1.8: Exact Equations

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Def: Suppose we are given a function $F(x,y)$ called a **potential function**. This function might describe, say the strength of an electric field, or potential energy, etc. Lets take lines of constant energy, $F(x,y) = C$, and compute their **total derivative**. The total derivative of a multivariable function is $dF(x_1, x_2, x_3, ...) = \frac{\partial F}{\partial x_1}dx_1 + \frac{\partial F}{\partial x_2}dx_2 + \frac{\partial F}{\partial x_3} + ...$

The total derivative of our potential function will be $dF = 0$. We can rewrite it as $dF = Mdx+Ndy = 0 \to dF = M+N\frac{dy}{dx} = 0$ and an equation in this form is called an **exact equation**.

An exact equation actually describes a vector fields with vectors consisting of $\vec{v} = (\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y})$, and is a conservative vector field, because by definition, it is the gradient of the potential function $F(x,y)$.

If we think of γ as a path starting at (x_1, y_1) and ending at (x_2, y_2), and want to find the work (change in energy) done to traverse this path, we get that $\int_\gamma = \vec{v}(\vec{r})d\vec{r} = \int_\gamma Mdx + Ndy = F(x_2, y_2) - F(x_1, y_1)$.

0.1 Solving Exact Equations

Differentiate the equation with respect to x to get $F(x,y) = \int(M)dx + A(y)$. $A(y)$ is a constant of integration with respect to x. Then, derive $A(y)$ and set $A'(y) = N$, then integrate N to get $A(y) = \int Ndy$, then substitute back into $F(x,y)$

0.2 Integrating Factor

Sometimes, $Mdx + Ndy = 0$ is not exact. However, maybe $u(x,y)Mdx + u(x,y)Ndy = 0$ is exact. In fact, a linear equation is always exact. Let $r(x) = e^{\int p(x)dx}$, and multiply a linear equation by $r(x)$ to get $r(x)p(x)y - r(x)f(x) + r(x)\frac{dy}{dx} = 0$. Then $M = r(x)p(x)y - r(x)f(x)$ so $\frac{\partial M}{\partial y} = r(x)p(y)$ and $N = r(x)$ so $\frac{\partial N}{\partial x} = r(x)p(x)$. Actually, linear equations are just a special case of exact equations.

How do we find $u(x,y)$? It should be a function such that $\frac{\partial uM}{\partial y} = \frac{\partial u}{\partial y}M + \frac{\partial M}{\partial y}u = \frac{\partial uN}{\partial x} = \frac{\partial u}{\partial x}N + \frac{\partial N}{\partial x}u$, therefore $(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x})u = \frac{\partial u}{\partial x}N - \frac{\partial u}{\partial y}N$.

Some equations that fulfill this are when u is a function of x or y alone, but not both. Then, by rearranging terms, we get $\frac{\partial u}{\partial x} = \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}$, and we can set that as our integrating factor.