2.1: Topological Spaces

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Def: A **topology** on a set x is a collection $\mathcal{T}$ of subsets of X having the following properties:

1. $\varnothing$ and X are in $\mathcal{T}$
2. The union of any number of elements in $\mathcal{T}$ is in $\mathcal{T}$
3. The intersection of any elements in $\mathcal{T}$ is in $\mathcal{T}$

Def: A **topological space** is an ordered pair of elements $(X, \mathcal{T})$, a set X with a toplology $\mathcal{T}$ on X.

Def: If X is a topological space with topology $\mathcal{T}$, then a subset $U \subset X$ is an **open set** if $U \in \mathcal{T}$.

There can be many topologies on a set.

Def: The collection of all subsets of X is called the **discrete topology**, and the collection of $\varnothing, X$ is called the **trivial topologies**. Both are valid topologies.

Def: Let $\mathcal{T}_f$ be a topology such that any subset U that fulfills the criteria: $X - U$ is finite or all of X is in $\mathcal{T}_f$. This is called the **finite complement topology**, because the complement of every member is finite (or all of X). $\varnothing$ and X are in $\mathcal{T}_f$, the complement of $\varnothing$ is all of X, and the complement of X is finite. $\mathcal{T}_f$ is complete under unions because $X - \bigcup U_a = \bigcap (X - U_a)$, where $U_a \in \mathcal{T}_f$, because $X - U_a$, the complements of membersof $\mathcal{T}_f$, are finite, and the intersection of finite sets are finite, so $\bigcup U_a$ is a complement of a finite set. Likewise, the same goes for intersections, and the union of finite sets is finite, so $\mathcal{T}_f$ is closed under union and intersection, therefore, it is a topology.

Def: Consider two topologies, $\mathcal{T}, \mathcal{T}'$ on a set X. If $\mathcal{T}' \supset \mathcal{T}$, then we say that $\mathcal{T}'$ is **finer** than $\mathcal{T}$. If the reverse is true, we say that $\mathcal{T}'$ is coarser than $\mathcal{T}$. If one set contains the other, we say the two are **comparable**. Remember, the superset is finer, the subset is coarser.