3.7: Multiple Eigenvalues

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1 Motivation

What do we do when we encounter repeated eigenvalues and defective eigenvalues?

2 Content

2.1 Repeated Eigenvalues

Definition: (Repeated Eigenvalues) Whenever there is a repeated root of the characteristic equation

$$\det(A - \lambda I) = 0$$

with multiplicity m, then we have a repeated eigenvalue. m is called the **algebraic multiplicity**. The number of linearly independent eigenvectors corresponding to this eigenvalue is a distinct number, called the **geometric multiplicity**. The geometric multiplicity is also the dimension of the corresponding **eigenspace** (the span of all eigenvectors associated with a corresponding eigenvalue).

The above definition makes sense if you think about it. If we have an eigenvalue with two associated eigenvectors, $\vec{v}_1$ and $\vec{v}_2$, then linear combinations of them also result in eigenvectors, because

$$A(\vec{v}_1 + \vec{v}_2) = A\vec{v}_1 + A\vec{v}_2 \text{ (by properties of matrix multiplication)} = \lambda\vec{v}_1 + \lambda\vec{v}_2 = \lambda(\vec{v}_1 + \vec{v}_2)$$

And so having multiple linearly independent eigenvectors means you get an eigenspace of that many dimensions.

If we get a matrix with repeated eigenvalues, just assign a unique eigenvector to each term, so for a matrix

$$A = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

which has eigenvalues $\lambda = 3$ and eigenvectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ our result looks like

$$\vec{x} = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{3t}$$

2.2 Defective Eigenvalues

Definition: Sometimes, the eigenvectors for our repeated eigenvalue are not linearly independent. In this case, our geometric multiplicity is less than our algebraic multiplicity, and we call the eigenvalue **defective**. The difference between the two is called the **defect**.

Suppose we have two linearly dependent eigenvalues $\vec{v}_1$ and $\vec{v}_2$. We can't form a general solution out of these, so lets just slap an x on one of them (in this case, its a t as our independent variable but I digress).

We get

$$\vec{x} = \vec{v}_1 e^{\lambda_1 t} + t\vec{v}_2 e^{\lambda_1 t}$$

Lets see if this is actually a solution. Differentiating, we get

$$\vec{x}' = (\lambda_1 \vec{v}_1 + \vec{v}_2)e^{\lambda_1 t} + \lambda_1 t \vec{v}_2 e^{\lambda_1 t}$$

Our solution must satisfy $A\vec{x} = \vec{x}'$ so plugging in our original equation, we get

$$\vec{x}' = A\vec{v}_1 e^{\lambda_1 t} + At\vec{v}_2 e^{\lambda_1 t}$$

Matching terms and then rearranging, we get that

$$(A - \lambda I)\vec{v}_1 = \vec{v}_2$$

and

$$(A - 3I)\vec{v}_2 = 0$$

and so our new vector has to satisfy these two equations.