4.9: Expected Value of Sums of Random Variables

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1 Motivation

We will try to give a rigorous proof as to why the expected value of the sums of random variables is equal to the sum of the expectations of individual variables.

2 Content

Proposition: Given a sample space S with elements s, the expected value a random variable X on the sample space is given by

$$E[X] = \sum_{s \in S} X(s)p(s)$$

Proof: Suppose X takes on the values x_i. For each x_i, let S_i be the set of all elements of s that make $X = x_i$. Then,

$$E[X] = \sum_i x_i P\{X = x_i\}$$

$$E[X] = \sum_i x_i p(S_i)$$

And since the probability of a subset of S is the union of the individual probabilities,

$$E[X] = \sum_i x_i \sum_{s \in S_i} p(s)$$

Since x_i gives the same value regardless of which $s \in S_i$ we pick, we can put it inside the summation, getting

$$E[X] = \sum_i \sum_{s \in S_i} x_i p(s)$$

Rewriting, we get

$$E[X] = \sum_i \sum_{s \in S_i} X(s)p(s)$$

And all is will partition the entire sample space, so this becomes

$$\sum_{s \in S} X(s)p(s)$$

Proposition:

$$E[\sum_{i=1}^{n} X_i] = \sum_{i=1}^{n} E[X_i]$$

Proof: Let Z be the random variable that is the sum of all X_i, that is, $Z = \sum_{i=1}^{n}$. Then,

$$E[Z] = \sum_{s \in S} Z(s)p(s)$$

By the definition of $Z(s)$, we substitute to get

$$E[Z] = \sum_{s \in S} (X_1(s) + X_2(s) + X_3(s) + ... + X_n(S))p(s)$$

We distribute $p(s)$ to get

$$E[Z] = \sum_{s \in S} X_1(s)p(s) + \sum_{s \in S} X_2(s)p(s) + \sum_{s \in S} X_3(s) + ... + \sum_{s \in S} X_n(s)$$

And this gives

$$E[Z] = E[X_1] + E[X_2] + E[X_3] + ... + E[X_n]$$