Rudin Chapter 3: Summation by Parts

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Theorem:

Given two sequences $\{a_n\}$ and $\{b_n\}$. We write A_n to denote the nth partial sum of $\{a_n\}$, and $A_{-1} = 0$. Then, if $0 \geq q \geq p$, we get

$$\sum_{n=p}^{q} a_n b_n = \sum_{n=p}^{q-1} A_n(b_n - b_{n+1}) + A_q b_q - A_{p-1} b_p$$

Proof:

We start with

$$\sum_{n=p}^{q} a_n b_n$$

Since $a_n = A_n - A_{n-1}$, we get

$$\sum_{n=p}^{q} (A_n - A_{n-1}) b_n$$

We then split the sum to get

$$\sum_{n=p}^{q} A_n b_n - \sum_{n=p} A_{n-1} b_n$$

and we can reindex the right sum to get

$$\sum_{n=p}^{q} A_n b_n - \sum_{n=p-1}^{q-1} A_n b_{n+1}$$

We now break out the last case of the left sum to get

$$\sum_{n=q}^{q-1} A_n b_n + A_q b_q - \sum_{n=p-1}^{q-1} A_n b_{n+1}$$

and we break out the first case of the right sum to get

$$\sum_{n=q}^{q-1} A_n b_n + A_q b_q - \sum_{n=p}^{q-1} A_n b_{n+1} - A_{p-1} b_q$$

Now that both of our sums have the correct indices, we can combine them to yield

$$\sum_{n=p}^{q} a_n b_n = \sum_{n=p}^{q-1} A_n(b_n - b_{n+1}) + A_q b_q - A_{p-1} b_p$$

Theorem:

If the following are true:

1. the partial sums of $\{a_n\}$ form a bounded sequence

2. $b_0 \geq b_1 \geq b_2 ...$
3. $\lim_{n \to \infty} b_n = 0$

then $\sum a_n b_n$ converges.

Proof:

We will use the Cauchy criterion.

Choose an M that is greater than or equal to all $|A_n|$. This is possible since the partial sums are bounded. Then, given $\epsilon > 0$, it is possible to choose N such that $b_N \leq \frac{\epsilon}{2M}$. For all p, q where $N \leq p \leq q$, we have

$$|\sum_{n=p}^{q} a_n b_n| = |\sum_{n=p}^{q-1} A_n(b_n - b_{n+1}) + A_q b_q - A_{q-1} b_p$$

Since M is greater than all A_n, we know that

$$M|\sum_{n=p}^{q-1}(b_n - b_{n+1}) + b_q + b_p$$

is greater than the prior (it is the result of "factoring" out all of the A_n and A_{n-1}). The above is equal to

$$2Mb_p \leq 2Mb_N \leq \epsilon$$

so the Cauchy criterion is fulfilled.

Theorem:

Suppose $\{c_n\}$ has the following proeprties:

1. $|c_1| \geq |c_2| \geq ...$
2. $c_{2m-1} \geq 0$, $c_{2m} \leq 0$ for all m
3. $\lim_{n \to \infty} c_n = 0$

This series is an **alternating series** and converges.

Proof:

Just apply the previous theorem, letting $a_n = (-1)^{n+1}$, $b_n = |c_n|$

Theorem:

Suppose the radius of convergence of the power series $\sum c_n z^n$ is 1, and suppose $c_0 \geq c_1 \geq c_2 ...$, and $\lim_{n \to \infty} c_n = 0$. Then, $\sum c_n z^n$ converges on every point of the complex circle $|z| = 1$ except for maybe $z = 1$

Proof:

Let $a_n = c_n$ and $b_n = z^n$. Then, $|B_n|$ forms a bounded sequence since $|\sum_{m=0}^{n} z^m| = |\frac{1-z^{n+1}}{1-z}|$ is bounded above by $|\frac{2}{1-z}|$, if $z \neq 1$ and $|z| = 1$, and so the hypotheses of the prior theorem are fulfilled.