Rudin Chapter 4: Limits of Functions

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August 30, 2025

Definition: (Limits of a Function)

If we have metric spaces X and Y where $E \subset X$ and a function $f : E \to Y$, and p being a limit point of E, then we write $f(x) \to q$ as $x \to p$ or $\lim_{x \to p} f(x) = q$ if:

For all $\epsilon > 0$, there exists a $\delta > 0$ where $d_y(f(x), q) < \epsilon$ for all $0 < d_x(x, p) < \delta$.

Basically, for every distance ϵ, we can find a distance δ so that all the points within δ of p also map to points that fall within ϵ of q.

Theorem:

$\lim_{x \to p} f(x) = q$ if and only if $\lim_{n \to \infty} f(p_n) = q$ for every sequence $\{p_n\}$ in E such that $p_n \neq p$, $\lim_{n \to \infty} p_n = p$

Proof:

We do the forward proof. What if $\lim_{n \to \infty} f(p_n) = q$ for every sequence with the above criteria, but $\lim_{x \to p} f(x) \neq q$. Then, there must be some $\epsilon > 0$ such that no $\delta > 0$ can be found where $d_x(x, p) < \delta$ implies that $d_y(f(x), q) < \epsilon$. That means for every $\delta > 0$, there must be k within δ of p where $d_y(f(x), q) > \epsilon$, otherwise we could choose that δ to fulfill the criteria. Then, make a sequence of all of these k. This sequence fulfills the criteria above but the mapped points do not tend towards q. This is a contradiction.

For the reverse proof, what if $\lim_{x \to p} f(x) = q$ but there was some sequence $\{a_n\}$ that fulfilled the above criteria and the mapped points $f(a_n)$ did not tend towards q? Then, for at least one $\epsilon > 0$, we can always find some a_n where $d_x(a_n, p) < \delta$ and $d_y(f(a_n), q) > \epsilon$, therfore, $\lim_{x \to p} f(x) \neq q$, a contradiction.

Corollary:

If $f(x)$ has a limit p, it is unique.

Proof:

Sequences can't converge to two points, so if a function had two limits, some sequences would converge to one point, and others would converge to the other, breaking the above theorem.

Theorem:

If $E \subset X$ and p is a limit point of E and f, g are complex valued functions, with limits of A and B respectively, then

1. $\lim_{x \to p} (f + g)(x) = A + B$
2. $\lim_{x \to p} (f \cdot g)(x) = AB$
3. $\lim_{x \to p} (\frac{f}{g})(x) = \frac{A}{B}$ if $B \neq 0$

Proof:

This follows from the addition, multiplication, and division of sequences and their limits