3.3: The Isomorphism Theorems

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Theorem: The First Isomorphism Theorem: If $\varphi : G \to H$ is a homomrophism of groups, then $\ker\varphi \trianglelefteq G$ and $G/\ker\varphi \cong \varphi G$

Corrolary 17: Let $\varphi : G \to H$ be a homomorphism of groups.

1. φ is injective iff $\ker\varphi = 1$
2. $|G : \ker\varphi| = |\varphi(G)|$

Theorem: The Second Isomorphism Theorem Let G be a group, let A and B be subgroups of G and assume $A \leq N_G(B)$. Then, AB is a subgroup of G, $B \trianglelefteq AB$, $A \cap B \trianglelefteq A$, and $AB/B \simeq A/(A \cap B)$. (Remember that $N_G(A)$ is the set of elements in G that commute with all elements in A)

Proof: Note: all elements of A do normalize B. Then, by a previous corrolary in 3.2, AB is a subgroup of G. Every element in G normalizes AB because $babb^{-1} = ba$, and by a previous theorem, $BA = AB$ if $A \leq N_G(B)$, so ba is equal to some element in AB.

To show everything else, lets establish a map $\varphi : A \to AB/B$ mapping elements of A to their equivalence classes in AB by $\varphi(a) = aB$. φ is a homomorphism because $\varphi(a_1a_2) = (a_1a_2)B = a_1Ba_2B = \varphi(a_1)\varphi(a_2)$. The kernel of the homomorphism is all elements in A that fulfill $aB = 1B$, or in other words, elements in A that are closed in B. The only elements that do this are ones that are in the group B, by definition of closure, so $\ker\varphi = A \cap B$. By the First Isomorphism Theorem, the kernel of φ is normal in G, so by extension, $\ker\varphi = A \cap B \trianglelefteq A$.

The First Isomorphism Theorem also tells us that $A/\ker\varphi \cong \varphi(A)$, and by substituting, we get $A/A \cap B \cong AB/B$.

This theorem is called the Diamond Isomorphism Theorem, because the lattice forms a diamond, with $A \cap B$ being a subgroup of both A and B, and those two being both subgroups of AB, which is a subgroup of G.