposition:

$$E[X] = \frac{r}{p}$$

$$\text{Var}(X) = \frac{r(1-p)}{p^2}$$

3 Hypergeometric Random Variable

Definition: (Hypergeometric Random Variable) Suppose we have an urn of N balls with m white and $N - m$ black balls, and we select a sample of n balls. Let a random variable X denote the number of white balls selected. Then, the density of X is given by

$$P\{X = i\} = \frac{\binom{m}{i}\binom{N-m}{n-i}}{\binom{N}{m}}$$

Proposition:

$$E[X] = \frac{nm}{N}$$

$$\text{Var}(X) = \frac{nm}{N}(\frac{(n-1)(m-1)}{N-1} + 1 - \frac{nm}{N})$$

4 Zipf (Zeta) Distribution

Definition: A random variable has a **zipf distribution** if

$$P\{X = k\} = \frac{C}{k^{\alpha+1}}$$

for some value $\alpha > 0$.