1.9: First Order PDEs

Alex L.

October 8, 2024

0.1 Method

Consider the equation

$$a(x,y)\frac{\partial u}{\partial x} + b(x,t)\frac{\partial u}{\partial t} + c(x,t)u = g(x,t) \quad u(x,0) = f(x) \quad -\infty < x < \infty, t > 0$$

Notice that our initial conditions are $u(x,0) = f(x)$, a function of x.

The method we will use is called the **method of characteristics**. The goal is to find lines of constant x or t along which the equation is an ODE.

0.2 Examples

Ex: Consider

$$\frac{\partial u}{\partial t} + \alpha\frac{\partial u}{\partial x} = 0 \quad u(x,0) = f(x)$$

This equation is called the transport equation. The idea is that data varies along certain coordinates, called **characteristic coordinates**. For example, maybe an equation which is radial in nature can be solved by changing into polar coordinates.

For this equation, we will change into characteristic coordinates (ζ, s), and let $\zeta = x - \alpha t$ and $s = t$

Def: Generalized Chain Rule: If $f(x,y)$ is a function, but x, y are themselves functions $x = x(s,t), y = y(s,t)$, then the entire function becomes $f(x(s,t), y(s,t))$ and the partial derivatives of f with respect to s, t become

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} \qquad \frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}$$

In fact, if you have a function $f = f(x_0, x_1, x_2, ..., x_n)$ in n variables and each of those variables are dependent on $x_n = x_n(y_0, y_1, y_2, ..., y_m)$; m other variables, then the chain rule goes:

$$\frac{\partial f}{\partial y_m} = \frac{\partial f}{\partial x_0}\frac{\partial x_0}{\partial y_m} + \frac{\partial f}{\partial x_1}\frac{\partial x_1}{\partial y_m} + ... + \frac{\partial f}{\partial x_n}\frac{\partial x_n}{\partial y_m}$$

for all y_m.

Applying generalized chain rule to our variables, we get

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial \zeta}\frac{\partial \zeta}{\partial t} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial t} = -\alpha\frac{\partial u}{\partial \zeta} + \frac{\partial u}{\partial s}$$

and

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \zeta}\frac{\partial \zeta}{\partial x} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial x} = \frac{\partial u}{\partial \zeta}$$

Then, the equation becomes $-\alpha\frac{\partial u}{\partial \zeta} + \frac{\partial u}{\partial s} + \alpha\frac{\partial u}{\partial \zeta} = 0$, and by simplifying, we get $\frac{\partial u}{\partial s} = 0$.

This means that there is some function $u = A(\zeta) = A(x - at)$. Our initial conditions stipulate that $t = 0$, so we end up with $f(x) = A(x)$, so $A = f$, and our particular solution is $u(x,t) = f(x - at)$.

Basically, by choosing a good change in variables and applying generalized chain rule, we get a easy differential equation to solve.