3.4: Eigenvalue Method

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Suppose we have a system of linear equations

$$\vec{x'} = P\vec{x}$$

where P is a square matrix with constant values. Then, like with the method for single linear equations, we try $\vec{x} = \vec{v}e^{\lambda t}$, where $\vec{v}$ is some vector of constants.

Plugging in, we get

$$\lambda\vec{v}e^{\lambda t} = P\vec{v}e^{\lambda t}$$

Dividing by $e^{\lambda t}$, we get

$$\lambda\vec{v} = P\vec{v}$$

which is the formula for eignenvalues and eigenvectors.

0.1 Finding Eigenvalues and Eigenvectors

Eigenvalues are values of λ that satisfy $\det(P - \lambda I) = 0$, and the corresponding eigenvector $\vec{v}$ for an eigenvalue is found by $(P - \lambda I)\vec{v} = \vec{0}$, which can be solved by row reduction with an augmented matrix.

0.2 Distinct Real Eigenvalues

If we have a system $\vec{x'} = P\vec{x}$, with P being a constant coefficient matrix, and eigenvalues $\lambda_1, \lambda_2, ...\lambda_n$, with eigenvectors $\vec{v}_1, \vec{v}_2, ..., \vec{v}_n$, then $\vec{v}_1 e^{\lambda_1 t}, \vec{v}_2 e^{\lambda_2 t}, ..., \vec{v}_n e^{\lambda_n t}$ are solutions to the system of equations, and the general solution is $\vec{x} = C_1\vec{v}_1 e^{\lambda_1 t} + C_2\vec{v}_2 e^{\lambda_2 t} + ... + C_n\vec{v}_n e^{\lambda_n t}$ and the fundamental matrix solution is $X(t) = \begin{bmatrix} \vec{v}_1 e^{\lambda_1 t} & \vec{v}_2 e^{\lambda_2 t} & ... & \vec{v}_n e^{\lambda_n t} \end{bmatrix}$

0.3 Complex Eigenvalues

If P is a real-valued matrix, then complex eigenvalues come in conjugate pairs. We can expand these and take the real and imaginary components, to get $\vec{x}_1 = C_1 \mathrm{Re}\ \vec{v}e^{(a+ib)t}$ and $x_2 = C_2 \mathrm{Im}\ \vec{v}e^{(a+ib)t}$, and plugging into Euler's formula, we get $x_1 = C_1 e^{at}\vec{v}\cos(bt)$ and $x_2 = C_2 e^{at}\vec{v}\sin(bt)$.