12.5: Bernoulli's Equation

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January 17, 2025

To derive Bernoulli's equation, lets try to find the work done on some fluid in a pipe.

Theorem: Change in work of a fluid in a pipe is given by

$$dW = dV(p_1 - p_2)$$

Proof: Consider fluid at two points. Over some small time interval dt, the fluid at point 1 moves

$$ds_1 = v_1 dt$$

and the fluid at point 2 moves

$$ds_2 = v_2 dt$$

Since the fluid is incompressible,

$$ds_1 A_1 = ds_2 A_2$$

If the fluid is not viscous, meaning there is no friction, then the only elements doing work are gravity and pressure.

If the pressure at 1 is p_1 and the pressure at 2 is p_2, then the force at 1 is $p_1 A$ and the force at 2 is $p_2 A$. Therefore, the total work done on the fluid is

$$dW = p_1 A ds_1 - p_2 A ds_2$$

since work is force times distance, but the force at point 2 opposes the displacement of the fluid at point 1.

Since $Ads_1 = dV$, we substitute to get our final equation

Since gravity is conservative, it doesn't change total energy, so we disregard it.

We can also calculate kinetic energy.

Theorem: Change in kinetic energy of a fluid is given by

$$dK = \frac{1}{2}\rho dV(v_2^2 - v_1^2)$$

Proof: Change in kinetic energy is given by

$$\frac{1}{2}m(v_2^2 - v_1^2)$$

and the mass of a fluid is equal to $\rho A_1 ds_1 = \rho dV_1$, so we get

$$dK = \frac{1}{2}\rho dV(v_2^2 - v_1^2)$$

Theorem: Change in gravitational potential energy of a fluid is given by

$$dU = \rho dV g(y_2 - y_1)$$

Proof: Do the same substitution for mass as before.

Theorem: Bernoulli's Equation is given by:

$$p_1 - p_2 = \frac{1}{2}\rho(v_2^2 - v_1^2) + \rho g(y_2 - y_1)$$

Proof: The change in work can be described via this formula:

$$dW = dK + dU$$

and plugging in our values from before we get

$$(p_1 - p_2)dV = \frac{1}{2}\rho dV(v_2^2 - v_1^2) + \rho dV g(y_2 - y_1)$$

and cancelling the dV yields our final equation.