Rudin Chatper 1: The Complex Numbers

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Definition: (Complex Numbers)

Complex numbers are ordered pairs (a, b) (ordered meaning $(a, b) \neq (b, a)$) where a, b are real numbers.

We say that complex numbers $x = (a, b)$ and $y = (c, d)$ are equal, $x = y$ if $a = c$ and $b = d$.

We define addition of complex numbers and multiplication of complex numbers as follows:

1. $x + y = (a + c, b + d)$
2. $x \cdot y = (ac - bd, ad + bc)$

Theorem:

The operations defined above turn the complex numbers into a field with $(0, 0)$ as the additive identity and $(1, 0)$ as the multiplicative identity.

Proof:

Closure under addition is pretty obvious, if we have complex numbers $x = (a, b)$ and $y = (c, d)$ then $x + y = (a + c, b + d)$. Since $a + c$ and $b + d$ are both reals (due to closure under the reals), the sum will also be a complex number.

Addition is commutative because $x + y = (a + c, b + d) = (c + a, d + b) = y + x$.

Addition is associative because $x + (y + z) = (a + b) + (c + e, d + f) = (a + c + e, b + d + f) = (a + c, b + d) + (e, f) = (x + y) + z$

$(0, 0)$ is the additive identity since $x + (0, 0) = (a + 0, b + 0) = (a, b) = x$ for all x in the complex numbers.

For any complex number $x = (a, b)$, we let $-x$, the additive inverse of x, be $(-a, -b)$. Then, $x + (-x) = (a + (-a), b + (-b)) = (0, 0)$ and $(0, 0)$ is the additive identity.

There is closure under multiplication because $x \cdot y = (ac - bd, ad + bc)$ and since both $ac - bd$ and $ad + bc$ are real numbers, the resulting product is a complex number.

Multiplication is commutative beacuse $x \cdot y = (ac - bd, ad + bc) = (ca - db, da + cb) = y \cdot x$

Multiplication is associative beacuse $x \cdot (y \cdot z) = (a, b) \cdot (ce - df, cf + de) = (ace - adf - bcf - bde, acf + ade + bce - bdf) = (ac - bd, ad + bc) \cdot (e, f) = (x \cdot y) \cdot z$

$(1, 0)$ is the multiplicative identity since $x * (1, 0) = (a, b) * (1, 0) = (a * 1 - b * 0, b * 1 + a * 0) = (a, b)$

For any complex number $x = (a, b)$, we choose $x^{-1} = (\frac{a}{a^2+b^2}, -\frac{b}{a^2+b^2})$, and this is an inverse since $x \cdot x^{-1} = (\frac{a^2}{a^2+b^2} + \frac{b^2}{a^2+b^2}, \frac{ab}{a^2+b^2} - \frac{ab}{a^2+b^2}) = (1, 0)$

Distributive law holds because $x * (y + z) = (a, b) * (c + e, d + f) = (ac + ae - bd - bf, ad + af + bc + be) = x * y + x * z$

Theorem:

For any real numbers a, b we have $(a, 0) + (b, 0) = (a + b, 0)$ and $(a, 0) \cdot (b, 0) = (a * b, 0)$

Proof:

$$(a, 0) + (b, 0) = (a + b, 0 + 0) = (a + b, 0)$$

$$(a, 0) \cdot (b, 0) = (ab - 0 * 0, a * 0 + b * 0) = (ab, 0)$$

Definition: (i)

$$i = (0, 1)$$

Note: $(a, 0) = a$

Theorem:

$$i^2 = -1$$

Proof:

$$i^2 = i \cdot i = (0, 1) \cdot (0, 1) = (0 * 0 - 1 * 1, 0 * 1 + 0 * 1) = (-1, 0) = -1$$

Theorem:

If a and b are real, then $(a, b) = a + bi$.

Proof:

$$a + bi = (a, 0) + (b, 0) \cdot (0, 1) = (a, 0) + (b * 0 - 0 * 1, b * 1 + 0 * 0) = (a, 0) + (0, b) = (a, b)$$

Definition: (Conjugate)

If $z = a + bi$ then the **complex conjugate** of z, $\bar{z} = a - bi$.

