Rudin Chapter 4: Continuous Functions

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Definition: (Continuity)

Suppose X and Y are metric spaces, $E \subset X$, $p \in E$, and $f : E \to Y$. Then, f is **continuous** at p if:

For every $\epsilon > 0$, there is a $\delta > 0$ such that $d_y(f(x), f(p)) < \epsilon$ for all points x where $d_x(x, p) < \delta$.

Basically, f is continuous at p if for all ϵ, there is some neighborhood of points around p in E that map to within ϵ of $f(p)$.

Theorem:

If we also assume that p is a limit point of E, then $f(x)$ is continuous at p if and only if $\lim_{x \to p} f(x) = f(p)$

Proof:

The definitions are basically identical.

Theorem:

Suppose X, Y, Z are metric spaces, $E \subset X$, $f : E \to Y$, $g : f(E) \to Z$, and $h : E \to Z$ where $h(x) = g(f(x))$. Then, if f is continuous at p and g is continuous at $f(p)$, h is continuous at p as well.

Proof:

Choose some ϵ around $h(p)$. Then, since g is continuous, we can find some δ around $f(p)$ that fulfills the criteria. We can then put that as our new ϵ, and find a δ around p that fulfills the continuity criteria for h.

Theorem:

A mapping $f : X \to Y$ is continuous on X if and only if $f^{-1}(V)$ is open in X for every open set V in Y.

Proof:

Remember that open sets in metric spaces are neighborhoods. We do the forward proof first. Suppose $f^{-1}(V)$ is open in X for every open set V in Y, but that f was not continuous. Then, for some p and some $\epsilon > 0$, there is no δ where the neighborhood of radius δ at p has the property that all of its elements map to within ϵ of $f(p)$. If we let the open set with radius ϵ centered at $f(p)$ be V, then $f^{-1}(V)$ cannot be open, as if it was open, it would form a neighborhood and we could select a valid δ so that f is continuous. Hence, a contradiction.

For the reverse proof, suppose f is continuous but there is at least one open set V that did not have an open preimage. However, we could choose ϵ to be the radius of this neighborhood, and see that there is no δ for which all the elements within δ of the preimage of the.center of V map to within ϵ of the center of V, violating our continuity criterion.

Theorem:

Let f, g be continuous complex functions on a metric space X. Then, $f + g$, fg and $\frac{f}{g}$ are continuous

Proof:

Just extrapolate the limit example to all points on the function.

Theorem:

Let $f_1 ... f_k$ be functions mapping a metric space X to the reals, and let $\mathbf{f}(x)$ map X to $\mathbb{R}^k$ by making an n-tuple like $(f_1(x), f_2(x), ... f_k(x))$. Then, if all $f_1 ... f_k$ are continuous, so is $\mathbf{f}$

Proof:

Every open set in $\mathbb{R}^k$ is a k-tuple, and for each individual slice of the k-tuple, the respective f_k inverse maps it to an open set by virtue of continuity. Therefore, the entire preimage of that k-tuple is the union of these open sets, which is itself an open set.