Rudin Chapter 4: Continuity and Compactness

Alex L.

September 19, 2025

Definition: (Boundedness)

A mapping $f : E \to \mathbb{R}^k$ is **bounded** if there is some k where $|f(x)| \leq k$ for all x in E.

Theorem:

Suppose that f is a continuous mapping of a compact metric space X to another metric space Y. Then $f(X)$ is compact.

Proof:

Let's first get our definitions in order. $f(x)$ is continuous meaning that for every $\epsilon > 0$, there is a p and δ such that all the points within δ of p are also within ϵ of $f(p)$.

Compactness means that all coverings of X also have a finite subcovering which also covers X.

Let V be an open cover of $f(X)$. Then the preimage of each open set in V is also open (as per a previous theorem in this chapter). We take the preimage of all these, and they do cover X, and we find a finite subcovering since X is compact. We then take the images of these and they do form a finite covering of $f(X)$.

Theorem:

Suppose $f : X \to \mathbb{R}$ is a continuous function and X is compact. Then, $\sup_{p \in X} f(p)$ and $\inf_{p \in X} f(p)$ are in $f(X)$.

Proof:

Since X is compact, so is $f(X)$, and compact subsets of $\mathbb{R}^k$ are closed and bounded by the Heine-Borel theorem. Since it is both closed and bounded, the range must contain its bounds.

Theorem:

Suppose that $f : X \to Y$ is a continuous $1-1$ mapping between a compact metric space X and a metric space Y. Then, the inverse mapping, f^{-1} is a continuous mapping from Y to X.

Proof:

One way we can ascertain the continuity of a function f is that it is only continuous if $f^{-1}(V)$ is open in X for every open set V in Y.

So in this case, to show that f^{-1} is continuous, we want to show that $f(V)$ is open in Y for every open V in X.

First, note that V^c is closed in X, since V is open. Closed subsets of compact spaces are compact, so V^c is compact. Since f is continuous, $f(V^c)$ is also compact, and closed in Y, since compact sets are closed.

Definition: (Uniformly Continuous)

Let f be a mapping of a metric space X into a metric space Y. f is **uniformly continuous** on X if for every $\epsilon > 0$ there exists a $\delta > 0$ such that $d_Y(f(p), f(q)) < \epsilon$ for all p and q where $d_X(p, q) < \delta$

Theorem:

Every uniformly continuous function is continuous.

Proof:

Think about what uniformly continuous means. If I choose some $\epsilon > 0$, there will be a $\delta > 0$ such that any two points less than δ apart have images less than ϵ apart. Then, choose some $f(p)$ in the image. All points that are less than ϵ from $f(p)$ are less than δ from p, otherwise the function wouldn't be uniformly continuous, so the continuity criterion is fulfilled.

Theorem:

Let f be a continous mapping of a compact metric space X to a metric space Y. Then, f is uniformly continuous on X.

Proof:

Choose some $\epsilon > 0$. Since f is continuous, we can associate every p in X to some positive number $\Phi(p)$ where $q \in X$ and $d_X(p,q) < \Phi(p)$ implies that $d_Y(f(p), f(q)) < \frac{\epsilon}{2}$. Basically, let $\Phi(p)$ be the distance that another point has to be from p such that the distance of their images is $\frac{\epsilon}{2}$.

Then, let $J(p)$ be the set of all points q such that $d(p,q) < \frac{1}{2}\Phi(p)$. Since p is in $J(p)$, the collection of all such $J(p)$ is a neighborhood and forms an open cover of X, and since X is compact, there is a subset of this collection that covers X. Let δ be the smallest radius in this collection of $J(p)$. $\delta > 0$ since there is a finite number of $J(p)$.

Now let p, q be in X such that $d(p,q) < \delta$. p must be in at least some $J(p_m)$, and $d_X(p, p_m) \leq d_X(p,q) + d_X(p, p_m)$, and $d_X(p,q) + d_X(p,p_m) < \delta + \frac{1}{2}\Phi(p_m) \leq \Phi(p_m)$

And since $d_X(p,q) < \Phi(p)$ implies $d_Y(f(p), f(q)) \leq \frac{\epsilon}{2}$, then $d_Y(f(p), f(q)) \leq d_Y(f(p), f(q)) + d_Y(f(p), f(p_m)) < \epsilon$, so f is uniformly continuous.

Theorem:

Let E be a noncompact set in $\mathbb{R}^1$. Then:

1. there exists a continuous function on E which is not bounded
2. there exists a continuous and bounded function on E which has no maximum.
3. If E is bounded, then there exists a continuous function on E which is not uniformly continuous

Proof:

Suppose E is bounded, so there is a limit point x_0 of E which is not a point of E. (A compact set is closed and bounded so if E is noncompact and bounded it must not be closed). $f(x) = \frac{1}{x - x_0}$ is not bounded, and not uniformly continuous, but is continuous. The function $g(x) = \frac{1}{1+(x-x_0)^2}$ is bounded but has no maximum. If E is unbounded, $f(x) = x$ is continuous and unbounded, and $h(x) = \frac{x^2}{1+x^2}$ is continuous and bounded but has no maxiumum.