4.7: Poisson Random Variable

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Definition: (Poisson Random Variable) The **poisson random variable** is a tool used to approximate binomial random variables with a large number of independent trials. It relies on a parameter $\lambda = np$ where n is the number of independent trials and p is the chance that each trial turns positive.

Lets derive the poisson random variable from the definition of a binomial random variable. Suppose X is a binomial random variable with n trials and p chance for each success. Then,

$$P\{X = i\} = \binom{n}{i} p^i (1-p)^{n-i}$$

Expanding the binomial, we get

$$P\{X = i\} = \frac{n!}{i!(n-i)!} p^i (1-p)^{n-i}$$

If we let $\lambda = np$ and substitute in, we get

$$P\{X = i\} = \frac{n!}{i!(n-i)!} (\frac{\lambda}{n})^i (1 - \frac{\lambda}{n})^{n-i}$$

And if we cancel $n!$ with $(n-i)!$ in our binomial and split the last term, we get

$$P\{X = i\} = \frac{n(n-1)(n-2)...(n-i+1)}{i!} \frac{\lambda^i}{n^i} \frac{(1-\frac{\lambda}{n})^n}{1-\frac{\lambda}{n}^i}$$

Rearranging, we get

$$P\{X = i\} = \frac{n(n-1)(n-2)...(n-i+1)}{n^i} \frac{\lambda^i}{i!} \frac{(1-\frac{\lambda}{n})^n}{1-\frac{\lambda}{n}^i}$$

Notice that when n is large and λ is moderate,

$$(1 - \frac{\lambda}{n})^n \cong e^{-\lambda}$$

$$\frac{n(n-1)(n-2)...(n-i+1)}{n^i} \cong 1$$

$$(1 - \frac{\lambda}{n})^i \cong 1$$

This means that under these conditions,

$$P\{X = i\} \cong e^{-\lambda} \frac{\lambda^i}{i!}$$

Some use cases of a poisson random variable include:

1. Determining the amount of misprints in a set of pages in a book.
2. The number of people who survive to age 100.
3. The amount of people whose birthdays match a given day.

Example: Suppose the probability an item produced by a machine is defective is $p = .1$. Out of a sample of 10 items, what is the probability it will contain at most 1 defective item?

Solution: Let X be a poisson random variable with $\lambda = np = .1 * 10 = 1$ of a sample having x defective items. We want to find $P\{X = 0\} + P\{X = 1\}$ Plugging in, we get that

$$P\{X = 0\} = e^{-1}\frac{\lambda^0}{0!} = \frac{1}{e}$$

$$P\{X = 1\} = e^{-1}\frac{\lambda^1}{1!} = \frac{1}{e}$$

So in total, the chance of having 1 defective part or less is

$$\frac{2}{e}$$

Since the poisson random variable is an approximation of the binomial random variable, the expected value is

$$E[X] = np = \lambda$$

and

$$\mathrm{Var}[X] = np = \lambda$$