3.2: Conditional Probabilities

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0.1 Motivation

Suppose we toss two dice. Each ordered pair of numbers on the dice has a $\frac{1}{36}$ chance of happening. Now, what if the first dice lands on 3? Well, now we only have six outcomes, $(3,1),(3,2),...,(3,6)$, and each has a $\frac{1}{6}$ chance of occuring, and the probability of all other outcomes is now 0.

0.2 Conditional Probabilities

Def: If we let E denote the event that the sum of rolling two fair dice is 8, and F denote the event that the first dice lands on a 3, then the probability that E will occur if F just happened is denoted

$$P(E|F)$$

and is read "the probability of E given F". These situations are called **conditional probabilities**.

There is a formula for conditional probabilities.

$$P(E|F) = \frac{P(E \cap F)}{P(F)}$$

0.3 Examples

Example 2a: A student is taking a one-hour-time-limit makeup examination. Suppose the probability that the student will finish the exam in less than x hours is $\frac{x}{2}$, for all $0 \leq x \leq 1$. Then, given that the student is still working after .75 hour, what is the conditional probability that the full hour is used?

Solution: The event E is when the full hour is used, and the event F is when .75 the student hasn't finished in .75 hours. $P(E)$ is equal to $1 - \frac{1}{2}$, as $\frac{1}{2}$ is the probability the student finishes in less than an hour, so $P(E) = \frac{1}{2}$. Likewise, $P(F)$ is the probability the student hasn't finished in .75 hours, which is equal to $1 - \frac{.75}{2} = .625$. However, $P(F)$ always happens when $P(E)$ happens, so $P(E \cap F) = P(E) = .5$. As such, we have $P(E|F) = \frac{.5}{.625} = .8$.

Example 2b: A fair coin is flipped twice. What is the probability that I flip two heads given I the first flip is heads? How about if at least one flip is heads?

Solution: Let $E = \{(H,H)\}$ be the event of flipping two heads, and let $F = \{(H,H),(H,T)\}$ be the event of the first coin flipping heads, and let $G = \{(H,H),(H,T),(T,H)\}$ be the event that at least one coin is heads. $P(E|F) = \frac{P(E \cap F)}{P(F)} = \frac{\{(H,H)\}}{\{(H,H),(H,T)\}} = \frac{1}{2}$

$P(E|G) = \frac{\{(H,H)\}}{\{(H,H),(H,T),(T,H)\}} = \frac{1}{3}$