Rudin Chapter 2: Perfect Sets

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July 2, 2025

Definition: (Perfect Sets)

A **perfect set** is a set where all the points in the set are limit points of the set

Theorem:

If P is a nonempty perfect set in $\mathbb{R}^k$, then P is uncountable.

Proof:

Since P has limit points (it's nonempty and a perfect set so there is at least one element in P, which is a limit point of P since P is perfect), P must have an infinite number of elements. If it didn't, for every point in P we could find a distance from a point in P where there were no other elements of P, which means that that P would have no limit points.

Now we do proof by contradiction. Suppose P was conuntable, and map points of P to natural numbers, so we have $p_1, p_2, p_3, ...$ Now let V_1 be any neighborhood of p_1, so it is the set of all points x where $|p_1 - x| < r$ for some r, and the closure of V_1, $\bar{V_1}$ is the set of all points y where $|p_1 - y| \leq r$.

Also suppose we choose the V_ns such that $V_n \cap P$ is nonempty. Since every point of P is also a limit point of P, there is a neighborhood of p_{n+1}, called V_{n+1}, where $V_{n+1} \subset V_n$, p_n is not in V_{n+1}, and $V_{n+1} \cap P$ is nonempty (we can find a neighborhood around V_{n+1} that is in V_n, doesn't contain p_n, and overlaps with P).

Put $K_n = \bar{V_n} \cap P$. Since $\bar{V_n}$ is closed and bounded, it is also compact. Since x_n is not in K_{n+1}, each subsequent K_n excludes at least another point of P, so the intersection of all of these is empty. But each K_n is nonempty, which is a contradiction of our earlier theorem that the intersection of nonempty nested sets is nonempty. This is a contradiction, so P must be uncountable.

Corollary:

Every interval $[a, b]$ is uncountable, and the real numbers are uncountable.

Example: (The Cantor Set)

We will construct a set which is perfect in $\mathbb{R}$ but has no segment (a segment is like $(2, 3)$ but an interval is like $[2, 3]$).

Let $E_0 = [0, 1]$. Now, remove the segment $(\frac{1}{3}, \frac{2}{3})$, and let $E_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$. Remove the middle thirds of these intervals, and repeat, so $E_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{3}{9}] \cup [\frac{4}{9}, \frac{5}{9}] \cup [\frac{6}{9}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$

So we have a sequence of compact sets (the Es are closed and bounded) such that $E_1 \supset E_2 \supset E_3 \supset ...$, and that E^n is the collective union of 2^n intervals, each with length $\frac{1}{3^n}$.

The set

$$P = \bigcap_{n=1}^{\infty} E_n$$

is the **Cantor set**. It is compact, since it is a subset of E_0 which is compact, and since it is an intersection of compact sets where the intersection of any finite subcollection is nonempty, P is nonempty as well.

No segment of the form $(\frac{3k+1}{3^m}, \frac{3k+2}{3^m})$ has any points in common with P, if k, m are positive integers (as these are the segments we cut out).

Every segment contains some subsegment of the form above, so P contains no segment. To show that P is perfect, we need to show that P has no isolated points. Let x be in P, and let S be some segment (a, b) containing x. Let I_n be the specific interval of E_n which contains x, and make sure that n is large enough so that $I_n \subset S$. Then let x_n be an endpoint of I_n where $x_n \neq x$. Then, x_n is in P. Since if x is in P, there is always some x_n present and arbitrarily close to x, for any neighborhood of x, there is always an x_n which is in P, so x is a limit point of P, and P is perfect.