8.13: Eigenvectors and Eigenvalues

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1 Motivation

We want to define eigenvectors and eigenvalues, and their properties and uses

2 Content

Definition: (Eigenvectors and Eigenvalues) **Eigenvectors** of a particular linear transformation A are vectors x that satisfy the equation

$$Ax = \lambda x$$

Instead of being rotated by A, they are scaled by an **eigenvalue** λ.

Proposition: If x_i and λ_i are the eigenvalues and eigenvectors of some matrix A,then the eigenvectors and values of A^{-1} are x_i and $\frac{1}{\lambda_i}$.

Proposition: If A is a normal matrix ($AA^\dagger = A^\dagger A$), then the eigenvalues of $A^\dagger$ are the complex conjugates of the eigenvalues of A.

Proof: We start with

$$(A - \lambda I)x = 0$$

If we take the hermitian conjugate of both sides, we get

$$((A - \lambda I)x)^\dagger = 0$$

And by the properties of hermitian conjugates, this turns into

$$x^\dagger (A - \lambda I)^\dagger = 0$$

Then, multiplying both sides by $(A - \lambda I)x$, we have

$$x^\dagger (A - \lambda I)^\dagger (A - \lambda I)x = 0$$

When expand $(A - \lambda I)^\dagger (A - \lambda I)$ we get $AA^\dagger + -\lambda^* A - \lambda A^\dagger + \lambda\lambda^*$. Each of these elements commute (since A is normal), so we get that $(A - \lambda I)^\dagger (A - \lambda I) = (A - \lambda I)(A - \lambda I)^\dagger$, and the entire equivalence becomes

$$x^\dagger (A - \lambda I)(A - \lambda I)^\dagger x = 0$$

Then, factoring out hermitian conjugates, we get

$$((A - \lambda I)^\dagger x)^\dagger (A - \lambda I)^\dagger x = 0$$

This gives us that

$$(A - \lambda I)^\dagger x = 0$$

And distributing the conjugate, we get

$$(A^\dagger - \lambda^* I)x = 0$$

Therefore, the eigenvalues of $A^\dagger$ are λ^*