Also, $\mathrm{Re}(z) = a$ and $\mathrm{Im}(z) = b$

Theorem:

Let z and w be complex numbers. Then,

1. $z \bar{+} w = \bar{z} + \bar{w}$
2. $z \bar{\cdot} w = \bar{z} \cdot \bar{w}$
3. $z + \bar{z} = 2\mathrm{Re}(z)$ and $z - \bar{z} = 2i\mathrm{Im}(z)$
4. $z \cdot \bar{z}$ is real and positive, except when $z = 0$

Proof:

Let $z = a + bi$ and $w = c + di$.

1. $\bar{z} + \bar{w} = a - bi + c - di = (a + c) - (bd)i = z \bar{+} w$
2. $\bar{z} \cdot \bar{w} = (a - bi) * (c - di) = ac - bd - (ad + bc)i = z \bar{\cdot} w$
3. $z + \bar{z} = a + bi + a - bi = 2a = 2\mathrm{Re}(z)$ and $z - \bar{z} = a + bi - a + bi = 2bi = 2i\mathrm{Im}(z)$
4. $z \cdot \bar{z} = (a + bi) * (a - bi) = a * a - b^2 i^2 + abi - abi = a^2 + b^2$

Definition: (Absolute Value of a Complex Number)

The **absolute value** of a complex number $|z| = (z \cdot \bar{z})^{\frac{1}{2}}$

Theorem:

1. $|z| > 0$ unless $z = 0$, then $|z| = 0$
2. $|\bar{z}| = |z|$
3. $|z \cdot w| = |z||w|$
4. $|\mathrm{Re}(z)| \leq |z|$
5. $|z + w| \leq |z| + |w|$

Proof:

1. Since $z \cdot \bar{z}$ is always non-negative, the square root of it will also be non-negative.
2. $|\bar{z}| = (\bar{z} \cdot \bar{\bar{z}})^{\frac{1}{2}} = (\bar{z} \cdot z)^{\frac{1}{2}} = (z \cdot \bar{z})^{\frac{1}{2}} = |z|$
3. $|z \cdot w| = ((z \cdot w) \cdot (z \bar{\cdot} w))^{\frac{1}{2}} = (z \cdot \bar{z})^{\frac{1}{2}} * (w \cdot \bar{w})^{\frac{1}{2}} = |z||w|$
4. $|z| = (a^2 + b^2)^{\frac{1}{2}}$, and this is always larger than $|\mathrm{Re}(z)| = (a^2)^{\frac{1}{2}} = a$
5. $|z + w| = ((a + c)^2 + (b + d)^2)^{\frac{1}{2}}$, and this is always smaller than $|z| + |w| = (a^2 + b^2)^{\frac{1}{2}} + (c^2 + d^2)^{\frac{1}{2}}$

Theorem:

If we have complex numbers $a_1, ...a_n$ and $b_1, ...b_n$, then $|\sum_{j=1}^{n} a_j b_j|^2 \leq |\sum_{j=1}^{n} |a_j|^2 \sum_{j=1}^{n} |b_j|^2$

Proof:

Let $A = \sum |a_j|^2$, $B = \sum |b_j|^2$ and $C = \sum a_j b)j$

Then, start with

$$\sum |Ba_J - Cb_j|^2 = \sum (Ba_j - Cb_j)(B\bar{a}_j - \bar{C}\bar{b}_j)$$

by the property that $z \cdot \bar{z} = |z|^2$

Then, we can multiply to get

$$B^2 \sum |a_j|^2 - B\bar{C} \sum a_j \bar{b}_j - BC \sum \bar{a}_j b_j + |C|^2 \sum |b_j|^2$$

We cancel the two BC terms (we can for some reason), to get

$$B^2 A - B|C|^2$$

$$B(AB - |C|^2)$$

Since we know all the terms in the initial sum are nonnegative, our result can't be negative either. Since B is postitive, this implies that $AB \geq |C|^2$ which is what we want.