8.1: Vector Spaces

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0.1 Vector Spaces

Definition: (Vector Space) A set of objects $a, b, c...$ forms a **linear vector space** if:

1. The set is closed under addition, and is commutative and associative: $a + b = b + a$ and $a + (b + c) = (a + b) + c$
2. The set is closed under multiplication with a scalar, and the multiplication is distributive over addition: $\lambda(a + b) = \lambda a + \lambda b$, $(\lambda + \mu)a = \lambda a + \mu a$, and $\lambda(\mu a) = (\lambda\mu)a$
3. There exists a **null vector** 0 such that $a + 0 = a$ for all a
4. Multiplication by unity leaves a vector unchanged: $a \times 1 = a$
5. All vectors have a corresponding negative vector such that $-a + a = 0$

If scalar multiplication only works with real numbers, we call it a **real vector space**. If it works with complex numbers, we call it a **complex vector space**.

Definition: (Span) The **span** of a set of vectors $v_0, v_1, v_2, ...$ is all of the vectors x that can be written as a linear combination of the set of vectors:

$$x = a_0 v_0 + a_1 v_1 + a_2 v_2 + a_3 v_3 + ...$$

Definition: (Linear Independence) A set of vectors is **linearly independent** if there is only one way to make the zero vector in a linear combination: if all coefficients are zero, or the trivial way. If there is a nontrivial way to make the zero vector, then at least one of the vectors in the span is redundant - removing it won't affect the span at all. We call a set of vectors with redundancies **linearly dependent**.

Definition: (Dimensions in a Vector Space) If we can make a set of N linearly independent vectors, but not a set of $N + 1$ linearly independent vectors, the vector space has N **dimensions**.

Definition: (Basis of a Vector Space) If V is an n dimensional vector space, then any set of n linearly independent vectors forms a **basis in** V.

Theorem: Every vector in a vector space can be expressed as a linear combination of basis vectors.

Proof: Suppose our basis is $\{v_0, v_1, v + 2, v_3, ..., v_n\}$. By definition, adding any other vector v_x will make the set linearly dependent, so $a_0 v_0 + a_1 v_1 + a_2 v + 2 + a_3 v_3 + ... + a_n v_n + x v_x = 0$ for some nontrivial coefficients. Then, by subtracting $x v_x$ to the other side, and dividing by x, we get $\frac{a_0}{x} v_0 + \frac{a_1}{x} v_1 + \frac{a_2}{x} v_2 + ... + \frac{a_n}{x} v_n = v_x$

0.2 Inner Products

Definition: (Inner Product) The **inner product** is an operation that turns two vectors into a scalar. It is written as $\langle a|b\rangle$ It has the following properties:

1. $\langle a|b\rangle = \langle b|a\rangle$
2. $\langle a + b|c\rangle = \langle a|c\rangle + \langle b|c\rangle$

3. $\langle \lambda a|b\rangle = \lambda\langle a|b\rangle$
4. $\langle a|a\rangle \geq 0$, and $\langle a|a\rangle = 0$ only when $a = 0$.

   Note: Some authors define an inner product to be the first three conditions, and if it satisfies the fourth, it is called a positive-definite inner product.

Definition: (Orthogonality) Two vectors a, b are **orthogonal** if $\langle a|b\rangle = 0$.

Definition: (Norm) The **norm** of a vector a is given by $||a|| = \sqrt{\langle a|a\rangle}$. The norm can be thought of as the length of a vector.

Definition: (Kronecker Delta) The **Kronecker Delta** is a function that takes in two vectors i, j and returns $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$.

Definition: (Orthonomal Basis) A basis is **orthonormal** if all basis vectors have length one and each basis vector is orthogonal to each other basis vector. In other words, for any two basis vectors i, j, $\langle i|j\rangle = \delta_{i,j}$.

If we have an N dimensional orthonormal basis $\hat{e}_1, \hat{e}_2, \hat{e}_3...$, we may express any vector $a = \sum_{i=1}^{N} a_i\hat{e}_i$ as the sum of its component vectors.

Furthermore, the inner product $\langle \hat{e}_j|a\rangle = a_j$ of a with a basis vector will give the corresponding component vector.

We can also write an inner product in terms of the component vectors in an orthonormal basis: $\langle a|b\rangle = \langle \sum_{i=1}^{n} a_i b_i \langle \hat{e}_i|\hat{e}_i\rangle + \sum_{i=1}^{N}\sum_{j\neq i}^{N} a_i * b_j\langle \hat{e}_i|\hat{e}_j\rangle\rangle = \sum_{i=1}^{N} a_i b_i$

When the basis is not orthonormal, we can't eliminate the scross terms like above, so instead we have $\sum_{i=1}^{N}\sum_{j=1}^{N} a_i\langle e_i|e_j\rangle b_j$

Proposition: (Inequalities) Here are some useful inequalities:

1. Schwarz's Inequality: $|\langle a|b\rangle vert \leq ||a|| \times ||b||$
2. The Triangle Inequality: $||a + b|| \leq ||a|| + ||b||$
3. Bessel's Inequality: $||a||^2 \geq \sum_i |\langle \hat{e}_i|a\rangle|^2$ (The length of a is always greater than any of its components).
4. The Parallelogram Inequality: $||a + b||^2 + ||a - b||^2 = 2(||a||^2 + ||b||^2)$