Rudin Chapter 3: The Root and Ratio Tests

Alex L.

August 19, 2025

Theorem: (The Root Test)

Given $\sum a_n$, let $\alpha = \limsup \sqrt[n]{|a_n|}$

Then, if $\alpha > 1$, the series diverges, if $\alpha < 1$, the series converges, and if $\alpha = 1$, the test gives no information.

Proof:

If $\alpha < 1$, we choose β so $\alpha < \beta < 1$. Then, we choose an N so that for all $n \geq N$, this means $\sqrt[n]{|a_n|} < \beta$, which means that $|a_n| < \beta^n$. Since $0 < \beta < 1$, $\sum \beta^n$ converges as $n \to \infty$. And $\sum a_n$ converges due to the comparison test.

If $\alpha > 1$, there is a sequence of indices $\{k_n\}$ where $\sqrt[k_n]{|a_{k_n}|} > \alpha > 1$, which means that there is an infinite number of a_{k_n} that are greater than 1, so the series diverges.

For the case of $\alpha = 1$, consider $\sum \frac{1}{n}$ and $\sum \frac{1}{n^2}$. In both these series, $\alpha = 1$, but the first diverges, and the second converges, so the test is inconclusive.

Theorem: (The Ratio Test)

The series $\sum a_n$ converges if $\limsup |\frac{a_{n+1}}{a_n}| < 1$ and diverges if $\limsup |\frac{a_{n+1}}{a_n}| \geq 1$ for all $n \geq N$, where N is some fixed integer.

Proof:

For the first case, it is possible to find a β such that $|\frac{a_{n+1}}{a_n} < \beta < 1$ for all $n \geq N$ after some integer N. This means that $a_{n+1} < \beta a_n$, and $|a_{n+p} < \beta^p |a_n|$. Also, terms like $|a_n|$ are always less than terms in the series $|a_N|\beta^{-N}\beta^n$ for $n \geq N$, so by the comparison test, this series converges since $\sum \beta^n$ converges.

For the second case, since the terms are increasing, the sequence does not tend to zero, so it cannot converge.

Theorem:

For any sequence $\{c_n\}$ of numbers, the following inequalities hold:

$$\liminf \frac{c_{n+1}}{c_n} \leq \liminf \sqrt[n]{c_n}$$

$$\limsup \sqrt[n]{c_n} \leq \limsup \frac{c_{n+1}}{c_n}$$