4.2: Summation of Series

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Def: An **arithmetic** series is a series where the difference between consecutive terms is constant, that is, the series looks like $a, a+d, a+2d, a+3d, a+4d$, etc.

The summation of an arithmetic series is $\frac{n}{2}(a_1+a_n)$, the first term plus the last term multiplied by half the size of the series. Evidently, an infinitely long arithmetic series will always diverge.

Def: A **geometric** series is a series where the ratio of consecutive terms remains constant, and it takes the form $a+ar+ar^2+ar^3+ar^4+....$

The summation of a geometric series is $\frac{a(1-r^n)}{1-r}$, where r is the ratio between terms, a is the starting term, and n is the size of the series. A geometric series may converge or diverge. If $|r|<1$, then the series will converge to $\frac{a}{1-r}$, and if it is greater than or equal to 1, it will diverge or oscillate.

Def: An **arithmetico-geometric** series is a combination of both an arithmetic and geometric series, in the form $a+(a+d)r+a+(a+2d)r^2+a+(a+3d)r^3+....$

The summation of the first n terms of an arithmetico-geometric series is equal to $\frac{a-(a+(n-1)d)r^n}{1-r}+\frac{rd(1-r^{n-1})}{(1-r)^2}$ and an infinite series with $|r|<1$ tends towards $\frac{a}{1-r}+\frac{rd}{(1-r)^2}$, and if $|r|\geq 1$, then the series oscillates or diverges.

0.1 The Difference Method

If we have a series $u_1+u_2+u_3+...+u_n$, where the terms can be expressed by $u_n=f(n)-f(n-1)$, then by expanding every term, we find that many of them cancel, until we are left with the sum of the first n terms being $S_n=f(n)-f(0)$.

0.2 Series with Natural Numbers

We can actually write series of squares and cubes of natural numbers using the difference method.

Take the function $f(n)=n(n+1)(2n+1)$, then $f(n-1)=(n-1)n(2n+1)$ and $f(n)-f(n-1)=6n^2$. We can then write the series of squares of natural numbers in this form, and by the difference method, the partial sum of the first n terms comes out to be $\frac{1}{6}n(n+1)(2n+1)$

The same can be done for cubes. Take $f(n)=(n(n+1))^2$, then $f(n-1)=((n-1)n)^2$, and $f(n)-f(n-1)=4n^3$, and the partial sum of the first n terms is $\frac{1}{4}n^2+n(+1)^2$

0.3 Transformation of a Series

You can multiply, divide, add, subtract, differentiate, or integrate a series to put it in a more solvable form, as long as you reverse all your changes when solving.