3.1: Introduction to Systems of ODEs

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Definition: (Systems of ODEs) A **system of differential equations** is when one or more differential equations are simultaneously true. There may be multiple dependent variables and independent variables as well.

Example:

$$y_1' = y_1$$
$$y_2' = y_1 - y_2$$

Solution: The general solution for the first equation is

$$y_1 = C_1 e^x$$

We then substitute into the second equation to get

$$y_2' = C_1 e^x - y_2$$

Then, rearranging, we get

$$y_2' + y_2 = C_1 e^x$$

This is a first order linear ODE, which we can use the integrating factor of e^x to solve. We get

$$e^x y_2 = \frac{C_1}{2} e^{2x} + C_2$$

0.1 Changing to first order

If we have a higher order differential equation like

$$y^{(n)} = F(y^{(n-1)}, ..., y, x)$$

we can define new variables such that

$$u_1' = u_2$$
$$u_2' = u_3$$
$$u_3' = u_4$$
$$u_{n-1}' = u_n$$
$$...$$
$$u_n' = F(u_n, u_{n-1}, ..., u, x)$$

Then, we solve the system, and once we're done, discard u_2 through u_n and set $y = u_1$. Then y is our solution to the equation.

Example:

$$x''' = 2x'' + 8x' + x + t$$

Solution: Let $u_1 = x$, $u_2 = x'$, $u_3 = x''$, and solving, we get $u_1' = u_2$, $u_2' = u_3$, and $u_3' = 2u_3 + 8u_2 + u_1 + t$.

0.2 Existence and Uniqueness

Theorem: (Picard's Theorem) Let a system of first order equations be

$$x_1' = F_1(x_1, x_2, x_3, ..., x_n, t)$$

$$x_2' = F_2(x_1, x_2, x_3, ..., x_n, t)$$

$$...$$

$$x_n' = F_n(x_1, x_2, x_3, ..., x_n, t)$$

If for everg $j = 1, 2, ..., n$ and every $k = 1, 2, ..., n$, F_j is continuous and each $\frac{\partial F_j}{\partial i}$ exists and is continuous in a neighborhood, then a solution to the system exists in that neighborhood.