3.6: Second Order Systems and Applications

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1 Motivation:

In this section, we want to learn how to solve more complex systems of springs and masses.

2 Content

Example: Suppose we have a system of three masses, m_1, m_2, and m_3, and a spring with constant k_1 connects m_1 to the wall, k_2 connects m_1 and m_2, k_3 connects m_2 and m_3, and k_4 connects m_3 to the other wall. Assume there are no forces other than the spring forces on the masses. Can you find equations governing the motion of the three masses?

Solution: Let x_1, x_2, x_3 be the displacement from the rest positions of each mass, with right being positive. Now we write the force equations for each mass:

For m_1, only k_1 and k_2 are acting on it, so the equation is

$$m_1 x_1'' = -k_1 x_1 + k_2(x_2 - x_1)$$

Since the force of a spring acts in the opposite direction of displacement, $k_1 x_1$ is negative and $k_2(x_2 - x_1)$ is positive.

For m_2, k_2 and k_3 act on it, so

$$m_2 x_2'' = -k_2(x_2 - x_1) + k_3(x_3 - x_2)$$

For m_3, k_3 and k_4 act on it, so

$$m_3 x_3'' = -k_3(x_3 - x_2) - k_4 x_3$$

Rearranging all three equations, we get:

$$m_1 x_1'' = -(k_1 + k_2)x_1 + k_2 x_2$$
$$m_2 x_2'' = k_2 x_1 - (k_2 + k_3)x_2 + k_3 x_3$$
$$m_3 x_3'' = k_3 x_2 - (k_3 + k_4)x_3$$

Putting these in matrix form, we get

$$M\vec{x}'' = K\vec{x}$$

where

$$M = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix}$$

$$K = \begin{bmatrix} -(k_1 + k_2) & k_2 & 0 \\ k_2 & -(k_2 + k_3) & k_3 \\ 0 & k_3 & -(k_3 + k_4) \end{bmatrix}$$

Now, we want to isolate $\vec{x}''$, so we left multiply by

$$M^{-1} = \begin{bmatrix} \frac{1}{m_1} & 0 & 0 \\ 0 & \frac{1}{m_2} & 0 \\ 0 & 0 & \frac{1}{m_3} \end{bmatrix}$$

to get
$$\vec{x}'' = M^{-1}K\vec{x}$$

Multiplying out the matrices, we get
$$A = \begin{bmatrix} \frac{-(k_1+k_2)}{m_1} & \frac{k_2}{m_1} & 0 \\ \frac{k_2}{m_2} & \frac{-(k_2+k_3)}{m_2} & \frac{k_3}{m_2} \\ 0 & \frac{k_3}{m_3} & \frac{-(k_3+k_4)}{m_3} \end{bmatrix}$$

The actual contents of the matrix are unimportant, but now our equation looks like
$$\vec{x}'' = A\vec{x}$$

Notice how all of the values in the coefficient matrix are constants, so we can solve this using the constant coefficients eigenvalue method.

Lets guess that the vector of our solutions looks like $\vec{x} = \vec{v}e^{\alpha t}$ where $\vec{v}$ is an eigenvector. Then, $\vec{x}'' = \alpha^2 e^{\alpha t}$. Plugging into our differential equation, we get that
$$\alpha^2 e^{\alpha t} = A\vec{v}e^{\alpha t}$$

We then divide by $e^{\alpha t}$ to get
$$\alpha^2 \vec{v} = A\vec{v}$$

Therefore, α^2 must be an eigenvalue of A and $\vec{v}$ must be an eigenvector of A.

Now, usually, α^2 will be negative, so $\alpha = \pm i\omega$, a complex number. Then, plugging back into our guess, we get that
$$\vec{x} = \vec{v}e^{\alpha t} = \vec{v}(\cos(\omega t) + i\sin(\omega t))$$

By taking the real and imaginary components, we get two solutions: $\vec{v}\cos(\omega t)$ and $\vec{v}\sin(\omega t)$

This solution works for any system of n masses and $n+1$ springs! You just make an n-dimensional matrix, and everything works!

Example: Suppose we have two masses, m_1 and m_2, with a spring connecting m_1 to the left wall, and a spring connecting m_2 to m_1. Given that $m_1 = 2, m_2 = 1, k_1 = 4, k_2 = 2$, wherehat are the equations of motion for this system?

Solution: Lets begin by defining x_1 to be the displacement from when m_1 is at rest (right is positive), and x_2 to be the displamenent from when m_2 is at rest.

Lets start by writing our force equations:
$$m_1 x_1'' = -k_1 x_1 + k_2(x_2 - x_1)$$
$$m_2 x_2'' = -k_2(x_2 - x_1)$$

Rearranging, we get
$$m_1 x_1'' = -(k_1 + k_2)x_1 + k_2 x_2$$
$$m_2 x_2'' = k_2 x_1 - k_2 x_2$$

Putting it in matrix form, we get
$$M\vec{x}'' = K\vec{x}$$

where
$$M = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}$$
$$K = \begin{bmatrix} -(k_1 + k_2) & k_2 \\ k_2 & -k_2 \end{bmatrix}$$

Now, we left-multiply by M^{-1} to get
$$\vec{x}'' = A\vec{x}$$

where
$$A = \begin{bmatrix} \frac{-(k_1+k_2)}{m_1} & \frac{k_2}{m_1} \\ \frac{k_2}{m_2} & \frac{-k_2}{m_2} \end{bmatrix}$$

Plugging in, we get

$$A = \begin{bmatrix} -3 & 1 \\ 2 & -2 \end{bmatrix}$$

The eigenvalues of this matrix are $\lambda = -1, -4$, and the corresponding eigenvectors are $\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. We can calculate that $\omega = 1, 2$, and our solutions are

$$\vec{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} (a_1 \cos(t) + b_1 \sin(t)) + \begin{bmatrix} 1 \\ -1 \end{bmatrix} (a_2 \cos(2t) + b_2 \sin(2t))$$

Our two natural frequencies of oscillation are 1 and 2, and the two terms in the solution are called normal modes.

2.1   Forced Oscillation

Suppose our system has a driving displacement, so that our equation is

$$\vec{x}'' = A\vec{x} + \vec{F} \cos(\omega t)$$

Remember that to form a solution to a non-homogeneous constant coefficient equation, we need a complementary solution and a particular solution. We'll just solve the particular solution right now.

Lets guess that our particular solution $\vec{x}_p = \vec{c}\cos(\omega t)$. We don't include a sin term because we only have a second derivative.

We calculate that $\vec{x}_p'' = -\omega^2 \vec{c}\cos(\omega t)$, and plugging in, we get

$$-\omega^2 \vec{c}\cos(\omega t) = A\vec{c}\cos(\omega t) + \vec{F}\cos(\omega t)$$

Rearranging, we get

$$(A + \omega^2 I)\vec{c} = -\vec{F}$$

therefore

$$\vec{c} = (A + \omega I)^{-1}(-\vec{F})$$