3.5: Transpositions and Alternating Groups

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October 28, 2024

The elements of S_n are the possible permutations of a set of size n.

Def: Each element of S_n can be written as a sequence of cycles of size 2, called **transpositions**. Imagine a cycle $(a_1, a_2, a_3, ..., a_n)$. This moves element number a_1 to element number a_2, a_2 to a_3, and so on. We can describe this with the following sequence of transpositions (read right-to-left): $(a_1 a_n)(a_1 a_{n-1})...(a_1 a_3)(a_1 a_2)$. In this sequence, a_1 is used as a placeholder, and successive elements are moved with their previous element, in a_1.

Ex: $\sigma = (1\ 12\ 8\ 10\ 4)(2\ 13)(5\ 11\ 7)(6\ 9)$ can be written as $(1\ 4)(1\ 10)(1\ 8)(1\ 12)(2\ 13)(5\ 7)(5\ 11)(6\ 9)$

0.1 The Alternating Group

Let $x_1, ..., x_n$ be independent variables, and let Δ be a polynomial defined as

$$\Delta = \prod_{1 \leq i < j \leq n} (x_i - x_j)$$

For example, when $n = 4$, we get $\Delta = (x_1 - x_2)(x_1 - x_3)(x_1 - x_4)(x_2 - x_3)(x_2 - x_4)(x_3 - x_4)$. Notice how the second term's number is always greater than the first term's number.

Now, lets define a function $\sigma(\Delta)$, which takes in a delta function and permutes each number according to an element of S_n. For example, if we chose $(1, 2, 3, 4)$ and our polynomial from above, we get $\sigma(\Delta) = (x_2 - x_3)(x_2 - x_4)(x_2 - x_1)(x_3 - x_4)(x_3 - x_1)(x_4 - x_1)$, as you can see, the number of every variable gets mapped to a new number according to our permutation.

However, now some of the first terms are larger than the second terms, for example, in our example, $(x_2 - x_1), (x_3 - x_1), (x_4 - x_1)$ are now all out of order. To fix this, we can factor out a minus sign and get $-(x_1 - x_2), -(x_1 - x_3), -(x_1 - x_4)$. We then multiply out these minus signs and get that the overall sign of $\sigma(\Delta)$ is now -1, but the individual terms haven't changed.

As a matter of fact, no matter what permutation you choose for σ, the result of $\sigma(\Delta) = \pm\Delta$.

For each σ in S_n, lets define a function that tells us if $\sigma(\Delta)$ is positive or negative. We will call this function $\epsilon(\sigma)$, or the sign function.

Def:

1. $\epsilon(\sigma)$ is called the sign of σ

2. σ is called an even permutation if $\epsilon(\sigma) = 1$ and an odd permutation if $\epsilon(\sigma) = -1$

Prop: The map $\epsilon : S_n \to \{1, -1\}$ is a homomorphism.

Proof: Let τ, σ be elements of S_n. Then, $\epsilon(\tau\sigma) = \tau \cdot \sigma(\Delta) = \prod_{1 \leq i < j \leq n}(x_{\tau\sigma(i)} - x_{\tau\sigma(j)})$. Lets evaluate just $\sigma(\Delta)$ first, and this will result in Δ, but with k factors of the form $(x_j - x_i)$, which we will flip. The end result is $\epsilon(\sigma)\Delta$, with all the values of Δ in order as they were originally. Then, we evaluate τ, and order the variables again, to get $\epsilon(\tau)\epsilon(\sigma)\Delta$, showing that ϵ is a homomorphism.

Prop: Transpositions are odd permutations and ϵ is a surjective homomorphism.

> **Proof:** Let $n = 4$, $\sigma = (1234)$, $\tau = (432)$ and so $\tau\sigma = (1324)$.
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> Then, $(\tau\sigma)(\Delta) = (1324)\Delta = (x_3 - x_4)(x_3 - x_2)(x_3 - x_1)(x_4 - x_2)(x_4 - x_1)(x_2 - x_1) = (-1)^5\Delta$
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> On the other hand, $\tau(\sigma\Delta) = (x_{\tau(2)} - x_{\tau(3)})(x_{\tau(2)} - x_{\tau(4)})(x_{\tau(2)} - x_{\tau(1)})(x_{\tau(3)} - x_{\tau(4)})(x_{\tau(3)} - x_{\tau(1)})(x_{\tau(4)} - x_{\tau(1)}) = (x_3 - x_4)(x_3 - x_2)(x_3 - x_1)(x_4 - x_2)(x_4 - x_1)(x_2 - x_1) = (-1)^5(\Delta)$
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> Now, lets take the case of the transpose $(1, 2)$. $\epsilon(1, 2) = -1$, because it flips only one factor, $(x_1 - x_2)$. Now note that $(i, j) = \lambda(1, 2)\lambda$. Since we have two lambdas, their sign cancels, and we just have $\epsilon(i, j) = \epsilon(1, 2) = -1$.

Definition: (Alternating Groups) The alternating group A_n is the kernel of the homomorphism ϵ, or the set of all even permutations. The order of $A_n = \frac{1}{2}|S_n| = \frac{1}{2}n!$

Proposition: A permutation σ is odd if and only if the number of cycles of even length in its decomposition is odd

Proof: An odd cycle times an odd cycle is an even cycle, and an even cycle times an odd cycle is odd. A cycle of size m decomposes into $m-1$ transposes, which are odd, so an even length cycle has an odd parity. An even number of even cycles results in an even permutation, so the number of even cycles must be odd for the permutation to be odd.