8.12: Special Types of Square Matrices

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1 Motivation:

We want to explore some special square matrices which have unique properties.

2 Content:

2.1 Diagonal Matrices

Definition: (Diagonal Matrices) A **diagonal matrix** is a square matrix with all of its nonzero terms in the main/leading diagonal. For example,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

is a diagonal matrix. Additionally, diagonal matrices can be denoted with

$$\text{diag}(A_{11}, A_{22}, A_{33}, ...A_{nn}) = \begin{bmatrix} A_{11} & 0 & 0 & ... & 0 \\ 0 & A_{22} & 0 & ... & 0 \\ 0 & 0 & A_{33} & ... & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & ... & A_{nn} \end{bmatrix}$$

Proposition: If A and B are n by n diagonal matrices, then:

1. $|A| = A_{11}A_{22}A_{33}...A_{nn}$
2. $A^{-1} = \text{diag}(\frac{1}{A_{11}}, \frac{1}{A_{22}}, \frac{1}{A_{33}}, ..., \frac{1}{A_{nn}})$
3. $AB = BA$

2.2 Upper and Lower Triangular Matrices

Definition: (Upper and Lower Triangular Matrices) An **upper triangular matrix** is a matrix with nonzero terms only on or above the leading diagonal. Likewise, a **lower diagonal matrix** is a matrix with nonzero terms only on or below the leading diagonal.

Proposition: If A is an n by n upper or lower triangular matrix, then

$$|A| = A_{11}A_{22}A_{33}...A_{nn}$$

2.3 Symmetric Matrices

Definition: (Symmetric Matrices) A **symmetric matrix** is a matrix such that $A = A^T$ and an **antisymmetric matrix** is a matrix where $A^T = -A$

Proposition:

1. We can write any square matrix as the sum of one symmetric and one antisymmetric matrix.
2. If a matrix is symmetric, then so is its transpose.

Proof:

1. Given a square matrix A, we can write
$$A = \frac{1}{2}(2A) + \frac{1}{2}A^T - \frac{1}{[}2]A^T$$
then, by grouping terms, we get
$$A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)$$
The first term is symmetric, and the second term is anti-symmetric.
2. Let A be a symmetric or antisymmetric matrix. Then:
$$(A^{-1})^T = (A^T)^{-1} = \pm A^{-1}$$

2.4 Orthogonal Matrices

Definition: (Orthogonal Matrices) **Orthogonal matrices** are matrices with the property that
$$A^T = A^{-1}$$

Proposition:

1. The inverse of an orthogonal matrix is orthogonal.
2. The determinant of an orthogonal matrix is always ± 1.

Proof: Suppose that A is an orthogonal matrix.

1. $(A^{-1})^T = (A^T)^{-1} = (A^{-1})^{-1}$
2. $|A^T A| = |A^T||A| = |A|^2 = |I| = 1$

Since the determinant of an orthognal matrix is always either 1 or -1. the linear transformation associated with an orthogonal matrix is always one that keeps vectors at the same length, and just rotates them, as the determinant of a matrix is the scale factor of the associated transformation.

2.5 Hermitian and Anti-Hermitian Matrices

Definition: (Hermitian Matrices) A **hermitian matrix** is a matrix where $A = A^\dagger$ and likewise, an **anti-hermitian** matrix is one where $A^\dagger = -A$.

Proposition: Any matrix can be written as the sum of a hermitian and an anti-hermitian matrix.

Proof: Suppose A is a matrix. Then, $A = \frac{1}{2}A + \frac{1}{2}A + \frac{1}{2}A^\dagger - \frac{1}{2}A^\dagger = \frac{1}{2}(A + A^\dagger) + \frac{1}{2}(A - A^\dagger)$. Notice that since $(A^\dagger)^\dagger = A$, $(A + A^\dagger)^\dagger = (A^\dagger + A)$, so $(A + A^\dagger)$ is a hermitian matrix, and likewise, $(A - A^\dagger)^\dagger = (A^\dagger - A) = -(A - A^\dagger)$, so $(A - A^\dagger)$ is an anti-hermitian matrix.

2.6 Unitary Matrices

Definition: (Unitary Matrices) A **unitary matrix** is a matrix such that $A^\dagger = A^{-1}$.

If a matrix is real, then $A^\dagger = A^T$, so if $A^T = A^{-1}$, like in an orthognal matrix, then it is also unitary.

Proposition: Suppose A is a unitary matrix.

1. The inverse of a unitary matrix is also unitary.
2. The determinant of a unitary matrix is always ± 1.

2.7 Normal Matrices

Definition: (Normal Matrices) A matrix A is a **normal matrix** if $AA^\dagger = A^\dagger A$.

Proposition:

Hermitian, unitary, symmetric, and orthogonal matrices are normal.

If A is normal then so is A^{-1}