3.3: Linear Systems of ODEs

Alex L.

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Definition: (Matrix and Vector Valued Functions) A **vector valued function** is a function in the form

$$\vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ \vdots \\ x_n(t) \end{bmatrix}$$

A **matrix valued function** is a function in the form

$$A(x) = \begin{bmatrix} a_{11}(t) & a_{12}(t) & a_{13}(t) & ... & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & a_{23}(t) & ... & a_{2n}(t) \\ a_{31}(t) & a_{32}(t) & a_{33}(t) & ... & a_{3n}(t) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1}(t) & a_{n2}(t) & a_{n3}(t) & ... & a_{nn}(t) \end{bmatrix}$$

Definition: (Systems of First Order Linear ODEs) A **first order linear system of ODEs** is a system that can be represented by

$$\vec{x'}(t) = P(t)\vec{x}(t) + \vec{f}(t)$$

Where $P(T)$ is a matrix valued function and $\vec{x'}(t)$, $\vec{x}(t)$, and $\vec{f}(t)$ are vector-valued functions.

If $P(t)$ is a matrix of constants, with no values depending on t, we say the system has **constant coefficients**.

If $\vec{f}(t) = \vec{0}$, the zero vector, then we say that the system is homogeneous.

Theorem: (Superposition) If $\vec{x'}(t) = P(t)\vec{x}(t)$ is a homogeneous linear system of ODEs, and $\vec{x_1}, \vec{x_2}, ..., \vec{x_n}$ are solutions and linearly independent, then $\vec{x} = C_1\vec{x_1} + C_2\vec{x_2} + ... + C_n\vec{x_n}$ is a general solution to the system.

The general solution to a homogeneous differential equation can be written as $X(t)\vec{c}$, where X is a matrix with columns of $\vec{x_1}, \vec{x_2}, ..., \vec{x_n}$, and $\vec{c}$ is a column vector with entries $c_1, c_2, ..., c_n$. In this form, $X(t)$ is called the fundamental matrix.