3.5: Conditional Probabilites are Probabilities

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Proposition: We want to show that a conditional probability satisfies all three axioms of a probability:

1. $0 \leq P(E|F) \leq 1$
2. $P(S|F) = 1$
3. If E_i, where $i = 1, 2, 3...$, are mutually exclusive events, then $P(\bigcup_1^\infty E_i|F) = \sum_1^\infty P(E_i|F)$

Proof: Let E and F be events in the probability space S. Then:

1. $0 \leq P(E|F)$ is equal to $0 \leq \frac{P(E \cap F)}{P(F)}$, which is true because the smallest that $P(E \cap F)$ can be is zero. $P(E|F) \leq 1$ is true because $(E \cap F) \subset F$, so $P(E \cap F) \leq P(F)$, so $\frac{P(E \cap F)}{P(F)} \leq 1$.
2. $P(S|F) = 1$ because $P(S \cap F) = P(F)$ so $\frac{P(S|F)}{P(F)} = \frac{P(F)}{P(F)} = 1$
3. $P(\bigcup_1^\infty E_i|F) = \frac{P(\bigcup_1^\infty (E_i) \cap F)}{P(F)} = \frac{P(\bigcup_1^\infty (E_i \cap F))}{P(F)} = \frac{\sum_1^\infty P(E_i \cap F)}{P(F)} = \sum_1^\infty P(E_i|F)$

This means that if we define $Q(E) = P(E|F)$, then $Q(E)$ is a probability function on the sample space S.

Now, lets see what happens when we have $Q(E_1|E_2) = \frac{Q(E_1 \cap E_2)}{Q(E_2)} = \frac{P(E_1 \cap E_2|F)}{P(E_2|F)} = \frac{\frac{P(E_1 \cap E_2 \cap F)}{P(F)}}{\frac{P(E_2 \cap F)}{P(F)}} = P(E_1|E_2 \cap F)$

The above equation is equivalent to $P(E_1|F) = P(E_1|E_2 \cap F)P(E_2|F) + P(E_1|E_2^C \cap F)P(E_2^C|F)$

Example: (5a) Consider Example 3a, which is concerned with an insurance company which believes that people can be divided into two distinct classes: those who are accident prone and those who are not. During any given year, an accident-prone person will have an accident with probability .4, whereas the corresponding figure for a person who is not prone to accidents is .2. What is the conditional probability that a new policyholder will have an accident in his or her second year of policy ownership, given that the policyholder has had an accident in the first year?

Solution: Let A be the chance someone is accident-prone, and A_i be the probability that someone has an accident in their ith year. We want to know $P(A_2|A_1)$, and we already know that $P(A_1) = .26$, $P(A) = .3$, $P(A_i|A) = .4$ and $P(A_i|A^C) = .2$. Our equation becomes $P(A_2|A_1) = P(A_2|A \cap A_1)P(A|A_1) + P(A_2|A^C \cap A_1)P(A^C|A_1)$

Now, lets find $P(A|A_1) = \frac{P(A_1|A)P(A)}{P(A_1)} = \frac{.4 * .3}{.26} = \frac{6}{13}$

Likewise, $P(A^C|A_1) = \frac{7}{13}$

$P(A_2|A \cap A_1) = .4$, as this is equal to $P(A_2|A)$, because A_2 and A_1 are independent events. This applies to $P(A_2|A^C A_1) = .2$.

Our entire equation becomes $P(A_2|A_1) = .4 * \frac{6}{13} + .2 * \frac{7}{13} = .29$