Rudin Chapter 3: Subsequences

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Definition: (Subsequences)

Given a sequence $\{p_n\}$, if we have another sequence $\{n_i\}$ of strictly increasing positive integers, so $n_1 < n_2 < n_3...$, then the sequence $\{p_{n_i}\}$ is called a ∼≊∼∥≊⋉ of $\{p_n\}$. If $\{p_{n_i}\}$ converges, then the number it converges to is called the **subsequential limit** of $\{p_{n_i}\}$

Theorem:

A sequence converges if and only if every subsequence of that sequence converges.

Proof:

Suppose we had a sequence $\{p_n\}$ and a subsequence of $\{p_n\}$ called $\{p_{n_i}\}$.

For the forward proof, we do a proof by contradiction. Suppose $\{p_n\}$ converged to p but $\{p_{n_i}\}$ did not converge. Then, since $\{p_{n_i}\}$ did not converge, there must be some $\epsilon > 0$ where we could not find an I where all $i > I$ had the property that $d(p, p_{n_i}) < \epsilon$, so in other words, there was no element in $\{p_{n_i}\}$ such that all successive elements were less than ϵ away from p. However, since $\{p_n\}$ converges, this property must exist for every ϵ in relation to elements of $\{p_n\}$, and since elements of $\{p_{n_i}\}$ are elements of $\{p_n\}$ this property must exist for them as well. Therefore, we have a contradition, in the sense that we must be able to find some I for which this property holds since $\{p_n\}$ is convergent, but we can find no such I since $\{p_{n_i}\}$ does not converge.

For the reverse proof, suppose that every $\{p_{n_i}\}$ converges to p. Then, we take some set of subsequences of $\{p_n\}$ where their union is $\{p_n\}$, and for every ϵ, we take the I where the property $i > I$ is $d(p_{n_i}, p) < \epsilon$ holds, and take the maxmimum of all these I. Then, the same property will hold true for the union, $\{p_n\}$.

Theorem:

If $\{p_n\}$ is a sequence in a compact metric space X, then some subsequence of $\{p_n\}$ must converge.

Every bounded sequence in $\mathbb{R}^k$ contains a convergent subsequence

Proof:

For the first part, we have two scenarios. If the range of $\{p_n\}$ is finite, meaning the sequence hits a finite amount of elements of X, then since the sequence itself is infinite, we must hit some element in the range an infinite number of times, so we can construct a sequence consisting of only hits on that element.

If the range is infinite, then the range is both infinite and compact (it is a subset of a compact space so it is compact with respect to X). This means that it has some limit point p, so we can just pick elements from successively smaller neighborhoods to make our subsequence.

Since every bounded subset of $\mathbb{R}^k$ can be contained within a compact subset of $\mathbb{R}^k$, the above applies.

Theorem:

The subsequential limits of a sequence $\{p_n\}$ in a metric space X form a closed subset of X.

Proof:

A closed set is a set which contains all of its limit points. Suppose our set of subsequential limits, E, has some limit point

p. We want to show that p is in E. Since p is a limit point of E, there is some x in E so that $d(p, x) < \frac{\epsilon}{2}$. Since x is the point of convergence for some subsequene, there is some p_n where $d(p_n, x) < \frac{\epsilon}{2}$, so for any ϵ, we can find a p_n less than ϵ from it.