3.5: Transpositions and Alternating Groups

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The elements of S_n are the possible permutations of a set of size n.

Def: Each element of S_n can be written as a sequence of cycles of size 2, called **transpositions**. Imagine a cycle $(a_1, a_2, a_3, ..., a_n)$. This moves element number a_1 to element number a_2, a_2 to a_3, and so on. We can describe this with the following sequence of transpositions (read right-to-left): $(a_1 a_n)(a_1 a_{n-1})...(a_1 a_3)(a_1 a_2)$. In this sequence, a_1 is used as a placeholder, and successive elements are moved with their previous element, in a_1.

Ex: $\sigma = (1\ 12\ 8\ 10\ 4)(2\ 13)(5\ 11\ 7)(6\ 9)$ can be written as $(1\ 4)(1\ 10)(1\ 8)(1\ 12)(2\ 13)(5\ 7)(5\ 11)(6\ 9)$

0.1 The Alternating Group

Let $x_1, ..., x_n$ be independent variables, and let Δ be a polynomial defined as

$$\Delta = \prod_{1 \leq i < j \leq n} (x_i - x_j)$$

For example, when $n = 4$, we get $\Delta = (x_1 - x_2)(x_1 - x_3)(x_1 - x_4)(x_2 - x_3)(x_2 - x_4)(x_3 - x_4)$. Notice how the second term's number is always greater than the first term's number.

Now, lets define a function $\sigma(\Delta)$, which takes in a delta function and permutes each number according to an element of S_n. For example, if we chose $(1, 2, 3, 4)$ and our polynomial from above, we get $\sigma(\Delta) = (x_2 - x_3)(x_2 - x_4)(x_2 - x_1)(x_3 - x_4)(x_3 - x_1)(x_4 - x_1)$, as you can see, the number of every variable gets mapped to a new number according to our permutation.

However, now some of the first terms are larger than the second terms, for example, in our example, $(x_2 - x_1), (x_3 - x_1), (x_4 - x_1)$ are now all out of order. To fix this, we can factor out a minus sign and get $-(x_1 - x_2), -(x_1 - x_3), -(x_1 - x_4)$. We then multiply out these minus signs and get that the overall sign of $\sigma(\Delta)$ is now -1, but the individual terms haven't changed.

As a matter of fact, no matter what permutation you choose for σ, the result of $\sigma(\Delta) = \pm\Delta$.

For each σ in S_n, lets define a function that tells us if $\sigma(\Delta)$ is positive or negative. We will call this function $\epsilon(\sigma)$, or the sign function.

Def:

1. $\epsilon(\sigma)$ is called the sign of σ
2. σ is called an even permutation if $\epsilon(\sigma) = 1$ and an odd permutation if $\epsilon(\sigma) = -1$

Prop: The map $\epsilon : S_n \to \{1, -1\}$ is a homomorphism.

Proof: Let τ, σ be elements of S_n. Then, $\epsilon(\tau\sigma) = \tau \cdot \sigma(\Delta) = \prod_{1 \leq i < j \leq n}(x_{\tau\sigma(i)} - x_{\tau\sigma(j)})$. Lets evaluate just $\sigma(\Delta)$ first, and this will result in Δ, but with k factors of the form $(x_j - x_i)$, which we will flip. The end result is $\epsilon(\sigma)\Delta$, with all the values of Δ in order as they were originally. Then, we evaluate τ, and order the variables again, to get $\epsilon(\tau)\epsilon(\sigma)\Delta$, showing that ϵ is a homomorphism.

Prop: Transpositions are odd permutations and ϵ is a surjective homomorphism.

Proof: A non-trivial group S_n will have