Rudin Chapter 3: The Number e

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Definition: (The Number e)

$$e = \sum_{n=0}^{\infty} \frac{1}{n!}$$

Theorem:

$\lim_{x \to \infty} (1 + \frac{1}{n})^n = e$

Proof:

Hint: Try establishing a sequence where $n = 1, 2, 3, ...$ and find the upper and lower limits of the sequence.

Let $s_n = \sum_{k=0}^{n} \frac{1}{k!}$ and $t_n = (1 + \frac{1}{n})^n$. If we apply the binomial theorem, we get that $t_n = 1 + 1 + \frac{1}{2!}(1 - \frac{1}{n}) + \frac{1}{3!}(1 - \frac{1}{n})(1 - \frac{2}{n}) + ... + \frac{1}{n!}(1 - \frac{1}{n})(1 - \frac{2}{n})...(1 - \frac{n-1}{n})$. Note that each term of this expanded sequence is less than s_n, so the lower limit of the sequence must be less than or equal to that of s_n, which is e.

Next, note that $t_n \geq 1 + 1 + \frac{1}{2!}(1 - \frac{1}{n}) + ... + \frac{1}{m!}(1 - \frac{1}{n})...(1 - \frac{m-1}{n})$ when $n \geq m$, and if we let $n \to \infty$, we get that the lower limit of t_n is greather than or equal to $1 + 1 + \frac{1}{2!} + ... + \frac{1}{m!}$, and when we let $m \to \infty$, we get that the lower limit of m is greater than or equal to e.

If both the upper and lower limits are greater than or equal to and less than or equal to e respectively, the sequence must tend to e.

Theorem:

e is irrational.

Proof:

Suppose e was rational. Then $e = \frac{p}{q}$ for some positive integers p, q (since e is positive). Then, $0 < q!(e - s_q) < \frac{1}{q}$ (since $e - s_q = \frac{1}{(q+1)!} + \frac{1}{(q+2)!} + ...$).

Since $q!e$ is an integer, and $q!s_q$ is an integer, then $q!(e - s_q)$ is an integer, and so $0 < q!(e - s_q) < \frac{1}{q} < 1$ implies the existence of an integer between 0 and 1.