1.1: The Field, Positivity, and Completeness Axioms

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Definition: (Field Axioms: Addition and Multiplication) **Addition** and **multiplication** are defined as mappings from $\mathbb{R} \times \mathbb{R} \to \mathbb{R}$ with the following axioms/properties:

1. Commutativity of Addition: $a + b = b + a$
2. Associativity of Addition: $(a + b) + c = a + (b + c)$
3. The Additive Identity: There is a number called 0 such that $a + 0 = a$
4. The Additive Inverse: For each real number a, there is another number b such that $a + b = 0$
5. Commutativity of Multiplication: $ab = ba$
6. Associativity of Multiplication: $a(bc) = (ab)c$
7. The Multiplicative Identity: There is a number called 1 such that $1a = a$
8. The Multiplicative Inverse: For each real number a except for 0, there exists a number b such that $ab = 1$
9. The Distributive Property: $a(b + c) = ab + ac$
10. The Normativity Assumption: $1 \neq 0$

If you recall from abstract algebra, the above properties define a field of addition and multiplication on the real numbers, hence why these assumptions are called the **field axioms**.

Definition: (The Positivity Axioms) We will now define what a **positive number** is using **positivity axioms**:

1. If a and b are positive, then $a + b$ and ab are positive
2. For a real number a, exactly one of three statements is true:

$$\text{a is positive}$$

$$-a \text{ (the additive inverse of } a\text{) is positive}$$

$$a = 0$$

We will now construct an ordering of real numbers. If $a - b$ is positive, then $a > b$. If $a - b$ is positive or zero, then $a \geq b$. If $a > b$ then $b < a$. If $a \geq b$, then $b \geq a$.

Now we will define what an interval means. An interval (a, b) is the set of all numbers x that simultaneously fulfill $a < x$ and $x < b$. Likewise, for $[a, b]$, this is the set of real numbers x that fulfill $a \leq x \leq b$.

Definition: (Boundedness) A sset of real numbers X has an **upper bound** if there exists b such that $x \leq b$ for all x in X. Likewise, a set has a lower bound if there exists an a such that $x \geq a$ for all x in X.

The smallest number that is an upper bound for a set X is called the **supremum** of X. Likewise, the largest number that is a lower bound for X is called the **infimum**.

Definition: (The Completeness Axiom) Let E be a nonempty set which is bounded above. Then, E has a supremum.

Theorem: Suppose that A is a nonempty set which is bounded below. Then, given the completeness axiom holds, A has an infimum.

Proof: Let L be the set of all numbers which bound A below (the numbers which are less than all elements of A). Then, we know that L has a supremum, which we will call b. Now, to show that b is an infimum of A we need to show that it is a lower bound of A and that all other lower bounds are smaller than it.

Firstly, b is a lower bound on A. Suppose we have an element a in A and that $a < b$. Then, all elements of L must be less than a, so a is now an upper bound of L and less than b. However, b is the least upper bound, so this can't happen, therefore, $b \leq a$.

Now, suppose that there was a lower bound of A, called l. Then, l would be in L, the set of lower bounds of A. Since b is the supremum of L, by definition, $l \leq b$.

We have proved our criteria, therefore there exists an infimum of A, b if A is lower bounded.

Definition: (The Triangle Inequality) We define the **absoute value** of a number x, $|x|$ to be x if $x \geq 0$ and $-x$ if $x < 0$.

The following axiom is called the **triangle inequality**:

$$|a+b| \leq |a| + |b|$$

Definition: (The Extended Real Numbers) We define the set of **extended real numbers** to be $\mathbb{R} + \{\infty, -\infty\}$. If an interval is not upper bounded, then we say its supremum is ∞ and if it isn't lower bounded, we say its infimum is $-\infty$.

Exercise: (1) For $a, b \neq 0$, show that $(ab)^{-1} = a^{-1}b^{-1}$

Solution: If we left multiply by ab on both sides, we get

$$(ab)(ab)^{-1} = aba^{-1}b^{-1}$$

Cancelling, we get

$$1 = aba^{-1}b^{-1}$$

We can use the commutative property to get

$$1 = aa^{-1}bb^{-1}$$

And cancelling, we get

$$1 = 1$$

Which is true, and since we did stuff to both sides, we maintain equality, so

$$(ab)^{-1} = a^{-1}b^{-1}$$

Exercise: (2) Verify the following:

1. For each real number $a \neq 0$, $a^2 > 0$. In particular, $1 > 0$ since $1 \neq 0$ and $1 = 1^2$.
2. For each positive number a its multiplicative inverse, a^{-1} also is positive.
3. If $a > b$, then $ac > bc$ if $c > 0$ and $ac < bc$ if $c < 0$

Solution:

1. Since a is nonzero, either a can be positive or $-a$ can be positive. If a is positive, then $a^2 = a * a$ and by the first positivity axiom, $a * a$ is positive, therefore a^2 is positive. For the negative case, notice that by the inverse distributive law,

$$ab + (-a)b = (a-a)b = 0$$

$$ab + a(-b) = a(b-b) = 0$$

Therefore,

$$ab + (-a)b = ab + a(-b)$$

And left subtracting by ab gets us that

$$(-a)b + a(-b)$$

Since for each real number r there is a $-r$ such that $r + -r = 0$, for the real number $-a$, there is a $- - a$ such that $-a + - - a = 0$. Then, adding a to both sides, we get $a + -a + - - a = a$. Cancelling, we get $- - a = a$. Therefore,

$$(-a)(-a) = - - aa = aa = a^2$$

. In particular, since $1 = 1^2$, and squares are always positive, 1 must be positive.

2. A number a times its multiplicative inverse a^{-1} always make a positive number, 1. Therefore, if a is positive, by the first positivity axiom, so must a^{-1}.

3. For the $c > 0$ case, $ac > bc$ indicates that $ac - bc$ is positive. Then, distributing, we get $(a - b)c$. Since c is positive and $a - b$ is positive (since $a > b$), by the first positivity axiom, $(a - b)c$ must be positive, so $ac - bc$ must be positive, and therefore, $ac > bc$.

   For the $c < 0$ case, we want to show that $ac - bc$ is negative. Distributing, we get that we want to show that $(a - b)c$ is negative.