3.3: The Isomorphism Theorems

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October 13, 2024

Theorem: The First Isomorphism Theorem: If $\varphi : G \to H$ is a homomrophism of groups, then $\ker\varphi \trianglelefteq G$ and $G/\ker\varphi \cong \varphi G$

Corrolary 17: Let $\varphi : G \to H$ be a homomorphism of groups.

1. φ is injective iff $\ker\varphi = 1$
2. $|G : \ker\varphi| = |\varphi(G)|$

Theorem: The Second Isomorphism Theorem Let G be a group, let A and B be subgroups of G and assume $A \leq N_G(B)$. Then, AB is a subgroup of G, $B \trianglelefteq AB$, $A \cap B \trianglelefteq A$, and $AB/B \simeq A/(A \cap B)$. (Remember that $N_G(A)$ is the set of elements in G that commute with all elements in A)

Proof: Note: all elements of A do normalize B. Then, by a previous corrolary in 3.2, AB is a subgroup of G. Every element in G normalizes AB because $babb^{-1} = ba$, and by a previous theorem, $BA = AB$ if $A \leq N_G(B)$, so ba is equal to some element in AB.

To show everything else, lets establish a map $\varphi : A \to AB/B$ mapping elements of A to their equivalence classes in AB by $\varphi(a) = aB$. φ is a homomorphism because $\varphi(a_1a_2) = (a_1a_2)B = a_1Ba_2B = \varphi(a_1)\varphi(a_2)$. The kernel of the homomorphism is all elements in A that fulfill $aB = 1B$, or in other words, elements in A that are closed in B. The only elements that do this are ones that are in the group B, by definition of closure, so $\ker\varphi = A \cap B$. By the First Isomorphism Theorem, the kernel of φ is normal in G, so by extension, $\ker\varphi = A \cap B \trianglelefteq A$.

The First Isomorphism Theorem also tells us that $A/\ker\varphi \cong \varphi(A)$, and by substituting, we get $A/A \cap B \cong AB/B$.

This theorem is called the Diamond Isomorphism Theorem, because the lattice forms a diamond, with $A \cap B$ being a subgroup of both A and B, and those two being both subgroups of AB, which is a subgroup of G.

Theorem: The Third Isomorphism Theorem: Let G be a group and let H and K be normal subgroups of G, and let $H \leq K$. Then, $K/H \trianglelefteq G/H$ and $(G/H)/(K/H) \cong G/K$

Proof: To prove the first part, we want to show that $kHgH(kH)^{-1} = gH$. Then, $kHgH = gHkH$, and evaluating the left side, we get that $kHgH = kH(k_1gk_1^{-1})H$ since K is normal in G. Then, by distributing, we get $kHk_1HgHk_1^{-1}H$, and $kHk_1H = k_2H$, so we get $k_2HgHk_1^{-1}H$. Then, we get $(k_2gk_1^{-1})H$, and since K is normal in G, $k_2g = gk_2$, so this becomes $(gk_2k_1^{-1})H$, which is equal to $gHkH$.

To prove the second part, lets define a function $\varphi : G/H \to G/K$ that maps elements $gH \to gK$. φ is well defined (equal inputs, equal outputs), because for some $g_1H = g_2H$, then $g_1 = g_2h$ due to them being in the same equivalence class. Then, h is in K because H is a subgroup of K, so $g_1 = g_2k$, so $g_1K = g_2K$. This function is surjective, but not injective. The kernel of φ is all of the elements like gH that map to $1K$, or all elements that satisfy $gK = 1K$. The elements in g, that when multiplied with K, result in K, are just the elements that are closed in K, also known as the elements of K itself. As such, the kernel of the homomorphism is just K/H. By the first isomorphism theorem, $(G/H)/(K/H) \cong G/K$

Theorem: The Fourth Isomorphism Theorem: Let G be a group and N be a normal subgroup of G. The set of all subgroups of G which contain N is isomorphic to the set of all subgroups of G/N.

Now, let A, B be subgroups of G which contain N, and let $\overline{A}, \overline{B}$ be subgroups of $\overline{G} = G/N$. The following relations are true:

1. $A \leq B$ if and only if $\overline{A} \leq \overline{B}$
2. If $A \leq B$, then $|B : A| = |\overline{B} : \overline{A}|$
3. $\overline{< A, B >}$, the quotient group version of the group generated by A and B, is equal to $< \overline{A}, \overline{B} >$.
4. $\overline{A \cap B} = \overline{A} \cap \overline{B}$
5. $A \trianglelefteq B$ if and only if $\overline{A} \trianglelefteq \overline{G}$

Proof: The preimage of a subgroup in G/N is a group in G, but since all subgroups of G/N must contain $1N$, the preimage will always contain N, therefore there is a bijective relationship between the two.